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SUPERSYMMETRIES AND COHOMOLOGY OF GRAPH COMPLEXES.

SERGUEI BARANNIKOV

1. INTRODUCTION.

I study the noncommutative generalisation of the Chern-Simons type construction of cohomology classes of graph complexes from [K2]. I propose the extension of the initial framework beyond the paradigm of homological algebra. My main initial ingredient is the h-invertible supersymmetry generator I acting on finite-dimensional algebra, whose square is nonzero in general, $I^2 \neq 0$. The second principal result is the construction of boundaries on the stable ribbon graph complex associated to a pair of certain operators. It is used to construct the Virasoro algebra action on the stable ribbon graph complexes in [B3].

Notations. For an element a from super vector space A I denote by $\pi a \in \Pi A$ the same element considered with inversed parity. $\text{char}(k) = 0$. The parity of a is denoted by \bar{a} .

Let A be finite-dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra with *odd* invariant inner product

$$\begin{aligned} g(\cdot, \cdot) &: S^2 A \rightarrow \Pi k \\ g(ab, c) &= g(a, bc) \\ g(a, b) &= (-1)^{\bar{a}\bar{b}} g(b, a) \end{aligned}$$

If the algebra has an identity then such an inner product is the same as an *odd* trace

$$l: A/[A, A] \rightarrow \Pi k, g(a, b) = l(ab)$$

In general the linear functional l is well defined via $l(ab) = g(a, b)$ on the image of the multiplication map $A^{\otimes 2} \rightarrow A$.

Examples: $q(n)$, H_{DR}^* on odd-dimensional manifold, Cl_1 .

2. SUPERSYMMETRY.

Let I be an *odd derivation* acting on A , preserving the inner product:

$$g(Ix, y) = (-1)^{\bar{x}+1} g(x, Iy)$$

or, if the algebra has an identity, equivalently,

$$l|_{\text{im}(I)} = 0$$

Examples: $[\Lambda, \cdot]$ with $\bar{\Lambda} = 1$, i_v with $\bar{v} = 0$, ...

Notice that the square of the derivation I is an even derivation of A , which is nonzero in the general case.

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Condition 1. *Let the kernel of the derivation I be isotropic with respect to the pairing:*

$$g(\cdot, \cdot)|_{\ker(I)} = 0$$

or, if the algebra has an identity, equivalently,

$$l|_{\ker(I)} = 0$$

Proposition 1. *Any such derivation is invertible in the following homotopic sense. There exists an odd self-adjoint operator \tilde{I} , acting on A , such that:*

$$(2.1) \quad [I, \tilde{I}] = Id$$

Proof. If $\ker(I) = 0$ then I is a linear isomorphism. Then I take simply the one-half of the inverse to I

$$\tilde{I} = \frac{1}{2}I^{-1}$$

In the general case, the condition (1) implies $\ker(I)$ belongs to the orthogonal complement of itself, or since I is anti-selfadjoint,

$$\ker(I) \subset \text{Im}(I)$$

Then the restriction of I on $\ker(I^2)$ induces an odd linear isomorphism

$$(\ker(I^2))/\ker(I) \stackrel{I}{\simeq} \ker(I)$$

One can choose subspaces L_i , so that $L_i \oplus \ker(I)_i = \ker(I^2)_i$, $i = 0, 1$, and that L_0 is orthogonal to L_1 . Define \tilde{I} on $\ker(I)$ as the inverse to the odd isomorphism $L \stackrel{I}{\simeq} \ker(I)$, where $L = L_0 \oplus L_1$. Extend \tilde{I} to $\ker(I^2)$ by $\tilde{I}|_L = 0$. Then $[I, \tilde{I}]|_{\ker(I^2)} = Id$. Let L' be the orthogonal complement to $\ker(I^2)$ in A . The inner product is non-degenerate on $\ker(I^2)$ and we have the direct sum decomposition

$$(2.2) \quad L' \oplus \ker(I^2) = A,$$

preserved by I . Then restriction of I on L' is a linear isomorphism and I extend \tilde{I} to A by

$$\tilde{I}|_{L'} = \frac{1}{2}I^{-1}|_{L'}.$$

It is easy to check that \tilde{I} is self-adjoint. □

Definition 1. *I call such a derivation I satisfying (2.1) h -invertible.*

This notion extends the notion of contractible differential to the case when $I^2 \neq 0$.

3. THE RIBBON GRAPH COMPLEX.

Recall the definition of the (even) ribbon graph complex (C_*, d) , see ([K2],[GK]).

Definition 2. *A ribbon graph G is a triple $(\text{Flag}(G), \sigma, \eta)$, where $\text{Flag}(G)$ is a finite set, whose elements are called flags, σ is a permutation from $\text{Aut}(\text{Flag}(G))$ with orbits of length greater than two, and η is a fixed-point free involution acting on $\text{Flag}(G)$.*

The vertices of the graph correspond to the cycles of the permutation σ . The set of vertices is denoted by $Vert(G)$. The subset of $Flag(G)$ corresponding to vertex v is denoted by $Flag(v)$. The set $Flag(v)$ has natural cyclic order since it is a cycle of σ . The cardinality of $Flag(v)$ is called the valence of v and is denoted $n(v)$. It is assumed that $n(v) \geq 3$. The edges of the graph are the pairs of flags forming a two-cycle of the involution η . The set of edges is denoted $Edge(G)$. The subset of edges which are loops is denoted by $Loop(G)$. The edges which are not loops are called "regular" and I denote the corresponding subset of $Edge(G)$ by $Edge_r(G)$. The two element subset of $Flag(G)$ corresponding to an edge $e \in Edge(G)$ is denoted $flag(e)$. I denote by S_G the Riemann surface associated with the ribbon graph G (see [K1], [B1] and references therein). I denote by $|G|$ the one-dimensional CW-complex which is the geometric realisation of the underlying graph G .

Definition 3. *The even ribbon graph complex is the vector space generated by equivalence classes of pairs $(G, or(G))$, where G is a connected ribbon graph, $or(G)$ is an orientation on the vector space*

$$\otimes_{v \in Vert(G)} (k^{Flag(v)} \oplus k)$$

and the relation $(G, -or(G)) = -(G, or(G))$ is imposed. The differential is

$$d(G, or(G)) = \sum_{e \in Edge_r(G)} (G/\{e\}, induced\ orientation)$$

The identification of my definition of the orientation with that one from ([K2]) follows from ([GK], 4.14). A choice of orientation can be fixed by a choice of a flag from $Flag(v)$ for all vertices v having *even* number of flags and a choice of an order on the set of such vertices. Two different choices are related by a set of some cyclic permutations on $Flag(v)$ for vertices v having even number of flags and a permutation on the set of such vertices. The two corresponding orientations differ by the sign equal to the product of signs of all these permutations.

The induced orientation on $G/\{e\}$ can be easily worked out. In the case when the edge e connects two vertices v_1 and v_2 with even number of flags, assume, that the order on the set of vertices with even number of flags is such that $v_1 < v_2$ and no other such vertices are between them, and that the flags $f_i \in Flag(v_i)$, $i = 1, 2$, fixing the orientation, form the edge e . Then the induced orientation for $G/\{e\}$ is defined by placing the new vertex v_0 , obtained from v_1 and v_2 after shrinking of e , to the place of v_1 and by choosing the element corresponding to $\sigma_G(f_1)$ in $Flag(v_0)$. In the case when the edge e connects two vertices v_1 and v_2 with odd number of flags, the induced orientation for $G/\{e\}$ is defined by placing the new vertex v_0 , to the lowest level in the hierarchy of vertices with even number of flags, and by choosing the flag corresponding to $\sigma_G(f)$ in $Flag(v_0)$, where f is either of the two flags forming e .

I denote via (C^*, δ) the complex dual to the (even) ribbon graph complex. We identify the generators of both complexes corresponding to oriented ribbon graphs when it does not seem to lead to a confusion.

4. THE PARTITION FUNCTION.

Since g is odd, the symmetric dual inner product g^{-1} is defined on $\Pi \text{Hom}(A, k)$, the dual space with parity inversed. If P denotes the isomorphism $A \rightarrow \Pi \text{Hom}(A, k)$

induced by g , then

$$g^{-1}(\varphi, \psi) = g(P^{-1}\varphi, P^{-1}\psi).$$

Denote by \tilde{I}^* the dual to \tilde{I} , odd symmetric operator, acting on $\Pi \text{Hom}(A, k)$:

$$\tilde{I}^*\varphi(a) = (-1)^{\overline{\varphi}}\varphi(\tilde{I}a),$$

one can check that

$$\tilde{I}^* = P\tilde{I}P^{-1},$$

since \tilde{I} is symmetric. The odd symmetric operator \tilde{I}^* defines the even symmetric pairing $g_{\tilde{I}}^{-1}$ on $\Pi \text{Hom}(A, k)$:

$$g_{\tilde{I}}^{-1}(\varphi, \psi) = g^{-1}(\tilde{I}^*\varphi, \psi), \quad g_{\tilde{I}}^{-1} \in (\Pi A)^{\otimes 2}$$

Definition 4. Given a ribbon graph G , the tensor product of the even symmetric tensors $(g_{\tilde{I}}^{-1})_e$, associated with every edge e of G , $(g_{\tilde{I}}^{-1})_e \in (\Pi A)^{\otimes \text{flag}(e)}$, defines the canonical element

$$g_{\tilde{I}, G}^{-1} = \otimes_{e \in \text{Edge}(G)} (g_{\tilde{I}}^{-1})_e, \quad g_{\tilde{I}, G}^{-1} \in (\Pi A)^{\otimes \text{Flag}(G)}.$$

Definition 5. Define the $(-1)^{n+1}$ -cyclically symmetric tensors $\alpha_n \in \text{Hom}((\Pi A)^{\otimes n}, k)$, $n \geq 2$

$$\alpha_n(\pi a_1, \dots, \pi a_n) = (-1)^{\sum_{i=1}^{n-1} \sum_{j=1}^i \overline{a_j}} l(a_1 \cdot a_2 \cdot \dots \cdot a_{n-1} \cdot a_n)$$

(the tensor from $V^{\otimes n}$ is $(-1)^{n+1}$ -cyclically symmetric if under the elementary cyclic shift it is multiplied by the Koszul sign resulting from parities of elements of V plus the sign of the cyclic permutation, which is $(-1)^{n+1}$)

Notice that the space $\text{Flag}(v)$ is canonically oriented for a vertex $v \in \text{Vert}(G)$ of odd valency, $n(v) \equiv 1 \pmod{2}$, thanks to the cyclic order. For a vertex $v \in \text{Vert}(G)$ of valency $n(v) \equiv 0 \pmod{2}$, let $or(v)$ denotes an orientation on the vector space $\text{Flag}(v)$. Because of cyclic order on $\text{Flag}(v)$, $or(v)$ is fixed canonically by a choice of an element $f \in \text{Flag}(v)$.

Definition 6. The tensors α_{2n+1} , $n \in \mathbb{N}$ are cyclically symmetric and define canonical elements

$$\alpha_v \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(v)}, k)$$

for all vertices of odd valence. The tensors α_{2n} , $n \in \mathbb{N}$, are cyclically anti-symmetric, they define canonical elements $\alpha_v \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(v)}, k)$ for all vertices of even valence equipped with a choice of orientation $or(v)$.

Proposition 2. Given an oriented ribbon graph G with orientation $or(G)$, the product over all vertices v of the $(-1)^{n(v)+1}$ -cyclically symmetric tensors α_v defines the canonical element

$$\alpha_{G, or(G)} = \otimes_{v \in \text{Vert}(G)} \alpha_v, \quad \alpha_{G, or(G)} \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(G)}, k).$$

Proof. The orientation $or(G)$ is a choice of an element of $\text{Flag}(v)$ for all vertices v of even valence, and a choice of an order on the total set of such vertices of even valencies. The first part gives a choice of orientation for all vertices of even valence. Notice that the parity of α_n is even for odd n , and odd for even n . Therefore the choice of order on the set of vertices of even valency the product $\otimes_{v \in \text{Vert}(G)} \alpha_v$ gives a well-defined element from $\text{Hom}((\Pi A)^{\otimes \text{Flag}(G)}, k)$. In more details, the orientation $or(G)$ fixes the signs in the definition of $\alpha_{G, or(G)} \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(G)}, k)$ by dictating for every vertex of even valence $n(v)$ which element a_i is to be placed the first inside $\alpha_{n(v)}$, and then in which order the tensors $\alpha_{n(v)}$ are to be multiplied,

the other choices are irrelevant because the sign of cyclic permutation is always positive for odd number of elements, and because α_n defines an even tensor for odd n . \square

Definition 7. *The partition function $Z_I^{G,or(G)}$ of an oriented ribbon graph G is the contraction of $\otimes_{v \in Vert(G)} \alpha_v$ with $\otimes_{e \in Edge(G)} (g_I^{-1})_e$. The sum over all equivalence classes of connected ribbon graphs defines the cochain Z_I*

$$(4.1) \quad Z_I = \sum_{[G]} Z_I^{G,or(G)}(G, or(G)), \quad Z_I \in (C^*, \delta)$$

Notice that the element $Z_I^{G,or(G)}(G, or(G))$ of (C^*, δ) does not depend on the choice of $or(G)$, since a change of orientation changes the signs of both $Z_I^{G,or(G)}$ and $(G, or(G))$.

5. ACTION OF I ON THE TENSORS g_I^{-1} AND α_n .

I denote by I^* the dual to I odd antisymmetric operator, acting on $\text{Hom}(\Pi A, k)$:

$$I^* \varphi(a) = (-1)^{\bar{\varphi}} \varphi(Ia)$$

The action of any endomorphism of the super vector space $\text{Hom}(\Pi A, k)$ is naturally extended to its tensor algebra $\oplus_n (\text{Hom}(\Pi A, k))^{\otimes n}$ as a derivation, by the Leibitz rule. The dual action on $\oplus_n (\Pi A)^{\otimes n}$ corresponds to the similar extension of the action of I on ΠA .

Proposition 3. *The result of action of the operator I on g_I^{-1} is*

$$I(g_I^{-1}) = g^{-1}$$

Proof.

$$I(g_I^{-1}) = g_I^{-1}(I^* \varphi, \psi) + (-1)^{\bar{\varphi}} g_I^{-1}(\varphi, I^* \psi) = g^{-1}([I^*, \tilde{I}^*] \varphi, \psi) = g^{-1}(\varphi, \psi)$$

\square

Consider the partition functions of an oriented ribbon graph G and the graph $G' = G/\{e'\}$ obtained from G by contracting the edge $e' \in Edge_r(G)$, with its induced orientation. The previous proposition shows that acting by I on the two-tensor associated with the edge e' gives

$$I(g_I^{-1})_{e'} = g_{e'}^{-1},$$

where $g_{e'}^{-1} \in (\Pi A)^{\otimes \{f, f'\}}$, $e' = (ff')$, denotes the two-tensor inverse to g , associated with the edge e' . I claim that inserting $g_{e'}^{-1}$ instead of $(g_I^{-1})_{e'}$ for the edge e' which is not a loop gives $Z_I^{G/\{e'\}, or(G/\{e'\})}$.

Proposition 4. *The partition function $Z_I^{G/\{e'\}, or(G/\{e'\})}$, for a regular edge e' , is equal to the contraction of $\otimes_{v \in Vert(G)} \alpha_v$ with $\otimes_{e \in Edge(G)} h_e$ where $h_e = (g_I^{-1})_e$ for $e \neq e'$ and $h_{e'} = g_{e'}^{-1}$.*

Proof. The part of the contraction involving $g_{e'}^{-1}$ is

$$\sum_{\mu, \nu} (-1)^{\bar{\nu}} \alpha_{k+1}(v_1 \dots v_k u_\mu) g^{-1}(u^\mu, u^\nu) \alpha_{n-k+1}(u_\nu v_{k+1} \dots v_n)$$

where $\{u_\mu\}$ is a basis in (ΠA) and $\{u^\mu\}$ is the dual basis, u_μ and u_ν represent the two flags of the edge e' , and

$$v_1 \otimes \dots \otimes v_k \otimes v_{k+1} \dots \otimes v_n \in (\Pi A)^{\otimes \text{Flag}(v)}$$

represents the flags corresponding to the new vertex of $G/\{e'\}$ to which the edge e' shrinks. Using the linear algebra identity

$$b = \sum_{\mu} u_{\mu} u^{\mu}(b) = \sum_{\mu\nu} (-1)^{\bar{\nu}} u_{\mu} g^{-1}(u^{\mu}, u^{\nu}) g(u_{\nu}, b).$$

with $b = v_{k+1} \dots v_n$ I get

$$\alpha_n(v_1 \dots v_n) = \sum_{\mu, \nu} (-1)^{\bar{\nu}} \alpha_{k+1}(v_1 \dots v_k u_{\mu}) g^{-1}(u^{\mu}, u^{\nu}) \alpha_{n-k+1}(u_{\nu} v_{k+1} \dots v_n)$$

i.e. the tensor α_v associated with the new vertex. \square

Next proposition shows that acting by I^* on α_n gives zero because I is a derivation of A , preserving g .

Proposition 5.

$$I^*(\alpha_n) = 0$$

Proof. It follows from

$$(5.2) \quad \sum_{i=1}^n (-1)^{\varepsilon_i} l(a_1 \dots (I a_i) \dots a_n) = l(I(a_1 \dots a_n)) = 0$$

where $\varepsilon_i = \sum_{j=1}^{i-1} \bar{a}_j$, $n \geq 2$. \square

6. THE BOUNDARY OF THE COCHAIN Z_I .

Combining the three propositions we get the following theorem.

Theorem 1. *The boundary of the cochain Z_I (4.1) is given by the sum over graph with loops, each loop contributing the term similar to the one described in proposition 4 for regular edges:*

$$\delta Z_I = \sum_{[G], \text{Loop}(G) \neq \emptyset} (G, \text{or}(G)) Z_I^{\text{loop}, (G, \text{or}(G))}$$

where

$$Z_I^{\text{loop}, (G, \text{or}(G))} = - \left(\left(\bigotimes_{e \in \text{Edge}_r(G)} (g_I^{-1})_e \right) I \left(\bigotimes_{l \in \text{Loop}(G)} (g_I^{-1})_l \right) \right) \left(\bigotimes_{v \in \text{Vert}(G)} \alpha_v \right).$$

Proof. The boundary of Z_I

$$\delta Z_I = \sum_{[G']} Z_I^{(G', \text{or}(G'))} \delta(G', \text{or}(G')) = \sum_{[G]} (G, \text{or}(G)) \sum_{e \in \text{Edge}_r(G)} Z_I^{(G/\{e\}, \text{or}(G/\{e\}))}$$

For any ribbon graph G we have

$$\begin{aligned}
 0 &\stackrel{(\text{Prop. 5})}{=} \left(\otimes_{e \in \text{Edge}(G)} (g_I^{-1})_e \right) I^* \left(\otimes_{v \in \text{Vert}(G)} \alpha_v \right) = \\
 &= I \left(\otimes_{e \in \text{Edge}(G)} (g_I^{-1})_e \right) \left(\otimes_{v \in \text{Vert}(G)} \alpha_v \right) \stackrel{(\text{Prop. 3,4})}{=} \\
 &= \left(\sum_{e \in \text{Edge}_r(G)} Z_I^{(G/\{e\}, \text{or}(G/\{e\}))} \right) + Z_I^{\text{loop}, (G, \text{or}(G))}
 \end{aligned}$$

□

7. COMPACTIFICATION AND THE *stable* RIBBON GRAPH COMPLEX (EVEN SCALAR PRODUCT CASE).

To remedy the problem with contribution of loops, since they are not contracted in the ribbon graph complex, one has to consider the complex in which all the edges, including the loops, are allowed to contract. This leads to the complex of *stable* ribbon graphs and the combinatorial compactification of the moduli space of Riemann surfaces, which was introduced by Kontsevich in [K1], two years before [K2], from completely different point of view in connection with the proof of Witten conjecture. As far as I know it was not until the paper ([B1]) that this compactification reappeared in a new context more than ten years later, in my work devoted to study of noncommutative Batalin-Vilkovisky equation. In loc.cit I have related the complex of *stable* ribbon graphs with some modular variant of cyclic associative algebras. This leads, using in particular the dual to the map from loc.cit., proposition 11.3, and the same propagator (g_I^{-1}) , to the construction of weights on the stable ribbon graphs, completing the cochain Z_I to a coboundary free cochain.

Definition 8. A *stable ribbon graph* \widehat{G} is a data $(\text{Flag}(\widehat{G}), \sigma, \eta, \lambda, \{\gamma_v\})$, where $\text{Flag}(\widehat{G})$ is a finite set, whose elements are called *flags*, λ is a partition on the set $\text{Flag}(\widehat{G})$, σ is a permutation from $\text{Aut}(\text{Flag}(\widehat{G}))$, stabilizing λ , η is a fixed-point free involution acting on $\text{Flag}(\widehat{G})$, and $\gamma_v \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer attached to every cluster v of λ . The clusters of the partition λ are the vertices of *stable ribbon graph* \widehat{G} . The edges are the orbits of the involution η . This data satisfy the *stability condition* ...

In particular to any vertex v corresponds the integer γ_v and a permutation σ_v on the subset of flags $\text{Flag}(v)$ from the cluster v , so that $\sigma = \prod_v \sigma_v$. I denote by $\text{Cycle}(v)$ the set of cycles of the permutation σ_v attached to a vertex $v \in \text{Vert}(\widehat{G})$, and by $\text{Cycle}(\widehat{G})$ the union of such sets which is the set of cycles of σ . For $c \in \text{Cycle}(\widehat{G})$ I denote by $\text{Flag}(c)$ the corresponding subset of $\text{Flag}(\widehat{G})$. Denote $b_v = |\text{Cycle}(v)|$

7.1. Contraction of edges on the stable ribbon graphs. In order to describe the action of the differential on the stable ribbon graph \widehat{G} one needs to describe the result of the contraction of an arbitrary edge $\widehat{G}/\{e\}$. The contraction of edges in the stable ribbon graphs is described combinatorially via compositions and contractions on permutations, see ([B1]) and references therein, representing the corresponding geometric operations on $S_{\widehat{G}}$. For the reader convenience I rephrase it here. For an edge ending at vertices with $\gamma_v = 0$, $b_v = 1$ this reproduces the standard contraction on ribbon graphs.

I identify the permutation $\sigma^{\widehat{G}}$ with multicyclic order on $Flag(\widehat{G})$, i.e. the collection of cyclic orders on the orbits of σ . Then the result of the contraction of an edge $e = (ff')$, $f, f' \in Flag(\widehat{G})$ is the stable ribbon graph $\widehat{G}/\{e\}$ such that

$$\begin{aligned} Flag(\widehat{G}/\{e\}) &= Flag(\widehat{G}) \setminus \{f, f'\}, \\ \sigma^{\widehat{G}/\{e\}} &= \left((ff') \circ \sigma^{\widehat{G}} \right) \Big|_{Flag(\widehat{G}) \setminus \{f, f'\}}, \end{aligned}$$

as a multicyclic order, i.e. $\sigma^{\widehat{G}/\{e\}}$ is the multi-cyclic order induced on $Flag(\widehat{G}) \setminus \{f, f'\}$ by the multi-cyclic order on $Flag(\widehat{G})$ represented by the product $(ff') \circ \sigma^{\widehat{G}}$, naturally

$$\eta^{\widehat{G}/\{e\}} = \eta^{\widehat{G}} \Big|_{Flag(\widehat{G}) \setminus \{f, f'\}}$$

as a map, i.e. the involution $\eta^{\widehat{G}/\{e\}}$ is the restriction of the involution $\eta^{\widehat{G}}$ on the subset $Flag(\widehat{G}) \setminus \{f, f'\}$. If an edge $e = (ff')$, $f, f' \in Flag(\widehat{G})$ is not a loop, i.e. f, f' are from two distinguished clusters v and v' of λ , then the clusters v and v' collide to the new cluster having $\gamma = \gamma_v + \gamma_{v'}$. If an edge e is a loop then no vertices must collide and the partition $\lambda^{\widehat{G}/\{e\}}$ on $Flag(\widehat{G}) \setminus \{f, f'\}$ is induced from $\lambda^{\widehat{G}}$, if the flags f, f' are from the same cycle of $\sigma_v^{\widehat{G}}$ then $\gamma_v^{\widehat{G}/\{e\}} = \gamma_v^{\widehat{G}}$ and if they are from different cycles of $\sigma_v^{\widehat{G}}$ then $\gamma_v^{\widehat{G}/\{e\}} = \gamma_v^{\widehat{G}}$. Lastly if f, f' are neighbors in a cycle of $\sigma_v^{\widehat{G}}$, i.e. say $\sigma_v^{\widehat{G}}(f) = f'$, then, by definition, $\widehat{G}/\{e\} = \emptyset$, so that such loop do not contribute to the boundary operator. This exception is dictated by the relation of the stable ribbon graphs with the combinatorial compactification of the moduli spaces from ([K1]), and leads to the interesting condition on the algebra A .

8. THE DIFFERENTIAL ON THE STABLE RIBBON GRAPH COMPLEX.

Definition 9. *The (even) stable ribbon graph complex is the vector space generated by equivalence classes of pairs $(\widehat{G}, or(\widehat{G}))$, where \widehat{G} is a stable ribbon graph, $or(\widehat{G})$ is an orientation on the vector space*

$$\otimes_{v \in Vert(\widehat{G})} (k^{Flag(v)} \oplus k^{Cycle(v)})$$

and the relation $(\widehat{G}, -or(\widehat{G})) = -(\widehat{G}, or(\widehat{G}))$ is imposed. The differential is

$$d(\widehat{G}, or(\widehat{G})) = \sum_{e \in Edge(\widehat{G})} (\widehat{G}/\{e\}, induced\ orientation)$$

Using the multi-cyclic order on $Flag(\widehat{G})$, as in section *** above, a choice of orientation is fixed by a choice of a flag from every cycle of σ of even length and a choice of order on the total set of such cycles. The induced orientation is also analogous to the case of usual ribbon graphs, see ([B1]) for details.

9. WEIGHTS ON STABLE RIBBON GRAPHS.

I start by constructing the tensor $\alpha_{\sigma_v, \gamma_v} \in \text{Hom}((IIA)^{\otimes Flag(v)}, k)$ for any given vertex $v \in Vert(\widehat{G})$ with permutation $\sigma_v \in \text{Aut}(Flag(v))$ and integer γ_v attached, together with a choice of orientation $or(v)$ from $k^{Flag(v)} \oplus k^{Cycle(v)}$. Let $\sigma_v = (\rho_1 \dots \rho_r) \dots (\tau_1 \dots \tau_t)$ be a representation of σ_v compatible with $or(v)$, in the

sense that the order $\rho_1 < \dots < \tau_t$ on flags together with the order $(\rho_1 \dots \rho_r) < \dots < (\tau_1 \dots \tau_t)$ on cycles gives $or(v)$, put

$$(9.1) \quad \alpha_{\sigma_v, \gamma_v} (\pi a_{\rho_1} \otimes \dots \otimes \pi a_{\tau_t}) = l \left((-1)^\epsilon \left(\sum_{\mu_1, \dots, \mu_{b_v-1}} e^{\mu_1} a_{\rho_1} \dots a_{\rho_r} e_{\mu_1} e^{\mu_2} \dots e_{\mu_{b_v-1}} a_{\tau_1} \dots a_{\tau_t} \right) \prod_{i=1}^{\gamma_v} \left(\sum_{\xi_i, \zeta_i} e^{\xi_i} e^{\zeta_i} e_{\xi_i} e_{\zeta_i} \right) \right)$$

where $\{e_\mu\}, \{e^\mu\}$ is a pair of dual bases in A , $g(e^\mu, e_\nu) = \delta_\nu^\mu$, ϵ is the Koszul sign (taking into account the passing of all π 's to the left and then putting a_i inside $(-1)^{\sum \mu_i} e^{\mu_1} e_{\mu_1} \dots e^{\mu_{b_v-1}} e_{\mu_{b_v-1}}$).

Proposition 6. *For any $a, b \in A$*

$$(9.2) \quad \sum_{\mu} (-1)^{\overline{e^\mu}(\overline{a}+\overline{b}+1)} e^\mu a b e_\mu = (-1)^{\overline{a}\overline{b}} \sum_{\nu} (-1)^{\overline{e^\nu}(\overline{a}+\overline{b}+1)} e^\nu b a e_\nu.$$

For any $a, b \in A$,

$$(9.3) \quad \sum_{\mu} (-1)^{\overline{e^\mu}(\overline{a}+1)} e^\mu a e_\mu b = (-1)^{\overline{a}\overline{b}} \sum_{\nu} (-1)^{\overline{e^\nu}(\overline{a}+1)} b e^\nu a e_\nu.$$

Proof. If

$$b e_\mu = \sum_{\nu} \beta_\mu^\nu e_\nu$$

then

$$\beta_\mu^\nu = g(e^\nu, b e_\mu) = g(e^\nu b, e_\mu)$$

so

$$e^\nu b = \sum_{\mu} \beta_\mu^\nu e^\mu.$$

Therefore,

$$\begin{aligned} \sum_{\mu} (-1)^{\overline{e^\mu}(\overline{a}+\overline{b}+1)} e^\mu a b e_\mu &= \sum_{\mu} (-1)^{\overline{e^\mu}(\overline{a}+\overline{b}+1)} e^\mu a \sum_{\nu} \beta_\mu^\nu e_\nu = \\ &= (-1)^{(\overline{a}+\overline{b}+1)\overline{b}} \sum_{\mu, \nu} (-1)^{\overline{e^\nu}(\overline{a}+\overline{b}+1)} \beta_\mu^\nu e^\mu a e_\nu = (-1)^{\overline{a}\overline{b}} \sum_{\nu} (-1)^{\overline{e^\nu}(\overline{a}+\overline{b}+1)} e^\nu b a e_\nu \end{aligned}$$

The proof of (9.3) is analogous. \square

Remark 1. *It follows that $\sum_{\mu} (-1)^{\overline{e^\mu}\overline{a}} e^\mu a e_\mu$ and $\sum_{\xi, \zeta} (-1)^{\overline{e^\xi}e^\zeta} e^\xi e^\zeta e_\xi e_\zeta$ are in the center of A .*

It follows from (9.2) and (9.3) that for every cycle $(\rho_1 \dots \rho_r)$ of σ_v the expression (9.1) is $(-1)^{r+1}$ -cyclically symmetric with respect to cyclic permutations of $\pi a_{\rho_1} \dots \pi a_{\rho_r}$. And that the expression (9.1) is invariant under the changing the order of cycles in the representation of σ , up to the sign taking into account that the total parity of $e^{\mu_1} a_{\rho_1} \dots a_{\rho_r} e_{\mu_1}$ differs from the parity of $\pi a_{\rho_1} \dots \pi a_{\rho_r}$ by the factor $(-1)^{r+1}$. Therefore, for a fixed choice of $or(v)$, the expression (9.1) gives a well-defined element $\alpha_{\sigma_v, \gamma_v} \in \text{Hom}((\Pi A)^{\otimes \text{Flag}(v)}, k)$. Notice that for a vertex with $\gamma_v = 0$, $b_v = 1$ this gives the tensor α_v , defined above.

A choice of orientation $or(\widehat{G})$ on stable ribbon graph can be identified with a choice of orientation $or(v)$ on every vertex plus a choice of order on the set of vertices with *odd* number of cycles of *even* length. These are precisely the vertices

for which the tensor $\alpha_{\sigma_v, \gamma_v}$ is odd. Therefore for a choice of $or(\widehat{G})$ the product $\otimes_{v \in Vert(\widehat{G})} \alpha_{\sigma_v, \gamma_v}$ gives a well-defined element from $Hom((\Pi A)^{\otimes Flag(\widehat{G})}, k)$.

Definition 10. A choice of orientation $or(\widehat{G})$ on a stable ribbon graph \widehat{G} allows to define the linear functional $\alpha_{\widehat{G}, or(\widehat{G})}$ on $(\Pi A)^{\otimes Flag(\widehat{G})}$:

$$\alpha_{\widehat{G}, or(\widehat{G})} = \otimes_{v \in Vert(\widehat{G})} \alpha_{\sigma_v, \gamma_v}$$

For the propagator I use the same bilinear form associated with h -inverse of the odd supersymmetry I .

Definition 11. Given a ribbon graph G , the tensor product of the even symmetric tensors $(g_I^{-1})_e$, associated with every edge e of G , $(g_I^{-1})_e \in (\Pi A)^{\otimes flag(e)}$, defines the canonical element

$$g_{I, \widehat{G}}^{-1} = \otimes_{e \in Edge(\widehat{G})} (g_I^{-1})_e, \quad g_{I, \widehat{G}}^{-1} \in (\Pi A)^{\otimes Flag(\widehat{G})}.$$

Definition 12. The partition function $\widehat{Z}_I^{\widehat{G}, or(\widehat{G})}$ of an oriented stable ribbon graph \widehat{G} is the contraction of $\otimes_{v \in Vert(\widehat{G})} \alpha_{\sigma_v, \gamma_v}$ with $\otimes_{e \in Edge(\widehat{G})} (g_I^{-1})_e$. As above, the element $\widehat{Z}_I^{\widehat{G}, or(\widehat{G})}(\widehat{G}, or(\widehat{G}))$ of (\widehat{C}^*, δ) does not depend on the choice of $or(\widehat{G})$. The sum over all equivalence classes of connected stable ribbon graphs defines the cochain \widehat{Z}_I

$$(9.4) \quad \widehat{Z}_I = \sum_{[G]} \widehat{Z}_I^{\widehat{G}, or(\widehat{G})}(\widehat{G}, or(\widehat{G})), \quad \widehat{Z}_I \in (\widehat{C}^*, \delta)$$

10. THE CANCELATION OF ANOMALY.

In this section I discuss the condition imposed on A in order that the weights introduced below behave correctly with respect to the contraction of a loop encircling a boundary component.

Condition 2. For any $a \in A$ the super trace of the operator of multiplication by a is zero:

$$(10.1) \quad \sum_{\mu} (-1)^{\overline{\mu}} g(u^{\mu}, a \cdot u_{\mu}) = 0$$

Below I describe different sufficient conditions for (10.1) to hold. All of them trivially hold in our basic examples.

Assume that $I(\sum_i [p_i, q_i]) = 0$, p_i, q_i - symplectic basis in L_0 for $g(I, \cdot)$ in particular, for example if $\ker(I)_1 = 0$.

Proposition 7. Assume that $I(\sum_i [p_i, q_i]) = 0$, where p_i, q_i - symplectic basis in L_0 for $g(I, \cdot)$ then (10.1) holds

Proposition 8. Assume that A_0 is semisimple, then (10.1) holds

Proof. The statement is trivial for $a \in A_1$. Let $\{x_{\mu} | x_{\mu} \in A_0\}$ be a linear basis in A_0 and $\{\xi_{\mu} | \xi_{\mu} \in A_1\}$ be the dual basis in A_1 , so that

$$g(x_{\mu}, \xi_{\nu}) = \delta_{\mu\nu}$$

Let us denote by K_a the operator of left multiplication by a acting on A and by R_a the operator of right multiplication. From

$$g(x_{\mu}, a\xi_{\mu}) = g(x_{\mu}a, \xi_{\mu})$$

it follows that

$$\mathrm{Tr} K_a|_{A_1} = \mathrm{Tr} R_a|_{A_0}$$

Let us show that $\mathrm{Tr} R_a|_{A_0} = \mathrm{Tr} K_a|_{A_0}$. If A_0 is semisimple then $t(a, b) = \mathrm{Tr} K_{ab}|_{A_0}$ is a nondegenerate invariant even inner product on A_0 and

$$\mathrm{Tr} K_a|_{A_0} = \sum_{\mu} t(\tilde{e}_{\mu}, a\tilde{e}_{\mu}) = \sum_{\mu} t(\tilde{e}_{\mu}a, \tilde{e}_{\mu}) = \mathrm{Tr} R_a|_{A_0}$$

where \tilde{e}_{μ} is the orthonormal basis for t in $A_0 \otimes_k \bar{k}$. \square

Proposition 9. *Assume that the factor algebra by the nil radical $A_0/\mathrm{nil}(A_0)$ is simple, then (10.1) holds.*

Proposition 10. *Contracting any loop whose flags are neighbors in the sense that $\sigma(f) = f'$, so that the loop encircles some boundary component of the surface $S(\hat{G})$, gives zero.*

$$\sum_{\mu\nu} (-1)^{\bar{\nu}} \alpha_{k+2}(v_1 \dots v_k u_{\mu} u_{\nu}) g^{-1}(u^{\mu}, u^{\nu}) = 0$$

Proof. Taking $a = v_1 \cdot \dots \cdot v_k$ the statement is reduced to (10.1). \square

11. ACTION OF I ON THE TENSORS $\alpha_{\sigma_v, \gamma_v}$ AND $g_{\hat{I}, \hat{G}}^{-1}$.

11.1. Action of I on the tensors $\alpha_{\sigma_v, \gamma_v}$.

Proposition 11.

$$(11.1) \quad I^*(\alpha_{\sigma_v, \gamma_v}) = 0$$

Proof. It follows from the fact that I is a derivation of the multiplication and of the bilinear form. It follows from

$$g(Ie^{\nu}, e_{\mu}) + (-1)^{\bar{e}^{\nu}} g(e^{\nu}, Ie_{\mu}) = 0$$

that if

$$Ie_{\mu} = \sum_{\nu} I_{\mu}^{\nu} e_{\nu}$$

then

$$Ie^{\nu} = (-1)^{\bar{e}^{\nu}+1} \sum_{\mu} I_{\mu}^{\nu} e^{\mu}.$$

Therefore

$$\sum_{\mu} (-1)^{\bar{e}^{\mu}(\bar{a}+1)} I(e^{\mu}) a e_{\mu} + \sum_{\nu} (-1)^{(\bar{e}^{\nu}+1)\bar{a}} e^{\nu} a I(e_{\nu}) = 0$$

for any $a \in A$. Now (11.1) follows from (5.2). \square

The immediate consequence is the invariance of $\alpha_{\hat{G}, \mathrm{or}(\hat{G})}$ under the action of derivation I .

Proposition 12. $I^* \alpha_{\hat{G}, \mathrm{or}(\hat{G})} = 0$

11.2. **Action of I on $g_{\widehat{I}, \widehat{G}}^{-1}$ and contractions of edges.** As it was shown in the

proposition 3, acting by I on the two-tensor associated with an edge e' gives

$$I(g_{\widehat{I}}^{-1})_{e'} = g_{e'}^{-1},$$

where $g_{e'}^{-1} \in (\Pi A)^{\otimes \{f, f'\}}$, $e' = (ff')$, denotes the two-tensor inverse to g . The next proposition shows that, inserting $g_{e'}^{-1}$ instead of $(g_{\widehat{I}}^{-1})_{e'}$ for *arbitrary* edge e' and contracting with $\alpha_{\widehat{G}, or(\widehat{G})}$ gives $\widehat{Z}_I^{\widehat{G}/\{e'\}, or(\widehat{G}/\{e'\})}$, the partition function of the oriented stable ribbon graph $\widehat{G}/\{e'\}$.

Proposition 13. *For arbitrary edge e' the partition function $\widehat{Z}_I^{\widehat{G}/\{e'\}, or(\widehat{G}/\{e'\})}$ is equal to the contraction of $\alpha_{\widehat{G}, or(\widehat{G})}$ with $\otimes_{e \in \text{Edge}(\widehat{G})} h_e$ where $h_e = (g_{\widehat{I}}^{-1})_e$ for $e \neq e'$ and $h_{e'} = g_{e'}^{-1}$.*

Proof. Consider first the case when e' is not a loop. Then the term of the contraction of $\otimes_{v \in \text{Vert}(\widehat{G})} \alpha_v$ with $\otimes_{e \in \text{Edge}(\widehat{G})} h_e$, involving $g_{e'}^{-1}$ is

$$\sum_{\mu, \nu} (-1)^{\overline{\nu}} \alpha_{\sigma_v, \gamma_v}(a_1, \dots, a_k, U_\mu) g^{-1}(U^\mu, U^\nu) \alpha_{\sigma_{v'}, \gamma_{v'}}(U_\nu, a_{k+1}, \dots, a_n)$$

where $\{U_\mu\}$ is a basis in (ΠA) and $\{U^\mu\}$ is the dual basis in $\text{Hom}((\Pi A), k)$, U_μ and U_ν represent the two flags of the edge e' . Using the identities (9.2) and (9.3) the tensor $\alpha_{\sigma_{v'}, \gamma_{v'}}$ can be represented in the form:

$$l \left((-1)^\epsilon \left(\sum_{\mu_1, \dots, \mu_{b_v-1}} U_\nu a_{\tau_2} \dots a_{\tau_t} e^{\mu_1} a_{\rho_1} \dots a_{\rho_r} e_{\mu_1} e^{\mu_2} \dots e_{\mu_{b_v-1}} \prod_{i=1}^{\gamma_v} \left(\sum_{\xi_i, \zeta_i} e^{\xi_i} e^{\zeta_i} e_{\xi_i} e_{\zeta_i} \right) \right) \right)$$

Using as above the linear algebra identity

$$b = \sum_{\mu} U_\mu U^\mu(b) = \sum_{\mu\nu} (-1)^{\overline{\nu}} U_\nu g^{-1}(U^\mu, U^\nu) g(U_\nu, b).$$

with

$$b = (-1)^\epsilon \sum_{\mu_1, \dots, \mu_{b_v-1}} a_{\tau_2} \dots a_{\tau_t} e^{\mu_1} a_{\rho_1} \dots a_{\rho_r} e_{\mu_1} e^{\mu_2} \dots e_{\mu_{b_v-1}} \prod_{i=1}^{\gamma_v} \left(\sum_{\xi_i, \zeta_i} e^{\xi_i} e^{\zeta_i} e_{\xi_i} e_{\zeta_i} \right)$$

I get the tensor $\alpha_{\sigma_v, \gamma_v}$ with b substituted instead of U_μ . Using again (9.2) and (9.3) to bring it to the standard form(9.1), I get precisely the tensor associated with the new vertex in $\widehat{G}/\{e'\}$, i.e. with the two permutations $\sigma_{v'}$ and σ_v merged at the flags of e' , with $\gamma_{v^{new}} = \gamma_v + \gamma_{v'}$ etc. Let's now $e' = (ff')$ is a loop, $f, f' \in \text{flag}(v)$. Then if f and f' are in the same cycle of σ_v , assuming that flags f, f' are not neighbors, $\sigma(f) \neq f', \sigma(f') \neq f$.

$$\begin{aligned} \sum_{\mu, \nu, \kappa} (-1)^{\overline{\nu}} g^{-1}(U^\mu, U^\nu) e^\kappa a_{\rho_1} \dots a_{\rho_i} U_\mu a_{\rho_{i+1}} \dots a_{\rho_{j-1}} U_\nu a_{\rho_j} \dots a_{\rho_r} e_\kappa = \\ = \sum_{\mu, \kappa} e^\kappa a_{\rho_1} \dots a_{\rho_i} e^\mu a_{\rho_{i+1}} \dots a_{\rho_{j-1}} e_\mu a_{\rho_j} \dots a_{\rho_r} e_\kappa \end{aligned}$$

Using (9.3) it is transformed to

$$\sum_{\mu, \kappa} e^\kappa a_{\rho_1} \dots a_{\rho_i} a_{\rho_j} \dots a_{\rho_r} e_\kappa e^\mu a_{\rho_{i+1}} \dots a_{\rho_{j-1}} e_\mu$$

And the tensor $\alpha_{\sigma_v, \gamma_v}$ is transformed to $\alpha_{\sigma_v^{new}, \gamma_v}$ corresponding to the permutation σ_v^{new} obtained from σ_v by dissecting one cycle into two at flags f and f' . Similarly if f and f' are in the two different cycles then the result is the tensor $\alpha_{\sigma_v^{new}, \gamma_v^{new}}$ corresponding to the permutation σ_v^{new} obtained from σ_v by merging the two cycles at flags f and f' and with $\gamma_v^{new} = \gamma_v + 1$. In both cases I get precisely the tensor corresponding to the tranformed vertex v in $\widehat{G}/\{e'\}$. Finally if f, f' are neighbors, say $\sigma(f) \neq f'$, then the insertion of $g_{e'}^{-1}$ instead of $(g_{\bar{I}}^{-1})_{e'}$ gives zero because of the property (10.1) imposed on A . \square

12. THE BOUNDARY OF THE EXTENDED COCHAIN \widehat{Z}_I .

Combining the results from the previous sections I get the following theorem.

Theorem 2. *The boundary of the cochain \widehat{Z}_I (9.4) is zero*

$$\delta \widehat{Z}_I = 0$$

Proof. The boundary of \widehat{Z}_I

$$\delta \widehat{Z}_I = \sum_{[\widehat{G}']} \widehat{Z}_I^{(\widehat{G}', or(\widehat{G}'))} \delta(\widehat{G}', or(\widehat{G}')) = \sum_{[\widehat{G}]} (\widehat{G}, or(\widehat{G})) \sum_{e \in Edge(\widehat{G})} \widehat{Z}_I^{(\widehat{G}/\{e\}, or(\widehat{G}/\{e\}))}$$

For any stable oriented ribbon graph \widehat{G} I have

$$\begin{aligned} & \left(\sum_{e \in Edge(\widehat{G})} \widehat{Z}_I^{(\widehat{G}/\{e\}, or(\widehat{G}/\{e\}))} \right) \stackrel{(Prop.13)}{=} \\ & = I \left(\otimes_{e \in Edge(\widehat{G})} (g_{\bar{I}}^{-1})_e \right) \left(\otimes_{v \in Vert(\widehat{G})} \alpha_{\sigma_v, \gamma_v} \right) = \\ & = \left(\otimes_{e \in Edge(\widehat{G})} (g_{\bar{I}}^{-1})_e \right) I^* \left(\otimes_{v \in Vert(\widehat{G})} \alpha_{\sigma_v, \gamma_v} \right) \stackrel{(Prop. 12)}{=} 0 \end{aligned} \quad \square$$

13.

14. GAUGE TRANSFORMATIONS AND COBOUNDARIES.

Same arguments as in the previuos section show that if one modifies \widetilde{I} via

$$\widetilde{I} \rightarrow \widetilde{I} + [I, B]$$

then the cohomology cochain is modified by a coboundary .

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