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Paul-Emile Paradan

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# SPIN-QUANTIZATION COMMUTES WITH REDUCTION

PAUL-EMILE PARADAN

ABSTRACT. In this paper, we prove that the “quantization commutes with reduction” phenomenon of Guillemin-Sternberg [10] applies in the context of the metaplectic correction.

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## 1. INTRODUCTION

Let  $K$  be a compact connected Lie group with Lie algebra  $\mathfrak{k}$ . An Hamiltonian  $K$ -manifold  $(M, \omega, \Phi)$  is Spin-prequantized if  $M$  carries an equivariant  $\text{Spin}^c$  structure  $P$  with determinant line bundle being a Kostant-Souriau line bundle over  $(M, 2\omega, 2\Phi)$ . Let  $\mathcal{D}_P$  be the  $\text{Spin}^c$  Dirac operator attached to  $P$ , where  $M$  is oriented by its symplectic form. The Spin quantization of  $(M, \omega, \Phi)$  corresponds to the equivariant index of the elliptic operator  $\mathcal{D}_P$ , and is denoted

$$\mathcal{Q}_{\text{spin}}^K(M) \in R(K).$$

Let  $\widehat{A}(M)(X)$  be the equivariant A-genus class: it is an equivariant analytic function from a neighborhood of  $0 \in \mathfrak{k}$  with value in the algebra of differential forms on  $M$ . The Atiyah-Segal-Singer index theorem [6] tell us that

$$(1.1) \quad \mathcal{Q}_{\text{spin}}^K(M)(e^X) := \int_M e^{i(\omega + \langle \Phi, X \rangle)} \widehat{A}(M)(X)$$

for  $X \in \mathfrak{k}$  small enough. It shows in particular that  $\mathcal{Q}_{\text{spin}}^K(M) \in R(K)$  does not depend of the choice of the Spin-prequantum data.

This notion of Spin-quantization is closely related to the notion of *metaplectic correction*. Suppose that  $(M, \omega, \Phi)$  carries a Kostant-Souriau line bundle  $L_\omega$ , and that the bundle of half-forms  $\kappa_J^{1/2}$  associated to an invariant almost complex structure  $J$  is well defined. In this case,  $(M, \omega, \Phi)$  is Spin-prequantized by the  $\text{Spin}^c$ -structure defined by  $J$  and twisted by the line bundle  $L_\omega \otimes \kappa_J^{1/2}$ . The crucial

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point here is that the corresponding Spin-quantization of  $(M, \omega, \Phi)$  does not depend of the choice of the almost complex structure. Note that the existence of the bundle of half-form  $\kappa_J^{1/2}$  is equivalent to the existence of a Spin structure on  $M$  [14].

The purpose of this article is to compute geometrically the multiplicities of  $\mathcal{Q}_{\text{spin}}^K(M) \in R(K)$  in a way similar to the famous “quantization commutes with reduction” phenomenon of Guillemin-Sternberg [10, 16, 17, 24, 18, 25, 12, 26, 21]. This question was partially resolved in a previous paper [19] under the condition that the infinitesimal stabilizers of the  $K$ -action are abelian. C. Teleman also obtained some results [23][Proposition 3.10] in the algebraic setting.

The striking difference with the standard Guillemin-Sternberg phenomenon is the *rho shift* that we explain now. Let  $T$  be a maximal torus of  $K$  with Lie algebra  $\mathfrak{t} \subset \mathfrak{k}$ . Let  $\mathfrak{t}_+^* \subset \mathfrak{t}^*$  be the closed Weyl chamber. We will look at  $\mathfrak{t}_+^*$  as a disjoint union of its open faces, the maximal one being its interior  $(\mathfrak{t}_+^*)^\circ$ . Let  $\rho \in (\mathfrak{t}_+^*)^\circ$  be the half sum of the positive roots. At each open face  $\tau$  of  $\mathfrak{t}_+^*$ , we associate the term  $\rho_\tau$  which is the half sum of the positive roots which are orthogonal to  $\tau$ . We note that  $\rho - \rho_\tau \in \tau$  is to the orthogonal projection of  $\rho$  on  $\tau$ .

For any  $\xi \in \mathfrak{t}_+^*$  and any face  $\tau$  containing  $\xi$  in its closure, we consider the *shifted symplectic reduction*

$$M_\xi^\tau := \Phi^{-1}(\xi + \rho - \rho_\tau)/K_\tau$$

where  $K_\tau$  is the common stabiliser of points in  $\tau$ . Note that  $\xi + \rho - \rho_\tau \in \tau$  when  $\xi \in \bar{\tau}$ .

We are particularly interested to the smallest face  $\sigma$  of the Weyl chamber so that the Kirwan polytope  $\Delta(M) := \Phi(M) \cap \mathfrak{t}_+^*$  is contained in the closure of  $\sigma$ . It is not hard to see that the Spin-prequantum data on  $(M, \omega, \Phi)$  descends to the shifted symplectic reduction  $M_\mu^\sigma$  when  $\mu$  is a dominant weight belonging to  $\bar{\sigma}$ . Then  $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma) \in \mathbb{Z}$  is naturally defined when  $\mu + \rho - \rho_\sigma$  is a regular value of the moment map. In general, the number  $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma)$  is defined by shift-desingularization (see Section 2.4).

By definition  $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma)$  vanishes when  $\mu + \rho - \rho_\sigma \notin \Delta(M)$ , but in fact we can strengthen this vanishing property:  $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma) = 0$  if  $\mu + \rho - \rho_\sigma$  does not belong to the *relative interior* of the Kirwan polytope  $\Delta(M)$ .

Recall that the irreducible representation  $V_\mu^K$  of  $K$  are parametrized by their highest weight  $\mu \in \hat{K} \subset \mathfrak{t}_+^*$ .

The main result of this paper is the following

**Theorem 1.1.** *Let  $(M, \omega, \Phi)$  be a compact Spin-prequantized Hamiltonian  $K$ -manifold. Let  $\sigma$  be the smallest face of the Weyl chamber so that  $\Delta(M) \subset \bar{\sigma}$ . We have*

$$\mathcal{Q}_{\text{spin}}^K(M) = \sum_{\mu \in \hat{K} \cap \bar{\sigma}} \mathcal{Q}_{\text{spin}}(M_\mu^\sigma) V_\mu^K.$$

Let us give some ideas about the proof. The representation  $V_\mu^K$  is equal to the Spin-quantization of the coadjoint orbit  $\mathcal{O}_\mu := K \cdot (\mu + \rho)$ . Then the shifting trick tells us that the multiplicity  $m_\mu$  of  $V_\mu^K$  in  $\mathcal{Q}_{\text{spin}}^K(M)$  is equal to

$$[\mathcal{Q}_{\text{spin}}^K(M) \otimes (V_\mu^K)^*]^K = [\mathcal{Q}_{\text{spin}}^K(M \times \bar{\mathcal{O}}_\mu)]^K,$$

where  $\overline{\mathcal{O}}_\mu$  is the coadjoint orbit with the opposite symplectic structure. As we did in [18, 19], we study the expression  $[\mathcal{Q}_{\text{spin}}^K(M \times \overline{\mathcal{O}}_\mu)]^K$  by localizing the Riemann-Roch character on the critical points of the square of the moment map

$$\Phi_\mu : M \times \overline{\mathcal{O}}_\mu \rightarrow \mathfrak{k}^*.$$

Here our treatment differs depending on whether the Kirwan polytope  $\Delta(M)$  intersects the interior of the Weyl chambers or not (i.e.  $\sigma = \mathfrak{t}_+^*$  or not).

When  $\sigma = \mathfrak{t}_+^*$ , we show that the multiplicity  $m_\mu$  is calculated using the Riemann-Roch character localized near the zero level set of the moment map  $\Phi_\mu$ . This case is (more or less) treated in [19].

The heart of this paper is when we work out the case  $\sigma \neq \mathfrak{t}_+^*$ . We have  $\Phi_\mu^{-1}(0) = \emptyset$ , but we show how to compute  $m_\mu$  using the Riemann-Roch character localized near

$$K \cdot (N^{\rho_\sigma} \cap \Phi_\mu^{-1}(-\rho_\sigma)).$$

Here  $N^{\rho_\sigma}$  denotes the submanifold of  $N = M \times \mathcal{O}_\mu$  where the infinitesimal action of  $\rho_\sigma$  vanishes.

**Notations.** Throughout the paper,  $K$  will denote a compact connected Lie group, and  $\mathfrak{k}$  its Lie algebra. We let  $T$  be a maximal torus in  $K$ , and  $\mathfrak{t}$  be its Lie algebra. The integral lattice  $\Lambda \subset \mathfrak{t}$  is defined as the kernel of  $\exp : \mathfrak{t} \rightarrow T$ , and the real weight lattice  $\Lambda^* \subset \mathfrak{t}^*$  is defined by  $\Lambda^* := \text{hom}(\Lambda, 2\pi\mathbb{Z})$ . Every  $\mu \in \Lambda^*$  defines a 1-dimensional  $T$ -representation, denoted  $\mathbb{C}_\mu$ , where  $t = \exp(X)$  acts by  $t^\mu := e^{i\langle \mu, X \rangle}$ . We fix a positive Weyl chamber  $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ . For any dominant weight  $\mu \in \widehat{K} := \Lambda^* \cap \mathfrak{t}_+^*$ , we denote by  $V_\mu^K$  the irreducible representation with highest weight  $\mu$ . We denote  $R(K)$  the representation ring of  $K$ . We denote  $R^{-\infty}(K) := \text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$  its dual. An element  $E \in R^{-\infty}(K)$  can be represented as an infinite sum  $E = \sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$ , with  $m_\mu \in \mathbb{Z}$ . The multiplicity  $m_0$  of the trivial representation is denoted  $[E]^K$ . If  $H$  is a closed subgroup of  $K$ , we have the induction map  $\text{Ind}_H^K : R^{-\infty}(H) \rightarrow R^{-\infty}(K)$  which is the dual of the restriction morphism  $R(K) \rightarrow R(H)$ . We see that  $[\text{Ind}_H^K(E)]^K = [E]^H$ .

## 2. SPIN-QUANTIZATION OF COMPACT HAMILTONIAN $K$ -MANIFOLDS

Let  $M$  be a *compact* Hamiltonian  $K$ -manifold with symplectic form  $\omega$  and moment map  $\Phi : M \rightarrow \mathfrak{k}^*$  characterized by the relation

$$(2.2) \quad \iota(X_M)\omega = -d\langle \Phi, X \rangle, \quad X \in \mathfrak{k},$$

where  $X_M(m) := \frac{d}{dt}|_{t=0} e^{-tX} \cdot m$  is the vector field on  $M$  generated by  $X \in \mathfrak{k}$ .

In the Kostant-Souriau framework [13, 22], a Hermitian line bundle  $L_\omega$  with an invariant Hermitian connection  $\nabla$  is a prequantum line bundle over  $(M, \omega, \Phi)$  if

$$(2.3) \quad \mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X \rangle \quad \text{and} \quad \nabla^2 = -i\omega,$$

for every  $X \in \mathfrak{k}$ . Here  $\mathcal{L}(X)$  is the infinitesimal action of  $X \in \mathfrak{k}$  on the sections of  $L_\omega \rightarrow M$ .  $(L_\omega, \nabla)$  is also called a Kostant-Souriau line bundle. Remark that conditions (2.3) imply, via the equivariant Bianchi formula, the relation (2.2).

**2.1. Spin-quantization: definitions.** Let  $J$  be any invariant almost complex structure on  $M$ , not necessarily compatible with the symplectic form  $\omega$ . Let

$$RR_J^K(M, -)$$

be the corresponding Riemann-Roch character [18]. We consider the complex tangent bundle  $(TM, J)$  and its complex dual  $T_{\mathbb{C}}^*M := \text{hom}_{\mathbb{C}}(TM, \mathbb{C})$ . We consider the complex line bundle

$$\kappa_J := \det T_{\mathbb{C}}^*M.$$

If  $(M, \omega, \Phi)$  is prequantized by  $L_\omega$ , a standard procedure (called the metaplectic correction in the geometric quantization literature) is to tensor  $L_\omega$  by the bundle of half-forms  $\kappa_J^{1/2}$  [27]. We may consider the equivariant index

$$(2.4) \quad \mathcal{Q}_J^K(M) := \epsilon_J RR_J^K(M, L_\omega \otimes \kappa_J^{1/2})$$

where  $\epsilon_J = \pm 1$  is the quotient of the orientations defined by  $\omega$  and by  $J$ . In Proposition 2.2 we check that  $\mathcal{Q}_J^K(M)$  has a meaning when the tensor product  $\tilde{L} = L_\omega \otimes \kappa_J^{1/2}$  is well defined (even if neither  $L_\omega$  nor  $\kappa_J^{1/2}$  exist).

The almost complex structure  $J$  defines a  $\text{Spin}^c$  structure  $P_J$  on  $M$  with determinant line bundle  $\det P_J = \kappa_J^{-1}$ . If we twist the  $\text{Spin}^c$  structure  $P_J$  by any complex line bundle  $\mathbb{L}$  we get a  $\text{Spin}^c$  structure  $P_{J, \mathbb{L}}$  with determinant line bundle

$$\det P_{J, \mathbb{L}} = \mathbb{L}^2 \otimes \kappa_J^{-1}.$$

See [14, 19].

We make the following basic observation.

**Proposition 2.1.** *Let  $(M, \omega, \Phi)$  be a Hamiltonian  $K$ -manifold. The following assertions are equivalent:*

- a) *For any invariant complex structure  $J$  there exists a  $K$ -equivariant line bundle  $\tilde{L}$  such that  $\tilde{L}^2 \otimes \kappa_J^{-1}$  is a prequantum line bundle over  $(M, 2\omega, 2\Phi)$ .*
- b) *There exist an invariant complex structure  $J$  and a  $K$ -equivariant line bundle  $\tilde{L}$  such that  $\kappa_J^{-1} \otimes \tilde{L}^2$  is a prequantum line bundle over  $(M, 2\omega, 2\Phi)$ .*
- c) *There exists an equivariant  $\text{Spin}^c$  structure  $P$  such that its determinant line bundle  $\det P$  is a prequantum line bundle over  $(M, 2\omega, 2\Phi)$ .*

When the previous assertions holds, we says that  $(M, \omega, \Phi)$  is Spin-prequantized, either by the  $\text{Spin}^c$ -structure  $P$ , or by the data  $(J, \tilde{L})$ .

**Proposition 2.2.** *Let  $(M, \omega, \Phi)$  be a Spin-prequantized Hamiltonian  $K$ -manifold. The equivariant index  $\mathcal{Q}_J^K(M) := \epsilon_J RR_J^K(M, \tilde{L})$  does not depend of the choice of the Spin-prequantum data  $(J, \tilde{L})$ . In fact  $\mathcal{Q}_J^K(M)$  coincides with the equivariant index of the  $\text{Spin}^c$  Dirac operator  $\mathcal{D}_P$  attached to the  $\text{Spin}^c$ -structure  $P$ .*

**Definition 2.3.** *Let  $(M, \omega, \Phi)$  be a Spin-prequantized Hamiltonian  $K$ -manifold. The Spin-quantization of  $(M, \omega, \Phi)$  is defined as the equivariant index  $\mathcal{Q}_J^K(M)$ , and is denoted*

$$\mathcal{Q}_{\text{spin}}^K(M) \in R(K).$$

*Proof of Propositions 2.1 and 2.2.* We have obviously  $a) \implies b)$ , and we get  $b) \implies c)$  by taking the  $\text{Spin}^c$  structure  $P_{J, \tilde{L}}$ . Let us prove  $c) \implies a)$ .

Let  $P$  be a  $\text{Spin}^c$ -structure on  $M$  such that its determinant line bundle  $\det P$  is a prequantum line bundle over  $(M, 2\omega)$ . Let  $\mathcal{S}_P$  be the corresponding bundle of spinors. Let  $P_J$  and  $\mathcal{S}_J$  be respectively the associated  $\text{Spin}^c$ -structure and the bundle of spinors on  $M$  associated to an invariant almost complex structure  $J$  on  $M$ . Since  $\mathcal{S}_P, \mathcal{S}_J$  are irreducible clifford modules, we have

$$(2.5) \quad \mathcal{S}_P \simeq \mathcal{S}_J \otimes \tilde{L}$$

where  $\tilde{L}$  is the line bundle defined by  $\tilde{L} := \text{hom}_{\text{cl}}(\mathcal{S}_J, \mathcal{S}_P)$ . From (2.5) we get that the line bundle

$$\begin{aligned} \det P &= \tilde{L}^2 \otimes \det P_J \\ &= \tilde{L}^2 \otimes \kappa_J^{-1}. \end{aligned}$$

is a prequantum line bundle over  $(M, 2\omega, 2\Phi)$ .

Let  $P$  be the  $\text{Spin}^c$  structure attached to a data  $(J, \tilde{L})$ . The symplectic orientation on  $M$  defines a decomposition on the bundle of spinors,  $\mathcal{S}_P = \mathcal{S}_P^+ \oplus \mathcal{S}_P^-$ , and the corresponding  $\text{Spin}^c$  Dirac operator  $\mathcal{D}_P$  maps  $\Gamma(\mathcal{S}_P^+)$  to  $\Gamma(\mathcal{S}_P^-)$ .

On the other hand the almost complex structure on  $M$  gives the decomposition  $\wedge T^*M \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j} T^*M$  of the bundle of differential form. The corresponding bundle of spinors is  $\mathcal{S}_J := \wedge^{0,\bullet} T^*M$  and the complex orientation induces the splitting  $\mathcal{S}_J = \mathcal{S}_J^+ \oplus \mathcal{S}_J^-$  with  $\mathcal{S}_J^\pm := \wedge^{0,\text{even/odd}} T^*M$ . The Dolbeault Dirac operator  $\bar{\partial}_{\tilde{L}} + \bar{\partial}_{\tilde{L}}^*$  maps  $\Gamma(\mathcal{S}_J^\pm \otimes \tilde{L})$  to  $\Gamma(\mathcal{S}_J^\mp \otimes \tilde{L})$ , and the Riemann-Roch character  $RR_J^K(M, \tilde{L})$  is defined as the equivariant index of the elliptic operator

$$\bar{\partial}_{\tilde{L}} + \bar{\partial}_{\tilde{L}}^* : \Gamma(\mathcal{S}_J^+ \otimes \tilde{L}) \longrightarrow \Gamma(\mathcal{S}_J^- \otimes \tilde{L})$$

If  $\epsilon_J = \pm 1$  is the quotient of the orientations defined by  $\omega$  and by  $J$ , one has that

$$\mathcal{S}_P^\pm = \mathcal{S}_J^{\pm \epsilon_J} \otimes \tilde{L}.$$

Hence  $\mathcal{Q}_J^K(M) = \epsilon_J RR_J^K(M, \tilde{L})$  is defined as the equivariant index of the Dolbeault Dirac operator  $\bar{\partial}_{\tilde{L}} + \bar{\partial}_{\tilde{L}}^*$  viewed as an elliptic operator  $\mathcal{D}_{\tilde{L}}^\pm$  from  $\Gamma(\mathcal{S}_P^+)$  to  $\Gamma(\mathcal{S}_P^-)$ .

Finally we know that  $\text{Index}^K(\mathcal{D}_P) = \text{Index}^K(\mathcal{D}_{\tilde{L}}^\pm)$  since the first order elliptic operators  $\mathcal{D}_P$  and  $\mathcal{D}_{\tilde{L}}^\pm$  have the same principal symbol [9].  $\square$

In the remaining part of this paper, we find convenient to work with the following

**Definition 2.4.** *A Hamiltonian  $K$ -manifold  $(M, \omega, \Phi)$  is Spin-prequantized by  $\tilde{L}$  if there exists an invariant almost complex structure  $J$  compatible with  $\omega$  such that  $\tilde{L}^2 \otimes \kappa_J^{-1}$  is a Kostant-Souriau line bundle over  $(M, 2\omega, 2\Phi)$ .*

We remark that  $\epsilon_J = 1$  when  $J$  is compatible with  $\omega$ . Moreover, the Riemann-Roch character  $RR_J^K(M, -)$  does not depend [18] on the choice of the compatible invariant almost complex structure  $J$ : we denote it simply by  $RR^K(M, -)$ .

Finally, when a Hamiltonian manifold  $(M, \omega, \Phi)$  is Spin-prequantized by the line bundle  $\tilde{L}$ , its Spin-quantization is defined by

$$\mathcal{Q}_{\text{spin}}^K(M) := RR^K(M, \tilde{L}).$$

**2.2. Functorial properties.** We summarize the functorial properties of  $\mathcal{Q}_{\text{spin}}$  in the next

**Proposition 2.5.** • *If  $(M, \omega, \Phi)$  is a Spin-prequantized Hamiltonian  $K$ -manifold, and  $H$  is a closed subgroup of  $K$  then the restriction of  $\mathcal{Q}_{\text{spin}}^K(M)$  to  $H$  is equal to  $\mathcal{Q}_{\text{spin}}^H(M)$ .*

• *If  $(M_j, \omega_j, \Phi_j)$  are Spin-prequantized Hamiltonian  $K_j$ -manifold, for  $j = 1, 2$ , then  $M_1 \times M_2$  is a Spin-prequantized Hamiltonian  $K_1 \times K_2$ -manifold and*

$$\mathcal{Q}_{\text{spin}}^{K_1 \times K_2}(M_1 \times M_2) = \mathcal{Q}_{\text{spin}}^{K_1}(M_1) \otimes \mathcal{Q}_{\text{spin}}^{K_2}(M_2)$$

in  $R(K_1 \times K_2) \simeq R(K_1) \otimes R(K_2)$ .

• *If  $(M, \omega_M, \Phi_M)$  and  $(N, \omega_N, \Phi_N)$  are Spin-prequantized Hamiltonian  $K$ -manifold, then  $M \times N$  is a Spin-prequantized Hamiltonian  $K$ -manifold and*

$$\mathcal{Q}_{\text{spin}}^K(M \times N) = \mathcal{Q}_{\text{spin}}^K(M) \cdot \mathcal{Q}_{\text{spin}}^K(N),$$

where  $\cdot$  denotes the product in  $R(K)$ .

• *A Spin-prequantization on  $(M, \omega, \Phi)$  induces a Spin-prequantization on  $\overline{M} := (M, -\omega, -\Phi)$ . The Spin-quantization of  $\overline{M}$  corresponds to the dual of the Spin-quantization of  $M$ :*

$$\mathcal{Q}_{\text{spin}}^K(\overline{M}) = [\mathcal{Q}_{\text{spin}}^K(M)]^*.$$

*Proof.* The first three points are direct consequences of the functorial properties of the index map. Let us prove the last point. One see that if  $(\tilde{L}, J)$  is a Spin-prequantum data for  $M$  then  $(\tilde{L}^{-1}, -J)$  is a Spin-prequantum data for  $\overline{M}$ . Then we have for  $X \in \mathfrak{k}$  small enough

$$\begin{aligned} \mathcal{Q}_{\text{spin}}^K(\overline{M})(e^X) &= \int_M e^{i(-\omega - \langle \Phi, X \rangle)} \widehat{A}(M)(X) \\ &= \overline{\int_M e^{i(\omega + \langle \Phi, X \rangle)} \widehat{A}(M)(X)} \quad [1] \\ &= \overline{\mathcal{Q}_{\text{spin}}^K(M)(e^X)}. \quad [2] \end{aligned}$$

The relation [1] is due to the fact that the differential form  $\widehat{A}(M)(X)$  has real coefficients. Since  $X \rightarrow \mathcal{Q}_{\text{spin}}^K(M)(e^X)$  are analytic functions, the identity [2] shows that  $\mathcal{Q}_{\text{spin}}^K(\overline{M})(k) = \overline{\mathcal{Q}_{\text{spin}}^K(M)(k)}$  for any  $k \in K$ . In other words the (virtual) representation  $\mathcal{Q}_{\text{spin}}^K(\overline{M})$  corresponds to the dual of the (virtual) representation  $\mathcal{Q}_{\text{spin}}^K(M)$ .  $\square$

**2.3. Spin-quantization of coadjoint orbits.** Let  $\mu \in \widehat{K}$  be a dominant weight. Let  $\sigma$  be a face of the Weyl chamber such that  $\mu \in \overline{\sigma}$ : hence the stabilizer subgroup  $K_\mu$  contains  $K_\sigma$ . We will restrict the one-dimensional representation  $\mathbb{C}_\mu$  of  $K_\mu$  to the subgroup  $K_\sigma$ .

Let  $\rho$  be half the sum of the positive roots, and let  $\rho_\sigma$  be half the sum of the positive roots which are orthogonal to  $\sigma$ . Note that  $\rho - \rho_\sigma$  belongs to  $\sigma$ , hence  $\mu + \rho - \rho_\sigma$  belongs also to  $\sigma$  for any  $\mu \in \overline{\sigma}$ . The coadjoint orbit

$$\mathcal{O}_\mu^\sigma := K \cdot (\mu + \rho - \rho_\sigma)$$

is Spin-prequantized by the compatible complex structure and the line bundle  $\tilde{L} = K \times_{K_\sigma} \mathbb{C}_\mu$ . We have

$$\begin{aligned} \mathcal{Q}_{\text{spin}}^K(\mathcal{O}_\mu^\sigma) &= RR^K(K/K_\sigma, K \times_{K_\sigma} \mathbb{C}_\mu) \\ &= V_\mu^K. \end{aligned}$$

Thanks to Proposition 2.5, we know that  $\mathcal{Q}_{\text{spin}}^K(\overline{\mathcal{O}_\mu^\sigma}) = (V_\mu^K)^*$ , where  $\overline{\mathcal{O}_\mu^\sigma}$  be the coadjoint orbit  $\mathcal{O}_\mu^\sigma$  with the opposite symplectic form.

We have seen that the same irreducible representations  $V_\mu^K$  can be realized as the Spin-quantization of the coadjoint orbits  $\mathcal{O}_\mu^\sigma$  where  $\sigma$  is a face of the Weyl chamber containing  $\mu$  in its closure.

**2.4. Spin-prequantization commutes with reduction.** We consider first the case of a Hamiltonian  $H$ -manifold  $(N, \omega, \Phi)$ , not necessarily compact, which is Spin-prequantized by  $\tilde{L}$ . We suppose that 0 is a regular value of  $\Phi$ . Let  $N_0 := \Phi^{-1}(0)/H$  be the orbifold reduced space with its canonical symplectic structure  $\omega_0$ .

**Lemma 2.6.** *The orbifold line bundle  $\tilde{\mathcal{L}}_0 := (\tilde{L}|_{\Phi^{-1}(0)})/H$  Spin-prequantizes  $(N_0, \omega_0)$ .*

*Proof.* The fiber  $\mathcal{Z} = \Phi^{-1}(0)$  is a smooth  $H$ -invariant submanifold of  $N$ . Let  $\pi : \mathcal{Z} \rightarrow \mathcal{Z}/H = N_0$  be the projection. Recall that the symplectic structure  $\omega_0$  on  $N_0$  is defined by the relation  $\pi^*(\omega_0) = \omega|_{\mathcal{Z}}$ . Let  $L_{2\omega}$  the Kostant-Souriau line bundle on  $(N, 2\omega, 2\Phi)$  such that

$$(2.6) \quad \tilde{L}^2 = L_{2\omega} \otimes \kappa_J.$$

Here  $J$  is a compatible invariant almost complex structure on  $N$ . We have  $TM|_{\mathcal{Z}} = T\mathcal{Z} \oplus J(\mathfrak{h}_{\mathcal{Z}})$  where  $\mathfrak{h}_{\mathcal{Z}} \subset T\mathcal{Z}$  is the trivial bundle given by the infinitesimal action of  $H$ . Since  $T\mathcal{Z} \simeq \pi^*(TN_0) \oplus \mathfrak{h}_{\mathcal{Z}}$  we get

$$TM|_{\mathcal{Z}} \simeq \pi^*(TN_0) \oplus \mathfrak{h}_{\mathcal{Z}} \oplus J(\mathfrak{h}_{\mathcal{Z}}).$$

Hence  $J$  induces a compatible almost complex structure  $J_0$  on  $(N_0, \omega_0)$ , such that  $(\kappa_J|_{\mathcal{Z}})/H = \kappa_{J_0}$ .

The line bundle  $L_{2\omega_0} = (L_{2\omega}|_{\mathcal{Z}})/H$  is a prequantum line bundle on  $(N_0, \omega_0)$ . Finally, if we restrict (2.6) to  $\mathcal{Z}$ , we get

$$\tilde{\mathcal{L}}_0^2 = L_{2\omega_0} \otimes \kappa_{J_0}.$$

after taking the quotient by  $H$ . We have proved that  $(J_0, \tilde{\mathcal{L}}_0)$  Spin-prequantizes  $(N_0, \omega_0)$ .  $\square$

For the rest of this section we consider a compact Hamiltonian  $K$ -manifold  $(M, \omega, \Phi)$ , that we suppose Spin-prequantized by the line bundle  $\tilde{L}$ .

Let  $\tau$  be a face of the Weyl chamber, and let  $K_\tau$  be the common stabilizer of points in  $\tau$ . Following Guillemin-Sternberg [11], we introduce the following  $K_\tau$ -invariant open subset of  $\mathfrak{k}_\tau^*$ :

$$U_\tau = K_\tau \cdot \{\xi \in \mathfrak{k}_\tau^* | K_\xi \subset K_\tau\} = K_\tau \cdot \bigcup_{\tau \subset \bar{\sigma}} \sigma.$$

By construction,  $U_\tau$  is a slice for the coadjoint action: this mean that the map  $K \times U_\tau, (k, \xi) \mapsto k \cdot \xi$  factors through an inclusion  $K \times_{K_\tau} U_\tau \hookrightarrow \mathfrak{k}^*$ .

The symplectic cross-section theorem [11] asserts that the pre-image  $Y_\tau = \Phi^{-1}(U_\tau)$  is a symplectic submanifold : we denote  $\omega_\tau$  the restriction of  $\omega$  to  $Y_\tau$ . The action of  $K_\tau$  on  $(Y_\tau, \omega_\tau)$  is Hamiltonian, where the restriction of  $\Phi$  to  $Y_\tau$  is a moment



map. Since  $\rho - \rho_\tau$  is a  $K_\tau$ -invariant element, we can use the translated moment map  $\Phi_\tau : Y_\tau \rightarrow \mathfrak{k}_\tau^*$  defined by

$$\Phi_\tau = \Phi|_{Y_\tau} - (\rho - \rho_\tau).$$

**Lemma 2.7.** *The symplectic slice  $(Y_\tau, \omega_\tau, \Phi_\tau)$  is Spin-prequantized by the line bundle  $\tilde{L}_\tau := \tilde{L}|_{Y_\tau}$ .*

*Proof.* We consider the open subset  $K \times_{K_\tau} Y_\tau$  of  $M$  and the projection  $\pi : K \times_{K_\tau} Y_\tau \rightarrow K/K_\tau$ . We can suppose that the Spin-prequantum data, when restricted to  $K \times_{K_\tau} Y_\tau$ , is given by  $(J, \tilde{L})$  where  $J$  is a compatible almost complex structure on  $K \times_{K_\tau} Y_\tau$  defined as the “sum” of the compatible almost complex structures  $J_o$  and  $J_\tau$ :  $J_o$  on  $K/K_\tau$  and  $J_\tau$  on  $Y_\tau$ .

When we restrict the identity  $L_{2\omega} = \tilde{L}^2 \otimes \kappa_J^{-1}$  to  $Y_\tau$  we get

$$(2.7) \quad L_{2\omega}|_{Y_\tau} = (\tilde{L}|_{Y_\tau})^2 \otimes \kappa_{J_\tau}^{-1} \otimes \mathbb{C}_{2(\rho - \rho_\tau)}.$$

We consider the following line bundle on  $Y_\tau$  :

$$L_{2\omega_\tau} := L_{2\omega}|_{Y_\tau} \otimes \mathbb{C}_{2(\rho - \rho_\tau)}^{-1}.$$

The relation (2.7) is then  $L_{2\omega_\tau} = (\tilde{L}|_{Y_\tau})^2 \otimes \kappa_{J_\tau}^{-1}$ . Since  $L_{2\omega_\tau}$  is a  $K_\tau$ -equivariant prequantum bundle over  $(Y_\tau, 2\omega_\tau, 2\Phi_\tau)$ , we conclude that  $(Y_\tau, \omega_\tau, \Phi_\tau)$  is Spin-prequantized by the data  $(J_\tau, \tilde{L}_\tau)$ .  $\square$

Let us consider the case where  $\tau = \sigma$  is the *smallest face* of the Weyl chamber so that moment polyhedron  $\Delta(M) := \Phi(M) \cap \mathfrak{k}_+^*$  is contained in the closure of  $\sigma$ . Then the symplectic slice  $Y_\sigma$  is equal to  $\Phi^{-1}(\sigma)$ , and the action of the subgroup  $[K_\sigma, K_\sigma]$  is trivial on it [15].

We will then consider the Hamiltonian action of the center  $Z_\sigma = Z(K_\sigma)$  on  $Y_\sigma$ . The map  $\Phi_\sigma : Y_\sigma \rightarrow \mathfrak{k}_\sigma^*$  takes values in  $\mathfrak{z}_\sigma^* = \mathbb{R}\sigma \subset \mathfrak{k}^*$  and corresponds to the moment map relative to the action of  $Z_\sigma$  on  $(Y_\sigma, \omega_\sigma)$ . We know that  $(Y_\sigma, \omega_\sigma, \Phi_\sigma)$  is Spin-prequantized by  $\tilde{L}_\sigma := \tilde{L}|_{Y_\sigma}$ .

For each dominant weights  $\mu$  which belongs to the closure of  $\sigma$ , we consider the symplectic reduction

$$\begin{aligned} M_\mu^\sigma &= \Phi^{-1}(\mathcal{O}_\mu^\sigma)/K \\ &= \Phi_\sigma^{-1}(\mu)/Z_\sigma. \end{aligned}$$

**For the rest of this section we fix a dominant weight  $\mu \in \bar{\sigma}$  such that  $\mu + \rho - \rho_\sigma \in \Delta(M)$ , and we explain how one defines the Spin-quantization of the (possibly singular) reduced spaces  $M_\mu^\sigma$ .**

Let  $\vec{\Delta} \subset \mathfrak{z}_\sigma^*$  be the rational vector subspace generated by  $\{a - b \mid a, b \in \Delta(M)\}$ . Let  $\mathfrak{z}_\sigma^\Delta \subset \mathfrak{z}_\sigma$  be the subspace orthogonal to  $\vec{\Delta}$ , and let  $Z_\sigma^\Delta \subset Z_\sigma$  be the corresponding subtorus.

**Lemma 2.8.** *The torus  $Z_\sigma^\Delta$  acts trivially on  $Y_\sigma$  and on the line bundle  $\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$ .*

*Proof.* By definition of  $\mathfrak{z}_\sigma^\Delta$ ,  $0 = d\langle \Phi_\sigma, X \rangle = -\iota(X_{Y_\sigma})\omega_\sigma$  on  $Y_\sigma$  for any  $X \in \mathfrak{z}_\sigma^\Delta$ . Hence the torus  $Z_\sigma^\Delta$  acts trivially on  $Y_\sigma$ . Let  $L_{2\omega_\sigma}$  be the Kostant-Souriau line bundle over  $(Y_\sigma, 2\omega_\sigma, 2\Phi_\sigma)$  so that  $\tilde{L}_\sigma^2 = L_{2\omega_\sigma} \otimes \kappa_{J_\sigma}$  (see Lemma 2.7). We have on the section of  $L_{2\omega_\sigma}$  the following equality of linear operators:

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle 2\Phi_\sigma, X \rangle, \quad \forall X \in \mathfrak{z}_\sigma.$$

If one takes  $X \in \mathfrak{z}_\sigma^\Delta$ , the function  $y \in Y_\sigma \mapsto \langle \Phi_\sigma(y), X \rangle$  is constant equal to  $\langle \mu, X \rangle$ . Finally

$$\mathcal{L}(X) - 2i\langle \mu, X \rangle = 0, \quad \forall X \in \mathfrak{z}_\sigma^\Delta$$

as an operator on the section of  $L_{2\omega_\sigma}$ . In other words, the torus  $Z_\sigma^\Delta$  acts trivially on  $L_{2\omega_\sigma} \otimes \mathbb{C}_{-2\mu} = (\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu})^2 \otimes \kappa_{J_\sigma}^{-1}$ . Since  $Z_\sigma^\Delta$  acts trivially on  $\kappa_{J_\sigma}$ , we conclude finally that  $Z_\sigma^\Delta$  acts trivially on the line bundle  $\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$ .  $\square$

Let  $Z'_\sigma \subset Z_\sigma^\Delta$  be another subtorus such that  $Z_\sigma = Z_\sigma^\Delta \times Z'_\sigma$ : the dual of its Lie algebra  $\mathfrak{z}'_\sigma$  is identified with  $\vec{\Delta} \subset \mathfrak{z}_\sigma^*$ . We look now at  $(Y_\sigma, \omega_\sigma)$  as a Hamiltonian  $Z'_\sigma$ -manifold with moment map

$$\Phi'_\sigma := \Phi_\sigma - \mu = \Phi|_{Y_\sigma} - (\mu + \rho - \rho_\sigma).$$

The  $Z'_\sigma$ -equivariant line bundle  $\tilde{L}'_\sigma := \tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$  Spin-prequantizes the Hamiltonian  $Z'_\sigma$ -manifold  $(Y_\sigma, \omega_\sigma, \Phi'_\sigma)$ .

If  $0 \in \vec{\Delta}$  is a regular value of  $\Phi'_\sigma$ , we know after Lemma 2.6 that the orbifold reduced space  $(M_\mu^\sigma, \omega_\mu^\sigma)$  is Spin-prequantized by the line bundle

$$\tilde{\mathcal{L}}_\mu^\sigma := \left( \tilde{L}|_{\Phi_\sigma^{-1}(\mu)} \otimes \mathbb{C}_{-\mu} \right) / Z'_\sigma,$$

and its Spin-quantization  $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma)$  is defined like in Definition 2.3. In the general case where  $0 \in \vec{\Delta}$  is not necessarily a regular value of  $\Phi'_\sigma$  we proceed by shift desingularization. For  $\varepsilon \in \vec{\Delta}$  small enough and generic we consider the orbifold reduced space

$$M_{\mu+\varepsilon}^\sigma := (\Phi'_\sigma)^{-1}(\varepsilon) / Z'_\sigma = \Phi_\sigma^{-1}(\mu + \varepsilon) / Z'_\sigma$$

and its orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma := \left( \tilde{L}|_{\Phi_\sigma^{-1}(\mu+\varepsilon)} \otimes \mathbb{C}_{-\mu} \right) / Z'_\sigma.$$

The following crucial fact is proved in Section 3.4.

**Theorem 2.9.** *The Riemann-Roch number  $RR(M_{\mu+\varepsilon}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma) \in \mathbb{Z}$  does not depend of the choice of a generic and small enough  $\varepsilon \in \vec{\Delta}$ .*

Thanks to the last Theorem we can define the quantization  $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) \in \mathbb{Z}$  of the (possibly singular) reduced space  $M_\mu^\sigma$  for  $\mu \in \widehat{K} \cap \bar{\sigma}$ .

**Definition 2.10.** *Let  $\mu \in \widehat{K} \cap \bar{\sigma}$ .*

- *If  $\mu + \rho - \rho_\sigma \in \Delta(M)$ , the integer  $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) \in \mathbb{Z}$  is defined as the Riemann-Roch character  $RR(M_{\mu+\varepsilon}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma)$  for  $\varepsilon \in \vec{\Delta}$  generic and small enough.*
- *If  $\mu + \rho - \rho_\sigma \notin \Delta(M)$ , we set  $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) = 0$*

**Remark 2.11.** *If  $\mu + \rho - \rho_\sigma$  does not belongs to the relative interior of  $\Delta(M)$ , we can choose  $\varepsilon$  so that  $\mu + \rho - \rho_\sigma + \varepsilon \notin \Delta(M)$ . Then the reduced space  $M_{\mu+\varepsilon}^\sigma$  is empty and the corresponding Riemann-Roch character  $RR(M_{\mu+\varepsilon}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma)$  vanishes. Hence  $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) = 0$ .*

## 3. SPIN-QUANTIZATION COMMUTES WITH REDUCTION

Let  $(M, \omega, \Phi)$  be a compact Hamiltonian  $K$ -manifold which is Spin prequantized. We are looking to a geometric interpretation of the multiplicity, denoted  $m_\mu$ , of the representation  $V_\mu^K$  into  $\mathcal{Q}_{\text{spin}}^K(M)$ .

The main result of this paper is the following

**Theorem 3.1.** *Let  $\sigma$  be the smallest face of the Weyl chamber so that  $\Phi(M) \cap \mathfrak{t}_+^* \subset \bar{\sigma}$ . For  $\mu \in \widehat{K}$ , we have*

$$m_\mu = \begin{cases} 0 & \text{if } \mu \notin \bar{\sigma}; \\ \mathcal{Q}_{\text{spin}}(M_\mu^\sigma) & \text{if } \mu \in \bar{\sigma}. \end{cases}$$

In this section we introduce the main tools needed for the proof of Theorem 3.1. In Section 3.1, we recall the notion of *transversally elliptic symbols*.

In Section 3.2, we recall the Witten's way of localization the Riemann-Roch character [18]. We recall in Proposition 3.7, the criterium observed in [19] for the vanishing of the invariant part of the localized Riemann-Roch character.

In Section 3.3, we recall an induction formula proved in [18, 19] for the localized Riemann-Roch character.

In Section 3.4, we prove Theorem 3.1 when  $K$  is a torus<sup>1</sup>. We give by the same way a proof of Theorem 2.9 which is essential to the definition of the Spin-quantization of the (possibly singular) reduced spaces  $M_\mu^\sigma$ .

**3.1. Elliptic and transversally elliptic symbols.** Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [7, 8] and Paradan-Vergne [20]. For a short introduction see [18].

Let  $\mathcal{X}$  be a *compact*  $K$ -manifold. Let  $p : T\mathcal{X} \rightarrow \mathcal{X}$  be the projection, and let  $(-, -)_\mathcal{X}$  be a  $K$ -invariant Riemannian metric. If  $E^0, E^1$  are  $K$ -equivariant complex vector bundles over  $\mathcal{X}$ , a  $K$ -equivariant morphism  $h \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$  is called a *symbol* on  $\mathcal{X}$ . The subset of all  $(x, v) \in T\mathcal{X}$  where<sup>2</sup>  $h(x, v) : E_x^0 \rightarrow E_x^1$  is not invertible is called the *characteristic set* of  $h$ , and is denoted by  $\text{Char}(h)$ .

In the following, the “product” of a symbol  $h$  by a complex vector bundle  $F \rightarrow M$ , is the symbol

$$h \otimes F$$

defined by  $h \otimes F(x, v) = h(x, v) \otimes \text{Id}_{F_x}$  from  $E_x^0 \otimes F_x$  to  $E_x^1 \otimes F_x$ . Note that  $\text{Char}(h \otimes F) = \text{Char}(h)$ .

Let  $T_K\mathcal{X}$  be the following subset of  $T\mathcal{X}$  :

$$T_K\mathcal{X} = \{(x, v) \in T\mathcal{X}, (v, X_\mathcal{X}(x))_\mathcal{X} = 0 \text{ for all } X \in \mathfrak{k}\}.$$

A symbol  $h$  is *elliptic* if  $h$  is invertible outside a compact subset of  $T\mathcal{X}$  (i.e.  $\text{Char}(h)$  is compact), and is  *$K$ -transversally elliptic* if the restriction of  $h$  to  $T_K\mathcal{X}$  is invertible outside a compact subset of  $T_K\mathcal{X}$  (i.e.  $\text{Char}(h) \cap T_{K_2}\mathcal{X}$  is compact). An elliptic symbol  $h$  defines an element in the equivariant  $\mathbf{K}$ -theory of  $T\mathcal{X}$  with compact support, which is denoted by  $\mathbf{K}_K(T\mathcal{X})$ , and the index of  $h$  is a virtual finite dimensional representation of  $K$ , that we denote  $\text{Index}_\mathcal{X}^K(h) \in R(K)$  [2, 3, 4, 5].

<sup>1</sup>This situation was already handled in [19].

<sup>2</sup>The map  $h(x, v)$  will be also denote  $h|_x(v)$

Let

$$R_{tc}^{-\infty}(K) \subset R^{-\infty}(K)$$

be the  $R(K)$ -submodule formed by all the infinite sum  $\sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$  where the map  $\mu \in \widehat{K} \mapsto m_\mu \in \mathbb{Z}$  has at most a *polynomial* growth. The  $R(K)$ -module  $R_{tc}^{-\infty}(K)$  is the Grothendieck group associated to the *trace class* virtual  $K$ -representations: we can associate to any  $V \in R_{tc}^{-\infty}(K)$ , its trace  $k \rightarrow \text{Tr}(k, V)$  which is a generalized function on  $K$  invariant by conjugation. Then the trace defines a morphism of  $R(K)$ -module

$$(3.8) \quad R_{tc}^{-\infty}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^K.$$

A  $K$ -transversally elliptic symbol  $h$  defines an element of  $\mathbf{K}_K(\text{T}_K \mathcal{X})$ , and the index of  $h$  is defined as a trace class virtual representation of  $K$ , that we still denote  $\text{Index}_{\mathcal{X}}^K(h) \in R_{tc}^{-\infty}(K)$ .

Remark that any elliptic symbol of  $\text{T}\mathcal{X}$  is  $K$ -transversally elliptic, hence we have a restriction map  $\mathbf{K}_K(\text{T}\mathcal{X}) \rightarrow \mathbf{K}_K(\text{T}_K \mathcal{X})$ , and a commutative diagram

$$(3.9) \quad \begin{array}{ccc} \mathbf{K}_K(\text{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_K(\text{T}_K \mathcal{X}) \\ \text{Index}_{\mathcal{X}}^K \downarrow & & \downarrow \text{Index}_{\mathcal{X}}^K \\ R(K) & \longrightarrow & R_{tc}^{-\infty}(K) . \end{array}$$

Using the *excision property*, one can easily show that the index map  $\text{Index}_{\mathcal{U}}^K : \mathbf{K}_K(\text{T}_K \mathcal{U}) \rightarrow R_{tc}^{-\infty}(K)$  is still defined when  $\mathcal{U}$  is a  $K$ -invariant relatively compact open subset of a  $K$ -manifold (see [18][section 3.1]).

Suppose that  $M$  is a  $K$ -manifold equipped with an invariant almost complex structure  $J$ . Let us recall the definition of the Riemann-Roch character  $RR_J^K(M, -)$ .

The complex vector bundle  $(\text{T}^*M)^{0,1}$  is  $K$ -equivariantly identified with the tangent bundle  $\text{T}M$  equipped with the complex structure  $J$ . We work with the Hermitian structure on  $(\text{T}M, J)$  defined by  $(v, w) := \Omega(v, Jw) - i\Omega(v, w)$  for  $v, w \in \text{T}M$ . The symbol

$$\text{Thom}(M, J) \in \Gamma(M, \text{hom}(p^*(\wedge_{\mathbb{C}}^{\text{even}} \text{T}M), p^*(\wedge_{\mathbb{C}}^{\text{odd}} \text{T}M)))$$

at  $(m, v) \in \text{T}M$  is equal to the Clifford map

$$(3.10) \quad \mathbf{c}_m(v) : \wedge_{\mathbb{C}}^{\text{even}} \text{T}_m M \longrightarrow \wedge_{\mathbb{C}}^{\text{odd}} \text{T}_m M,$$

where  $\mathbf{c}_m(v).w = v \wedge w - \iota(v)w$  for  $w \in \wedge_{\mathbb{C}}^{\bullet} \text{T}_m M$ . Here  $\iota(v) : \wedge_{\mathbb{C}}^{\bullet} \text{T}_m M \rightarrow \wedge_{\mathbb{C}}^{\bullet-1} \text{T}_m M$  denotes the contraction map. Since  $\mathbf{c}_m(v)^2 = -\|v\|^2 \text{Id}$ , the map  $\mathbf{c}_m(v)$  is invertible for all  $v \neq 0$ . Hence the characteristic set of  $\text{Thom}(M, J)$  corresponds to the 0-section of  $\text{T}M$ .

Let  $E$  be a  $K$ -equivariant complex vector bundle over  $M$ . It is a classical fact that the principal symbol of the Dolbeault-Dirac operator  $\bar{\partial}_E + \bar{\partial}_E^*$  is equal to the following elliptic symbol<sup>3</sup>

$$\mathbf{c}_E := \text{Thom}(M, J) \otimes E,$$

see [9]. Since  $M$  is compact, the symbol  $\mathbf{c}_E$  is elliptic and then defines an element of the equivariant  $\mathbf{K}$ -group of  $\text{T}M$ .

<sup>3</sup>Here we use an identification  $\text{T}^*M \simeq \text{T}M$  given by an invariant Riemannian metric.

**Definition 3.2.** *The Riemann-Roch character  $RR_J^K(M, E) \in R(K)$  is defined equivalently*

- *as the topological index of  $\mathbf{c}_E \in \mathbf{K}_K(TM)$ , or*
- *as the analytical index of the Dolbeault-Dirac operator  $\bar{\partial}_E + \bar{\partial}_E^*$ .*

**3.2. Localization of the Riemann-Roch character.** Let  $(M, \omega, \Phi)$  a compact Hamiltonian  $K$ -manifold Spin-prequantized by  $(\tilde{L}, J)$  where  $J$  is a *compatible* almost complex structure on  $M$ . The Riemann-Roch character attached to  $J$  is just denoted  $RR^K(M, -)$ .

By definition the Spin-quantization of  $(M, \omega, \Phi)$  is

$$\mathcal{Q}_{\text{spin}}^K(M) := RR^K(M, \tilde{L}) \in R(K).$$

We recall the Witten's deformation of the Riemann-Roch character [18, 19]. We use in all this paper an isomorphism  $\mathfrak{k}^* \simeq \mathfrak{k}$  defined by a  $K$ -invariant scalar product on  $\mathfrak{k}^*$ . In order to simplify the notation, we use the same symbol for  $\xi \in \mathfrak{k}^*$  and its corresponding element in  $\mathfrak{k}$ .

The moment map  $\Phi$  is seen as an equivariant map from  $M$  to  $\mathfrak{k}$ . We define the *Kirwan vector field* on  $M$  :

$$(3.11) \quad \kappa_m = (\Phi(m))_M(m), \quad m \in M.$$

**Definition 3.3.** *The symbol  $\mathbf{c}_{\tilde{L}} = \text{Thom}(M, J) \otimes \tilde{L}$  pushed by the vector field  $\kappa$  is the symbol  $\mathbf{c}_{\tilde{L}}^\kappa$  defined by the relation*

$$\mathbf{c}_{\tilde{L}}^\kappa|_m(v) = \text{Thom}(M, J) \otimes \tilde{L}|_m(v - \kappa_m)$$

for any  $(m, v) \in TM$ .

Note that  $\mathbf{c}_{\tilde{L}}^\kappa|_m(v)$  is invertible except if  $v = \kappa_m$ . If furthermore  $v$  belongs to the subset  $T_K M$  of tangent vectors orthogonal to the  $K$ -orbits, then  $v = 0$  and  $\kappa_m = 0$ . Indeed  $\kappa_m$  is tangent to  $K \cdot m$  while  $v$  is orthogonal.

Since  $\kappa$  is the Hamiltonian vector field of the function  $\frac{1}{2}\|\Phi\|^2$ , the set of zeros of  $\kappa$  coincides with the set  $\text{Cr}(\|\Phi\|^2)$  of critical points of  $\|\Phi\|^2$ . Finally we have

$$\begin{aligned} \text{Char}(\mathbf{c}_{\tilde{L}}^\kappa) \cap T_K M &\simeq \text{Cr}(\|\Phi\|^2) \\ &= \bigcup_{\beta \in \mathcal{B}} \underbrace{K \cdot (M^\beta \cap \Phi^{-1}(\beta))}_{C_\beta} \end{aligned}$$

where  $\mathcal{B} \subset \mathfrak{k}^*$  is a finite subset parametrizing coadjoint orbits  $K \cdot \beta$ .

We are interested to the restriction  $\mathbf{c}_{\tilde{L}}^\kappa|_U$  of the elliptic symbol on an invariant open subset  $U \subset M$ . Note that the set  $\text{Char}(\mathbf{c}_{\tilde{L}}^\kappa|_U) \cap T_K U \simeq \text{Cr}(\|\Phi\|^2) \cap U$  is compact when

$$(3.12) \quad \partial U \cap \text{Cr}(\|\Phi\|^2) = \emptyset.$$

When (3.12) holds we denote

$$(3.13) \quad \mathcal{Q}_\Phi^K(U) := \text{Index}_U^K(\mathbf{c}_{\tilde{L}}^\kappa|_U) \in R_{tc}^{-\infty}(K)$$

the equivariant index of the transversally elliptic symbol  $\mathbf{c}_{\tilde{L}}^\kappa|_U$ .

For any  $\beta \in \mathcal{B}$ , we consider a relatively compact open invariant neighborhood  $U_\beta$  of  $C_\beta$  such that  $\text{Cr}(\|\Phi\|^2) \cap \overline{U_\beta} = C_\beta$ .

**Definition 3.4.** We denote

$$\mathcal{Q}_\beta^K(M) \in R_{tc}^{-\infty}(K)$$

the index of the transversally elliptic symbol  $\mathbf{c}_L^\kappa|_{U_\beta}$ .

Everything can be defined if we replace the line bundle  $\tilde{L}$  by any equivariant complex vector bundle  $E$ . We can consider the pushed symbol  $\mathbf{c}_E^\kappa$ , and the localized Riemann-Roch characters

$$RR_\Phi^K(U, E) := \text{Index}_U^K(\mathbf{c}_E^\kappa|_U) \quad \text{and} \quad RR_\beta^K(M, E) := \text{Index}_{U_\beta}^K(\mathbf{c}_E^\kappa|_{U_\beta}).$$

A direct application of the excision property [18] gives that

$$(3.14) \quad \mathcal{Q}_{\text{spin}}^K(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}_\beta^K(M).$$

If we work with  $RR_\Phi^K(U, E)$ , we have

$$(3.15) \quad RR_\Phi^K(U, E) = \sum_{\beta \in \mathcal{B} \cap \Phi(U)} RR_\beta^K(U, E).$$

The decomposition (3.14) and (3.15) will be used in the next chapters when one want to compute the multiplicity, denoted  $[\mathcal{Q}_{\text{spin}}^K(M)]^K$ , of the trivial representation in  $\mathcal{Q}_{\text{spin}}^K(M)$ . We have

$$[\mathcal{Q}_{\text{spin}}^K(M)]^K = \sum_{\beta \in \mathcal{B}} [\mathcal{Q}_\beta^K(M)]^K.$$

and we finish this section by recalling a *criterion* under which one has  $[\mathcal{Q}_\beta^K(M)]^K = 0$ .

Let  $\beta$  be a non-zero element in  $\mathfrak{k}$ : let  $\mathbb{T}_\beta \subset K$  be the torus generated by  $\beta$ . For  $m \in M^\beta$ , let  $\alpha_1^m, \dots, \alpha_p^m$  be the real infinitesimal weights for the action of  $\mathbb{T}_\beta$  on the fibers of  $\mathbb{T}_m M$  (we equip the fibers of  $\mathbb{T}_m M / \mathbb{T}_m M^\beta$  with a  $\mathbb{T}_\beta$ -invariant complex structure).

**Definition 3.5.** Let us denote by  $\mathbf{Tr}_\beta|\mathbb{T}_m M|$  the following positive number

$$\mathbf{Tr}_\beta|\mathbb{T}_m M| := \sum_{i=1}^l |\langle \alpha_i^m, \beta \rangle|.$$

Note that  $m \in M^\beta \mapsto \mathbf{Tr}_\beta|\mathbb{T}_m M|$  is constant along a connected component of  $M^\beta$ . We see also that the expression  $\mathbf{Tr}_\beta|E|$  is well defined for any  $H$ -equivariant real vector bundle  $E \rightarrow P$ , when  $\beta \in \mathfrak{h}$  acts trivially on  $P$ .

**Example 3.6.** The map  $\beta \in \mathfrak{k} \mapsto \mathbf{Tr}_\beta|\mathfrak{k}|$  is invariant under the adjoint action. When  $\beta$  belongs to the Weyl chamber, one has  $\mathbf{Tr}_\beta|\mathfrak{k}| = 2(\rho, \beta)$ . Note that  $\mathbf{Tr}_\beta|\mathfrak{k}| \leq 2\|\rho\| \|\beta\|$  for any  $\beta \in \mathfrak{k}$ .

We have proved in [19] the following nice criterium.

**Proposition 3.7.** Let  $\beta \neq 0$  in  $\mathcal{B}$ . The multiplicity of the trivial representation in  $\mathcal{Q}_\beta^K(M)$  is equal to zero if

$$(3.16) \quad \|\beta\|^2 + \frac{1}{2}\mathbf{Tr}_\beta|\mathbb{T}_m M| > \mathbf{Tr}_\beta|\mathfrak{k}|, \quad \forall m \in M^\beta \cap \Phi^{-1}(\beta).$$

**Remark 3.8.** Note that condition (3.16) is equivalent to

$$(3.17) \quad \|\Phi(m)\|^2 + \frac{1}{2} \mathbf{Tr}_{\Phi(m)} |\mathbb{T}_m M| > \mathbf{Tr}_{\Phi(m)} |\mathfrak{k}|, \quad \forall m \in C_\beta.$$

If the critical set  $C_\beta$  decomposes in a finite disjoint union of closed  $K$ -invariant subset  $C_\beta = \cup_j C_\beta^j$ , we consider invariant open neighborhood  $U^j$  of  $C_\beta^j$  such that  $\overline{U_\beta^j} \cap \text{Cr}(\|\Phi\|^2) = C_\beta^j$ , and we define

$$\mathcal{Q}_{C_\beta^j}^K(M) := \text{Index}_{U_\beta^j}^K(\mathbf{c}^\kappa|_{U_\beta^j}) \in R_{tc}^{-\infty}(K)$$

Then the generalized character  $\mathcal{Q}_\beta^K(M)$  is equal to the sum  $\sum_j \mathcal{Q}_{C_\beta^j}^K(M)$  and Proposition 3.7 tells us that  $[\mathcal{Q}_{C_\beta^j}^K(M)]^K = 0$  if (3.17) holds on  $C_\beta^j$ .

**3.3. Induction formulas.** Let  $H$  be a compact connected Lie group. Let  $H \cdot a$  be a coadjoint orbit. Let  $(N, \omega_N, \Phi_N)$  be an Hamiltonian  $H$ -manifold which is not assumed to be compact. But we assume that  $\Phi_N$  is *proper* near  $H \cdot a$ : the pullback  $\Phi_N^{-1}(\mathcal{C})$  is compact if  $\mathcal{C} \subset \mathfrak{h}^*$  is a small enough compact invariant neighborhood of  $H \cdot a$ .

Let  $H_a$  be the stabilizer of  $a \in \mathfrak{h}^*$ , and let  $Y_a$  be a symplectic slice near  $H \cdot a$ :  $Y_a$  is a  $H_a$ -invariant symplectic manifold of  $N$  such that  $\Phi_N(Y_a) \subset \mathfrak{h}_a^*$  and such that  $H \times_{H_a} Y_a$  is diffeomorphic to an invariant open neighborhood of  $\Phi_N^{-1}(H \cdot a)$ . We will work with the following moment map on  $Y_a$ :

$$\Phi_{Y_a} = \Phi_N|_{Y_a} - a.$$

Let  $N \times \overline{H \cdot a}$  be the Hamiltonian  $H$ -manifold, with moment map  $\Phi(n, \xi) = \Phi_N(n) - \xi$ . Let

$$RR_0^H(N \times \overline{H \cdot a}, -)$$

be the Riemann-Roch character localized near the compact subset  $\Phi^{-1}(0) \subset N \times \overline{H \cdot a}$ . Let

$$RR_0^{H_a}(Y_a, -)$$

be the Riemann-Roch character localized near the compact subset  $\Phi_{Y_a}^{-1}(0) = \Phi_N^{-1}(a) \subset Y_a$ .

Let  $\text{Ind}_{H_a}^H : R^{-\infty}(H_a) \rightarrow R^{-\infty}(H)$  be the induction map. If  $E$  and  $F$  are respectively  $H$ -equivariant complex vector bundles on  $N$  and  $H \cdot a$ , we denote  $E \boxtimes F$  their product. We have proved in [18] (see also Proposition 4.13 in [19]) the following induction formula

**Proposition 3.9.** For any equivariant complex vector bundles  $E \rightarrow N$  and  $F \rightarrow H \cdot a$ , we have

$$RR_0^H(N \times \overline{H \cdot a}, E \boxtimes F) = \text{Ind}_{H_a}^H \left[ RR_0^{H_a}(Y_a, E|_{Y_a} \otimes F|_{\{a\}}) \right].$$

The last Proposition gives in particular that

$$(3.18) \quad [RR_0^H(N \times \overline{H \cdot a}, E \boxtimes F)]^H = [RR_0^{H_a}(Y_a, E|_{Y_a} \otimes F|_{\{a\}})]^{H_a}.$$

**3.4. The torus case.** Let  $T$  be a compact torus, and let  $(M, \omega, \Phi)$  be a compact Hamiltonian  $T$ -manifold which is Spin-prequantized by the data  $(J, \tilde{L})$ . We suppose that  $J$  is compatible with  $\omega$ . The irreducible representation of  $T$  is parametrized by the lattice  $\hat{T} \subset \mathfrak{t}^*$ : at each  $\mu \in \hat{T}$  we associate the one-dimensional representation  $\mathbb{C}_\mu$ .

We write  $\mathcal{Q}_{\text{spin}}^T(M) = \sum_{\mu \in \hat{T}} m_\mu \mathbb{C}_\mu$ , and one wants to show that the multiplicity  $m_\mu$  is equal to the Spin-quantization of the (possibly singular) reduced space  $M_\mu := \Phi^{-1}(\mu)/T$ .

We fix once for all  $\mu \in \hat{T}$ . And we apply the Witten deformation procedure to the Hamiltonian  $T$ -manifold  $(M, \omega, \Phi - \mu)$  which is Spin-prequantized by  $(J, \tilde{L} \otimes \mathbb{C}_{-\mu})$ . We have

$$m_\mu = \sum_{\beta \in \mathcal{B}^\mu} \left[ RR_\beta^T(M, \tilde{L} \otimes \mathbb{C}_{-\mu}) \right]^T$$

where  $\mathcal{B}^\mu$  parametrizes the critical points of  $\|\Phi - \mu\|^2$ . Here the criterion (3.16) holds for any non-zero  $\beta$  since the Lie algebra  $\mathfrak{t}$  is abelian. We have then

$$m_\mu = \left[ RR_0^T(M, \tilde{L} \otimes \mathbb{C}_{-\mu}) \right]^T.$$

In particular  $m_\mu = 0$  if  $\mu \notin \Phi(M)$ . When  $\mu \in \Phi(M)$ , we consider a small neighborhood  $U$  of  $\Phi^{-1}(\mu) \subset M$  so that  $\bar{U} \cap \text{Cr}(\|\Phi - \mu\|^2) = \Phi^{-1}(\mu)$ . We know then that

$$(3.19) \quad m_\mu = \left[ RR_{\Phi^{-1}(\mu)}^T(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu}) \right]^T.$$

**3.4.1. First case:  $\mu$  is a regular value of  $\Phi$ .** We consider the orbifold reduced space  $M_\mu = \Phi^{-1}(\mu)/T$  which is equipped with a canonical symplectic form  $\omega_\mu$ . Let  $RR(M_\mu, -)$  be the Riemann-Roch character attached to a compatible almost complex structure. We prove in [18] that for any complex vector bundle  $E \rightarrow U$

$$(3.20) \quad \left[ RR_{\Phi^{-1}(\mu)}^T(U, E) \right]^T = RR(M_\mu, \mathcal{E})$$

where  $\mathcal{E} = E|_{\Phi^{-1}(\mu)}/T$  is the induced orbifold bundle on  $M_\mu$ . If we take  $E = \tilde{L}|_U \otimes \mathbb{C}_{-\mu}$  on sees (thanks to Lemma 2.6) that

$$\tilde{\mathcal{L}}_\mu = (\tilde{L}|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/T$$

is an orbifold line bundle which Spin-prequantizes  $(M_\mu, \omega_\mu)$ , and (3.20) gives together with (3.19) that

$$m_\mu = RR(M_\mu, \tilde{\mathcal{L}}_\mu) = \mathcal{Q}_{\text{spin}}(M_\mu).$$

**3.4.2. Second case :  $\mu$  is a not (necessarilly) a regular value of  $\Phi$ .** Let  $\vec{\Delta} \subset \mathfrak{t}^*$  be the rational vector subspace generated by  $\{a-b \mid a, b \in \Phi(M)\}$ . We work here with a weight  $\mu \in \Phi(M)$  so that the polytope  $\Phi(M)$  lives in the affine subspace  $\mu + \vec{\Delta}$ . Let  $\mathfrak{t}_\Delta \subset \mathfrak{t}$  be the subspace orthogonal to  $\Delta$ , and let  $T_\Delta \subset T$  be the corresponding subtorus.

**Lemma 3.10.** *The group  $T_\Delta$  acts trivially on  $M$  and on the line bundle  $\tilde{L} \otimes \mathbb{C}_{-\mu}$ .*

*Proof.* See the proof of Lemma 2.8.  $\square$



Let  $T' \subset T$  be another subtorus such that  $T = T_\Delta \times T'$ : the dual of its Lie algebra  $\mathfrak{t}'$  is identified with  $\overrightarrow{\Delta} \subset \mathfrak{t}^*$ . We look now at  $(M, \omega)$  as a Hamiltonian  $T'$ -manifold with moment map

$$\Phi' := \Phi - \mu : M \longrightarrow \overrightarrow{\Delta} = (\mathfrak{t}')^*$$

The  $T'$ -equivariant line bundle  $\tilde{L}' := \tilde{L} \otimes \mathbb{C}_{-\mu}$  Spin-prequantizes the Hamiltonian  $T'$ -manifold  $(M, \omega, \Phi')$ . Let  $U$  be a small neighborhood of  $\Phi'^{-1}(0)$  in  $M$ . The generalized character  $RR_{\Phi'-\mu}^T(U, \tilde{L}'|_U \otimes \mathbb{C}_{-\mu})$  belongs to  $R^{-\infty}(T')$  and corresponds to the localized Riemann-Roch character

$$RR_{\Phi'}^{T'}(U, \tilde{L}'|_U).$$

We deform the moment map  $\Phi'$  in  $\Phi' - \varepsilon$  where  $\varepsilon$  is a small element in  $\overrightarrow{\Delta}$ . We have proved in [19][Proposition 4.14] the following

**Lemma 3.11.** • *If  $\varepsilon$  is small enough, the critical set of  $\|\Phi' - \varepsilon\|^2$  does not intersect  $\partial U$ , so that the localized Riemann-Roch character  $RR_{\Phi'-\varepsilon}^{T'}(U, -)$  is well defined.*

- *We have  $RR_{\Phi'}^{T'}(U, \tilde{L}'|_U) = RR_{\Phi'-\varepsilon}^{T'}(U, \tilde{L}'|_U)$  if  $\varepsilon$  is small enough.*

Now we are left to the computation of  $m_\mu = \left[ RR_{\Phi'-\varepsilon}^{T'}(U, \tilde{L}'|_U) \right]^{T'}$  when  $\varepsilon \in \overrightarrow{\Delta}$  is small enough. We start with the decomposition

$$RR_{\Phi'-\varepsilon}^{T'}(U, \tilde{L}'|_U) = \sum_{\beta \in \mathcal{B}_\varepsilon} RR_{\Phi'-\varepsilon, \beta}^{T'}(U, \tilde{L}'|_U)$$

where  $RR_{\Phi'-\varepsilon, \beta}^{T'}(U, -)$  denotes the Riemann-Roch character localized near the compact subset  $U^\beta \cap (\Phi')^{-1}(\beta + \varepsilon)$ . We have proved in [19][Lemma 4.16] the following

**Lemma 3.12.** *If  $\varepsilon$  is small enough we have  $\left[ RR_{\Phi'-\varepsilon, \beta}^{T'}(U, \tilde{L}'|_U) \right]^{T'} = 0$  when  $\beta \neq 0$ .*

At  $\varepsilon \in \overrightarrow{\Delta}$  small enough and generic we associate the orbifold  $M_{\mu+\varepsilon} = \Phi^{-1}(\mu + \varepsilon)/T'$  which is equipped with the orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon} = \left( \tilde{L}|_{\Phi^{-1}(\mu+\varepsilon)} \otimes \mathbb{C}_{-\mu} \right) / T'.$$

Let  $RR(M_{\mu+\varepsilon}, -)$  be the Riemann-Roch map associated to a compatible almost complex structure. If we use (3.20) together with the Lemmas 3.11 and 3.12 we get

**Theorem 3.13.** *The multiplicity  $m_\mu$  is equal to the Riemann-Roch number  $RR(M_{\mu+\varepsilon}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) \in \mathbb{Z}$  where  $\varepsilon \in \overrightarrow{\Delta}$  is small and generic.*

We prove here that the quantity  $RR(M_{\mu+\varepsilon}, \tilde{\mathcal{L}}_{\mu+\varepsilon})$  does not depend of the choice of  $\varepsilon$  small and generic: it is the definition of the Spin quantization, denoted  $\mathcal{Q}_{\text{spin}}(M_\mu)$ , of the (possibly singular) reduced space  $M_\mu$ .

**3.4.3. Proof of Theorem 2.9.** The same kind of proof work for Theorem 2.9. We consider an invariant relatively compact neighborhood  $U_{\sigma, \mu}$  of  $\Phi_\sigma^{-1}(\mu) = \Phi^{-1}(\mu + \rho - \rho_\sigma)$  in the slice  $Y_\sigma$  so that  $\text{Cr}(\|\Phi_\sigma - \mu\|^2) \cap \overline{U_{\sigma, \mu}} = \Phi_\sigma^{-1}(\mu)$ . Thanks to Lemmas 3.11 and 3.12, we know that the Riemann-Roch character

$$RR_{\Phi_\sigma - \mu - \varepsilon}^{Z'_\sigma}(U_{\sigma, \mu}, \tilde{L}') \in R^{-\infty}(Z'_\sigma)$$

are well defined for  $\varepsilon \in \overrightarrow{\Delta}$  small enough, and they do not depend of the choice of  $\varepsilon$ . If  $\varepsilon_1, \varepsilon_2 \in \overrightarrow{\Delta}$  are small enough regular values of  $\Phi_\sigma - \mu$  we get thanks to (3.20) that

$$\begin{aligned} RR(M_{\mu+\varepsilon_1}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon_1}^\sigma) &= \left[ RR_{\Phi_\sigma - \mu - \varepsilon_1}^{Z'_\sigma}(U_{\sigma, \mu}, \tilde{L}') \right]^{Z'_\sigma} \\ &= \left[ RR_{\Phi_\sigma - \mu - \varepsilon_2}^{Z'_\sigma}(U_{\sigma, \mu}, \tilde{L}') \right]^{Z'_\sigma} \\ &= RR(M_{\mu+\varepsilon_2}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon_2}^\sigma). \end{aligned}$$

#### 4. PROOF OF THEOREM 3.1

Let  $(M, \omega, \Phi)$  be a compact Hamiltonian  $K$ -manifold which is Spin prequantized. Let  $\sigma$  be the smallest face of the Weyl chamber so that  $\Phi(M) \cap \mathfrak{t}_+^* \subset \bar{\sigma}$ . Let  $\mu$  be a dominant weight, and let  $m_\mu$  be the multiplicity of  $V_\mu^K$  in  $\mathcal{Q}_{\text{spin}}^K(M)$ .

Let  $\mathcal{O}_\mu$  be the coadjoint orbit  $K \cdot (\mu + \rho)$ . Since the dual representation  $(V_\mu^K)^*$  can be realized<sup>4</sup> as  $\mathcal{Q}_{\text{spin}}^K(\overline{\mathcal{O}_\mu})$ , we know by the shifting trick that

$$m_\mu = [\mathcal{Q}_{\text{spin}}^K(M \times \overline{\mathcal{O}_\mu})]^K.$$

Now we work with the Hamiltonian  $K$ -manifold  $N = M \times \overline{\mathcal{O}_\mu}$  with moment map  $\Phi_N(m, \xi) = \Phi(m) - \xi$ . The Witten's deformation on  $N$  gives  $\mathcal{Q}_{\text{spin}}^K(M \times \overline{\mathcal{O}_\mu}) = \sum_{\beta \in \mathcal{B}^\mu} \mathcal{Q}_\beta^K(M \times \overline{\mathcal{O}_\mu})$  where  $\mathcal{B}^\mu$  is a finite set parametrizing  $\text{Cr}(\|\Phi_N\|^2)$ . We have then

$$(4.21) \quad m_\mu = \sum_{\beta \in \mathcal{B}^\mu} [\mathcal{Q}_\beta^K(M \times \overline{\mathcal{O}_\mu})]^K.$$

We remark that 0 does not appears in  $\mathcal{B}^\mu$  when  $\sigma \neq \mathfrak{t}_+^*$ , since  $\mu + \rho \notin \Phi(M)$ .

The main point of this section is the following

**Proposition 4.1.** • *If  $\mu \notin \bar{\sigma}$ , the identity (3.17) holds on  $C_\beta$  for any  $\beta \in \mathcal{B}^\mu$ . Hence  $m_\mu = 0$ .*

• *If  $\mu \in \bar{\sigma}$ , the identity (3.17) holds on  $C_\beta$  for any  $\beta \neq -\rho_\sigma$ . Then*

$$m_\mu = [\mathcal{Q}_{-\rho_\sigma}^K(M \times \overline{\mathcal{O}_\mu})]^K.$$

When  $\sigma = \mathfrak{t}_+^*$ , we have  $\rho_\sigma = 0$  and Proposition 4.1 tell us that the multiplicity  $m_\mu$  is equal to  $[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K$  for any  $\mu \in \widehat{K}$ . In particular  $m_\mu = 0$  if  $\mu + \rho \notin \Phi(M)$ .

When  $\sigma \neq \mathfrak{t}_+^*$  and  $\mu \in \bar{\sigma}$ , we precise Proposition 4.1 as follow. The generalized character  $\mathcal{Q}_{-\rho_\sigma}^K(M \times \overline{\mathcal{O}_\mu})$  is defined as the index of a transversally elliptic symbol living in a neighborhood of

$$C_{-\rho_\sigma} = K(N^{\rho_\sigma} \cap \Phi_N^{-1}(-\rho_\sigma)).$$

Let  $K_{\rho_\sigma}$  be the stabilizer subgroup of  $\rho_\sigma$ . Let  $W(K_{\rho_\sigma}) \subset W$  be the Weyl subgroup of  $K_{\rho_\sigma}$ . A direct computation gives that

$$C_{-\rho_\sigma} = \bigcup_{\bar{w} \in W(K_{\rho_\sigma}) \setminus W} C_{-\rho_\sigma, \bar{w}}$$

with

$$C_{-\rho_\sigma, \bar{w}} = K(M^{\rho_\sigma} \cap \Phi^{-1}(w(\mu + \rho) - \rho_\sigma) \times \{w(\mu + \rho)\}).$$

<sup>4</sup> $\overline{\mathcal{O}_\mu}$  is the coadjoint orbit  $\mathcal{O}_\mu$  with the opposite symplectic structure.

We are particularly interested in the component  $C_{-\rho_\sigma, \bar{e}}$ . Let us denote  $C_{-\rho_\sigma, out}$  the union of the  $C_{-\rho_\sigma, \bar{w}}$  for  $\bar{w} \neq \bar{e}$ . We have a decomposition

$$(4.22) \quad C_{-\rho_\sigma} = C_{-\rho_\sigma, \bar{e}} \cup C_{-\rho_\sigma, out}$$

into closed invariant disjoint subsets. Then the generalized character  $\mathcal{Q}_{-\rho_\sigma}^K(M \times \overline{\mathcal{O}_\mu})$  is equal to the sum

$$\mathcal{Q}_{-\rho_\sigma, \bar{e}}^K(M \times \overline{\mathcal{O}_\mu}) + \mathcal{Q}_{-\rho_\sigma, out}^K(M \times \overline{\mathcal{O}_\mu})$$

where both terms correspond to the specialization of the transversally elliptic symbol to the neighborhood of each part of the decomposition (4.22).

**Proposition 4.2.** *Suppose that  $\sigma \neq \mathfrak{t}_+^*$  and that  $\mu \in \bar{\sigma}$ . The identity (3.17) holds on the subset  $C_{-\rho_\sigma, out}$ , and then*

$$m_\mu = [\mathcal{Q}_{-\rho_\sigma, \bar{e}}^K(M \times \overline{\mathcal{O}_\mu})]^K.$$

Note that  $C_{-\rho_\sigma, e} = \emptyset$  if  $\mu + \rho - \rho_\sigma \notin \Phi(M)$ . At this stage we know then that  $m_\mu = 0$  if  $\mu + \rho - \rho_\sigma$  does not belong to the image of the moment map.

**4.1. Proofs of Propositions 4.1 and 4.2.** Let  $N = M \times \overline{\mathcal{O}_\mu}$  and let  $\|\Phi_N\|^2 : N \rightarrow \mathbb{R}$  be the square of the moment map. Recall that we denote by  $\sigma$  the smallest face of the Weyl chamber so that  $\Phi(M) \cap \mathfrak{t}_+^* \subset \bar{\sigma}$ .

We want to prove that for any  $n = (m, \xi) \in \text{Cr}(\|\Phi_N\|^2)$  the vector  $\beta := \Phi(m) - \xi$  satisfies

$$(I) \quad \|\beta\|^2 + \frac{1}{2} \text{Tr}_\beta |T_n N| \geq \text{Tr}_\beta |\mathfrak{k}|.$$

Afterwards we will discuss the case of equality in (I).

The tangent space  $T_\xi \mathcal{O}_\mu$  is equal to the  $\mathfrak{k}_\xi$ -module  $\mathfrak{k}/\mathfrak{k}_\xi$ : then

$$\begin{aligned} \text{Tr}_\beta |T_\xi \mathcal{O}_\mu| &= \text{Tr}_\beta |\mathfrak{k}| - \text{Tr}_\beta |\mathfrak{k}_\xi| \\ &= \text{Tr}_\beta |\mathfrak{k}|, \end{aligned}$$

since  $\beta$  belongs to the abelian subalgebra  $\mathfrak{k}_\xi$ . Using that  $\text{Tr}_\beta |T_n N| = \text{Tr}_\beta |T_m M| + \text{Tr}_\beta |\mathfrak{k}|$ , we see that (I) is equivalent to

$$(II) \quad \|\beta\|^2 + \frac{1}{2} \text{Tr}_\beta |T_m M| \geq \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}|.$$

The module  $\mathfrak{k}/\mathfrak{k}_m$  is naturally a subspace of  $T_m M$ . Let  $E_m$  be a  $K_m$ -equivariant supplement to  $\mathfrak{k}/\mathfrak{k}_m$  in  $T_m M$ . Using that  $\text{Tr}_\beta |T_m M| = \text{Tr}_\beta |\mathfrak{k}/\mathfrak{k}_m| + \text{Tr}_\beta |E_m|$ , we see that (II) is equivalent to

$$(III) \quad \|\beta\|^2 + \frac{1}{2} \text{Tr}_\beta |E_m| \geq \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}_m|.$$

Thanks to the inclusion  $\mathfrak{k}_m \subset \mathfrak{k}_{\Phi(m)}$ , we see that (I)  $\Leftrightarrow$  (II)  $\Leftrightarrow$  (III) are induced by the following inequality

$$(IV) \quad \|\beta\|^2 \geq \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}_{\Phi(m)}|.$$

**Lemma 4.3.** • *For any  $(m, \xi) \in \text{Cr}(\|\Phi_N\|^2)$  the vector  $\beta := \Phi(m) - \xi$  satisfies the inequality (IV).*

• *Let  $(m, \xi) \in \text{Cr}(\|\Phi_N\|^2)$  such that  $\beta := \Phi(m) - \xi$  satisfies the  $\|\beta\|^2 = \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}_{\Phi(m)}|$ . Then there exists a face  $\tau$  of  $\sigma$  such that*

- (1)  $\mu \in \bar{\tau}$
- (2)  $(m, \xi)$  belongs to the  $K$ -orbit of  $\Phi^{-1}(\mu + \rho - \rho_\tau) \times \{\mu + \rho\} \subset N$ .

(3)  $\beta$  belongs to the coadjoint orbit  $K \cdot (-\rho_\tau)$ .

*Proof.* Up to the multiplication of  $(m, \xi)$  by an element of  $K$ , we can assume that  $\beta \in \mathfrak{t}^*$ . Up to the multiplication of  $n = (m, \xi)$  by an element of the stabilizer subgroup  $K_\beta := \{k \in K \mid \text{Ad}(k)\beta = \beta\}$  we can assume that  $n = (m, w(\mu + \rho))$  with  $m \in M^\beta$  and  $\Phi(m) = \beta + w(\mu + \rho) \in \mathfrak{t}^*$ .

Up to the multiplication of  $n = (m, w(\mu + \rho))$  by an element of the Weyl group, we can assume that  $\Phi(m)$  belongs to the Weyl chamber: let  $\tau$  be the face of  $\sigma$  containing  $\Phi(m)$  so that  $K_{\Phi(m)} = K_\tau$ .

So we have to prove that for  $\Phi(m) = a \in \tau$  and  $w \in W$  the vector  $\beta = a - w(\mu + \rho)$  satisfies the relation

$$(4.23) \quad \|\beta\|^2 \geq \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau|.$$

The inequality (4.23) is the consequence of three basic inequalities.

We have

$$(4.24) \quad \|a - w(\mu + \rho)\| \geq \|a - (\mu + \rho)\|$$

for any  $w \in W$ , and (4.24) is strict unless  $w \in W(K_\tau)$ . In order to prove (4.24), we consider the function  $\xi \in K \cdot (\mu + \rho) \mapsto \|\xi - a\|^2 = \|a\|^2 + \|\xi\|^2 - 2(\xi, a)$ . It is a classical result of symplectic geometry that the function  $\xi \mapsto (\xi, a)$  has a unique maximum on the coadjoint orbit  $K \cdot (\mu + \rho)$  which is reached on an orbit of the stabilizer subgroup  $K_a = K_\tau$ . Since  $a$  and  $\mu + \rho$  belongs to the Weyl chamber, one checks easily that this maximum is obtained on the orbit  $K_\tau(\mu + \rho)$ . Hence  $\|k(\mu + \rho) - a\|^2 \geq \|\mu + \rho - a\|^2$  for any  $k \in K$  with equality only if  $k \in K_\tau$ . Inequality (4.24) is proved by taking  $k = w$ .

On the other hand we have

$$(4.25) \quad \begin{aligned} \|\mu + \rho - a\| &\geq \frac{(\mu + \rho - a, \rho_\tau)}{\|\rho_\tau\|} \\ &= \frac{1}{\|\rho_\tau\|} \underbrace{(\mu, \rho_\tau)}_{\geq 0} + \frac{1}{\|\rho_\tau\|} \underbrace{(\rho - \rho_\tau - a, \rho_\tau)}_{=0} + \frac{1}{\|\rho_\tau\|} (\rho_\tau, \rho_\tau) \\ &\geq \|\rho_\tau\|. \end{aligned}$$

Note that (4.25) is strict unless  $\mu \in \bar{\tau}$  and  $\mu + \rho - a = \rho_\tau$ . The third inequality is

$$(4.26) \quad \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau| \leq \|\rho_\tau\| \|\beta\|.$$

See Example 3.6. If we put (4.24), (4.25) and (4.26) together we have

$$\|\beta\|^2 \geq \|\beta\| \|a - (\mu + \rho)\| \geq \|\beta\| \|\rho_\tau\| \geq \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau|,$$

and the equality  $\|\beta\|^2 = \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau|$  holds if and only if we have the equality in (4.24), (4.25) and (4.26).

But equalities in (4.24) and (4.25) gives that  $w \in W(K_\tau)$ ,  $\mu \in \bar{\tau}$  and  $a = \mu + \rho - \rho_\tau \in \tau$ . Then  $(m, w(\mu + \rho)) = w(m', \mu + \rho)$  with  $\Phi(m') = w^{-1}(\mu + \rho - \rho_\tau) = \mu + \rho - \rho_\tau$  and  $\beta = \mu + \rho - \rho_\tau - w(\mu + \rho) = -w\rho_\tau$ . We have then

$$\frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau| = \frac{1}{2} \mathbf{Tr}_{\rho_\tau} |\mathfrak{k}_\tau| = \|\rho_\tau\|^2$$

which is the equality in (4.26). □

Since the strict inequality in **(IV)** implies the strict inequality in **(I)**, Lemma 4.3 tells us that the identity (3.17) holds on  $C_\beta$  for all  $\beta \in \mathcal{B}^\mu$  when  $\mu \notin \bar{\sigma}$ . When  $\mu \in \bar{\sigma}$  the identity (3.17) holds

- (1) on  $C_\beta$  for the  $\beta$  which are not in  $K \cdot (-\rho_\tau)$ , where  $\tau$  is a face of  $\sigma$  such that  $\mu \in \bar{\tau}$ ,
- (2) on  $C_{-\rho_\sigma, w}$  for all the  $\bar{w} \neq \bar{e}$ .

The proof of Proposition 4.1 is completed by

**Lemma 4.4.** *Let  $\tau$  be a face of  $\sigma$ , distinct from  $\sigma$ , such that  $\mu \in \bar{\tau}$ . Then the identity (3.17) holds for  $C_\beta$  for  $\beta = -\rho_\tau$ .*

*Proof.* Let  $\beta = -\rho_\tau$ . The critical set  $C_{-\rho_\tau} := K(N^{\rho_\tau} \cap \Phi_N^{-1}(-\rho_\tau))$  admits the decomposition  $C_{-\rho_\tau} = \cup_{w \in W} C_{-\rho_\tau, w}$  where

$$C_{-\rho_\tau, w} = K(M^{\rho_\tau} \cap \Phi^{-1}(w(\mu + \rho) - \rho_\tau) \times \{w(\mu + \tau)\}).$$

It is not hard to see that  $C_{-\rho_\tau, w}$  intersects  $C_{-\rho_\tau, e}$  only if  $w \in W(K_{\rho_\tau})$ , and that  $C_{-\rho_\tau, w} = C_{-\rho_\tau, e}$  when  $w \in W(K_{\rho_\tau})$ . We know then from Lemma 4.3 that the strict inequality in **(IV)** holds on  $C_{-\rho_\tau, w}$  for  $w \notin W(K_{\rho_\tau})$ .

Let us consider now the case where  $m \in M^{\rho_\tau} \cap \Phi^{-1}(\mu + \rho - \rho_\tau)$ . We know that the equality holds in **(IV)** for  $(m, \mu + \rho)$ . The equality in **(I)** for  $(m, \mu + \rho)$  is then equivalently to

$$(4.27) \quad \mathbf{Tr}_\beta |E_m| + \mathbf{Tr}_\beta |\mathfrak{k}_\tau / \mathfrak{k}_m| = 0.$$

Let us prove that (4.27) can not hold. The image of  $m$  by the moment map belongs to  $\tau$ . Then  $m$  belongs to the symplectic slice  $Y_\tau \subset M$ . A neighborhood  $m$  is then  $K \times_{K_\tau} Y_\tau$ . So the tangent space at  $m$  decomposes in two manners

$$\begin{aligned} \mathbf{T}_m M &= \mathfrak{k} / \mathfrak{k}_\tau \oplus \mathbf{T}_m Y_\tau \\ &= \mathfrak{k} / \mathfrak{k}_m \oplus E_m \end{aligned}$$

If (4.27) holds we see that  $\mathbf{Tr}_\beta |\mathbf{T}_m Y_\tau| = \mathbf{Tr}_\beta |E_m| = 0$ , which means that  $\beta = -\rho_\tau$  acts trivially on the tangent space  $\mathbf{T}_m Y_\tau$ . Hence it would imply that  $\rho_\tau$  acts trivially on the manifold  $Y_\tau$ . Since  $Y_\sigma \subset Y_\tau$ , the action of  $\rho_\tau$  on the principal slice  $Y_\sigma$  is also trivial.

We know that  $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$  acts trivially on  $Y_\sigma$ : since  $\rho_\sigma \in [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$ , the infinitesimal action of  $\rho_\sigma$  is trivial on  $Y_\sigma$ . Finally if (4.27) holds, we have that

$$\rho_{\tau/\sigma} := \rho_\tau - \rho_\sigma \in \mathbb{R}\sigma$$

acts trivially on  $Y_\sigma$ . Note that  $\rho_{\tau/\sigma}$  is a sum of weights which are orthogonal to  $\tau$ .

The moment polytope of  $M$ ,  $\Delta(M)$ , which is equal to the closure of  $\Phi(Y_\sigma) \subset \sigma$  is a convex polytope. Since the action of  $\rho_{\tau/\sigma}$  is trivial on  $Y_\sigma$  we know that the map  $\xi \in \Delta(M) \mapsto (\xi, \rho_{\tau/\sigma})$  is constant.

Finally we can use the last information in our hands:  $\mu + \rho - \rho_\tau = \Phi(m)$  belongs to  $\Delta(M)$ . Then for  $\xi \in \Delta(M)$  we have

$$(\xi, \rho_{\tau/\sigma}) = (\mu + \rho - \rho_\tau, \rho_{\tau/\sigma}) = 0,$$

since  $\mu + \rho - \rho_\tau \in \tau$  and  $\rho_{\tau/\sigma} \in \tau^\perp$ . It is contradictory with the fact that  $(\xi, \rho_{\tau/\sigma}) = (\xi, \rho_\sigma) > 0$  for any  $\xi \in \sigma$ .

We have finally proved that when  $(m, \xi) \in N^{\rho_\tau} \cap \Phi_N^{-1}(-\rho_\tau)$  the vector  $\beta = \Phi(m) - \xi$  satisfies  $\|\beta\|^2 + \frac{1}{2} \mathbf{Tr}_\beta |\mathbf{T}_m M| > \mathbf{Tr}_\beta |\mathfrak{k}|$ .  $\square$

**4.2. Computation of the multiplicities when  $\sigma = \mathfrak{t}_+^*$ .** In this section we suppose that the moment polytope  $\Delta(M) = \Phi(M) \cap \mathfrak{t}_+^*$  intersects the interior of the Weyl chamber. Let  $\Delta(M)^\circ \subset (\mathfrak{t}_+^*)^\circ$  be the relative interior of the moment polytope. We know that  $m_\mu = [\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K$  for any  $\mu \in \widehat{K}$ . In Definition 2.10, we have defined the number  $\mathcal{Q}(M_{\mu^+}^{\mathfrak{t}_+^*})$  has follows. If  $\mu + \rho \notin \Delta(M)^\circ$ , we set  $\mathcal{Q}(M_{\mu^+}^{\mathfrak{t}_+^*}) = 0$ . If  $\mu + \rho \in \Delta(M)^\circ$ , we consider, for  $\varepsilon$  generic and small enough, the orbifold reduced space  $M_{\mu+\varepsilon}^{\mathfrak{t}_+^*} := \Phi^{-1}(\mu + \varepsilon + \rho)/T$  and the orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon} = \left( \tilde{L}|_{\Phi^{-1}(\mu+\varepsilon+\rho)} \otimes \mathbb{C}_{-\mu} \right) / T.$$

The Spin quantization  $\mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}) \in \mathbb{Z}$  is defined as the Riemann-Roch number

$$RR(M_{\mu+\varepsilon}^{\mathfrak{t}_+^*}, \tilde{\mathcal{L}}_{\mu+\varepsilon}).$$

The main result of this section is the following

**Theorem 4.5.** *The number  $[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K$  is equal to  $\mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*})$ .*

*Proof.* When  $\mu + \rho \notin \Delta(M)$ , we see that  $\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu}) = 0$  since the moment map on  $M \times \overline{\mathcal{O}_\mu}$  does not goes through  $0 \in \mathfrak{t}^*$ . We have then  $[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K = \mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}) = 0$ .

We consider now a dominant weight  $\mu$  such that  $\mu + \rho \in \Delta(M)$ . Let  $Y = \Phi^{-1}((\mathfrak{t}_+^*)^\circ)$  be the symplectic slice with its canonical symplectic form  $\omega_Y$ . The action of  $T$  on  $(Y, \omega_Y)$  is Hamiltonian with moment map  $\Phi_Y := \Phi|_Y - \rho$ . We know that  $\tilde{L}|_Y$  Spin-prequantizes  $(Y, \omega_Y, \Phi_Y)$  (see Lemma 2.7).

We consider the Riemann-Roch character  $RR_0^T(Y, \tilde{L}|_Y \otimes \mathbb{C}_{-\mu})$  which is localised near  $(\Phi_Y - \mu)^{-1}(0) \subset Y$ . Thanks to the induction formula (3.18), we know that

$$\begin{aligned} m_\mu &= [\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K &= \left[ RR_0^K(M \times \overline{\mathcal{O}_\mu}, \tilde{L} \boxtimes \mathbb{C}_{[-\mu]}) \right]^K \\ & &= \left[ RR_0^T(Y, \tilde{L}|_Y \otimes \mathbb{C}_{-\mu}) \right]^T \\ & &= \left[ RR_{\Phi_Y - \mu}^T(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu}) \right]^T \end{aligned}$$

where  $U$  is a small neighborhood of  $\Phi_Y^{-1}(\mu)$  in  $Y$ .

The computation of the expression  $[RR_{\Phi_Y - \mu}^T(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})]^T$  is identical to what we have done in Section 3.4. For  $\varepsilon$  small enough and generic, we get

$$\begin{aligned} [RR_{\Phi_Y - \mu}^T(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})]^T &= [RR_{\Phi_Y - \mu - \varepsilon}^T(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu})]^T \\ &= RR(M_{\mu+\varepsilon}^{\mathfrak{t}_+^*}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) \\ &= \mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}). \end{aligned}$$

When  $\mu + \rho$  does not belong to the relative interior of  $\Delta(M)$ , we can choose  $\varepsilon$  so that  $\mu + \rho + \varepsilon \notin \Delta(M)$ , and then  $RR(M_{\mu+\varepsilon}^{\mathfrak{t}_+^*}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) = \mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}) = 0$ .  $\square$

**4.3. Computation of the multiplicities when  $\sigma \neq \mathfrak{t}_+^*$ .** Let  $\mu \in \bar{\sigma}$  so that  $\mu + \rho - \rho_\sigma \in \sigma$ . In the rest of this section the term  $\beta$  is  $-\rho_\sigma$ .

Let  $\mathcal{Q}_{\beta,\bar{e}}^K(M \times \overline{\mathcal{O}_\mu})$  be the generalized character defined as the index of a transversally elliptic symbol defined in a neighborhood of

$$C_{\beta,\bar{e}} = K(M^\beta \cap \Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\}) \subset N.$$

We have proved in the last section that  $m_\mu = \left[ \mathcal{Q}_{\beta,\bar{e}}^K(M \times \overline{\mathcal{O}_\mu}) \right]^K$ .

First we notice that the character  $\mathcal{Q}_{\beta,\bar{e}}^K(M \times \overline{\mathcal{O}_\mu})$  corresponds to the Riemann-Roch character  $RR_{\beta,\bar{e}}^K(N, \tilde{L}_N)$  localized with the Kirwan vector field near  $C_{\beta,\bar{e}} \subset \text{Cr}(\|\Phi_N\|^2)$ . We can look at  $N$  as a  $K_\beta$ -Hamiltonian manifold, and consider the Riemann-Roch character

$$RR_{\beta,e}^{K_\beta}(N, -)$$

localized with the Kirwan vector field near  $C'_\beta := K_\beta(\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\})$ .

We have prove in [18] that

$$(4.28) \quad RR_{\beta,e}^K(N, \tilde{L}_N) = \text{Ind}_{K_\beta}^K \left( RR_{\beta,e}^{K_\beta}(N, \tilde{L}_N) \wedge_{\mathbb{C}}^\bullet (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}} \right)$$

where  $\text{Ind}_{K_\beta}^K : R^{-\infty}(K_\beta) \rightarrow R^{-\infty}(K)$  is the induction map, and  $(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$  is the complexification of the real  $K_\beta$ -module  $\mathfrak{k}/\mathfrak{k}_\beta$ . It gives that

$$\left[ RR_{\beta,e}^K(N, \tilde{L}_N) \right]^K = \left[ RR_{\beta,e}^{K_\beta}(N, \tilde{L}_N) \wedge_{\mathbb{C}}^\bullet (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}} \right]^{K_\beta}.$$

Let  $Y_\sigma$  be the principal symplectic slice of  $M$ . Recall that the subgroup  $[K_\sigma, K_\sigma]$  acts trivially on  $Y_\sigma$  and that  $\rho_\sigma$  belongs to  $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$ : hence

$$\Phi^{-1}(\mu + \rho - \rho_\sigma) \subset Y_\sigma \subset M^\beta$$

and then  $C'_\beta = \Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\}$ . We are looking at a  $K_\beta$ -invariant neighborhood  $\mathcal{U}$  of  $C'_\beta$  in  $N^\beta$ . We consider the open neighborhood  $K \times_{K_\sigma} Y_\sigma$  of  $\Phi^{-1}(\mu + \rho - \rho_\sigma)$  in  $M$ . Since  $K_\beta \cap K_\sigma = T$ , one sees that

$$K_\beta \times_T Y_\sigma \subset (K \times_{K_\sigma} Y_\sigma)^\beta$$

is a  $K_\beta$ -invariant neighborhood of  $Y_\sigma$  in  $M^\beta$ . Then we can take

$$\mathcal{U} := (K_\beta \times_T Y_\sigma) \times K_\beta(\mu + \rho) \subset N^\beta.$$

We look at  $\mathcal{U}$  as a Hamiltonian  $K_\beta$ -manifold with moment map  $\Phi_{\mathcal{U}}([k, y], \xi) = k\Phi(y) - \xi \in \mathfrak{k}_\beta^*$ . The set  $C'_\beta$  is a connected component of critical points of  $\text{Cr}(\|\Phi_{\mathcal{U}}\|^2)$ , and we consider the Riemann-Roch character

$$RR_{\beta}^{K_\beta}(\mathcal{U}, -)$$

localized with the Kirwan vector field near  $C'_\beta \subset \mathcal{U}$ .

Let  $\mathcal{N}$  be the normal bundle of  $\mathcal{U}$  in  $N$ . We have  $\mathcal{N} = \mathcal{N}_1 \boxtimes \mathcal{N}_2$  where  $\mathcal{N}_1$  is the normal bundle of  $K_\beta \times_T Y_\sigma$  in  $K \times_{K_\sigma} Y_\sigma$  and  $\mathcal{N}_2$  is the normal bundle of  $K_\beta(\mu + \rho)$  in  $K(\mu + \rho)$ . One computes that  $\mathcal{N}_1 = K_\beta \times_T N_1$  where

$$N_1 = \sum_{\substack{\alpha > 0 \\ \alpha|_\sigma \neq 0, (\alpha, \beta) \neq 0}} \mathfrak{k}(\alpha),$$

and that  $\mathcal{N}_2 = K_\beta \times_T N_2$  where

$$N_2 = \sum_{\substack{\alpha < 0 \\ (\alpha, \beta) \neq 0}} \mathfrak{k}(\alpha).$$

We decompose  $\mathcal{N}$  in the sum of the polarized bundle  $\mathcal{N}^{+,\beta}$  and  $\mathcal{N}^{-,\beta}$ . Similarly let  $\mathcal{N}_{\mathbb{C}}$  the complexified bundle, and its polarized  $\beta$ -positive part  $\mathcal{N}_{\mathbb{C}}^{+,\beta}$ .

Let  $S(\mathcal{N}_{\mathbb{C}}^{+,\beta}) = \sum_{k \geq 0} S^k(\mathcal{N}_{\mathbb{C}}^{+,\beta})$  be the symmetric algebra vector bundle associated to  $\mathcal{N}_{\mathbb{C}}^{+,\beta}$ . Let us compute the rank  $n_{\beta,+}$  of the polarized vector bundle vector  $\mathcal{N}^{+,\beta}$ . We have

$$\begin{aligned} n_{\beta,+} &= \#\{\alpha > 0 \mid (\alpha, \beta) > 0 \text{ and } \alpha|_{\sigma} \neq 0\} + \#\{\alpha < 0 \mid (\alpha, \beta) > 0\} \\ &= \#\{\alpha > 0 \mid (\alpha, \beta) > 0\} + \#\{\alpha < 0 \mid (\alpha, \beta) > 0\} \\ &= \frac{1}{2} \dim(K/K_{\beta}). \end{aligned} \quad (1)$$

In (1) we use that  $\alpha|_{\sigma} = 0$  imposes  $(\alpha, \rho - \rho_{\sigma}) = 0$ . Then  $(\alpha, \beta) = -(\alpha, \rho) < 0$  for  $\alpha > 0$ . Let  $\det \mathcal{N}^{+,\beta}$  be the determinant line bundle associated to  $\mathcal{N}^{+,\beta}$ .

Thanks to the results in [18][Section 6.3], we know that

$$(4.29) \quad RR_{\beta,e}^{K_{\beta}}(N, \tilde{L}_N) = (-1)^{n_{\beta}^+} RR_{\beta}^{K_{\beta}} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes S(\mathcal{N}_{\mathbb{C}}^{+,\beta}) \right).$$

Hence we know that  $m_{\mu} = \left[ RR_{\beta,e}^{K_{\beta}}(N, \tilde{L}_N) \right]^K$  is equal to  $(-1)^{n_{\beta}^+}$  times

$$(4.30) \quad \begin{aligned} & \left[ RR_{\beta}^{K_{\beta}} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes S(\mathcal{N}_{\mathbb{C}}^{+,\beta}) \right) \wedge_{\mathbb{C}}^{\bullet} (\mathfrak{k}/\mathfrak{k}_{\beta})_{\mathbb{C}} \right]^{K_{\beta}} \\ &= \sum_{k \geq 0} \left[ RR_{\beta}^{K_{\beta}} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k(\mathcal{N}_{\mathbb{C}}^{+,\beta}) \right) \wedge_{\mathbb{C}}^{\bullet} (\mathfrak{k}/\mathfrak{k}_{\beta})_{\mathbb{C}} \right]^{K_{\beta}} \end{aligned}$$

Let  $E \rightarrow \mathcal{U}$  be any  $K_{\beta}$ -equivariant Hermitian vector bundle. Since  $\beta$  acts trivially on  $\mathcal{U}$  we can look at the Lie derivative  $\mathcal{L}(\beta)$  on  $E$ . Then  $\frac{1}{i}\mathcal{L}(\beta)$  defines for each  $x \in \mathcal{U}$  a Hermitian endomorphism of  $E_x$ . Let us denote

$$\frac{1}{i}\mathcal{L}(\beta) > 0$$

when all its eigenvalue on the fibers of  $E$  are strictly positive.

We made in [18] the crucial observation

**Lemma 4.6.** *If  $\frac{1}{i}\mathcal{L}(\beta) > 0$ , then  $\left[ RR_{\beta}^{K_{\beta}}(\mathcal{U}, E) \right]^{K_{\beta}} = 0$ .*

Let us compute the Lie action  $\mathcal{L}(\beta)$  on the fibers of the bundle  $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k(\mathcal{N}_{\mathbb{C}}^{+,\beta})$ . It is easy to check (see [19]) that on  $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta}$  the Lie action  $\frac{1}{i}\mathcal{L}(\beta)$  is equal to

$$\|\beta\|^2 + \frac{1}{2} \mathbf{Tr}_{\beta} |\mathcal{N}|$$

Look now at the Lie derivative  $\mathcal{L}(\beta)$  on  $\wedge_{\mathbb{C}}^{\bullet}(\mathfrak{k}/\mathfrak{k}_{\beta})_{\mathbb{C}}$ . As a  $T$ -module  $\wedge_{\mathbb{C}}^{\bullet}(\mathfrak{k}/\mathfrak{k}_{\beta})_{\mathbb{C}}$  is equal to

$$\begin{aligned} \prod_{(\alpha, \beta) \neq 0} (1 - e^{i\alpha}) &= \prod_{(\alpha, \beta) < 0} (1 - e^{i\alpha}) \prod_{(\alpha, \beta) > 0} (1 - e^{i\alpha}) \\ &= (-1)^{1/2 \dim(K/K_{\beta})} e^{-i\delta_{\beta}} \left( \prod_{(\alpha, \beta) > 0} (1 - e^{i\alpha}) \right)^2 \end{aligned}$$

with  $\delta_{\beta} = \sum_{(\alpha, \beta) > 0} \alpha$ . Notice that  $e^{-i\delta_{\beta}}$  defines a character of the group  $K_{\beta}$  that we denote  $\mathbb{C}_{-\delta_{\beta}}$ . We have proved then that

$$\wedge_{\mathbb{C}}^{\bullet}(\mathfrak{k}/\mathfrak{k}_{\beta})_{\mathbb{C}} = (-1)^{n_{\beta,+}} \mathbb{C}_{-\delta_{\beta}} \oplus R$$



where the Lie derivative  $\frac{1}{i}\mathcal{L}(\beta)$  on  $\mathbb{C}_{-\delta_\beta}$  is equal to  $-(\delta_\beta, \beta) = -\mathbf{Tr}_\beta|\mathfrak{k}|$  and the Lie derivative  $\frac{1}{i}\mathcal{L}(\beta)$  on the  $\mathfrak{k}_\beta$ -module  $R$  is  $> -\mathbf{Tr}_\beta|\mathfrak{k}|$ .

Since  $\|\beta\|^2 + \frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}| = \mathbf{Tr}_\beta|\mathfrak{k}|$ , we can conclude that the Lie derivative  $\frac{1}{i}\mathcal{L}(\beta)$

- (1) is equal to zero on  $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \otimes \mathbb{C}_{-\delta_\beta}$ ,
- (2) is  $> 0$  on  $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \otimes R$ ,
- (3) is  $> 0$  on  $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \otimes S^k(\mathcal{N}_{\mathbb{C}}^{+, \beta}) \otimes \wedge_{\mathbb{C}}^\bullet(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$  for any  $k \geq 1$ .

With Lemma 4.6, we see that the sum (4.30) restricts to

$$(-1)^{n_{\beta,+}} \left[ RR_\beta^{K_\beta} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta}$$

At this stage we have proved that the multiplicity  $m_\mu$  is equal to

$$(4.31) \quad \left[ RR_\beta^{K_\beta} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta}.$$

On the symplectic slice  $(Y_\sigma, \omega_\sigma)$ , we have the moment map  $\Phi_\sigma - \mu$  relative to the action of  $Z_\sigma$ . The data  $(Y_\sigma, \omega_\sigma, \Phi_\sigma - \mu)$  is Spin-prequantized by the line bundle  $\tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}$ . Let

$$(4.32) \quad RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \in R^{-\infty}(Z_\sigma)$$

be the Riemann-Roch character localized near  $(\Phi_\sigma - \mu)^{-1}(0) = \Phi^{-1}(\mu + \rho - \rho_\sigma) \subset Y_\sigma$ .

We conclude the computation of the multiplicity  $m_\mu$  with the

**Lemma 4.7.** *We have*

$$\begin{aligned} m_\mu &= \left[ RR_\beta^{K_\beta} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta} \\ &= \left[ RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^{Z_\sigma} \quad (1) \\ &= \mathcal{Q}_{\text{spin}}(M_\mu^\sigma). \quad (2) \end{aligned}$$

*Proof.* Let us prove that (1) is a consequence of the induction formula of Proposition 3.9. First we notice that the data  $(Y_\sigma, \omega_\sigma, \Phi_\sigma - \mu, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu})$  is naturally equipped with an action of the maximal torus, but with a trivial action of  $T/Z_\sigma$ . So the generalized character (4.32) coincides with

$$RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \in R^{-\infty}(T).$$

Let us consider the Hamiltonian  $K_\beta$ -manifold  $\mathcal{U} := (K_\beta \times_T Y_\sigma) \times \overline{K_\beta(\mu + \rho)}$ . Since  $K_\beta$  acts trivially on  $\rho_\sigma$  the map  $\xi \mapsto \xi + \rho_\sigma$  realizes a  $K_\beta$ -equivariant symplectomorphism between the coadjoint orbits  $\overline{K_\beta(\mu + \rho)}$  and

$$\overline{\mathcal{O}} := \overline{K_\beta(\mu + \rho - \rho_\sigma)}.$$

The manifold  $\mathcal{U}$  is then symplectomorphic to  $(K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}$ . Moreover, one sees that the generalized Riemann-Roch character  $RR_\beta^{K_\beta}(\mathcal{U}, -)$  coincides with the Riemann-Roch character

$$RR_0^{K_\beta}((K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}, -)$$

localized on  $C_0 := K_\beta(\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho - \rho_\sigma\})$ .

Since  $K_\beta \cap K_\sigma = T$ , the Hamiltonian  $T$ -manifold  $Y_\sigma$  corresponds to the symplectic slice of the Hamiltonian  $K_\beta$ -manifold  $K_\beta \times_T Y_\sigma$ .

The bundle  $\det \mathcal{N}^{+, \beta}$  over  $(K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}$  is equal to the product of  $K_\beta \times_T \mathbb{C}_{\delta_1} \rightarrow K_\beta \times_T Y_\sigma$  with  $K_\beta \times_T \mathbb{C}_{\delta_2} \rightarrow \overline{\mathcal{O}}$ , where

$$\delta_1 = \sum_{\substack{\alpha > 0 \\ (\alpha, \beta) > 0}} \alpha \quad \text{and} \quad \delta_2 = \sum_{\substack{\alpha < 0 \\ (\alpha, \beta) > 0}} \alpha.$$

The line bundle  $\tilde{L}_N$  is equal to the product of  $\tilde{L}$  with  $K \times_T \mathbb{C}_{-\mu}$ . Then the restrictions of the line bundle  $\det \mathcal{N}^{+, \beta}$  and  $\tilde{L}_N$  to  $Y_\sigma \times \{\mu + \rho - \rho_\sigma\}$  are respectively equal to, the trivial line bundle  $\mathbb{C}_{\delta_1 + \delta_2} = \mathbb{C}_{\delta_\beta}$ , and to the line bundle  $\tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}$ .

Finally the induction formula of Proposition 3.9 gives that

$$\begin{aligned} RR_\beta^{K_\beta} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) &= RR_0^{K_\beta} \left( (K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) \\ &= \text{Ind}_T^{K_\beta} \left( RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \otimes \mathbb{C}_{\delta_\beta} \right) \\ &= \text{Ind}_T^{K_\beta} \left( RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma}) \right) \otimes \mathbb{C}_{\delta_\beta}. \end{aligned}$$

Hence

$$\begin{aligned} \left[ RR_\beta^{K_\beta} \left( \mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta} &= \left[ RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^T \\ &= \left[ RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^{Z_\sigma}. \end{aligned}$$

Equality (2), i.e.

$$\left[ RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^{Z_\sigma} = \mathcal{Q}_{\text{spin}}(M_\mu^\sigma),$$

has been proved in Section 3.4. □

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INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER (I3M), UNIVERSITÉ MONTPELLIER 2

*E-mail address:* Paul-Emile.Paradan@math.univ-montp2.fr