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Paul-Emile Paradan

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SPIN-QUANTIZATION COMMUTES WITH REDUCTION

PAUL-EMILE PARADAN

ABSTRACT. In this paper, we prove that the “quantization commutes with reduction” phenomenon of Guillemin-Sternberg [10] applies in the context of the metaplectic correction.

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1. INTRODUCTION

An Hamiltonian K -manifold (M, ω, Φ) is Spin-prequantized if M carries an equivariant Spin^c structure P with determinant line bundle being a Kostant-Souriau line bundle over $(M, 2\omega, 2\Phi)$. Let \mathcal{D}_P be the Spin^c Dirac operator attached to P , where M is oriented by its symplectic form. The Spin quantization of (M, ω, Φ) corresponds to the equivariant index of the elliptic operator \mathcal{D}_P , and is denoted

$$\mathcal{Q}_{\text{spin}}^K(M) \in R(K).$$

Let $\widehat{A}(M)(X)$ be the equivariant A-genus class: it is an equivariant analytic function from a neighborhood of $0 \in \mathfrak{k}$ with value in the algebra of differential forms on M . The Atiyah-Segal-Singer index theorem [6] tell us that

$$(1.1) \quad \mathcal{Q}_{\text{spin}}^K(M)(e^X) := \int_M e^{i(\omega + \langle \Phi, X \rangle)} \widehat{A}(M)(X)$$

for $X \in \mathfrak{k}$ small enough. It shows in particular that $\mathcal{Q}_{\text{spin}}^K(M) \in R(K)$ does not depend of the choice of the Spin-prequantum data.

This notion of Spin-quantization is closely related to the notion of *metaplectic correction*. Suppose that (M, ω, Φ) carries a Kostant-Souriau line bundle L_ω , and that the bundle of half-form $\kappa_J^{1/2}$ associated to an invariant almost complex structure J is well defined. In this case, (M, ω, Φ) is Spin-prequantized by the Spin^c -structure defined by J and twisted by the line bundle $L_\omega \otimes \kappa_J^{1/2}$. The crucial point here is that the corresponding Spin-quantization of (M, ω, Φ) does not depend

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of the choice of the almost complex structure. Note that the existence of the bundle of half-form $\kappa_J^{1/2}$ is equivalent to the existence of a Spin structure on M [14].

The purpose of this article is to compute geometrically the multiplicities of $\mathcal{Q}_{\text{spin}}^K(M) \in R(K)$ in a way similar to the famous “quantization commutes with reduction” phenomenon of Guillemin-Sternberg [10, 16, 17, 24, 18, 25, 12, 26, 21]. This question was partially resolved in a previous paper [19] under the condition that the infinitesimal stabilizers of the K -action are abelian. C. Teleman also obtained some results [23][Proposition 3.10] in the algebraic setting.

The striking difference with the standard Guillemin-Sternberg phenomenon is the *rho shift* that we explain now. Let T be a maximal torus of K with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$. Let $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ be the closed Weyl chamber. We will look at \mathfrak{t}_+^* as a disjoint union of its open faces, the maximal one being its interior $(\mathfrak{t}_+^*)^\circ$. Let $\rho \in (\mathfrak{t}_+^*)^\circ$ be the half sum of the positive roots. At each open face τ of \mathfrak{t}_+^* , we associate the term ρ_τ which is the half sum of the positive roots which are orthogonal to τ . We note that $\rho - \rho_\tau \in \tau$ corresponds to the orthogonal projection of ρ on τ .

For any $\xi \in \mathfrak{t}_+^*$ and any face τ containing ξ in its closure, we consider the *shifted symplectic reduction*

$$M_\xi^\tau := \Phi^{-1}(\xi + \rho - \rho_\tau)/K_\tau$$

where K_τ is the common stabiliser of points in τ . Note that $\xi + \rho - \rho_\tau \in \tau$ when $\xi \in \bar{\tau}$.

We are particularly interested to the smallest face σ of the Weyl chamber so that the Kirwan polytope $\Delta(M) := \Phi(M) \cap \mathfrak{t}_+^*$ is contained in the closure of σ . It is not hard to see that the Spin-prequantum data on (M, ω, Φ) descends to the shifted symplectic reduction M_μ^σ when μ is a dominant weight belonging to $\bar{\sigma}$. Then $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma) \in \mathbb{Z}$ is naturally defined when $\mu + \rho - \rho_\sigma$ is a regular value of the moment map. In general, the number $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma)$ is defined by shift-desingularization.

By definition $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma)$ vanishes when $\mu + \rho - \rho_\sigma \notin \Delta(M)$, but in fact we can strengthen this vanishing property: $\mathcal{Q}_{\text{spin}}(M_\mu^\sigma) = 0$ if $\mu + \rho - \rho_\sigma$ does not belong to the *relative interior* of the Kirwan polytope $\Delta(M)$.

Recall that the irreducible representation V_μ^K of K are parametrized by their highest weight $\mu \in \widehat{K} \subset \mathfrak{t}_+^*$.

The main result of this paper is the following

Theorem 1.1. *Let (M, ω, Φ) be a compact Spin-prequantized Hamiltonian K -manifold. Let σ be the smallest face of the Weyl chamber so that $\Delta(M) \subset \bar{\sigma}$. We have*

$$\mathcal{Q}_{\text{spin}}^K(M) = \sum_{\mu \in \widehat{K} \cap \bar{\sigma}} \mathcal{Q}_{\text{spin}}(M_\mu^\sigma) V_\mu^K$$

Let us give some ideas about the proof. The representation V_μ^K is equal to the Spin-quantization of the coadjoint orbit $\mathcal{O}_\mu := K \cdot (\mu + \rho)$. Then the shifting trick tells us that the multiplicity m_μ of V_μ^K in $\mathcal{Q}_{\text{spin}}^K(M)$ is equal to

$$[\mathcal{Q}_{\text{spin}}^K(M) \otimes (V_\mu^K)^*]^K = [\mathcal{Q}_{\text{spin}}^K(M \times \bar{\mathcal{O}}_\mu)]^K,$$

where $\overline{\mathcal{O}}_\mu$ is the coadjoint orbit with the opposite symplectic structure. As we did in [18, 19], we study the expression $[\mathcal{Q}_{\text{spin}}^K(M \times \overline{\mathcal{O}}_\mu)]^K$ by localizing the Riemann-Roch character on the critical points of the square of the moment map

$$\Phi_\mu : M \times \overline{\mathcal{O}}_\mu \rightarrow \mathfrak{k}^*.$$

Here our treatment differs depending on whether the Kirwan polytope $\Delta(M)$ intersects the interior of the Weyl chambers or not (i.e. $\sigma = \mathfrak{t}_+^*$ or not).

When $\sigma = \mathfrak{t}_+^*$, we show that the multiplicity m_μ is calculated using the Riemann-Roch character localized near the zero level set of the moment map Φ_μ . This case is (more or less) treated in [19].

The heart of this paper is when we work out the case $\sigma \neq \mathfrak{t}_+^*$. We have $\Phi_\mu^{-1}(0) = \emptyset$, but we show how to compute m_μ using the Riemann-Roch character localized near

$$K \cdot (N^{\rho_\sigma} \cap \Phi_\mu^{-1}(-\rho_\sigma)).$$

Here N^{ρ_σ} denotes the submanifold of $N = M \times \mathcal{O}_\mu$ where the infinitesimal action of ρ_σ vanishes.

Notations. Throughout the paper, K will denote a compact connected Lie group, and \mathfrak{k} its Lie algebra. We let T be a maximal torus in K , and \mathfrak{t} be its Lie algebra. The integral lattice $\Lambda \subset \mathfrak{t}$ is defined as the kernel of $\exp : \mathfrak{t} \rightarrow T$, and the real weight lattice $\Lambda^* \subset \mathfrak{t}^*$ is defined by $\Lambda^* := \text{hom}(\Lambda, 2\pi\mathbb{Z})$. Every $\mu \in \Lambda^*$ defines a 1-dimensional T -representation, denoted \mathbb{C}_μ , where $t = \exp(X)$ acts by $t^\mu := e^{i\langle \mu, X \rangle}$. We fix a positive Weyl chamber $\mathfrak{t}_+^* \subset \mathfrak{t}^*$. For any dominant weight $\mu \in \widehat{K} := \Lambda^* \cap \mathfrak{t}_+^*$, we denote by V_μ^K the irreducible representation with highest weight μ . We denote $R(K)$ the representation ring of K . We denote $R^{-\infty}(K) := \text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$ its dual. An element $E \in R^{-\infty}(K)$ can be represented as an infinite sum $E = \sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$, with $m_\mu \in \mathbb{Z}$. The multiplicity m_0 of the trivial representation is denoted $[E]^K$. If H is a closed subgroup of K , we have the induction map $\text{Ind}_H^K : R^{-\infty}(H) \rightarrow R^{-\infty}(K)$ which is the dual of the restriction morphism $R(K) \rightarrow R(H)$. We see that $[\text{Ind}_H^K(E)]^K = [E]^H$.

2. SPIN-QUANTIZATION OF COMPACT HAMILTONIAN K -MANIFOLDS

Let M be a *compact* Hamiltonian K -manifold with symplectic form ω and moment map $\Phi : M \rightarrow \mathfrak{k}^*$ characterized by the relation

$$(2.2) \quad \iota(X_M)\omega = -d\langle \Phi, X \rangle, \quad X \in \mathfrak{k},$$

where $X_M(m) := \left. \frac{d}{dt} \right|_{t=0} e^{-tX} \cdot m$ is the vector field on M generated by $X \in \mathfrak{k}$.

In the Kostant-Souriau framework [13, 22], a Hermitian line bundle L_ω with an invariant Hermitian connection ∇ is a prequantum line bundle over (M, ω, Φ) if

$$(2.3) \quad \mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X \rangle \quad \text{and} \quad \nabla^2 = -i\omega,$$

for every $X \in \mathfrak{k}$. Here $\mathcal{L}(X)$ is the infinitesimal action of $X \in \mathfrak{k}$ on the sections of $L_\omega \rightarrow M$. (L_ω, ∇) is also called a Kostant-Souriau line bundle. Remark that conditions (2.3) imply, via the equivariant Bianchi formula, the relation (2.2).

2.1. Spin-quantization: definitions. Let J be any invariant almost complex structure on M , not necessarily compatible with the symplectic form ω . Let

$$RR_J^K(M, -)$$

be the corresponding Riemann-Roch character [18]. We consider the complex tangent bundle (TM, J) and its complex dual $T_{\mathbb{C}}^*M := \text{hom}_{\mathbb{C}}(TM, \mathbb{C})$. We consider the line bundle

$$\kappa_J := \det T_{\mathbb{C}}^*M.$$

If (M, ω, Φ) is prequantized by L_ω , a standard procedure (called the metaplectic correction in the geometric quantization literature) is to tensor L_ω by the bundle of half-forms $\kappa_J^{1/2}$ [27]. We may consider the equivariant index

$$(2.4) \quad \mathcal{Q}_J^K(M) := \epsilon_J RR_J^K(M, L_\omega \otimes \kappa_J^{1/2})$$

where $\epsilon_J = \pm 1$ is the quotient of the orientations defined by ω and by J . Note that $\mathcal{Q}_J^K(M)$ is well defined when the tensor product $\tilde{L} = L_\omega \otimes \kappa_J^{1/2}$ has a meaning (even if neither L_ω nor $\kappa_J^{1/2}$ exist).

The almost complex structure J defines a Spin^c structure P_J on M with determinant line bundle $\det P_J = \kappa_J^{-1}$. If we twist the Spin^c structure P_J by any complex line bundle \mathbb{L} we get a Spin^c structure $P_{J, \mathbb{L}}$ with determinant line bundle

$$\det P_{J, \mathbb{L}} = \mathbb{L}^2 \otimes \kappa_J^{-1}.$$

See [14, 19].

We make the following basic observation.

Proposition 2.1. *Let (M, ω, Φ) be an Hamiltonian K -manifold. The following assertions are equivalent:*

- a) *For any invariant complex structure J there exists a K -equivariant line bundle \tilde{L} such that $\tilde{L}^2 \otimes \kappa_J^{-1}$ is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.*
- b) *There exist an invariant line complex structure J and a K -equivariant line bundle \tilde{L} such that $\kappa_J^{-1} \otimes \tilde{L}^2$ is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.*
- c) *There exists an equivariant Spin^c structure P such that its determinant line bundle $\det P$ is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.*

When the previous assertions holds, we says that (M, ω, Φ) is Spin -prequantized, either by the Spin^c -structure P , or by the data (J, \tilde{L}) .

Proposition 2.2. *Let (M, ω, Φ) be a Spin -prequantized Hamiltonian K -manifold. The equivariant index $\mathcal{Q}_J^K(M) := \epsilon_J RR_J^K(M, \tilde{L})$ does not depend of the choice of the Spin -prequantum data (J, \tilde{L}) . In fact $\mathcal{Q}_J^K(M)$ coincides with the equivariant index of the Spin^c Dirac operator \mathcal{D}_P attached to the Spin^c -structure P .*

Definition 2.3. *Let (M, ω, Φ) be a Spin -prequantized Hamiltonian K -manifold. The Spin -quantization of (M, ω, Φ) is defined as the equivariant index $\mathcal{Q}_J^K(M)$, and is denoted*

$$\mathcal{Q}_{\text{spin}}^K(M) \in R(K)$$

Proof of Propositions 2.1 and 2.2. We have obviously $a) \implies b)$, and we get $b) \implies c)$ by taking the Spin^c structure $P_{J, \tilde{L}}$. Let us prove $c) \implies a)$.

Let P be a Spin^c -structure on M such that its determinant line bundle $\det P$ is a prequantum line bundle over $(M, 2\omega)$. Let \mathcal{S}_P be the corresponding bundle of spinors. Let P_J and \mathcal{S}_J be respectively the associated Spin^c -structure and the bundle of spinors on M associated to an invariant almost complex structure J on M . Since $\mathcal{S}_P, \mathcal{S}_J$ are irreducible clifford modules, we have

$$(2.5) \quad \mathcal{S}_P \simeq \mathcal{S}_J \otimes \tilde{L}$$

where \tilde{L} is the line bundle defined by $\tilde{L} := \text{hom}_{\text{cl}}(\mathcal{S}_J, \mathcal{S}_P)$. From (2.5) we get that the line bundle

$$\begin{aligned} \det P &= \tilde{L}^2 \otimes \det P_J \\ &= \tilde{L}^2 \otimes \kappa_J^{-1}. \end{aligned}$$

is a prequantum line bundle over $(M, 2\omega, 2\Phi)$.

Let P be the Spin^c structure attached to a data (J, \tilde{L}) . The symplectic orientation on M defines a decomposition on the bundle of spinors, $\mathcal{S}_P = \mathcal{S}_P^+ \oplus \mathcal{S}_P^-$, and the corresponding Spin^c Dirac operator \mathcal{D}_P maps $\Gamma(\mathcal{S}_P^+)$ to $\Gamma(\mathcal{S}_P^-)$.

On the other hand the almost complex structure on M gives the decomposition $\wedge T^*M \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j} T^*M$ of the bundle of differential form. The corresponding bundle of spinors is $\mathcal{S}_J := \wedge^{0,\bullet} T^*M$ and the complex orientation induces the splitting $\mathcal{S}_J = \mathcal{S}_J^+ \oplus \mathcal{S}_J^-$ with $\mathcal{S}_J^+ := \wedge^{0,\text{even}} T^*M$. The Dolbeault Dirac operator $\bar{\partial}_{\tilde{L}} + \bar{\partial}_{\tilde{L}}^*$ maps $\Gamma(\mathcal{S}_J^+ \otimes \tilde{L})$ to $\Gamma(\mathcal{S}_J^- \otimes \tilde{L})$, and the Riemann Roch character $RR_J^K(M, \tilde{L})$ is defined as the equivariant index of the elliptic operator

$$\bar{\partial}_{\tilde{L}} + \bar{\partial}_{\tilde{L}}^* : \Gamma(\mathcal{S}_J^+ \otimes \tilde{L}) \longrightarrow \Gamma(\mathcal{S}_J^- \otimes \tilde{L})$$

If $\epsilon_J = \pm 1$ is the quotient of the orientations defined by ω and by J , one has that

$$\mathcal{S}_P^\pm = \mathcal{S}_J^{\pm \epsilon_J} \otimes \tilde{L}.$$

Hence $\mathcal{Q}_J^K(M) = \epsilon_J RR_J^K(M, \tilde{L})$ is defined as the equivariant index of the Dolbeault Dirac operator $\bar{\partial}_{\tilde{L}} + \bar{\partial}_{\tilde{L}}^*$ viewed as an elliptic operator $\mathcal{D}_{\tilde{L}}^+$ from $\Gamma(\mathcal{S}_P^+)$ to $\Gamma(\mathcal{S}_P^-)$.

Finally we know that $\text{Index}^K(\mathcal{D}_P) = \text{Index}^K(\mathcal{D}_{\tilde{L}}^+)$ since the first order elliptic operators \mathcal{D}_P and $\mathcal{D}_{\tilde{L}}^+$ have the same principal symbol [9]. \square

In the remaining part of this paper, we find convenient to work with the following

Definition 2.4. *An Hamiltonian K -manifold (M, ω, Φ) is Spin-prequantized by \tilde{L} if there exists an invariant almost complex structure J compatible with ω such that $\tilde{L}^2 \otimes \kappa_J^{-1}$ is a Kostant-Souriau line bundle over $(M, 2\omega, 2\Phi)$.*

We remark that $\epsilon_J = 1$ when J is compatible with ω . Moreover, the Riemann-Roch character $RR_J^K(M, -)$ does not depend [18] on the choice of the compatible invariant almost complex structure J : we denote it simply by $RR^K(M, -)$.

Finally, when an Hamiltonian manifold (M, ω, Φ) is Spin-prequantized by the line bundle \tilde{L} , its Spin-quantization is defined by

$$\mathcal{Q}_{\text{spin}}^K(M) := RR^K(M, \tilde{L}).$$

2.2. Functorial properties. We summarize the functorial properties of $\mathcal{Q}_{\text{spin}}$ in the next

Proposition 2.5. • *If (M, ω, Φ) is a Spin-prequantized Hamiltonian K -manifold, and H is a closed subgroup of K then the restriction of $\mathcal{Q}_{\text{spin}}^K(M)$ to H is equal to $\mathcal{Q}_{\text{spin}}^H(M)$.*

• *If (M_j, ω_j, Φ_j) are Spin-prequantized Hamiltonian K_j -manifold, for $j = 1, 2$, then $M_1 \times M_2$ is a Spin-prequantized Hamiltonian $K_1 \times K_2$ -manifold and*

$$\mathcal{Q}_{\text{spin}}^{K_1 \times K_2}(M_1 \times M_2) = \mathcal{Q}_{\text{spin}}^{K_1}(M_1) \otimes \mathcal{Q}_{\text{spin}}^{K_2}(M_2)$$

in $R(K_1 \times K_2) \simeq R(K_1) \otimes R(K_2)$.

• *If (M, ω_M, Φ_M) and (N, ω_N, Φ_N) are Spin-prequantized Hamiltonian K -manifold, then $M \times N$ is a Spin-prequantized Hamiltonian K -manifold and*

$$\mathcal{Q}_{\text{spin}}^K(M \times N) = \mathcal{Q}_{\text{spin}}^K(M) \cdot \mathcal{Q}_{\text{spin}}^K(N),$$

where \cdot denotes the product in $R(K)$.

• *A Spin-prequantization on (M, ω, Φ) induces a Spin-prequantization on $\overline{M} := (M, -\omega, -\Phi)$. The Spin-quantization of \overline{M} corresponds to the dual of the Spin-quantization of M :*

$$\mathcal{Q}_{\text{spin}}^K(\overline{M}) = [\mathcal{Q}_{\text{spin}}^K(M)]^*.$$

Proof. The first three points are direct consequences of the functorial properties of the index map. Let us prove the last point. One see that if (\tilde{L}, J) is a Spin-prequantum data for M then $(\tilde{L}^{-1}, -J)$ is a Spin-prequantum data for \overline{M} . Then we have for $X \in \mathfrak{k}$ small enough

$$\begin{aligned} \mathcal{Q}_{\text{spin}}^K(\overline{M})(e^X) &= \int_M e^{i(-\omega - \langle \Phi, X \rangle)} \widehat{A}(M)(X) \\ &= \overline{\int_M e^{i(\omega + \langle \Phi, X \rangle)} \widehat{A}(M)(X)} \quad [1] \\ &= \overline{\mathcal{Q}_{\text{spin}}^K(M)(e^X)}. \quad [2] \end{aligned}$$

The relation [1] is due to the fact that the differential form $\widehat{A}(M)(X)$ has real coefficients. Since $X \rightarrow \mathcal{Q}_{\text{spin}}^K(M)(e^X)$ are analytic functions, the identity [2] shows that $\mathcal{Q}_{\text{spin}}^K(\overline{M})(k) = \overline{\mathcal{Q}_{\text{spin}}^K(M)(k)}$ for any $k \in K$. In other words the (virtual) representation $\mathcal{Q}_{\text{spin}}^K(\overline{M})$ corresponds to the dual of the (virtual) representation $\mathcal{Q}_{\text{spin}}^K(M)$. \square

2.3. Spin-quantization of coadjoint orbits. Let $\mu \in \widehat{K}$ be a dominant weight. Let σ be a face of the Weyl chamber such that $\mu \in \overline{\sigma}$: hence the stabilizer subgroup K_μ contains K_σ . We will restrict the one-dimensional representation \mathbb{C}_μ of K_μ to the subgroup K_σ .

Let ρ be half the sum of the positive roots, and let ρ_σ be half the sum of the positive roots which are orthogonal to σ . Note that $\rho - \rho_\sigma$ belongs to σ , hence $\mu + \rho - \rho_\sigma$ belongs also to σ for any $\mu \in \overline{\sigma}$. The coadjoint orbit

$$\mathcal{O}_\mu^\sigma := K \cdot (\mu + \rho - \rho_\sigma)$$

is Spin-prequantized by the compatible complex structure and the line bundle $\tilde{L} = K \times_{K_\sigma} \mathbb{C}_\mu$. We have

$$\begin{aligned} \mathcal{Q}_{\text{spin}}^K(\mathcal{O}_\mu^\sigma) &= RR^K(K/K_\sigma, K \times_{K_\sigma} \mathbb{C}_\mu) \\ &= V_\mu^K. \end{aligned}$$

Thanks to Proposition 2.5, we know that $\mathcal{Q}_{\text{spin}}^K(\overline{\mathcal{O}_\mu^\sigma}) = (V_\mu^K)^*$, where $\overline{\mathcal{O}_\mu^\sigma}$ be the coadjoint orbit \mathcal{O}_μ^σ with the opposite symplectic form.

We have seen that the same irreducible representations V_μ^K can be realized as the Spin-quantization of the coadjoint orbits \mathcal{O}_μ^σ where σ is a face of the Weyl chamber containing μ in its closure.

2.4. Spin-prequantization commutes with reduction. We consider first the case of an Hamiltonian H -manifold (N, ω, Φ) , not necessarily compact, which is Spin-prequantized by \tilde{L} . We suppose that 0 is a regular value of Φ . Let $N_0 := \Phi^{-1}(0)/H$ be the orbifold reduced space with its canonical symplectic structure ω_0 .

Lemma 2.6. *The orbifold line bundle $\tilde{\mathcal{L}}_0 := (\tilde{L}|_{\Phi^{-1}(0)})/H$ Spin-prequantizes (N_0, ω_0) .*

Proof. The fiber $\mathcal{Z} = \Phi^{-1}(0)$ is a smooth H -invariant submanifold of N . Let $\pi : \mathcal{Z} \rightarrow \mathcal{Z}/H = N_0$ be the projection. Recall that the symplectic structure ω_0 on N_0 is defined by the relation $\pi^*(\omega_0) = \omega|_{\mathcal{Z}}$. Let $L_{2\omega}$ the Kostant-Souriau line bundle on $(N, 2\omega, 2\Phi)$ such that

$$(2.6) \quad \tilde{L}^2 = L_{2\omega} \otimes \kappa_J.$$

Here J is a compatible invariant almost complex structure on N . We have $\text{TM}|_{\mathcal{Z}} = \text{T}\mathcal{Z} \oplus J(\mathfrak{h}_{\mathcal{Z}})$ where $\mathfrak{h}_{\mathcal{Z}} \subset \text{T}\mathcal{Z}$ is the trivial bundle given by the infinitesimal action of H . Since $\text{T}\mathcal{Z} \simeq \pi^*(\text{TN}_0) \oplus \mathfrak{h}_{\mathcal{Z}}$ we get

$$\text{TM}|_{\mathcal{Z}} \simeq \pi^*(\text{TN}_0) \oplus \mathfrak{h}_{\mathcal{Z}} \oplus J(\mathfrak{h}_{\mathcal{Z}}).$$

Hence J induces a compatible almost complex structure J_0 on (N_0, ω_0) , such that $(\kappa_J|_{\mathcal{Z}})/H = \kappa_{J_0}$.

The line bundle $L_{2\omega_0} = (L_{2\omega}|_{\mathcal{Z}})/H$ is a prequantum line bundle on (N_0, ω_0) . Finally, if we restrict (2.6) to \mathcal{Z} , we have after taking the quotient by H that

$$\tilde{\mathcal{L}}_0^2 = L_{2\omega_0} \otimes \kappa_{J_0}.$$

We have proved that $(J_0, \tilde{\mathcal{L}}_0)$ Spin-prequantizes (N_0, ω_0) . \square

For the rest of this section we consider a compact Hamiltonian K -manifold (M, ω, Φ) , that we suppose Spin-prequantized by the line bundle \tilde{L} .

Let τ be a face of the Weyl chamber, and let K_τ be the common stabilizer of points in τ . Following Guillemin-Sternberg [11], we introduce the following K_τ -invariant open subset of \mathfrak{k}_τ^* :

$$U_\tau = K_\tau \cdot \{\xi \in \mathfrak{k}_+^* | K_\xi \subset K_\tau\} = K_\tau \cdot \bigcup_{\tau \subset \bar{\sigma}} \sigma.$$

By construction, U_τ is a slice for the coadjoint action: this mean that the map $K \times U_\tau, (k, \xi) \mapsto k \cdot \xi$ factors through an inclusion $K \times_{K_\tau} U_\tau \hookrightarrow \mathfrak{k}^*$.

The symplectic cross-section theorem [11] asserts that the pre-image $Y_\tau = \Phi^{-1}(U_\tau)$ is a symplectic submanifold : we denote ω_τ the restriction of ω to Y_τ . The action of

K_τ on (Y_τ, ω_τ) is Hamiltonian, where the restriction of Φ to Y_τ is a moment map. We will work with the translated moment map $\Phi_\tau : Y_\tau \rightarrow \mathfrak{k}_\tau^*$ defined by

$$\Phi_\tau = \Phi|_{Y_\tau} - (\rho - \rho_\tau).$$

Lemma 2.7. *The symplectic slice $(Y_\tau, \omega_\tau, \Phi_\tau)$ is Spin-prequantized by the line bundle $\tilde{L}_\tau := \tilde{L}|_{Y_\tau}$.*

Proof. We consider the open subset $K \times_{K_\tau} Y_\tau$ of M and the projection $\pi : K \times_{K_\tau} Y_\tau \rightarrow K/K_\tau$. We can suppose that the Spin-prequantum data, when restricted to $K \times_{K_\tau} Y_\tau$, is given by (J, \tilde{L}) where J is a compatible almost complex structure on $K \times_{K_\tau} Y_\tau$ defined as the “sum” of the compatible almost complex structures J_o and J_τ : J_o on K/K_τ and J_τ on Y_τ .

When we restrict the identity $L_{2\omega} = \tilde{L}^2 \otimes \kappa_J^{-1}$ to Y_τ we get

$$(2.7) \quad L_{2\omega}|_{Y_\tau} = (\tilde{L}|_{Y_\tau})^2 \otimes \kappa_{J_\tau}^{-1} \otimes \mathbb{C}_{2(\rho - \rho_\tau)}.$$

We consider the following line bundle on Y_τ :

$$L_{2\omega_\tau} := L_{2\omega}|_{Y_\tau} \otimes \mathbb{C}_{2(\rho - \rho_\tau)}^{-1}.$$

The relation (2.7) is then $L_{2\omega_\tau} = (\tilde{L}|_{Y_\tau})^2 \otimes \kappa_{J_\tau}^{-1}$. Since $L_{2\omega_\tau}$ is a K_τ -equivariant prequantum bundle over $(Y_\tau, 2\omega_\tau, 2\Phi_\tau)$, we conclude that $(Y_\tau, \omega_\tau, \Phi_\tau)$ is Spin-prequantized by the data (J_τ, \tilde{L}_τ) . \square

Let us consider the case where $\tau = \sigma$ is the *smallest face* of the Weyl chamber so that moment polyhedron $\Delta(M) := \Phi(M) \cap \mathfrak{k}_+^*$ is contained in the closure of σ . Then the symplectic slice Y_σ is equal to $\Phi^{-1}(\sigma)$, and the action of the subgroup $[K_\sigma, K_\sigma]$ is trivial on it [15].

We will then consider the Hamiltonian action of the center $Z_\sigma = Z(K_\sigma)$ on Y_σ . The map $\Phi_\sigma : Y_\sigma \rightarrow \mathfrak{k}_\sigma^*$ takes values in $\mathfrak{z}_\sigma^* = \mathbb{R}\sigma \subset \mathfrak{k}^*$ and corresponds to the moment map relative to the action of Z_σ on $(Y_\sigma, \omega_\sigma)$. We know that $(Y_\sigma, \omega_\sigma, \Phi_\sigma)$ is Spin-prequantized by $\tilde{L}_\sigma := \tilde{L}|_{Y_\sigma}$.

For each dominant weights μ which belongs to the closure of σ , we consider the symplectic reduction

$$\begin{aligned} M_\mu^\sigma &= \Phi^{-1}(\mathcal{O}_\mu^\sigma)/K \\ &= \Phi_\sigma^{-1}(\mu)/Z_\sigma. \end{aligned}$$

For the rest of this section we fix a dominant weight $\mu \in \bar{\sigma}$ such that $\mu + \rho - \rho_\sigma \in \Delta(M)$, and we explain how one defines the Spin-quantization of the (possibly singular) reduced spaces M_μ^σ .

Let $\vec{\Delta} \subset \mathfrak{z}_\sigma^*$ be the rational vector subspace generated by $\{a - b \mid a, b \in \Delta(M)\}$. Let $\mathfrak{z}_\sigma^\Delta \subset \mathfrak{z}_\sigma$ be the subspace orthogonal to $\vec{\Delta}$, and let $Z_\sigma^\Delta \subset Z_\sigma$ be the corresponding subtorus.

Lemma 2.8. *The torus Z_σ^Δ acts trivially on Y_σ and on the line bundle $\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$.*

Proof. By definition of $\mathfrak{z}_\sigma^\Delta$, $0 = d\langle \Phi_\sigma, X \rangle = -\iota(X_{Y_\sigma})\omega_\sigma$ on Y_σ for any $X \in \mathfrak{z}_\sigma^\Delta$. Hence the torus Z_σ^Δ acts trivially on Y_σ . Let $L_{2\omega_\sigma}$ be the Kostant-Souriau line bundle over $(Y_\sigma, 2\omega_\sigma, 2\Phi_\sigma)$ so that $\tilde{L}_\sigma^2 = L_{2\omega_\sigma} \otimes \kappa_{J_\sigma}$ (see Lemma 2.7). We have on the section of $L_{2\omega_\sigma}$ the following equality of linear operators:

$$\mathcal{L}(X) - \nabla_{X_M} = i\langle 2\Phi_\sigma, X \rangle, \quad \forall X \in \mathfrak{z}_\sigma.$$

If one takes $X \in \mathfrak{z}_\sigma^\Delta$, the function $y \in Y_\sigma \mapsto \langle \Phi_\sigma(y), X \rangle$ is constant equal to $\langle \mu, X \rangle$. Finally

$$\mathcal{L}(X) - 2i\langle \mu, X \rangle = 0, \quad \forall X \in \mathfrak{z}_\sigma^\Delta$$

as an operator on the section of $L_{2\omega_\sigma}$. In other words, the torus Z_σ^Δ acts trivially on $L_{2\omega_\sigma} \otimes \mathbb{C}_{-2\mu} = (\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu})^2 \otimes \kappa_{J_\sigma}^{-1}$. Since Z_σ^Δ acts trivially on κ_{J_σ} , we conclude finally that Z_σ^Δ acts trivially on the line bundle $\tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$. \square

Let $Z'_\sigma \subset Z_\sigma^\Delta$ be another subtorus such that $Z_\sigma = Z_\sigma^\Delta \times Z'_\sigma$: the dual of its Lie algebra \mathfrak{z}'_σ is identified with $\vec{\Delta} \subset \mathfrak{z}_\sigma^*$. We look now at $(Y_\sigma, \omega_\sigma)$ as an Hamiltonian Z'_σ -manifold with moment map

$$\Phi'_\sigma := \Phi_\sigma - \mu = \Phi|_{Y_\sigma} - (\mu + \rho - \rho_\sigma).$$

The Z'_σ -equivariant line bundle $\tilde{L}'_\sigma := \tilde{L}_\sigma \otimes \mathbb{C}_{-\mu}$ Spin-prequantizes the Hamiltonian Z'_σ -manifold $(Y_\sigma, \omega_\sigma, \Phi'_\sigma)$.

If $0 \in \vec{\Delta}$ is a regular value of Φ'_σ , we know after Lemma 2.6 that the orbifold reduced space $(M_\mu^\sigma, \omega_\mu^\sigma)$ is Spin-prequantized by the line bundle

$$\tilde{\mathcal{L}}_\mu^\sigma := \left(\tilde{L}|_{\Phi_\sigma^{-1}(\mu)} \otimes \mathbb{C}_{-\mu} \right) / Z'_\sigma,$$

and its Spin-quantization $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma)$ is defined like in Definition 2.3. In the general case where $0 \in \vec{\Delta}$ is not necessarily a regular value of Φ'_σ we proceed by shift desingularization. For $\varepsilon \in \vec{\Delta}$ small enough and generic we consider the orbifold reduced space

$$M_{\mu+\varepsilon}^\sigma := (\Phi'_\sigma)^{-1}(\varepsilon) / Z'_\sigma = \Phi_\sigma^{-1}(\mu + \varepsilon) / Z'_\sigma$$

and its orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma := \left(\tilde{L}|_{\Phi_\sigma^{-1}(\mu+\varepsilon)} \otimes \mathbb{C}_{-\mu} \right) / Z'_\sigma.$$

The following crucial fact is proved in Section 3.4.

Theorem 2.9. *The Riemann-Roch number $RR(M_{\mu+\varepsilon}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma) \in \mathbb{Z}$ does not depend of the choice of a generic and small enough $\varepsilon \in \vec{\Delta}$.*

Thanks to the last Theorem we can define the quantization $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) \in \mathbb{Z}$ of the (possibly singular) reduced space M_μ^σ for $\mu \in \widehat{K} \cap \bar{\sigma}$.

Definition 2.10. *Let $\mu \in \widehat{K} \cap \bar{\sigma}$.*

- *If $\mu + \rho - \rho_\sigma \in \Delta(M)$, the integer $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) \in \mathbb{Z}$ is defined as the Riemann-Roch character $RR(M_{\mu+\varepsilon}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma)$ for $\varepsilon \in \vec{\Delta}$ generic and small enough.*
- *If $\mu + \rho - \rho_\sigma \notin \Delta(M)$, we set $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) = 0$*

Remark 2.11. *If $\mu + \rho - \rho_\sigma$ does not belongs to the relative interior of $\Delta(M)$, we can choose ε so that $\mu + \rho - \rho_\sigma + \varepsilon \notin \Delta(M)$. Then the reduced space $M_{\mu+\varepsilon}^\sigma$ is empty and the corresponding Riemann-Roch character $RR(M_{\mu+\varepsilon}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon}^\sigma)$ vanishes. Hence $\mathcal{Q}_{\text{Spin}}(M_\mu^\sigma) = 0$.*

3. SPIN-QUANTIZATION COMMUTES WITH REDUCTION

We look at the multiplicities of the representation $\mathcal{Q}_{\text{spin}}^K(M)$

$$\mathcal{Q}_{\text{spin}}^K(M) = \sum_{\mu \in \widehat{K}} m_\mu V_\mu^K.$$

The main result of this paper is the following

Theorem 3.1. *Let (M, ω, Φ) be a compact Hamiltonian K -manifold which is Spin prequantized. Let σ be the smallest face of the Weyl chamber so that $\Phi(M) \cap \mathfrak{t}_+^* \subset \bar{\sigma}$. We have*

$$\mathcal{Q}_{\text{spin}}^K(M) = \sum_{\mu \in \bar{\sigma}} \mathcal{Q}_{\text{spin}}(M_\mu^\sigma) V_\mu^K$$

In particular the multiplicity m_μ vanishes if $\mu \notin \bar{\sigma}$.

In this section we introduce the main tools needed for the proof of Theorem 3.1.

In Section 3.1, we recall the notion of *transversally elliptic symbols*.

In Section 3.2, we recall the Witten's way of localization the Riemann-Roch character [18]. We recall in Proposition 3.7, the criterium observed in [19] for the vanishing of the invariant part of the localized Riemann-Roch character.

In Section 3.3, we recall the induction formula proved in [18, 19] for the localized Riemann-Roch character.

In Section 3.4, we prove Theorem 3.1 when K is a torus¹. We give by the same way a proof of Theorem 2.9 which is essential to the definition of the Spin-quantization of the (possibly singular) reduced spaces M_μ^σ .

3.1. Elliptic and transversally elliptic symbols. Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [7, 8] and Paradan-Vergne [20]. For a short introduction see [18].

Let \mathcal{X} be a compact K -manifold. Let $p : T\mathcal{X} \rightarrow \mathcal{X}$ be the projection, and let $(-, -)_\mathcal{X}$ be a K -invariant Riemannian metric. If E^0, E^1 are K -equivariant complex vector bundles over \mathcal{X} , a K -equivariant morphism $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$ is called a *symbol* on \mathcal{X} . The subset of all $(x, v) \in T\mathcal{X}$ where² $\sigma(x, v) : E_x^0 \rightarrow E_x^1$ is not invertible is called the *characteristic set* of σ , and is denoted by $\text{Char}(\sigma)$.

In the following, the product of a symbol σ by a complex vector bundle $F \rightarrow M$, is the symbol

$$\sigma \otimes F$$

defined by $\sigma \otimes F(x, v) = \sigma(x, v) \otimes \text{Id}_{F_x}$ from $E_x^0 \otimes F_x$ to $E_x^1 \otimes F_x$. Note that $\text{Char}(\sigma \otimes F) = \text{Char}(\sigma)$.

Let $T_K\mathcal{X}$ be the following subset of $T\mathcal{X}$:

$$T_K\mathcal{X} = \{(x, v) \in T\mathcal{X}, (v, X_\mathcal{X}(x))_\mathcal{X} = 0 \text{ for all } X \in \mathfrak{k}\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $T\mathcal{X}$ (i.e. $\text{Char}(\sigma)$ is compact), and is *K -transversally elliptic* if the restriction of σ to $T_K\mathcal{X}$ is invertible outside a compact subset of $T_K\mathcal{X}$ (i.e. $\text{Char}(\sigma) \cap T_K\mathcal{X}$ is compact). An elliptic symbol σ defines an element in the equivariant \mathbf{K} -theory of $T\mathcal{X}$ with

¹This situation was already handled in [19].

²The map $\sigma(x, v)$ will be also denote $\sigma|_x(v)$

compact support, which is denoted by $\mathbf{K}_K(\mathbb{T}\mathcal{X})$, and the index of σ is a virtual finite dimensional representation of K , that we denote $\text{Index}_{\mathcal{X}}^K(\sigma) \in R(K)$ [2, 3, 4, 5].

Let

$$R_{tc}^{-\infty}(K) \subset R^{-\infty}(K)$$

be the $R(K)$ -submodule formed by all the infinite sum $\sum_{\mu \in \widehat{K}} m_{\mu} V_{\mu}^K$ where the map $\mu \in \widehat{K} \mapsto m_{\mu} \in \mathbb{Z}$ has at most a *polynomial* growth. The $R(K)$ -module $R_{tc}^{-\infty}(K)$ is the Grothendieck group associated to the *trace class* virtual K -representations: we can associate to any $V \in R_{tc}^{-\infty}(K)$, its trace $k \rightarrow \text{Tr}(k, V)$ which is a generalized function on K invariant by conjugation. Then the trace defines a morphism of $R(K)$ -module

$$(3.8) \quad R_{tc}^{-\infty}(K) \hookrightarrow \mathcal{C}^{-\infty}(K)^K.$$

A K -transversally elliptic symbol σ defines an element of $\mathbf{K}_K(\mathbb{T}_K\mathcal{X})$, and the index of σ is defined as a trace class virtual representation of K , that we still denote $\text{Index}_{\mathcal{X}}^K(\sigma) \in R_{tc}^{-\infty}(K)$.

Remark that any elliptic symbol of $\mathbb{T}\mathcal{X}$ is K -transversally elliptic, hence we have a restriction map $\mathbf{K}_K(\mathbb{T}\mathcal{X}) \rightarrow \mathbf{K}_K(\mathbb{T}_K\mathcal{X})$, and a commutative diagram

$$(3.9) \quad \begin{array}{ccc} \mathbf{K}_K(\mathbb{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_K(\mathbb{T}_K\mathcal{X}) \\ \text{Index}_{\mathcal{X}}^K \downarrow & & \downarrow \text{Index}_{\mathcal{X}}^K \\ R(K) & \longrightarrow & R_{tc}^{-\infty}(K) . \end{array}$$

Using the *excision property*, one can easily show that the index map $\text{Index}_{\mathcal{U}}^K : \mathbf{K}_K(\mathbb{T}_K\mathcal{U}) \rightarrow R_{tc}^{-\infty}(K)$ is still defined when \mathcal{U} is a K -invariant relatively compact open subset of a K -manifold (see [18][section 3.1]).

Suppose that M is a K -manifold equipped with an invariant almost complex structure J . Let us recall the definition of the Riemann-Roch character $RR_J^K(M, -)$.

The complex vector bundle $(\mathbb{T}^*M)^{0,1}$ is K -equivariantly identified with the tangent bundle $\mathbb{T}M$ equipped with the complex structure J . Let h be the Hermitian structure on $(\mathbb{T}M, J)$ defined by : $h(v, w) = \Omega(v, Jw) - i\Omega(v, w)$ for $v, w \in \mathbb{T}M$. The symbol

$$\text{Thom}(M, J) \in \Gamma(M, \text{hom}(p^*(\wedge_{\mathbb{C}}^{\text{even}} \mathbb{T}M), p^*(\wedge_{\mathbb{C}}^{\text{odd}} \mathbb{T}M)))$$

at $(m, v) \in \mathbb{T}M$ is equal to the Clifford map

$$(3.10) \quad \mathbf{c}_m(v) : \wedge_{\mathbb{C}}^{\text{even}} \mathbb{T}_m M \longrightarrow \wedge_{\mathbb{C}}^{\text{odd}} \mathbb{T}_m M,$$

where $\mathbf{c}_m(v).w = v \wedge w - \iota(v)w$ for $w \in \wedge_{\mathbb{C}}^{\bullet} \mathbb{T}_m M$. Here $\iota(v) : \wedge_{\mathbb{C}}^{\bullet} \mathbb{T}_m M \rightarrow \wedge_{\mathbb{C}}^{\bullet-1} \mathbb{T}_m M$ denotes the contraction map relative to h . Since $\mathbf{c}_m(v)^2 = -\|v\|^2 \text{Id}$, the map $\mathbf{c}_m(v)$ is invertible for all $v \neq 0$. Hence the characteristic set of $\text{Thom}(M, J)$ corresponds to the 0-section of $\mathbb{T}M$.

Let E be a K -equivariant complex vector bundle over M . It is a classical fact that the principal symbol of the Dolbeault-Dirac operator $\overline{\partial}_E + \overline{\partial}_E^*$ is equal to the following elliptic symbol³

$$\mathbf{c}_E := \text{Thom}(M, J) \otimes E,$$

³Here we use an identification $\mathbb{T}^*M \simeq \mathbb{T}M$ given by an invariant Riemannian metric.

see [9]. Since M is compact, the symbol \mathbf{c}_E is elliptic and then defines an element of the equivariant \mathbf{K} -group of TM .

Definition 3.2. *The Riemann-Roch character $RR_J^K(M, E) \in R(K)$ is defined equivalently*

- as the topological index of $\mathbf{c}_E \in \mathbf{K}_K(TM)$, or
- as the analytical index of the Dolbeault-Dirac operator $\bar{\partial}_E + \bar{\partial}_E^*$.

3.2. Localization of the Riemann-Roch character. Let (M, ω, Φ) a compact Hamiltonian K -manifold Spin-prequantized by (\tilde{L}, J) where J is a *compatible* almost complex structure on M . The Riemann-Roch character attached to J is just denoted $RR^K(M, -)$.

By definition the Spin-quantization of (M, ω, Φ) is

$$\mathcal{Q}_{\text{spin}}^K(M) := RR^K(M, \tilde{L}) \in R(K).$$

We recall the Witten's deformation of the Riemann-Roch character [18, 19]. We use in all this paper an isomorphism $\mathfrak{k}^* \simeq \mathfrak{k}$ defined by a K -invariant scalar product on \mathfrak{k}^* . In order to simplify the notation, we use the same symbol for $\xi \in \mathfrak{k}^*$ and its corresponding element in \mathfrak{k} .

The moment map is seen as an equivariant map from M to \mathfrak{k} . We define the *Kirwan vector field* on M :

$$(3.11) \quad \kappa_m = (\Phi(m))_M(m), \quad m \in M.$$

Definition 3.3. *The symbol $\mathbf{c}_{\tilde{L}} = \text{Thom}(M, J) \otimes \tilde{L}$ pushed by the vector field κ is the symbol $\mathbf{c}_{\tilde{L}}^\kappa$ defined by the relation*

$$\mathbf{c}_{\tilde{L}}^\kappa|_m(v) = \text{Thom}(M, J) \otimes \tilde{L}|_m(v - \kappa_m)$$

for any $(m, v) \in TM$.

Note that $\mathbf{c}_{\tilde{L}}^\kappa|_m(v)$ is invertible except if $v = \kappa_m$. If furthermore v belongs to the subset $T_K M$ of tangent vectors orthogonal to the K -orbits, then $v = 0$ and $\kappa_m = 0$. Indeed κ_m is tangent to $K \cdot m$ while v is orthogonal.

Since κ is the Hamiltonian vector field of the function $\frac{1}{2}\|\Phi\|^2$, the set of zeros of κ coincides with the set $\text{Cr}(\|\Phi\|^2)$ of critical points of $\|\Phi\|^2$. Finally we have

$$\begin{aligned} \text{Char}(\mathbf{c}_{\tilde{L}}^\kappa) \cap T_K M &\simeq \text{Cr}(\|\Phi\|^2) \\ &= \bigcup_{\beta \in \mathcal{B}} \underbrace{K \cdot (M^\beta \cap \Phi^{-1}(\beta))}_{C_\beta} \end{aligned}$$

where $\mathcal{B} \subset \mathfrak{k}^*$ is a finite subset parametrizing coadjoint orbits $K \cdot \beta$.

We are interested to the restriction $\mathbf{c}_{\tilde{L}}^\kappa|_U$ of the elliptic symbol on an invariant open subset $U \subset M$. Note that the set $\text{Char}(\mathbf{c}_{\tilde{L}}^\kappa|_U) \cap T_K U \simeq \text{Cr}(\|\Phi\|^2) \cap U$ is compact when

$$(3.12) \quad \partial U \cap \text{Cr}(\|\Phi\|^2) = \emptyset.$$

When (3.12) holds we denote

$$(3.13) \quad \mathcal{Q}_\Phi^K(U) := \text{Index}_U^K(\mathbf{c}_{\tilde{L}}^\kappa|_U) \in R_{tc}^{-\infty}(K)$$

the equivariant index of the transversally elliptic symbol $\mathbf{c}_{\tilde{L}}^\kappa|_U$.

For any $\beta \in \mathcal{B}$, we consider a relatively compact open invariant neighborhood U_β of C_β such that $\text{Cr}(\|\Phi\|^2) \cap \overline{U_\beta} = C_\beta$.

Definition 3.4. We denote

$$\mathcal{Q}_\beta^K(M) \in R_{tc}^{-\infty}(K)$$

the index of the transversally elliptic symbol $\mathbf{c}_{\tilde{L}}^\kappa|_{\mathcal{U}_\beta}$.

Everything can be defined if we replace the line bundle \tilde{L} by any equivariant complex vector bundle E . We can consider the pushed symbol \mathbf{c}_E^κ , and the localized Riemann-Roch characters

$$RR_\Phi^K(U, E) := \text{Index}_U^K(\mathbf{c}_E^\kappa|_U) \quad \text{and} \quad RR_\beta^K(M, E) := \text{Index}_{U_\beta}^K(\mathbf{c}_E^\kappa|_{U_\beta}).$$

A direct application of the excision property [18] gives that

$$(3.14) \quad \mathcal{Q}_{\text{spin}}^K(M) = \sum_{\beta \in \mathcal{B}} \mathcal{Q}_\beta^K(M).$$

If we work with $RR_\Phi^K(U, E)$, we have

$$(3.15) \quad RR_\Phi^K(U, E) = \sum_{\beta \in \mathcal{B} \cap \Phi(U)} RR_\beta^K(U, E).$$

The decomposition (3.14) and (3.15) will be used in the next chapters when one want to compute the multiplicity, denoted $[\mathcal{Q}_{\text{spin}}^K(M)]^K$, of the trivial representation in $\mathcal{Q}_{\text{spin}}^K(M)$. We have

$$[\mathcal{Q}_{\text{spin}}^K(M)]^K = \sum_{\beta \in \mathcal{B}} [\mathcal{Q}_\beta^K(M)]^K.$$

and we will see a *criterion* under which one has $[\mathcal{Q}_\beta^K(M)]^K = 0$.

Let β be a non-zero element in \mathfrak{k} : let $\mathbb{T}_\beta \subset K$ be the torus generated by β . For $m \in M^\beta$, let $\alpha_1^m, \dots, \alpha_p^m$ be the real infinitesimal weights for the action of \mathbb{T}_β on the fibers of $\mathbb{T}_m M$ (we equip the fibers of $\mathbb{T}_m M / \mathbb{T}_m M^\beta$ with a \mathbb{T}_β -invariant complex structure).

Definition 3.5. Let us denote by $\mathbf{Tr}_\beta|\mathbb{T}_m M|$ the following positive number

$$\mathbf{Tr}_\beta|\mathbb{T}_m M| := \sum_{i=1}^l |\langle \alpha_i^m, \beta \rangle|.$$

Note that $m \in M^\beta \mapsto \mathbf{Tr}_\beta|\mathbb{T}_m M|$ is constant along a connected component of M^β . We see also that the expression $\mathbf{Tr}_\beta|E|$ is well defined for any H -equivariant real vector bundle $E \rightarrow P$, when $\beta \in \mathfrak{h}$ acts trivially on P .

Example 3.6. The map $\beta \in \mathfrak{k} \mapsto \mathbf{Tr}_\beta|\mathfrak{k}|$ is invariant under the adjoint action. When β belongs to the Weyl chamber, one has $\mathbf{Tr}_\beta|\mathfrak{k}| = 2(\rho, \beta)$. Note that $\mathbf{Tr}_\beta|\mathfrak{k}| \leq 2\|\rho\| \|\beta\|$ for any $\beta \in \mathfrak{k}$.

We have proved in [19] the following nice criterium.

Proposition 3.7. Let $\beta \neq 0$ in \mathcal{B} . The multiplicity of the trivial representation in $\mathcal{Q}_\beta^K(M)$ is equal to zero if

$$(3.16) \quad \|\beta\|^2 + \frac{1}{2}\mathbf{Tr}_\beta|\mathbb{T}_m M| > \mathbf{Tr}_\beta|\mathfrak{k}|, \quad \forall m \in M^\beta \cap \Phi^{-1}(\beta).$$

Remark 3.8. Note that condition (3.16) is equivalent to

$$(3.17) \quad \|\Phi(m)\|^2 + \frac{1}{2} \mathbf{Tr}_{\Phi(m)} |\mathbb{T}_m M| > \mathbf{Tr}_{\Phi(m)} |\mathfrak{k}|, \quad \forall m \in C_\beta.$$

If the critical set C_β decomposes in a finite disjoint union of closed K -invariant subset $C_\beta = \cup_j C_\beta^j$, we consider invariant open neighborhood U^j of C_β^j such that $\overline{U_\beta^j} \cap \text{Cr}(\|\Phi\|^2) = C_\beta^j$, and we define

$$\mathcal{Q}_{C_\beta^j}^K(M) := \text{Index}_{U_\beta^j}^K(\mathbf{c}^\kappa|_{U_\beta^j}) \in R_{tc}^{-\infty}(K)$$

Then the generalized character $\mathcal{Q}_\beta^K(M)$ is equal to the sum $\sum_j \mathcal{Q}_{C_\beta^j}^K(M)$ and Proposition 3.7 tells us that $[\mathcal{Q}_{C_\beta^j}^K(M)]^K = 0$ if (3.17) holds on C_β^j .

3.3. Induction formulas. Let H be a compact connected Lie group. Let $H \cdot a$ be a coadjoint orbit. Let (N, ω_N, Φ_N) be an Hamiltonian H -manifold which is not assumed to be compact. But we assume that Φ_N is *proper* near $H \cdot a$: the pullback $\Phi_N^{-1}(\mathcal{C})$ is compact if $\mathcal{C} \subset \mathfrak{h}^*$ is a small enough compact invariant neighborhood of $H \cdot a$.

Let H_a be the stabilizer of $a \in \mathfrak{h}^*$, and let Y_a be a symplectic slice near $H \cdot a$: Y_a is a H_a -invariant symplectic manifold of N such that $\Phi_N(Y_a) \subset \mathfrak{h}_a^*$ and such that $H \times_{H_a} Y_a$ is diffeomorphic to an invariant open neighborhood of $\Phi_N^{-1}(H \cdot a)$. We will work with the following moment map on Y_a :

$$\Phi_{Y_a} = \Phi_N|_{Y_a} - a.$$

Let $N \times \overline{H \cdot a}$ be the Hamiltonian H -manifold, with moment map $\Phi(n, \xi) = \Phi_N(n) - \xi$. Let

$$RR_0^H(N \times \overline{H \cdot a}, -)$$

be the Riemann-Roch character localized near the compact subset $\Phi^{-1}(0) \subset N \times \overline{H \cdot a}$. Let

$$RR_0^{H_a}(Y_a, -)$$

be the Riemann-Roch character localized near the compact subset $\Phi_{Y_a}^{-1}(0) = \Phi_N^{-1}(a) \subset Y_a$.

Let $\text{Ind}_{H_a}^H : R^{-\infty}(H_a) \rightarrow R^{-\infty}(H)$ be the induction map. If E and F are respectively H -equivariant complex vector bundles on N and $H \cdot a$, we denote $E \boxtimes F$ their product. We have proved in [18] (see also Proposition 4.13 in [19]) the following induction formula

Proposition 3.9. For any equivariant complex vector bundles $E \rightarrow N$ and $F \rightarrow H \cdot a$, we have

$$RR_0^H(N \times \overline{H \cdot a}, E \boxtimes F) = \text{Ind}_{H_a}^H \left[RR_0^{H_a}(Y_a, E|_{Y_a} \otimes F|_{\{a\}}) \right].$$

The last Proposition gives in particular that

$$(3.18) \quad [RR_0^H(N \times \overline{H \cdot a}, E \boxtimes F)]^H = [RR_0^{H_a}(Y_a, E|_{Y_a} \otimes F|_{\{a\}})]^{H_a}.$$

3.4. The torus case. Let T be a compact torus, and let (M, ω, Φ) be a compact Hamiltonian T -manifold which is Spin-prequantized by the data (J, \tilde{L}) . We suppose that J is compatible with ω . The irreducible representation of T is parametrized by the lattice $\hat{T} \subset \mathfrak{t}^*$: at each $\mu \in \hat{T}$ we associate the one-dimensional representation \mathbb{C}_μ .

We write

$$\mathcal{Q}_{\text{spin}}^T(M) = \sum_{\mu \in \hat{T}} m_\mu \mathbb{C}_\mu,$$

and one wants to show that the multiplicity m_μ is equal to the Spin-quantization of the (possibly singular) reduced space $M_\mu := \Phi^{-1}(\mu)/T$.

We fix once for all $\mu \in \hat{T}$. And we apply the Witten deformation procedure to the Hamiltonian T -manifold $(M, \omega, \Phi - \mu)$ which is Spin-prequantized by $(J, \tilde{L} \otimes \mathbb{C}_{-\mu})$. We have

$$m_\mu = \sum_{\beta \in \mathcal{B}^\mu} \left[RR_\beta^T(M, \tilde{L} \otimes \mathbb{C}_{-\mu}) \right]^T$$

where \mathcal{B}^μ parametrizes the critical points of $\|\Phi - \mu\|^2$. Here the criterion (3.16) holds for any non-zero β since the Lie algebra \mathfrak{t} is abelian. We have then

$$m_\mu = \left[RR_0^T(M, \tilde{L} \otimes \mathbb{C}_{-\mu}) \right]^T.$$

In particular $m_\mu = 0$ if $\mu \notin \Phi(M)$. When $\mu \in \Phi(M)$, we consider a small neighborhood U of $\Phi^{-1}(\mu) \subset M$ so that $\overline{U} \cap \text{Cr}(\|\Phi - \mu\|^2) = \Phi^{-1}(\mu)$. We know then that

$$(3.19) \quad m_\mu = \left[RR_{\Phi^{-1}(\mu)}^T(U, \tilde{L}|_U \otimes \mathbb{C}_{-\mu}) \right]^T.$$

3.4.1. First case: μ is a regular value of Φ . We consider the orbifold reduced space $M_\mu = \Phi^{-1}(\mu)/T$ which is equipped with a canonical symplectic form ω_μ . Let $RR(M_\mu, -)$ be the Riemann-Roch character attached to a compatible almost complex structure. We prove in [18] that for any complex vector bundle $E \rightarrow U$

$$(3.20) \quad \left[RR_{\Phi^{-1}(\mu)}^T(U, E) \right]^T = RR(M_\mu, \mathcal{E})$$

where $\mathcal{E} = E|_{\Phi^{-1}(\mu)}/T$ is the induced orbifold bundle on M_μ . If we take $E = \tilde{L}|_U \otimes \mathbb{C}_{-\mu}$ on sees that

$$\tilde{\mathcal{L}}_\mu = (\tilde{L}|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/T$$

is an orbifold line bundle which Spin-prequantizes (M_μ, ω_μ) , and (3.20) gives together with (3.19) that

$$m_\mu = RR(M_\mu, \tilde{\mathcal{L}}_\mu) = \mathcal{Q}_{\text{spin}}(M_\mu).$$

3.4.2. Second case : μ is a not (necessarilly) a regular value of Φ . Let $\vec{\Delta}$ be the rationally vector subspace generated by $\{a - b \mid a, b \in \Phi(M)\} \subset \mathfrak{t}^*$. We work here with a weigh $\mu \in \Phi(M)$ so that the polytope $\Phi(M)$ lives in the affine subspace $\mu + \vec{\Delta}$. Let $\mathfrak{t}_\Delta \subset \mathfrak{t}$ be the subspace orthogonal to Δ , and let $T_\Delta \subset T$ be the corresponding subtorus.

Lemma 3.10. *The group T_Δ acts trivially on M and on the line bundle $\tilde{L} \otimes \mathbb{C}_{-\mu}$.*

Proof. See the proof of Lemma 2.8. \square

Let $T' \subset T$ be another subtorus such that $T = T_\Delta \times T'$: the dual of its Lie algebra \mathfrak{t}' is identified with $\overrightarrow{\Delta} \subset \mathfrak{t}^*$. We look now at (M, ω) as an Hamiltonian T' -manifold with moment map

$$\Phi' := \Phi - \mu : M \longrightarrow \overrightarrow{\Delta} = (\mathfrak{t}')^*$$

The T' -equivariant line bundle $\tilde{L}' := \tilde{L} \otimes \mathbb{C}_{-\mu}$ Spin-prequantizes the Hamiltonian T' -manifold (M, ω, Φ') . Let U be a small neighborhood of $\Phi'^{-1}(0)$ in M . The generalized character $RR_{\Phi'-\mu}^T(U, \tilde{L}'|_U \otimes \mathbb{C}_{-\mu})$ belongs to $R^{-\infty}(T')$ and corresponds to the localized Riemann-Roch character

$$RR_{\Phi'}^{T'}(U, \tilde{L}'|_U).$$

We deform the moment map Φ' in $\Phi' - \varepsilon$ where ε is a small element in $\overrightarrow{\Delta}$. We have proved in [19][Proposition 4.14] the following

Lemma 3.11. • *If ε is small enough, the critical set of $\|\Phi' - \varepsilon\|^2$ does not intersect ∂U , so that the localized Riemann-Roch character $RR_{\Phi'-\varepsilon}^{T'}(U, \tilde{L}'|_U)$ is well defined.*

- *We have $RR_{\Phi'}^{T'}(U, \tilde{L}'|_U) = RR_{\Phi'-\varepsilon}^{T'}(U, \tilde{L}'|_U)$ if ε is small enough.*

Now we are left to the computation of $m_\mu = \left[RR_{\Phi'-\varepsilon}^{T'}(U, \tilde{L}'|_U) \right]^{T'}$ for ε is small enough. We start with the decomposition

$$RR_{\Phi'-\varepsilon}^{T'}(U, \tilde{L}'|_U) = \sum_{\beta \in \mathcal{B}_\varepsilon} RR_{\Phi'-\varepsilon, \beta}^{T'}(U, \tilde{L}'|_U)$$

where $RR_{\Phi'-\varepsilon, \beta}^{T'}(U, \tilde{L}'|_U)$ denotes the Riemann-Roch character localized near the compact subset $U^\beta \cap (\Phi')^{-1}(\beta + \varepsilon)$. We have proved in [19][Lemma 4.16] the following

Lemma 3.12. *If ε is small enough we have $\left[RR_{\Phi'-\varepsilon, \beta}^{T'}(U, \tilde{L}'|_U) \right]^{T'} = 0$ when $\beta \neq 0$.*

At $\varepsilon \in \overrightarrow{\Delta}$ small enough and generic we associate the orbifold $M_{\mu+\varepsilon} = \Phi^{-1}(\mu + \varepsilon)/T'$ which is equipped with the orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon} = \left(\tilde{L}|_{\Phi^{-1}(\mu+\varepsilon)} \otimes \mathbb{C}_{-\mu} \right) / T'.$$

Let $RR(M_{\mu+\varepsilon}, -)$ be the Riemann-Roch map associated to a compatible almost complex structure.

If we use (3.20) together with the Lemmas 3.11 and 3.12 we get

Theorem 3.13. *The multiplicity m_μ is equal to the Riemann-Roch number $RR(M_{\mu+\varepsilon}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) \in \mathbb{Z}$ where $\varepsilon \in \overrightarrow{\Delta}$ small and generic.*

We prove here that the quantity $RR(M_{\mu+\varepsilon}, \tilde{\mathcal{L}}_{\mu+\varepsilon})$ does not depend of the choice of ε small and generic: it is the definition of the Spin quantization, denoted $\mathcal{Q}_{\text{spin}}(M_\mu)$, of the (possibly singular) reduced space M_μ .

3.4.3. Proof of Theorem 2.9. The same kind of proof work for Theorem 2.9. We consider an invariant relatively compact neighborhood $U_{\sigma, \mu}$ of $\Phi_\sigma^{-1}(\mu) = \Phi^{-1}(\mu + \rho - \rho_\sigma)$ in the slice Y_σ so that $\text{Cr}(\|\Phi_\sigma - \mu\|^2) \cap \overline{U_{\sigma, \mu}} = \Phi_\sigma^{-1}(\mu)$. Thanks to Lemmas 3.11 and 3.12, we know that the Riemann-Roch character

$$RR_{\Phi_\sigma - \mu - \varepsilon}^{Z'_\sigma}(U_{\sigma, \mu}, \tilde{L}') \in R^{-\infty}(Z'_\sigma)$$

are well defined for $\varepsilon \in \overrightarrow{\Delta}$ small enough, and they do not depend of the choice of ε . If $\varepsilon_1, \varepsilon_2 \in \overrightarrow{\Delta}$ are small enough regular values of $\Phi_\sigma - \mu$ we get thanks to (3.20) that

$$\begin{aligned} RR(M_{\mu+\varepsilon_1}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon_1}^\sigma) &= \left[RR_{\Phi_\sigma - \mu - \varepsilon_1}^{Z'_\sigma}(U_{\sigma, \mu}, \tilde{L}') \right]^{Z'_\sigma} \\ &= \left[RR_{\Phi_\sigma - \mu - \varepsilon_2}^{Z'_\sigma}(U_{\sigma, \mu}, \tilde{L}') \right]^{Z'_\sigma} \\ &= RR(M_{\mu+\varepsilon_2}^\sigma, \tilde{\mathcal{L}}_{\mu+\varepsilon_2}^\sigma). \end{aligned}$$

4. PROOF OF THEOREM 3.1

Let (M, ω, Φ) be a compact Hamiltonian K -manifold which is Spin prequantized. Let σ be the smallest face of the Weyl chamber so that $\Phi(M) \cap \mathfrak{t}_+^* \subset \bar{\sigma}$. Let μ be a dominant weight, and let m_μ be the multiplicity of V_μ^K in $\mathcal{Q}_{\text{spin}}^K(M)$.

Let \mathcal{O}_μ be the coadjoint orbit $K \cdot (\mu + \rho)$. Since the dual representation $(V_\mu^K)^*$ can be realized⁴ as $\mathcal{Q}_{\text{spin}}^K(\overline{\mathcal{O}_\mu})$, we know by the shifting trick that

$$m_\mu = [\mathcal{Q}_{\text{spin}}^K(M \times \overline{\mathcal{O}_\mu})]^K$$

Now we work with the K -Hamiltonian manifold $N = M \times \overline{\mathcal{O}_\mu}$ with moment map $\Phi_N(m, \xi) = \Phi(m) - \xi$. The Witten deformation on N gives $\mathcal{Q}_{\text{spin}}^K(M \times \overline{\mathcal{O}_\mu}) = \sum_{\beta \in \mathcal{B}^\mu} \mathcal{Q}_\beta^K(M \times \overline{\mathcal{O}_\mu})$ where \mathcal{B}^μ is a finite set parametrizing $\text{Cr}(\|\Phi_N\|^2)$. We have then

$$(4.21) \quad m_\mu = \sum_{\beta \in \mathcal{B}^\mu} [\mathcal{Q}_\beta^K(M \times \overline{\mathcal{O}_\mu})]^K.$$

We remark that 0 does not appears in \mathcal{B}^μ when $\sigma \neq \mathfrak{t}_+^*$, since $\mu + \rho \notin \Phi(M)$.

The main point of this section is the following

Proposition 4.1. • *If $\mu \notin \bar{\sigma}$, the identity (3.16) holds on C_β for any $\beta \in \mathcal{B}^\mu$. Hence $m_\mu = 0$.*

• *If $\mu \in \bar{\sigma}$, the identity (3.16) holds on C_β for any $\beta \neq -\rho_\sigma$. Then*

$$m_\mu = [\mathcal{Q}_{-\rho_\sigma}^K(M \times \overline{\mathcal{O}_\mu})]^K.$$

When $\sigma = \mathfrak{t}_+^*$, we have $\rho_\sigma = 0$ and Proposition 4.1 tell us that the multiplicity m_μ is equal to $[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K$ for any $\mu \in \widehat{K}$. In particular $m_\mu = 0$ if $\mu + \rho \notin \Phi(M)$.

When $\sigma \neq \mathfrak{t}_+^*$ and $\mu \in \bar{\sigma}$, we precise Proposition 4.1 as follow. The generalized character $\mathcal{Q}_{-\rho_\sigma}^K(M \times \overline{\mathcal{O}_\mu})$ is defined as the index of a transversally elliptic symbol living in a neighborhood of

$$C_{-\rho_\sigma} = K(N^{\rho_\sigma} \cap \Phi_N^{-1}(-\rho_\sigma)).$$

Let K_{ρ_σ} be the stabilizer subgroup of ρ_σ . Let $W(K_{\rho_\sigma}) \subset W$ be the Weyl subgroup of K_{ρ_σ} . A direct computation gives that

$$C_{-\rho_\sigma} = \bigcup_{\bar{w} \in W(K_{\rho_\sigma}) \setminus W} C_{-\rho_\sigma, \bar{w}}$$

with

$$C_{-\rho_\sigma, \bar{w}} = K(M^{\rho_\sigma} \cap \Phi^{-1}(w(\mu + \rho) - \rho_\sigma) \times \{w(\mu + \rho)\}).$$

⁴ $\overline{\mathcal{O}_\mu}$ is the coadjoint orbit \mathcal{O}_μ with the opposite symplectic structure.

We are particularly interested in the component $C_{-\rho_\sigma, \bar{e}}$. Let us denote $C_{-\rho_\sigma, out}$ the union of the $C_{-\rho_\sigma, \bar{w}}$ for $\bar{w} \neq \bar{e}$. We have a decomposition

$$(4.22) \quad C_{-\rho_\sigma} = C_{-\rho_\sigma, \bar{e}} \cup C_{-\rho_\sigma, out}$$

into closed invariant disjoint subsets. Then the generalized character $\mathcal{Q}_{-\rho_\sigma}^K(M \times \overline{\mathcal{O}_\mu})$ is equal to the sum

$$\mathcal{Q}_{-\rho_\sigma, \bar{e}}^K(M \times \overline{\mathcal{O}_\mu}) + \mathcal{Q}_{-\rho_\sigma, out}^K(M \times \overline{\mathcal{O}_\mu})$$

where both terms correspond to the specialization of the transversally elliptic symbol to the neighborhood of each part of the decomposition (4.22).

Proposition 4.2. *Suppose that $\sigma \neq \mathfrak{t}_+^*$ and that $\mu \in \bar{\sigma}$. The identity (3.16) holds on the subset $C_{-\rho_\sigma, out}$, and then*

$$m_\mu = [\mathcal{Q}_{-\rho_\sigma, \bar{e}}^K(M \times \overline{\mathcal{O}_\mu})]^K.$$

Note that $C_{-\rho_\sigma, e} = \emptyset$ if $\mu + \rho - \rho_\sigma \notin \Phi(M)$. At this stage we know then that $m_\mu = 0$ if $\mu + \rho - \rho_\sigma$ does not belong to the image of the moment map.

4.1. Proofs of Propositions 4.1 and 4.2. Let $N = M \times \overline{\mathcal{O}_\mu}$ and let $\|\Phi_N\|^2 : N \rightarrow \mathbb{R}$ be the square of the moment map. Recall that we denote by σ the smallest face of the Weyl chamber so that $\Phi(M) \cap \mathfrak{t}_+^* \subset \bar{\sigma}$.

We want to prove that for any $n = (m, \xi) \in \text{Cr}(\|\Phi_N\|^2)$ the vector $\beta := \Phi(m) - \xi$ satisfies

$$(I) \quad \|\beta\|^2 + \frac{1}{2} \text{Tr}_\beta |T_n N| \geq \text{Tr}_\beta |\mathfrak{k}|.$$

Afterwards we will discuss the case of equality in (I).

The tangent space $T_\xi \mathcal{O}_\mu$ is equal to the \mathfrak{k}_ξ -module $\mathfrak{k}/\mathfrak{k}_\xi$: then

$$\begin{aligned} \text{Tr}_\beta |T_\xi \mathcal{O}_\mu| &= \text{Tr}_\beta |\mathfrak{k}| - \text{Tr}_\beta |\mathfrak{k}_\xi| \\ &= \text{Tr}_\beta |\mathfrak{k}|, \end{aligned}$$

since β belongs to the abelian subalgebra \mathfrak{k}_ξ . Using that $\text{Tr}_\beta |T_n N| = \text{Tr}_\beta |T_m M| + \text{Tr}_\beta |\mathfrak{k}|$, we see that (I) is equivalent to

$$(II) \quad \|\beta\|^2 + \frac{1}{2} \text{Tr}_\beta |T_m M| \geq \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}|.$$

The module $\mathfrak{k}/\mathfrak{k}_m$ is naturally a subspace of $T_m M$. Let E_m be a K_m -equivariant supplement to $\mathfrak{k}/\mathfrak{k}_m$ in $T_m M$. Using that $\text{Tr}_\beta |T_m M| = \text{Tr}_\beta |\mathfrak{k}/\mathfrak{k}_m| + \text{Tr}_\beta |E_m|$, we see that (II) is equivalent to

$$(III) \quad \|\beta\|^2 + \frac{1}{2} \text{Tr}_\beta |E_m| \geq \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}_m|.$$

Thanks to the inclusion $\mathfrak{k}_m \subset \mathfrak{k}_{\Phi(m)}$, we see that (I) \Leftrightarrow (II) \Leftrightarrow (III) are induced by the following inequality

$$(IV) \quad \|\beta\|^2 \geq \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}_{\Phi(m)}|.$$

Lemma 4.3. • *For any $(m, \xi) \in \text{Cr}(\|\Phi_N\|^2)$ the vector $\beta := \Phi(m) - \xi$ satisfies the inequality (IV).*

• *Let $(m, \xi) \in \text{Cr}(\|\Phi_N\|^2)$ such that $\beta := \Phi(m) - \xi$ satisfies the $\|\beta\|^2 = \frac{1}{2} \text{Tr}_\beta |\mathfrak{k}_{\Phi(m)}|$. Then there exists a face τ of σ such that*

- (1) $\mu \in \bar{\tau}$
- (2) (m, ξ) belongs to the K -orbit of $\Phi^{-1}(\mu + \rho - \rho_\tau) \times \{\mu + \rho\} \subset N$.

(3) β belongs to the coadjoint orbit $K \cdot (-\rho_\tau)$.

Proof. Up to the multiplication of (m, ξ) by an element of K , we can assume that $\beta \in \mathfrak{t}^*$. Up to the multiplication of $n = (m, \xi)$ by an element of the stabilizer subgroup $K_\beta := \{k \in K \mid \text{Ad}(k)\beta = \beta\}$ we can assume that $n = (m, w(\mu + \rho))$ with $m \in M^\beta$ and $\Phi(m) = \beta + w(\mu + \rho) \in \mathfrak{t}^*$.

Up to the multiplication of $n = (m, w(\mu + \rho))$ by an element of the Weyl group, we can assume that $\Phi(m)$ belongs to the Weyl chamber: let τ be the face of σ containing $\Phi(m)$ so that $K_{\Phi(m)} = K_\tau$.

So we have to prove that for $\Phi(m) = a \in \tau$ and $w \in W$ the vector $\beta = a - w(\mu + \rho)$ satisfies the relation

$$(4.23) \quad \|\beta\|^2 \geq \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau|$$

The inequality (4.23) is the consequence of three basic inequalities.

We have

$$(4.24) \quad \|a - w(\mu + \rho)\| \geq \|a - (\mu + \rho)\|.$$

for any $w \in W$, and (4.24) is strict unless $w \in W(K_\tau)$. In order to prove (4.24), we consider the function $\xi \in K \cdot (\mu + \rho) \mapsto \|\xi - a\|^2 = \|a\|^2 + \|\xi\|^2 - 2(\xi, a)$. It is a classical result of symplectic geometry that the function $\xi \mapsto (\xi, a)$ has a unique maximum on the coadjoint orbit $K \cdot (\mu + \rho)$ which is reached on an orbit of the stabilizer subgroup $K_a = K_\tau$. Since a and $\mu + \rho$ belongs to the Weyl chamber, one checks easily that this maximum is obtained on the orbit $K_\tau(\mu + \rho)$. Hence $\|k(\mu + \rho) - a\|^2 \geq \|\mu + \rho - a\|^2$ for any $k \in K$ with equality only if $k \in K_\tau$. Inequality (4.24) is proved by taking $k = w$.

On the other hand we have

$$(4.25) \quad \begin{aligned} \|\mu + \rho - a\| &\geq \frac{(\mu + \rho - a, \rho_\tau)}{\|\rho_\tau\|} \\ &= \frac{1}{\|\rho_\tau\|} \underbrace{(\mu, \rho_\tau)}_{\geq 0} + \frac{1}{\|\rho_\tau\|} \underbrace{(\rho - \rho_\tau - a, \rho_\tau)}_{=0} + \frac{1}{\|\rho_\tau\|} (\rho_\tau, \rho_\tau) \\ &\geq \|\rho_\tau\|. \end{aligned}$$

Note that (4.25) is strict unless $\mu \in \bar{\tau}$ and $\mu + \rho - a = \rho_\tau$. The third inequality is

$$(4.26) \quad \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau| \leq \|\rho_\tau\| \|\beta\|.$$

See Example 3.6. If we put (4.24), (4.25) and (4.26) together we have

$$\|\beta\|^2 \geq \|\beta\| \|a - (\mu + \rho)\| \geq \|\beta\| \|\rho_\tau\| \geq \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau|,$$

and the equality $\|\beta\|^2 = \frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau|$ holds if and only if we have the equality in (4.24), (4.25) and (4.26).

But equalities in (4.24) and (4.25) gives that $w \in W(K_\tau)$, $\mu \in \bar{\tau}$ and $a = \mu + \rho - \rho_\tau \in \tau$. Then $(m, w(\mu + \rho)) = w(m', \mu + \rho)$ with $\Phi(m') = w^{-1}(\mu + \rho - \rho_\tau) = \mu + \rho - \rho_\tau$ and $\beta = \mu + \rho - \rho_\tau - w(\mu + \rho) = -w\rho_\tau$. We have then

$$\frac{1}{2} \mathbf{Tr}_\beta |\mathfrak{k}_\tau| = \frac{1}{2} \mathbf{Tr}_{\rho_\tau} |\mathfrak{k}_\tau| = \|\rho_\tau\|^2$$

which is the equality in (4.26). □

Since the strict inequality in **(IV)** implies the strict inequality in **(I)**, Lemma 4.3 tells us that the identity (3.16) holds on C_β for all $\beta \in \mathcal{B}^\mu$ when $\mu \notin \bar{\sigma}$. When $\mu \in \bar{\sigma}$ the identity (3.16) holds

- (1) on C_β for the β which are not in $K \cdot (-\rho_\tau)$, where τ is a face of σ such that $\mu \in \bar{\tau}$,
- (2) on $C_{-\rho_\sigma, w}$ for all the $w \neq \bar{e}$.

The proof of Proposition 4.1 is completed by

Lemma 4.4. *Let τ be a face of σ , distinct from σ , such that $\mu \in \bar{\tau}$. Then the identity (3.16) holds for $\beta \in K \cdot (-\rho_\tau)$.*

Proof. Let $\beta = -\rho_\tau$. The critical set $C_{-\rho_\tau} := K(N^{\rho_\tau} \cap \Phi_N^{-1}(-\rho_\tau))$ admits the decomposition $C_{-\rho_\tau} = \cup_{w \in W} C_{-\rho_\tau, w}$ where

$$C_{-\rho_\tau, w} = K(M^{\rho_\tau} \cap \Phi^{-1}(w(\mu + \rho) - \rho_\tau) \times \{w(\mu + \tau)\}).$$

It is not hard to see that $C_{-\rho_\tau, w}$ intersects $C_{-\rho_\tau, e}$ only if $w \in W(K_{\rho_\tau})$, and that $C_{-\rho_\tau, w} = C_{-\rho_\tau, e}$ when $w \in W(K_{\rho_\tau})$. We know then from Lemma 4.3 that the strict inequality in **(IV)** holds on $C_{-\rho_\tau, w}$ for $w \notin W(K_{\rho_\tau})$.

Let us consider now the case where $m \in M^{\rho_\tau} \cap \Phi^{-1}(\mu + \rho - \rho_\tau)$. We know that the equality holds in **(IV)** for $(m, \mu + \rho)$. The equality in **(I)** for $(m, \mu + \rho)$ is then equivalent to

$$(4.27) \quad \mathbf{Tr}_\beta |E_m| + \mathbf{Tr}_\beta |\mathfrak{k}_\tau / \mathfrak{k}_m| = 0.$$

Let us prove that (4.27) can not hold. The image of m by the moment map belongs to τ . Then m belongs to the symplectic slice $Y_\tau \subset M$. A neighborhood m is then $K \times_{K_\tau} Y_\tau$. So the tangent space at m decomposes in two manners

$$\begin{aligned} \mathbf{T}_m M &= \mathfrak{k} / \mathfrak{k}_\tau \oplus \mathbf{T}_m Y_\tau \\ &= \mathfrak{k} / \mathfrak{k}_m \oplus E_m \end{aligned}$$

If (4.27) holds we see that $\mathbf{Tr}_\beta |\mathbf{T}_m Y_\tau| = \mathbf{Tr}_\beta |E_m| = 0$, which means that $\beta = -\rho_\tau$ acts trivially on the tangent space $\mathbf{T}_m Y_\tau$. Hence it would imply that ρ_τ acts trivially on the manifold Y_τ . Since $Y_\sigma \subset Y_\tau$, the action of ρ_τ on the principal slice Y_σ is also trivial.

We know that $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$ acts trivially on Y_σ : since $\rho_\sigma \in [\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$, the action of ρ_σ is trivial on Y_σ . Finally if (4.27) holds, we have that

$$\rho_{\tau/\sigma} := \rho_\tau - \rho_\sigma \in \mathbb{R}\sigma$$

acts trivially on Y_σ .

The moment polytope of M , $\Delta(M)$, which is equal to the closure of $\Phi(Y_\sigma) \subset \sigma$ is a convex polytope. Since the action of $\rho_{\tau/\sigma}$ is trivial on Y_σ we know that the map $\xi \in \Delta(M) \mapsto (\xi, \rho_{\tau/\sigma})$ is constant.

Finally we can use the last information in our hands: $\mu + \rho - \rho_\tau = \Phi(m)$ belongs to $\Delta(M)$. Then for $\xi \in \Delta(M)$ we have

$$(\xi, \rho_{\tau/\sigma}) = (\mu + \rho - \rho_\tau, \rho_{\tau/\sigma}) = 0.$$

It is contradictory since $(\xi, \rho_{\tau/\sigma}) > 0$ for any $\xi \in \sigma$.

We have finally proved that when $(m, \xi) \in N^{\rho_\tau} \cap \Phi_N^{-1}(-\rho_\tau)$ the vector $\beta = \Phi(m) - \xi$ satisfies $\|\beta\|^2 + \frac{1}{2} \mathbf{Tr}_\beta |\mathbf{T}_m M| > \mathbf{Tr}_\beta |\mathfrak{k}|$. \square

4.2. Computation of the multiplicities when $\sigma = \mathfrak{t}_+^*$. In this section we suppose that the moment polytope $\Delta(M) = \Phi(M) \cap \mathfrak{t}_+^*$ intersects the interior of the Weyl chamber. Let $\Delta(M)^\circ \subset (\mathfrak{t}_+^*)^\circ$ be the relative interior of the moment polytope. We know that $m_\mu = [\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K$ for any $\mu \in \widehat{K}$. In Definition 2.10, we have defined the number $\mathcal{Q}(M_{\mu^+}^{\mathfrak{t}_+^*})$ has follows. If $\mu + \rho \notin \Delta(M)^\circ$, we set $\mathcal{Q}(M_{\mu^+}^{\mathfrak{t}_+^*}) = 0$. If $\mu + \rho \in \Delta(M)^\circ$, we consider, for ε generic and small enough, the orbifold reduced space $M_{\mu+\varepsilon}^{\mathfrak{t}_+^*} := \Phi^{-1}(\mu + \varepsilon + \rho)/T$ and the orbifold line bundle

$$\tilde{\mathcal{L}}_{\mu+\varepsilon} = \left(\tilde{\mathcal{L}}|_{\Phi^{-1}(\mu+\varepsilon+\rho)} \otimes \mathbb{C}_{-\mu} \right) / T.$$

The Spin quantization $\mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}) \in \mathbb{Z}$ is defined as the Riemann-Roch number

$$RR(M_{\mu+\varepsilon}^{\mathfrak{t}_+^*}, \tilde{\mathcal{L}}_{\mu+\varepsilon}).$$

The main result of this section is the following

Theorem 4.5. *The number $[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K$ is equal to $\mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*})$.*

We have proved Theorem 3.1 when the moment polytope of M intersects the interior of the Weyl chamber :

$$\mathcal{Q}_{\text{spin}}^K(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}) V_\mu^K.$$

The rest of this section is devoted to the proof of Theorem 4.5. When $\mu + \rho \notin \Delta(M)$, we see that $\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu}) = 0$ since the moment map on $M \times \overline{\mathcal{O}_\mu}$ does not goes through $0 \in \mathfrak{t}^*$. We have then $[\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K = \mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}) = 0$.

We consider now a dominant weight μ such that $\mu + \rho \in \Delta(M)$. Let $Y = \Phi^{-1}((\mathfrak{t}_+^*)^\circ)$ be the symplectic slice with its canonical symplectic form ω_Y . The action of T on (Y, ω_Y) is Hamiltonian with moment map $\Phi_Y := \Phi|_Y - \rho$. We know that $\tilde{\mathcal{L}}|_Y$ Spin-prequantizes (Y, ω_Y, Φ_Y) (see Lemma 2.7).

We consider the Riemann-Roch character $RR_0^T(Y, \tilde{\mathcal{L}}|_Y \otimes \mathbb{C}_{-\mu})$ which is localised near $(\Phi_Y - \mu)^{-1}(0) \subset Y$. Thanks to the induction formula (3.18), we know that

$$\begin{aligned} m_\mu = [\mathcal{Q}_0^K(M \times \overline{\mathcal{O}_\mu})]^K &= \left[RR_0^K(M \times \overline{\mathcal{O}_\mu}, \tilde{\mathcal{L}} \boxtimes \mathbb{C}_{[-\mu]}) \right]^K \\ &= \left[RR_0^T(Y, \tilde{\mathcal{L}}|_Y \otimes \mathbb{C}_{-\mu}) \right]^T \\ &= \left[RR_{\Phi_Y - \mu}^T(U, \tilde{\mathcal{L}}|_U \otimes \mathbb{C}_{-\mu}) \right]^T \end{aligned}$$

where U is a small neighborhood of $\Phi_Y^{-1}(\mu)$ in Y .

The computation of the expression $[RR_{\Phi_Y - \mu}^T(U, \tilde{\mathcal{L}}|_U \otimes \mathbb{C}_{-\mu})]^T$ is identical to what we have done in Section 3.4. For ε small enough and generic, we get

$$\begin{aligned} [RR_{\Phi_Y - \mu}^T(U, \tilde{\mathcal{L}}|_U \otimes \mathbb{C}_{-\mu})]^T &= [RR_{\Phi_Y - \mu - \varepsilon}^T(U, \tilde{\mathcal{L}}|_U \otimes \mathbb{C}_{-\mu})]^T \\ &= RR(M_{\mu+\varepsilon}^{\mathfrak{t}_+^*}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) \\ &= \mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}). \end{aligned}$$

When $\mu + \rho$ does not belong to the relative interior of $\Delta(M)$, we can choose ε so that $\mu + \rho + \varepsilon \notin \Delta(M)$, and then $RR(M_{\mu+\varepsilon}^{\mathfrak{t}_+^*}, \tilde{\mathcal{L}}_{\mu+\varepsilon}) = \mathcal{Q}_{\text{spin}}(M_{\mu^+}^{\mathfrak{t}_+^*}) = 0$.

4.3. Computation of the multiplicities when $\sigma \neq \mathfrak{t}_+^*$. Let $\mu \in \bar{\sigma}$ so that $\mu + \rho - \rho_\sigma \in \sigma$. In the rest of this section β will be $-\rho_\sigma$.

Let $\mathcal{Q}_{\beta, \bar{e}}^K(M \times \overline{\mathcal{O}_\mu})$ be the generalized character defined as the index of a transversally elliptic symbol defined in a neighborhood of

$$C_{\beta, \bar{e}} = K(M^\beta \cap \Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\}) \subset N.$$

We have proved in the last section that $m_\mu = \left[\mathcal{Q}_{\beta, \bar{e}}^K(M \times \overline{\mathcal{O}_\mu}) \right]^K$.

First we notice that the character $\mathcal{Q}_{\beta, \bar{e}}^K(M \times \overline{\mathcal{O}_\mu})$ coincide with the Riemann-Roch character $RR_{\beta, \bar{e}}^K(N, \tilde{L}_N)$ localized with the Kirwan vector field near $C_{\beta, \bar{e}} \subset \text{Cr}(\|\Phi_N\|^2)$. We can look at N as a K_β -Hamiltonian manifold, and consider the Riemann-Roch character

$$RR_{\beta, e}^{K_\beta}(N, -)$$

localized with the Kirwan vector field near $K_\beta(\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\})$.

We have prove in [18] that

$$(4.28) \quad RR_{\beta, e}^K(N, \tilde{L}_N) = \text{Ind}_{K_\beta}^K \left(RR_{\beta, e}^{K_\beta}(N, \tilde{L}_N) \wedge_{\mathbb{C}}^\bullet (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}} \right)$$

where $\text{Ind}_{K_\beta}^K : R^{-\infty}(K_\beta) \rightarrow R^{-\infty}(K)$ is the induction map, and $(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ is the complexification of the real K_β -module $\mathfrak{k}/\mathfrak{k}_\beta$. It gives that

$$\left[RR_{\beta, e}^K(N, \tilde{L}_N) \right]^K = \left[RR_{\beta, e}^{K_\beta}(N, \tilde{L}_N) \wedge_{\mathbb{C}}^\bullet (\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}} \right]^{K_\beta}.$$

Let Y_σ be the principal symplectic slice of M . Recall that the subgroup $[K_\sigma, K_\sigma]$ acts trivially on Y_σ and that ρ_σ belongs to $[\mathfrak{k}_\sigma, \mathfrak{k}_\sigma]$: hence

$$\Phi^{-1}(\mu + \rho - \rho_\sigma) \subset Y_\sigma \subset M^\beta.$$

We are looking at a K_β -invariant neighborhood \mathcal{U} of $\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\}$ in N^β . We consider the open neighborhood $K \times_{K_\sigma} Y_\sigma$ of $\Phi^{-1}(\mu + \rho - \rho_\sigma)$ in M . Since $K_\beta \cap K_\sigma = T$, one sees that

$$K_\beta \times_T Y_\sigma \subset (K \times_{K_\sigma} Y_\sigma)^\beta$$

is a K_β -invariant neighborhood of Y_σ in M^β . Then we take

$$\mathcal{U} := (K_\beta \times_T Y_\sigma) \times K_\beta(\mu + \rho) \subset N^\beta.$$

We look at \mathcal{U} as an Hamiltonian K_β -manifold with moment map $\Phi_{\mathcal{U}}([k, y], \xi) = k\Phi(y) - \xi \in \mathfrak{k}_\beta^*$. The set $C'_\beta := K_\beta(\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho\})$ is a connected component of critical points of $\text{Cr}(\|\Phi_{\mathcal{U}}\|^2)$, and we consider the Riemann-Roch character

$$RR_{\beta}^{K_\beta}(\mathcal{U}, -)$$

localized with the Kirwan vector field near $C'_\beta \subset \mathcal{U}$.

Let \mathcal{N} be the normal bundle of \mathcal{U} in N . We have $\mathcal{N} = \mathcal{N}_1 \boxtimes \mathcal{N}_2$ where \mathcal{N}_1 is the normal bundle of $K_\beta \times_T Y_\sigma$ in $K \times_{K_\sigma} Y_\sigma$ and \mathcal{N}_2 is the normal bundle of $K_\beta(\mu + \rho)$ in $K(\mu + \rho)$. One computes that $\mathcal{N}_1 = K_\beta \times_T N_1$ where

$$N_1 = \sum_{\substack{\alpha > 0 \\ \alpha|_\sigma \neq 0, (\alpha, \beta) \neq 0}} \mathfrak{k}(\alpha)$$

and that $\mathcal{N}_2 = K_\beta \times_T N_2$ where

$$N_2 = \sum_{\substack{\alpha < 0 \\ (\alpha, \beta) \neq 0}} \mathfrak{k}(\alpha)$$

We decompose \mathcal{N} in the sum of the polarized bundle $\mathcal{N}^{+, \beta}$ and $\mathcal{N}^{-, \beta}$. Similarly let $\mathcal{N}_\mathbb{C}$ the complexified bundle, and its polarized β -positive part $\mathcal{N}_\mathbb{C}^{+, \beta}$.

Let $S(\mathcal{N}_\mathbb{C}^{+, \beta}) = \sum_{k \geq 0} S^k(\mathcal{N}_\mathbb{C}^{+, \beta})$ be the symmetric algebra vector bundle associated to $\mathcal{N}_\mathbb{C}^{+, \beta}$. Let us compute the rank $n_{\beta, +}$ of the polarized vector bundle vector $\mathcal{N}^{+, \beta}$. We have

$$\begin{aligned} n_{\beta, +} &= \#\{\alpha > 0 \mid (\alpha, \beta) > 0 \text{ and } \alpha|_\sigma \neq 0\} + \#\{\alpha < 0 \mid (\alpha, \beta) > 0\} \\ &= \#\{\alpha > 0 \mid (\alpha, \beta) > 0\} + \#\{\alpha < 0 \mid (\alpha, \beta) > 0\} \quad (1) \\ &= \frac{1}{2} \dim(K/K_\beta). \end{aligned}$$

In (1) we use that $\alpha|_\sigma = 0$ imposes $(\alpha, \rho - \rho_\sigma) = 0$. Then $(\alpha, \beta) = -(\alpha, \rho) < 0$ for $\alpha > 0$. Let $\det \mathcal{N}^{+, \beta}$ be the determinant line bundle associated to $\mathcal{N}^{+, \beta}$.

Thanks to the results in [18][Section 6.3], we know that

$$(4.29) \quad RR_{\beta, e}^{K_\beta}(N, \tilde{L}_N) = (-1)^{n_\beta^+} RR_{\beta}^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \otimes S(\mathcal{N}_\mathbb{C}^{+, \beta}) \right).$$

Hence we know that $m_\mu = \left[RR_{\beta, e}^{K_\beta}(N, \tilde{L}_N) \right]^K$ is equal to $(-1)^{n_\beta^+}$ times

$$\begin{aligned} &\left[RR_{\beta}^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \otimes S(\mathcal{N}_\mathbb{C}^{+, \beta}) \right) \wedge_{\mathbb{C}}^{\bullet} (\mathfrak{k}/\mathfrak{k}_\beta)_\mathbb{C} \right]^{K_\beta} \\ (4.30) \quad &= \sum_{k \geq 0} \left[RR_{\beta}^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \otimes S^k(\mathcal{N}_\mathbb{C}^{+, \beta}) \right) \wedge_{\mathbb{C}}^{\bullet} (\mathfrak{k}/\mathfrak{k}_\beta)_\mathbb{C} \right]^{K_\beta} \end{aligned}$$

Let $E \rightarrow \mathcal{U}$ be any K_β -equivariant Hermitian vector bundle. Since β acts trivially on \mathcal{U} we can look at the Lie derivative $\mathcal{L}(\beta)$ on E . Then $\frac{1}{i}\mathcal{L}(\beta)$ defines for each $x \in \mathcal{U}$ an Hermitian endomorphism of E_x . Let us denote

$$\frac{1}{i}\mathcal{L}(\beta) > 0$$

when all its eigenvalue on the fibers of E are strictly positive.

We made in [18] the crucial observation

Lemma 4.6. *If $\frac{1}{i}\mathcal{L}(\beta) > 0$, then $\left[RR_{\beta}^{K_\beta}(\mathcal{U}, E) \right]^{K_\beta} = 0$.*

Let us compute the Lie action $\mathcal{L}(\beta)$ on the fibers of the bundle $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \otimes S^k(\mathcal{N}_\mathbb{C}^{+, \beta})$. It is easy to check (see [19]) that on $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta}$ the Lie action $\frac{1}{i}\mathcal{L}(\beta)$ is equal to

$$\|\beta\|^2 + \frac{1}{2} \mathbf{Tr}_\beta |\mathcal{N}|$$

Look now at the Lie derivative $\mathcal{L}(\beta)$ on $\wedge_{\mathbb{C}}^{\bullet} (\mathfrak{k}/\mathfrak{k}_\beta)_\mathbb{C}$. As a T -module $\wedge_{\mathbb{C}}^{\bullet} (\mathfrak{k}/\mathfrak{k}_\beta)_\mathbb{C}$ is equal to

$$\begin{aligned} \prod_{(\alpha, \beta) \neq 0} (1 - e^{i\alpha}) &= \prod_{(\alpha, \beta) < 0} (1 - e^{i\alpha}) \prod_{(\alpha, \beta) > 0} (1 - e^{i\alpha}) \\ &= (-1)^{1/2 \dim(K/K_\beta)} e^{-i\delta_\beta} \left(\prod_{(\alpha, \beta) > 0} (1 - e^{i\alpha}) \right)^2 \end{aligned}$$

with $\delta_\beta = \sum_{(\alpha, \beta) > 0} \alpha$. Notice that $e^{-i\delta_\beta}$ defines a character of the group K_β that we denote $\mathbb{C}_{-\delta_\beta}$. We have proved then that

$$\wedge_{\mathbb{C}}^{\bullet}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}} = (-1)^{n_{\beta,+}} \mathbb{C}_{-\delta_\beta} \oplus R$$

where the Lie derivative $\frac{1}{i}\mathcal{L}(\beta)$ on $\mathbb{C}_{-\delta_\beta}$ is equal to $-(\delta_\beta, \beta) = -\mathbf{Tr}_\beta|\mathfrak{k}|$ and the Lie derivative $\frac{1}{i}\mathcal{L}(\beta)$ on the \mathfrak{k}_β -module R is $> -\mathbf{Tr}_\beta|\mathfrak{k}|$.

Since $\|\beta\|^2 + \frac{1}{2}\mathbf{Tr}_\beta|\mathcal{N}| = \mathbf{Tr}_\beta|\mathfrak{k}|$, we can conclude that the Lie derivative $\frac{1}{i}\mathcal{L}(\beta)$

- (1) is equal to zero on $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes \mathbb{C}_{-\delta_\beta}$,
- (2) is > 0 on $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes R$,
- (3) is > 0 on $\tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \otimes S^k(\mathcal{N}_{\mathbb{C}}^{+,\beta}) \otimes \wedge_{\mathbb{C}}^{\bullet}(\mathfrak{k}/\mathfrak{k}_\beta)_{\mathbb{C}}$ for any $k \geq 1$.

With Lemma 4.6, we see that the sum (4.30) restricts to

$$(-1)^{n_{\beta,+}} \left[RR_\beta^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta}$$

At this stage we have proved that the multiplicity m_μ is equal to

$$(4.31) \quad \left[RR_\beta^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta}.$$

On the symplectic slice $(Y_\sigma, \omega_\sigma)$, we have the moment map $\Phi_\sigma - \mu$ relative to the action of Z_σ . The data $(Y_\sigma, \omega_\sigma, \Phi_\sigma - \mu)$ is Spin-prequantized by the line bundle $\tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}$. Let

$$(4.32) \quad RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \in R^{-\infty}(Z_\sigma)$$

be the Riemann-Roch character localized near $(\Phi_\sigma - \mu)^{-1}(0) = \Phi^{-1}(\mu + \rho - \rho_\sigma) \subset Y_\sigma$.

We conclude the computation of the multiplicity m_μ with the

Lemma 4.7. *We have*

$$\begin{aligned} m_\mu &= \left[RR_\beta^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+,\beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta} \\ &= \left[RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^{Z_\sigma} \quad (1) \\ &= \mathcal{Q}_{\text{spin}}(M_\mu^\sigma). \quad (2) \end{aligned}$$

Proof. Let us prove that (1) is a consequence of the induction formula of Proposition 3.9. First we notice that the data $(Y_\sigma, \omega_\sigma, \Phi_\sigma - \mu, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu})$ is naturally equipped with an action of the maximal torus, but with a trivial action of T/Z_σ . So the generalized character (4.32) coincides with

$$RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \in R^{-\infty}(T).$$

Let us consider the Hamiltonian K_β -manifold $\mathcal{U} := (K_\beta \times_T Y_\sigma) \times \overline{K_\beta(\mu + \rho)}$. Since K_β acts trivially on ρ_σ the map $\xi \mapsto \xi + \rho_\sigma$ realizes a K_β -equivariant symplectomorphic between the coadjoint orbits $\overline{K_\beta(\mu + \rho)}$ and

$$\overline{\mathcal{O}} := \overline{K_\beta(\mu + \rho - \rho_\sigma)}.$$

The manifold \mathcal{U} is then symplectomorphic to $(K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}$. Moreover, one sees that the generalized Riemann-Roch character $RR_\beta^{K_\beta}(\mathcal{U}, -)$ coincides with the Riemann-Roch character

$$RR_0^{K_\beta}((K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}, -)$$

localized on $C_0 := K_\beta(\Phi^{-1}(\mu + \rho - \rho_\sigma) \times \{\mu + \rho - \rho_\sigma\})$.

Since $K_\beta \cap K_\sigma = T$, the Hamiltonian T -manifold Y_σ corresponds to the symplectic slice of the Hamiltonian K_β -manifold $K_\beta \times_T Y_\sigma$.

The bundle $\det \mathcal{N}^{+, \beta}$ over $(K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}$ is equal to the product of $K_\beta \times_T \mathbb{C}_{\delta_1} \rightarrow K_\beta \times_T Y_\sigma$ with $K_\beta \times_T \mathbb{C}_{\delta_2} \rightarrow \overline{\mathcal{O}}$, where

$$\delta_1 = \sum_{\substack{\alpha > 0 \\ (\alpha, \beta) > 0}} \alpha \quad \text{and} \quad \delta_2 = \sum_{\substack{\alpha < 0 \\ (\alpha, \beta) > 0}} \alpha.$$

The line bundle \tilde{L}_N is equal to the product of \tilde{L} with $K \times_T \mathbb{C}_{-\mu}$. Then the restrictions of the line bundle $\det \mathcal{N}^{+, \beta}$ and \tilde{L}_N to $Y_\sigma \times \{\mu + \rho - \rho_\sigma\}$ are respectively equal to, the trivial line bundle $\mathbb{C}_{\delta_1 + \delta_2} = \mathbb{C}_{\delta_\beta}$, and to the line bundle $\tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}$.

Finally the induction formula of Proposition 3.9 gives that

$$\begin{aligned} RR_\beta^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) &= RR_0^{K_\beta} \left((K_\beta \times_T Y_\sigma) \times \overline{\mathcal{O}}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) \\ &= \text{Ind}_T^{K_\beta} \left(RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \otimes \mathbb{C}_{\delta_\beta} \right) \\ &= \text{Ind}_T^{K_\beta} \left(RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma}) \right) \otimes \mathbb{C}_{\delta_\beta}. \end{aligned}$$

Hence

$$\begin{aligned} \left[RR_\beta^{K_\beta} \left(\mathcal{U}, \tilde{L}_N|_{\mathcal{U}} \otimes \det \mathcal{N}^{+, \beta} \right) \otimes \mathbb{C}_{-\delta_\beta} \right]^{K_\beta} &= \left[RR_0^T(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^T \\ &= \left[RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^{Z_\sigma}. \end{aligned}$$

Equality (2), i.e.

$$\left[RR_0^{Z_\sigma}(Y_\sigma, \tilde{L}|_{Y_\sigma} \otimes \mathbb{C}_{-\mu}) \right]^{Z_\sigma} = \mathcal{Q}_{\text{spin}}(M_\mu^\sigma)$$

has been proved in Section 3.4. □

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INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER (I3M), UNIVERSITÉ MONTPELLIER 2

E-mail address: Paul-Emile.Paradan@math.univ-montp2.fr