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# The first rational Chebyshev knots\*

P. -V. Koseleff, D. Pecker, F. Rouillier

November 3, 2009

## Abstract

A Chebyshev knot  $\mathcal{C}(a, b, c, \varphi)$  is a knot which has a parametrization of the form  $x(t) = T_a(t)$ ;  $y(t) = T_b(t)$ ;  $z(t) = T_c(t + \varphi)$ , where  $a, b, c$  are integers,  $T_n(t)$  is the Chebyshev polynomial of degree  $n$  and  $\varphi \in \mathbf{R}$ . We show that any two-bridge knot is a Chebyshev knot with  $a = 3$  and also with  $a = 4$ . For every  $a, b, c$  integers ( $a = 3, 4$  and  $a, b$  coprime), we describe an algorithm that gives all Chebyshev knots  $\mathcal{C}(a, b, c, \varphi)$ . We deduce a list of minimal Chebyshev representations of two-bridge knots with small crossing number.

**keywords:** Polynomial curves; two-bridge knots; Chebyshev curves; real roots isolation; computer algebra; algorithms

**Mathematics Subject Classification 2000:** 14H50, 57M25, 14P99

## 1 Introduction

A Chebyshev knot  $\mathcal{C}(a, b, c, \varphi)$  is a knot which has a parametrization of the form  $x(t) = T_a(t)$ ;  $y(t) = T_b(t)$ ;  $z(t) = T_c(t + \varphi)$ , where  $a, b$  are coprime integers,  $c$  is an integer,  $T_n(t)$  is the Chebyshev polynomial of degree  $n$  and  $\varphi \in \mathbf{R}$ . Chebyshev knots are polynomial analogues of Lissajous knots, which admit parametrizations of the form  $x = \cos(at)$ ;  $y = \cos(bt + \varphi)$ ;  $z = \cos(ct + \psi)$ , where  $a, b, c$  are pairwise coprime integers. These knots, first defined in [BHJS], have been studied by many authors: V. F. R. Jones, J. Przytycki, C. Lamm, J. Hoste and L. Zirbel [JP, La, HZ].

It is known that every knot may be obtained from a polynomial embedding  $\mathbf{R} \rightarrow \mathbf{R}^3 \subset \mathbf{S}^3$ , where  $\mathbf{S}^3$  is the one-point compactification of  $\mathbf{R}^3$  ([Va, DOS]). In [KP1] we have shown that every knot is a Chebyshev knot.

A two-bridge knot (or a rational knot) is a knot which is isotopic to a compact space curve such that the  $x$ -coordinate has only two maxima and two minima. When  $a = 3$  or  $a = 4$  the Chebyshev knot  $\mathcal{C}(a, b, c, \varphi)$  is a two-bridge knot  $K$ . Its projection onto the  $(x, y)$ -plane is the Chebyshev curve  $\mathcal{C}(a, b) : T_b(x) = T_a(y)$  and we have a regular diagram of  $K$  that is in Conway normal form when  $a = 3$  (see [Con, Mu]). This gives us an easy way to identify these knots using their classical Schubert invariant.

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\*Expanded version of a conference presented at Mega 09 (Barcelona, June 2009).

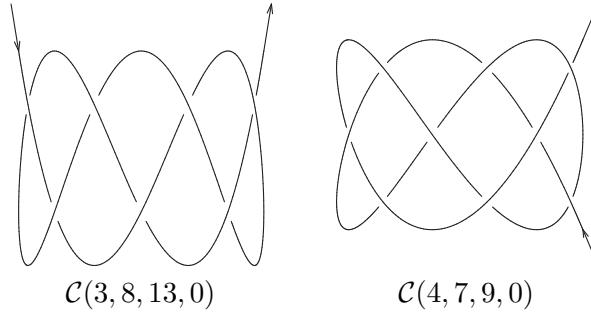


Figure 1: Two Chebyshev knots:  $\bar{7}_7$  and  $7_5$  (Conway-Rolfsen numbering)

In [KP2] we gave an explicit parametrization of the torus knots  $T(2, n)$  and other infinite families of knots. In [KP2], we gave a complete classification of harmonic knots  $\mathcal{C}(a, b, c, 0)$  where  $a = 3$  and  $a = 4$  (see also [FF]). We have shown in [KP2] that every rational knot of crossing number  $N$  admits a polynomial parametrization  $x = T_3(t)$ ,  $y = T_b(t)$ ,  $z = C(t)$  with  $b + \deg C = 3N$ .

It is proved in [KP1] that every two-bridge knot is a Chebyshev knot with  $a = 3$ . We first showed that every two-bridge knot has a Chebyshev curve  $\mathcal{C}(3, b)$  as projection. Then we used a density argument based on Kronecker's theorem to show that given a knot  $K$  whose  $\mathcal{C}(a, b)$  is a plane projection, there exists  $c$  and  $\varphi$  such that  $\mathcal{C}(a, b, c, \varphi)$  is isotopic to  $K$ .

Our aim is to give an algorithm that determines a Chebyshev parametrization  $\mathcal{C}(a, b, c, \varphi)$ , (with  $a = 3$  and also with  $a = 4$ ), for any rational knot.

1. Rational knots are classified by their Schubert fraction. For  $a = 3$  and  $a = 4$  we determine the minimal integer  $b$  such that the Chebyshev curve  $\mathcal{C}(a, b) : x = T_a(t)$ ,  $y = T_b(t)$  is a plane projection of  $K$ . This algorithm is based on continued fraction expansions.
2. Let  $\mathcal{Z}_{a,b,c}$  be the set of  $\varphi$  such that  $\mathcal{C}(a, b, c, \varphi)$  is singular.  $\mathcal{Z}_{a,b,c}$  is finite. The knot type of  $K(\varphi) = \mathcal{C}(a, b, c, \varphi)$  is constant over any interval of  $\mathbf{R} - \mathcal{Z}_{a,b,c}$ . Then we determine a rational number in each component of  $\mathbf{R} - \mathcal{Z}_{a,b,c}$ .
3. We determine the Schubert fraction of the knot  $\mathcal{C}(a, b, c, r)$  by evaluating the (under/over) nature of the crossings. This amounts to evaluating the signs of polynomials at the real solutions of a zero-dimensional system.

These algorithms are based on three basic black boxes:

- **PhiProjection**( $P_a(S, T), P_b(S, T), Q_c(S, T, \varphi) \in \mathbf{Q}[\varphi]$ )  
 $\rightarrow R \in \mathbf{Q}[\varphi]$ , such that  $Z(R) = \pi_\varphi(\{P_a = 0, P_b = 0, Q_c = 0\})$
- **PhiSampling**( $R \in \mathbf{Q}[\varphi]$ )  
 $\rightarrow r_0, \dots, r_N \in \mathbf{Q}$  such that  $r_0 < \varphi_1 < r_1 < \dots < \varphi_N < r_N$ , where  $\varphi_1 < \dots < \varphi_N$  are the real roots of  $R$

- **SignSolve**( $r \in \mathbf{Q}$ ,  $P_a(S, T)$ ,  $P_b(S, T)$ ,  $D_{a,b,c}(S, T, \varphi) \in Q[S, T, \varphi]$ )  
 $\rightarrow [\text{sign}(D_{a,b,c}(S, T, r))], (S, T) \in \{P_a(S, T) = P_b(S, T) = 0\}$

As a conclusion we give a list of Chebyshev parametrizations for the first 95 rational knots of crossing number not greater than 10. They admit polynomial parametrizations whose plane projections have few crossing points. This is not the case with Lissajous knots (see [BDHZ]).

## 2 Geometry of Chebyshev knots

Chebyshev curves were defined in [Fi]. Their double points are easier to study than those of Lissajous curves. The classical Chebyshev polynomials  $T_n$  are defined by  $T_n(t) = \cos n\theta$ , where  $\cos \theta = t$ . These polynomials satisfy the linear recurrence  $T_0 = 1$ ,  $T_1 = t$ ,  $T_{n+1} = 2tT_n - T_{n-1}$ ,  $n \geq 1$ , from which we deduce that  $T_n$  is a polynomial of degree  $n$  and leading coefficient  $2^{n-1}$ .

**Proposition 2.1 ([KP1]).** *Let  $a$  and  $b$  be relatively prime integers. The affine Chebyshev curve  $\mathcal{C}(a, b)$  defined by*

$$\mathcal{C}(a, b) : T_b(x) - T_a(y) = 0$$

*admits the parametrization  $x = T_a(t)$ ,  $y = T_b(t)$ .  $\mathcal{C}$  has  $\frac{1}{2}(a-1)(b-1)$  singular points which are crossing points. The pairs  $(t, s)$  giving a crossing point are*

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi, \quad s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi,$$

*where  $k, h$  are positive integers such that  $\frac{k}{a} + \frac{h}{b} < 1$ .*

When  $a$  and  $b$  are coprime integers, the projection of  $\mathcal{C}(a, b, c, \varphi)$  onto the  $(x, y)$ -plane is the plane Chebyshev curve  $\mathcal{C}(a, b)$ . The curve  $\mathcal{C}(a, b, c, \varphi)$  is a knot if and only if  $\mathcal{C}(a, b, c, \varphi)$  has no double point.

We thus deduce

**Proposition 2.2.** *Let  $a, b$  and  $c$  be integers ( $a$  and  $b$  being relatively prime). The number of Chebyshev knots  $\mathcal{C}(a, b, c, \varphi)$  is at most  $\frac{1}{2}(a-1) \times (b-1) \times (c-1) + 1$ .*

*Proof.*  $\mathcal{C}(a, b, c, \varphi)$  is a singular space curve iff there exists  $s \neq t \in \mathbf{R}$  such that

$$\frac{T_a(t) - T_a(s)}{t - s} = 0, \quad \frac{T_b(t) - T_b(s)}{t - s} = 0, \quad \frac{T_c(t + \varphi) - T_c(s + \varphi)}{t - s} = 0.$$

From Prop. 2.1, the set  $\left\{ \{s, t\}, \frac{T_a(t) - T_a(s)}{t - s} = 0, \frac{T_b(t) - T_b(s)}{t - s} = 0 \right\}$  has  $\frac{1}{2}(a-1)(b-1)$  elements. For each of these elements, the set  $\left\{ \varphi \in \mathbf{R}, \frac{T_c(t + \varphi) - T_c(s + \varphi)}{t - s} = 0 \right\}$  has at most  $c-1$  elements because the leading monomial of  $\left[ \frac{T_c(t + \varphi) - T_c(s + \varphi)}{t - s} \right]$  is  $c2^{c-1}\varphi^{c-1}$ .

Consequently the set  $\mathcal{Z}_{a,b,c}$  of *critical values*  $\varphi$  has at most  $\frac{1}{2}(a-1) \times (b-1) \times (c-1)$  elements. For any  $\varphi \in \mathbf{R} - \mathcal{Z}_{a,b,c}$ , the curve  $\mathcal{C}(a,b,c,\varphi)$  defines a knot. We claim that in any interval included in  $\mathbf{R} - \mathcal{Z}_{a,b,c}$ , the knots are the same because the nature of the crossings is constant.  $\square$

**Remark 2.3.** We see that  $\mathcal{C}(a,b,c,-\varphi)$  is the reverse of  $\mathcal{C}(a,b,c,\varphi)$  (see [Mu]). If  $a = 3$  or  $a = 4$ , they define the same knot.  $\mathcal{Z}_{a,b,c}$  is symmetrical about the origine.

### 3 Knot diagrams

We shall study the diagram of the curve  $\mathcal{C}(a,b,c,\varphi)$ , that is to say the plane projection  $\mathcal{C}(a,b)$  onto the  $(x,y)$ -plane and the nature (under/over) of the crossings over the double points of  $\mathcal{C}(a,b)$ . There are two cases of crossing: the right twist and the left twist (see [Mu], P. 178). In the following lemma, we see that the nature of the crossing is given by

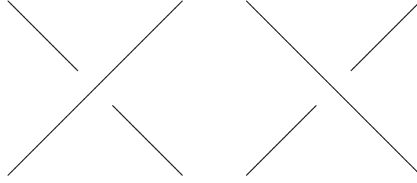


Figure 2: The right twist and the left twist

the sign of a symmetrical polynomial.

**Lemma 3.1.** *Let  $s \neq t$  be parameters such that  $T_a(t) = T_a(s)$  and  $T_b(t) = T_b(s)$  and consider the diagram of the curve  $\mathcal{C}(a,b,c,\varphi)$ . Let*

$$D(s,t,\varphi) = \left( \frac{T_c(t+\varphi) - T_c(s+\varphi)}{t-s} \right) \left( \frac{T_{b-a}(t) - T_{b-a}(s)}{t-s} \right). \quad (1)$$

*Then  $D(s,t,\varphi) > 0$  if and only if the crossing is a right twist.*

*Proof.* Let  $(s,t)$  be the parameters of a double point of  $\mathcal{C}(a,b)$ . The crossing is a right twist if and only if

$$\left( z(t) - z(s) \right) \left( \frac{y'(t)}{x'(t)} - \frac{y'(s)}{x'(s)} \right) > 0.$$

Using Prop. 2.1, we get  $s = \cos \sigma$  and  $t = \cos \tau$ . By simple computation we get  $\sin \tau x'(t) = -\sin \sigma x'(s)$ ,  $\sin \tau y'(t) = \sin \sigma y'(s)$  and  $\frac{y'(t)}{x'(t)} = -\frac{y'(s)}{x'(s)}$ . The slopes of the corresponding tangents are opposite. We therefore deduce that (using  $x \sim y$  for  $\text{sign}(x) = \text{sign}(y)$ )

$$x'(t)y'(t) \sim -x'(s)y'(s) \sim \frac{y'(t)}{x'(t)} - \frac{y'(s)}{x'(s)} \sim x'(t)y'(t) - x'(s)y'(s).$$

But

$$x'(t)y'(t) = ab \frac{\sin a\tau \sin b\tau}{\sin^2 \tau} \sim 2 \sin a\tau \sin b\tau = T_{a-b}(t) - T_{a+b}(t).$$

Consequently  $x'(t)y'(t) - x'(s)y'(s) \sim (T_{a-b}(t) - T_{a+b}(t)) - (T_{b-a}(s) - T_{b-a}(s))$ . On the other hand, using the identities  $T_{b+a} + T_{b-a} = 2T_aT_b$ ,  $T_a(t) = T_a(s)$  and  $T_b(t) = T_b(s)$ , we conclude that  $x'(t)y'(t) - x'(s)y'(s) \sim T_{b-a}(t) - T_{b-a}(s)$  and the announced result.  $\square$

A two-bridge knot (or link) admits a diagram in Conway's normal form. This form, denoted by  $C(a_1, a_2, \dots, a_n)$  where  $a_i$  are integers, is explained by the following picture (see [Con], [Mu] p. 187). The number of twists is denoted by the integer  $|a_i|$ , and the sign of  $a_i$  is

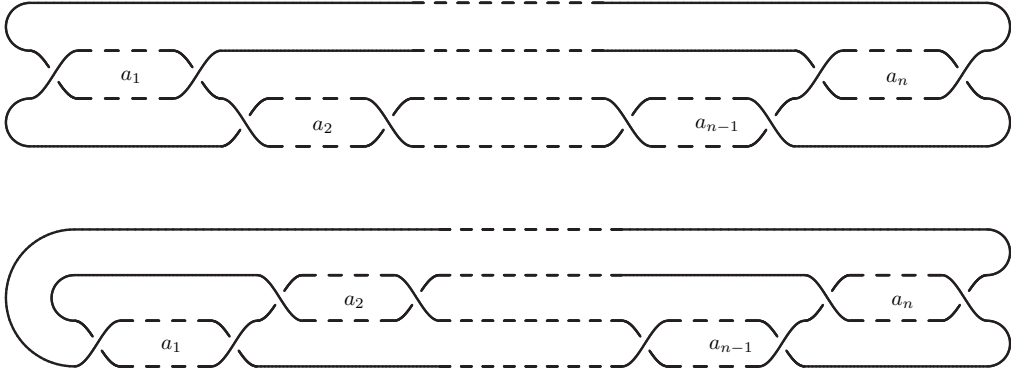


Figure 3: Conway's normal forms,  $n$  odd,  $n$  even

defined as follows: if  $i$  is odd, then the right twist is positive, if  $i$  is even, then the right twist is negative. On Figure 3 the  $a_i$  are positive (the  $a_1$  first twists are right twists).

The two-bridge links are classified by their Schubert fractions

$$\frac{\alpha}{\beta} = [a_1, \dots, a_n], \quad \alpha > 0,$$

where  $[a_1, \dots, a_n]$  is the continued fraction expansion  $a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\dots + \frac{1}{a_n}}}}$ .

We shall denote by  $S(\frac{\alpha}{\beta})$  the two-bridge link with Schubert fraction  $\frac{\alpha}{\beta}$ . The two-bridge links  $S(\frac{\alpha}{\beta})$  and  $S(\frac{\alpha'}{\beta'})$  are equivalent if and only if  $\alpha = \alpha'$  and  $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$ . The integer  $\alpha$  is odd for a knot, and even for a two-component link. If  $K = S(\frac{\alpha}{\beta})$ , its mirror image is  $\overline{K} = S(\frac{\alpha}{-\beta})$  (see [Mu]).

We shall study knots with a Chebyshev diagram  $\mathcal{C}(3, b) : x = T_3(t), y = T_b(t)$ . It is remarkable that such a diagram is already in Conway normal form (see Figure 4). Consequently, the Schubert fraction of such a knot is given by a continued fraction of the form  $[\pm 1, \pm 1, \dots, \pm 1]$ .

For example we obtain the torus knot  $7_1 = C(-1, -1, -1, 1, 1, 1, -1, -1, -1)$ , and the knots  $6_3 = C(1, 1, 1, 1, 1, 1)$ ,  $6_1 = C(1, 1, 1, 1, -1, -1, -1)$ .

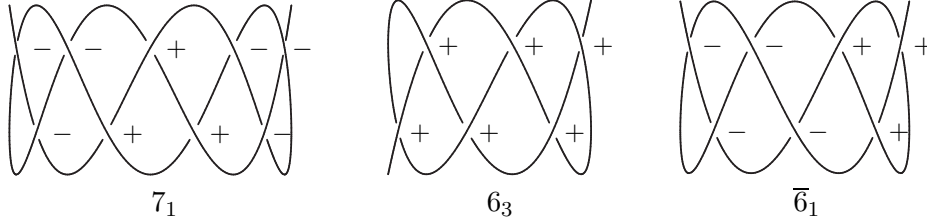


Figure 4: Chebyshev diagrams

We get for the knot  $7_1$  (resp.  $6_3, \bar{6}_1$ ) the fractions  $\frac{7}{-6} \sim 7$ , (resp.  $\frac{13}{8}, \frac{9}{-5}$ ).

The *crossing number* of a knot is the smallest number of double points in any plane projection of any isotopic knot. The crossing number of a two-bridge knot  $S(\frac{\alpha}{\beta})$  is the sum of the integers in the regular continued fraction expansion of  $\frac{\alpha}{\beta} > 1$ .

In the case of Chebyshev knots  $\mathcal{C}(3, b, c, \varphi)$  with  $a = 3$ , the Conway notation is given by  $C(\varepsilon_1, \dots, \varepsilon_{b-1})$  where  $\varepsilon_i = -(-1)^i \text{sign}(D_i)$ , the  $D_i$  being ordered by their abscissae.

In the case of Chebyshev knots  $\mathcal{C}(4, b, c, \varphi)$  with  $a = 4$ , we obtain diagrams like Figure 5. Let  $a_i$  (resp.  $b_i, c_i$ ) be the signs of  $D_i$  (resp.  $-D_i$ ) corresponding to the crossing points

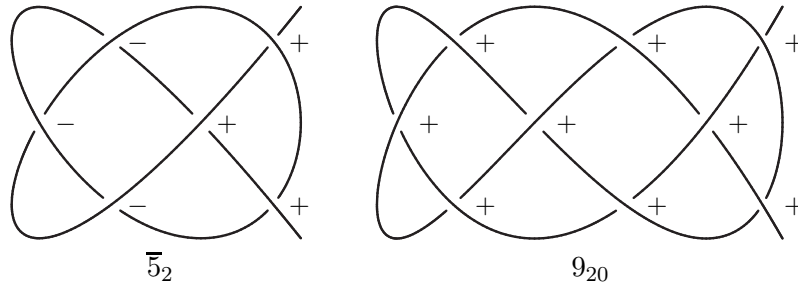


Figure 5:  $a = 4$

with  $y = 0$  (resp.  $y < 0, y > 0$ ). Following Murasugi ([Mu]), the Conway normal form for such a knot is  $C(a_1, b_1 + c_1, a_2, b_2 + c_2, \dots, a_n, b_n + c_n)$ . Figure 5 shows the examples  $\bar{5}_2$ :  $C(-1, -2, 1, 2)$  and  $9_{20}$ :  $C(1, 2, 1, 2, 1, 2)$ . We thus deduce that the Conway notation for a knot  $\mathcal{C}(4, b, c, \varphi)$  is  $C(a_1, b_1, \dots, a_n, b_n)$  where  $a_i = \pm 1, b_i = 0, \pm 2$ .

In conclusion, we see that, in the particular case when  $a = 3$  or  $a = 4$ , the knot  $\mathcal{C}(a, b, c, r)$  is determined by its Schubert fraction  $\frac{\alpha}{\beta}$  corresponding to the nature of the crossings over the double points of the projection  $\mathcal{C}(a, b)$ .

On the other hand, we show that any rational number  $\frac{\alpha}{\beta}$  may be expressed as continued fractions corresponding to Chebyshev diagrams  $\mathcal{C}(a, b)$  with  $a = 3$  and  $a = 4$ .

Algorithm 3.2. Let  $\frac{\alpha}{\beta}$  be a rational number.

1. There exists a sequence  $\varepsilon_1, \dots, \varepsilon_n$ ,  $\varepsilon_i = \pm 1$ , such that  $\frac{\alpha}{\beta} = [\varepsilon_1, \dots, \varepsilon_n]$ .
2. If  $\beta$  is even, there exists a sequence  $\varepsilon_1, \dots, \varepsilon_{2n}$ ,  $\varepsilon_i = \pm 1$ , such that  $\frac{\alpha}{\beta} = [\varepsilon_1, 2\varepsilon_2, \dots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$ .

*Proof.* Let us prove the existence by induction on the height  $h(\frac{\alpha}{\beta}) = \max(|\alpha|, |\beta|)$ .

1. If  $h = 2$  then  $\frac{\alpha}{\beta} = 1 = [1]$  and the result is true.

If  $\alpha > \beta$ , we have  $\frac{\alpha}{\beta} = [1, \frac{\beta}{\alpha - \beta}]$ . Since  $h(\frac{\beta}{\alpha - \beta}) < h(\frac{\alpha}{\beta})$ , we get our continued fraction by induction.

If  $\beta > \alpha$  we have  $\frac{\alpha}{\beta} = [1, -1, -\frac{\beta - \alpha}{\alpha}]$ . And we also get the continued fraction.

This completes the construction of our continued fraction expansion  $[\pm 1, \dots, \pm 1]$ .

2. If  $h(\frac{\alpha}{\beta}) = 2$ , then  $\alpha = 1$  and  $\beta = 2$  and we have  $r = [1, -2]$ .

If  $\alpha > 2\beta > 0$  then we write  $\frac{\alpha}{\beta} = [1, 2, -1, 2, \frac{\alpha - 2\beta}{\beta}]$ . We have  $h(\frac{\alpha - 2\beta}{\beta}) < h(\frac{\alpha}{\beta})$  and we conclude by induction.

If  $\beta < \alpha < 2\beta$  then we write  $\frac{\alpha}{\beta} = [1, 2, \frac{\alpha - \beta}{3\beta - 2\alpha}]$ . We have  $|3\beta - 2\alpha| \leq \alpha$  and  $|\alpha - \beta| < \alpha$  and we conclude by induction.

If  $\beta > \alpha > 0$  we write  $\frac{\alpha}{\beta} = [1, -2, \frac{\alpha - \beta}{2\alpha - \beta}]$ . From  $|2\alpha - \beta| \leq \beta$  we have  $h(\frac{\alpha - \beta}{2\alpha - \beta}) < h(\frac{\alpha}{\beta})$  and we conclude by induction.

The existence of a continued fraction  $[1, \pm 2, \dots, \pm 1, \pm 2]$  is proved.  $\square$

Note that we have proved in [KP2] that the continued fraction expansion  $\frac{\alpha}{\beta} = [\varepsilon_1, \dots, \varepsilon_n]$ ,  $\varepsilon_i = \pm 1$ , is unique if there is no two consecutive sign changes and  $\varepsilon_{n-1}\varepsilon_n > 0$ . We also



proved that the continued fraction expansion  $\frac{\alpha}{\beta} = [\varepsilon_1, 2\varepsilon_2, \dots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$ ,  $\varepsilon_i = \pm 1$ , is unique if there is no three consecutive sign changes.

**Corollary 3.3.** *Every two-bridge knot has a Chebyshev diagram  $\mathcal{C}(3, b)$ ,  $b \not\equiv 0 \pmod{3}$ . Every two-bridge knot has a Chebyshev diagram  $\mathcal{C}(4, b)$ ,  $b \equiv 1 \pmod{2}$ .*

*Proof.* Let us consider a knot  $K = S(\frac{\alpha}{\beta})$ . Using Algorithm 3.2, we can write  $\frac{\alpha}{\beta} = [\varepsilon_1, \dots, \varepsilon_n]$ ,  $\varepsilon_i = \pm 1$ . One can see that (see [KP2])  $n \equiv 2 \pmod{3}$  iff  $\alpha$  is even and, since  $K$  is a knot, this is not the case.  $K$  is isotopic to  $C(\varepsilon_1, \dots, \varepsilon_n)$  which corresponds to a Chebyshev diagram  $\mathcal{C}(3, n+1) : x = T_3(t), y = T_{n+1}(t)$ .

Using Algorithm 3.2, we can write  $\frac{\alpha}{\beta} = [\varepsilon_1, 2\varepsilon_2, \dots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$ ,  $\varepsilon_i = \pm 1$ . The knot  $K$  is isotopic to  $C(\varepsilon_1, 2\varepsilon_2, \dots, \varepsilon_{2n-1}, 2\varepsilon_{2n})$  which corresponds to a Chebyshev diagram  $\mathcal{C}(4, 2n+1) : x = T_3(t), y = T_{2n+1}(t)$ .  $\square$

**Corollary 3.4.** *Every two-bridge knot is a Chebyshev knot  $\mathcal{C}(3, b, c, \varphi)$ . Every two-bridge knot is a Chebyshev knot  $\mathcal{C}(4, b, c, \varphi)$ .*

*Proof.* Using a density argument (Kronecker theorem), we proved in [KP1] that if  $\varphi$  is small enough, then there exists  $c$  such that  $\mathcal{C}(3, b, c, \varphi) = C(\varepsilon_1, \dots, \varepsilon_n)$ . The case  $a = 4$  is similar.  $\square$

Unfortunately, this last corollary will not provide  $c$  and  $\varphi$  and not even any bound for  $c$ . We want to give the minimal Chebyshev parametrization for every rational knot with a small crossing number. We shall describe all rational knots  $\mathcal{C}(a, b, c, \varphi)$  with given  $a, b, c$ .

## 4 Description of Chebyshev knots

Let us consider the curve  $\mathcal{C}(a, b, c, \varphi)$  with  $a = 3$  or  $a = 4$ . From section 3., we know that

1. The curve is singular iff it has double points.
2. If the curve is not singular, the knot  $\mathcal{C}(a, b, c, \varphi)$  is determined by the sequence of crossings of the projection  $\mathcal{C}(a, b)$ .

We will use the symmetric variables  $S = s + t$  and  $T = st$ . Let us define

$$P_n(S, T) = \frac{T_n(t) - T_n(s)}{(t - s)}, \quad Q_n(S, T, \varphi) = \frac{T_n(t + \varphi) - T_n(s + \varphi)}{(t - s)}. \quad (2)$$

**Lemma 4.1.** *There exists  $R_{a,b,c} \in \mathbf{Q}[\varphi]$  with degree  $\deg R_{a,b,c} \leq \frac{1}{2}(a-1)(b-1)(c-1)$  such that  $\mathcal{Z}_{a,b,c} = Z(R_{a,b,c})$ .*

*Proof.* From Prop. 2.1,  $\{P_a(S, T) = 0, P_b(S, T) = 0\}$  is 0-dimensional and has degree  $\frac{1}{2}(a-1)(b-1)$ .  $V_{a,b,c} = \{P_a(S, T) = 0, P_b(S, T) = 0, Q_c(S, T, \varphi) = 0\}$  is 0-dimensional and has degree  $\frac{1}{2}(a-1)(b-1)(c-1)$  because  $Q_c$  has  $c(2\varphi)^{c-1}$  as leading term. From lemma 2.2,  $\mathcal{Z}_{a,b,c} = \pi_\varphi(V_{a,b,c})$  is finite. It is for example  $Z(R)$  where  $\langle R \rangle = V_{a,b,c} \cap \mathbf{Q}[\varphi]$ .  $\square$

We obtain  $R_{a,b,c}$  by using the black-box `PhiProjection`( $P_a, P_b, Q_c$ ).

The set  $\mathcal{Z}_{a,b,c} = \{\varphi_1, \dots, \varphi_N\}$ , for which  $\mathcal{C}(a, b, c, \varphi)$  is singular, is exactly  $\pi_\varphi(V_{a,b,c})$ , from Prop. 2.2. From section 3., the knot  $K(r) = \mathcal{C}(a, b, c, r)$  is constant over any interval  $]\varphi_i, \varphi_{i+1}[$ . We obtain  $r_i$  in  $]\varphi_i, \varphi_{i+1}[$  by using `PhiSampling`( $R_{a,b,c}$ ).

In the case when  $a = 3$  or  $a = 4$ , and  $r \in \mathbf{R} - \mathcal{Z}_{a,b,c}$ , the knot  $K(r)$  is uniquely determined by its Schubert fraction. Let  $(S_i, T_i)_{i=1, \dots, n}$  be the points of  $\{P_a(S, T) = 0, P_b(S, T) = 0\}$ . The Conway notation of the knot  $K(r)$  is deduced from the sequence  $[\text{sign}(D(S_i, T_i, r))]_{i=1, \dots, n} = \text{SignSolve}(r, P_a, P_b, Q_c \cdot P_{b-a})$ .

We will now show how to determine these quantities in the case  $a = 3$  and  $a = 4$ .

#### 4.1 Case $a = 3$

We get  $P_3(S, T) = 4T - 4S^2 + 3$  that is  $T = S^2 - \frac{3}{4}$ . The set  $\{(S, T), P_3(S, T) = 0, P_b(S, T) = 0\}$  has cardinal  $b - 1$ . Its elements satisfy  $P_b(S, S^2 - \frac{3}{4}) = 0, T = S^2 - \frac{3}{4}$ . We deduce that  $\deg(P_b(S, S^2 - \frac{3}{4})) = b - 1$ . The set of critical values  $\varphi$  is

$$\mathcal{Z}_{3,b,c} = \pi_\varphi\left(\left\{P_b\left(S, S^2 - \frac{3}{4}\right) = 0, Q_c\left(S, S^2 - \frac{3}{4}, \varphi\right) = 0\right\}\right).$$

It is exactly the roots of the polynomial of degree  $(b-1)(c-1)$ :

$$R_{3,b,c} = \text{Res}_S\left(P_b\left(S, S^2 - \frac{3}{4}\right), Q_c\left(S, S^2 - \frac{3}{4}, \varphi\right)\right).$$

Let  $A(S)$  be a crossing point corresponding to parameter  $S$  (and  $T = S^2 - \frac{3}{4}$ ). Its abscissa is  $T_3(t) = T_3(s) = -T_3(S) = S(3 - 4S^2)$ .

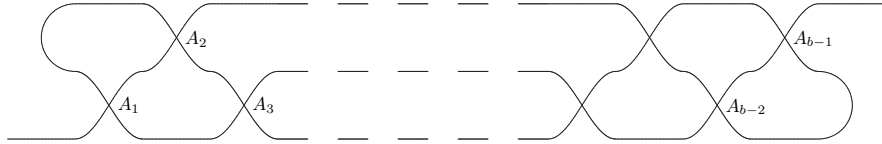


Figure 6:  $\mathcal{C}(3, b)$ ,  $b$  even

We define the order relation  $A(S) <_3 A(S')$  if  $T_3(S) > T_3(S')$ . The Conway notation of  $\mathcal{C}(3, b, c, r)$  is  $C(D(A_1), -D(A_2), \dots, (-1)^{b-1}D(A_{b-1}))$  where  $A_1 <_3 A_2 <_3 \dots <_3 A_{b-1}$  and  $D(A)$  is  $D(s, t, r)$  defined in Formula 1.

## 4.2 Case $a = 4$

We get  $P_4(S, T) = 8S(S^2 - 2T - 1)$ . We thus obtain two families of double points

$$\mathcal{A} : \{S = 0, P_b(0, T) = 0\}, \quad \mathcal{B} : \{T = \frac{1}{2}(S^2 - 1), P_b(S, \frac{1}{2}(S^2 - 1)) = 0\}.$$

Let  $n = \frac{1}{2}(b - 1)$ . We have  $|\mathcal{A}| = n$  from which we deduce that  $\deg_T P_b(0, T) = n$ . From  $|\mathcal{B}| = 2n$ , we deduce also that  $\deg_S P_b(S, \frac{1}{2}(S^2 - 1)) = 2n$ . As the leading coefficient of  $Q_c(S, T, \varphi)$  is  $c(2\varphi)^{c-1}$  we deduce that  $R_1(\varphi) = \text{Res}_T(P_b(0, T), Q_c(0, T, \varphi))$  has degree  $n(c - 1)$  and  $R_2(\varphi) = \text{Res}_S(P_b(S, \frac{1}{2}(S^2 - 1)), Q_c(S, \frac{1}{2}(S^2 - 1), \varphi))$  has degree  $2n(c - 1)$ .  $\mathcal{Z}_{4,b,c}$  is the set of real roots of  $R_{4,b,c} = R_1 \times R_2$ .

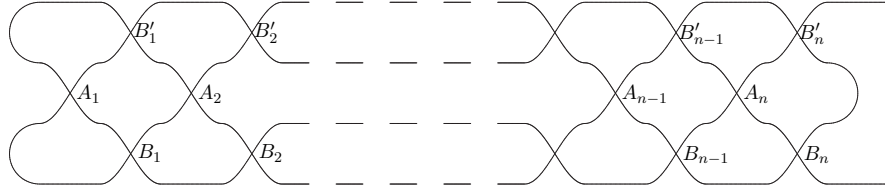


Figure 7:  $\mathcal{C}(4, 2n + 1)$

The abscissa of  $A(T) \in \mathcal{A}$  is given by  $T_4(t) = T_4(s) = 1 + 8T + 8T^2$ . The abscissa of  $B(S) \in \mathcal{B}$  is given by  $T_4(t) = T_4(s) = -1 + 4S^2 - 2S^4$ . We have to sort separately the crossing points of  $\mathcal{A}$  and  $\mathcal{B}$  by increasing abscissae:  $A_1, \dots, A_n$  and  $B_1, B'_1, \dots, B_n, B'_n$ . Note that  $B_i$  and  $B'_i$  have the same abscissa. The Conway notation for the knot we obtain is then

$$C\left(D(A_1), -(D(B_1) + D(B'_1)), \dots, D(A_n), -(D(B_n) + D(B'_n))\right),$$

where  $D(A)$  (resp.  $D(B)$ ) is  $D(s, t, r)$  defined in Formula 1.

**Remark 4.2.** We have here  $\deg R_{a,b,c} = \frac{1}{2}(a - 1)(b - 1)(c - 1)$ . We could have computed also  $R_{a,b,c}$  by eliminating  $S$  and  $T$  using Gröbner Basis (see [CLOS]). It may happen that  $\deg R_{a,b,c} < \frac{1}{2}(a - 1)(b - 1)(c - 1)$ .

## 4.3 Computation of the polynomials

As  $T_n$  satisfies the linear recurrence of order 2:  $T_{n+1} + T_{n-1} = 2tT_n$  we deduce that  $Q_n$  (Form. 2) satisfies the linear recurrence of order 4:

$$\begin{aligned} Q_0 &= 0, Q_1 = 1, Q_2 = 2S + 4\varphi, Q_3 = -4T + 12\varphi S + 4S^2 + 12\varphi^2 - 3. \\ Q_{n+4} &= 2(S + 2\varphi)(Q_{n+3} + Q_{n+1}) - 2(2\varphi^2 + 2T + 2\varphi S + 1)Q_{n+2} - Q_c. \end{aligned} \quad (3)$$

For  $P_n(S, T) = Q_n(S, T, 0)$  we find

$$\begin{aligned} P_0 &= 0, P_1 = 1, P_2 = 2S, P_3 = -4T + 4S^2 - 3. \\ P_{n+4} &= 2S(P_{n+3} + P_{n+1}) - (4T + 2)P_{n+2} - P_n. \end{aligned} \quad (4)$$

In the particular case when  $a = 3$  or  $a = 4$  we have to compute  $Q_n(S, T, \varphi)$  where  $T = S^2 - \frac{3}{4}$  or  $T = \frac{1}{2}(S^2 - 1)$  or  $S = 0$ . These polynomials satisfy also linear recurrences.

As  $T_m(T_n) = T_{mn}$  we deduce that  $Q_n|Q_{nm}$ . We can therefore obtain factors of the polynomials  $Q_n(S, T, \varphi)$ .

In the particular case where  $a = 3$  or  $a = 4$ , we obtain our resultants  $R_{a,b,c} = \text{PhiProjection}(P_a, P_b, Q_c)$  by computing the resultants between factors of  $P_b(S, T)$  and factors of  $Q_c(S, T, \varphi)$  that depend only on  $S$  and  $\varphi$  or on  $T$  and  $\varphi$ .

We have to determine the Schubert fraction of any knot of the type  $\mathcal{C}(a, b, c, r)$  where  $r$  is a given rational number in  $\mathbf{Q} - \mathcal{Z}_{a,b,c}$ . Such rational number is given by  $\text{PhiSampling}(R_{a,b,c})$ .

## 5 Examples

### 5.1 The family of knots $\mathcal{C}(3, 5, 7, \varphi)$

We get  $P_3(S, T) = \frac{T_3(t) - T_3(s)}{t - s} = 4(T - S^2 + \frac{3}{4})$ , so  $T = S^2 - \frac{3}{4}$  and

$$\begin{aligned} P_5(S, S^2 - \frac{3}{4}) &= -16S^4 + 12S^2 - 1, \\ Q_7(S, S^2 - \frac{3}{4}, \varphi) &= 64S^6 - 16(84\varphi^2 + 5)S^4 - 112\varphi(1 + 20\varphi^2)S^3 + 24(42\varphi^2 + 1)S^2 \\ &\quad + 28\varphi(48\varphi^4 + 3 + 80\varphi^2)S - 1 + 1120\varphi^4 + 448\varphi^6 + 84\varphi^2. \end{aligned}$$

$\mathcal{C}(3, 5, 7, \varphi)$  is singular iff  $\varphi$  is a root of  $R_{3,5,7} = \text{Res}_S(P_5(S, S^2 - \frac{3}{4}), Q_7(S, S^2 - \frac{3}{4}, \varphi))$ .

$R_{3,5,7}$  has degree  $24 = \frac{1}{2}(3-1)(5-1)(7-1)$  and 12 real roots  $\varphi_1, \dots, \varphi_{12}$ . We choose 13 rational values  $r_0 < \varphi_1 < r_1 < \dots < \varphi_{12} < r_{12}$ .

Let us determine now the nature of  $\mathcal{C}(3, 5, 7, r)$ . We have to evaluate  $D(s, t, \varphi) = Q_7(S, S^2 - \frac{3}{4}, r) \cdot P_2(S, S^2 - \frac{3}{4})$  when  $P_5(S, S^2 - \frac{3}{4}) = 0$ . Let  $S_1 < S_2 < S_3 < S_4$  be the 4 real roots of  $P_5$ . They correspond to parameters  $(s_1, t_1), \dots, (s_4, t_4)$  such that  $s_i + t_i = S_i$ ,  $s_i t_i = T_i = S_i^2 - \frac{3}{4}$ . We have  $T_3(-S_2) < T_3(-S_1) < T_3(-S_4) < T_3(-S_3)$  and the knot  $\mathcal{C}(3, 5, 7, r)$  is given by the continued fraction expansion  $\alpha/\beta = [D_2, -D_1, D_4, -D_3]$  where  $D_i = \text{sign}(Q_7(S_i, S_i^2 - \frac{3}{4}, r) \cdot P_2(S_i, S_i^2 - \frac{3}{4}))$ . We obtain

$r_7 = 0$	$r_8 = \frac{1}{15}$	$r_9 = \frac{1}{5}$	$r_{10} = \frac{1}{4}$	$r_{11} = \frac{1}{2}$	$r_{12} = \frac{2}{3}$	$r_{13} = 1$
$\frac{\alpha}{\beta} = \frac{5}{3}$	$\frac{\alpha}{\beta} = -\frac{1}{3}$	$\frac{\alpha}{\beta} = -1$	$\frac{\alpha}{\beta} = -1$	$\frac{\alpha}{\beta} = 1$	$\frac{\alpha}{\beta} = 1$	$\frac{\alpha}{\beta} = 1$

The only non trivial knot is obtained for  $r = 0$ . It is the figure-eight knot  $4_1 = S(\frac{5}{2})$ .

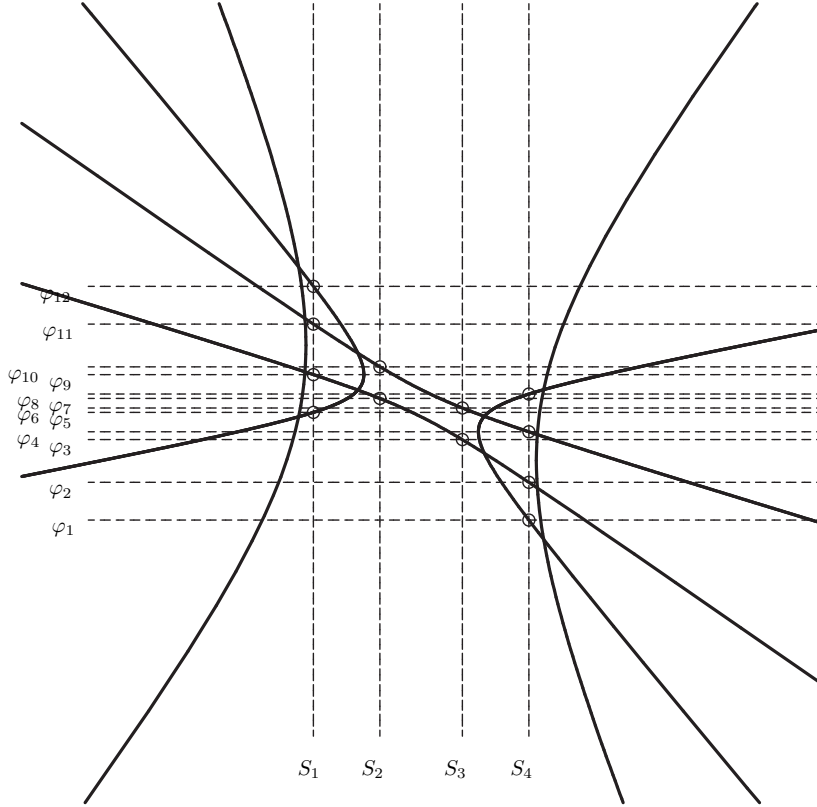


Figure 8: The curve  $Q_7(S, S^2 - \frac{3}{4}, \varphi) = 0$  and the lines  $P_5(S, S^2 - \frac{3}{4}) = 0$

## 5.2 A more complicated example, the family of knots $\mathcal{C}(3, 14, 292, \varphi)$

$P_{14}(S, S^2 - \frac{3}{4})$  is a product of 4 factors of degrees  $[6, 3, 3, 1]$ . We have  $292 = 4 \cdot 73$ . We know that  $Q_{73}|Q_{146}|Q_{292}$ .  $Q_{73}(S, S^2 - \frac{3}{4}, \varphi)$  is irreducible and has degree 72.  $Q_{146}$  is a product of polynomials with degrees  $[72, 72, 1]$ . At the end  $Q_{292}$  is a product of 5 factors with degrees  $[144, 72, 72, 2, 1]$ . We compute  $R_{3,14,292}$  as the product of 20 resultants between factors of  $P_{14}$  and  $Q_{292}$ .  $R_{3,14,292}(\varphi)$  has degree 3783 and exactly 2185 distinct real roots. We compute the 1093 Schubert fractions  $\mathcal{C}(3, 14, 292, r_i)$  where  $r_i > 0$ . We obtain 275 non trivial knots and eventually 34 distinct knots. One of these has crossing number greater than 10, it is the knot  $\overline{12}_{518} = S(\frac{157}{34})$ .

## 5.3 A much more complicated example, the family of knots $\mathcal{C}(4, 13, 267, \varphi)$

$P_{13}(S, T)$  is irreducible so as  $P_{13}(0, T)$  (that has degree 6) and  $P_{13}(S, \frac{1}{2}(S^2 - 1))$  (that has degree 12). We have  $267 = 3 \cdot 89$ . We know that  $Q_{89}|Q_{267}$ .  $Q_{267}$  is the product of  $Q_3$ ,  $Q_{89}$  and a polynomial of degree 176 in  $\varphi$ .  $Q_{267}(S, T, \varphi)$  is a product of polynomials of degrees  $[176, 88, 2]$ .

We thus obtain  $R_1 = \text{Res}_T(P_{13}(0, T), Q_{267}(0, T))$  as a product of polynomials of de-

rees [1056, 528, 12] and  $R_2 = \text{Res}_S\left(P_{13}(S, \frac{1}{2}(S^2 - 1)), Q_{267}(S, \frac{1}{2}(S^2 - 1))\right)$  as a product of polynomials of degrees [2112, 1056, 24].

$R_{4,13,267} = R_1 \times R_2$  has degree 4788. It has 2882 distinct real roots. We compute the 1442 Schubert fractions  $\mathcal{C}(4, 13, 267, r_i)$  where  $r_i > 0$ . We obtain 710 non trivial knots. 72 of these are distinct knots whose crossing numbers take all values between 3 and 16.

## 6 Results

In this section we present some results we have obtained using certified implementations of the three black-boxes on which our algorithms are based. They are easily implementable in any high level language.

### 6.1 Implementations

There are numerous choices for the implementations, but our requirements are strict: we must certify all the results since our goal is to obtain a classification; bearing in mind that the systems of polynomial equations have thousands of roots.

- **PhiProjection**( $P_a(S, T), P_b(S, T), Q_c(S, T, \varphi) \in \mathbf{Q}[\varphi]$ )  
 $\rightarrow R \in \mathbf{Q}[\varphi]$ , such that  $Z(R) = \pi_\varphi(\{P_a = 0, P_b = 0, Q_c = 0\})$

A straightforward way is to compute a Gröbner basis of  $\langle P_a(S, T), P_b(S, T), Q_c(S, T, \varphi) \rangle$  for a so called *elimination order* (see [CLOS]). Triangular decompositions provide a suitable alternative, or, more basically, iterative resultants in *generic* situations. Resultants can be used efficiently for our problem, the system being sufficiently generic. Our choice is then an *ad-hoc* method based on resultants computations in the same spirit as in section 4.

- **PhiSampling**( $R \in \mathbf{Q}[\varphi]$ )  
 $\rightarrow r_0, \dots, r_N \in \mathbf{Q}$  such that  $r_0 < \varphi_1 < r_1 < \dots < \varphi_N < r_N$ , where  $\varphi_1 < \dots < \varphi_N$  are the real roots of  $R$

Such a function can easily be implemented using any solver that is able to *isolate* real roots of univariate polynomials (say providing non overlapping intervals with rational bounds around all the real roots). It must be able to discriminate multiple roots from clusters of roots, real roots from complex roots with a small imaginary part (which excludes many numerical methods and most of implementations using hardware floats). One can use methods based on Sturm sequences or the Descartes rule of signs (see [BPR] for an overview), but also many strategies using interval analysis. Due to the high degree of the polynomials, our choice is to use algorithms based on the Descartes rule of signs using multi-precision interval arithmetic as in [RZ].

- **SignSolve**( $r \in \mathbf{Q}, P_a(S, T), P_b(S, T), D_{a,b,c}(S, T, \varphi) \in \mathbf{Q}[S, T, \varphi]$ )  
 $\rightarrow [\text{sign}(D_{a,b,c}(S, T, r))], (S, T) \in \{P_a(S, T) = P_b(S, T) = 0\}$

The determination of the sign of a polynomial over a zero-dimensional system is difficult to certify when using numerical method. There are few exact/certified existing methods/implementations for this problem. The strategy is naturally linked to the implementation of the function `PhiProjection` since the zero-dimensional system to be considered by `SignSolve` is a subsystem of the one which is to be considered by `PhiProjection`. One can use the *generalized Hermite method* for zero-dimensional systems as in [PRS], which makes use of Gröbner bases. One can use also any method that first rewrites the system as a rational parametrization (as in [Rou] or [GLS]) and then apply any algorithm that computes the sign of an univariate polynomial at a real algebraic number. This last step can be done by extending methods based on Sturm theorem or based on the Descartes rule of signs. Due to our implementation of `PhiProjection` and to the degrees of the polynomials, we base our implementation on the Descartes rule of signs.

For the experiments, we used the MAPLE environment. `PhiProjection` is based on resultants computation (see 4.1 and 4.2). We use the MAPLE function `Isolate` for `PhiSampling` (without constraints) and for `SignSolve` (with constraints). In the univariate case, this function is based on the algorithm described in [RZ]. Other computations have been straightforwardly implemented according to the descriptions proposed in section 4.

## 6.2 Experiments

Let us remind in the next table the number  $\mathcal{K}_N$  of two-bridge knots with crossing number  $N$ , up to mirror symmetry (see [ES] for a formula).

$N$	3	4	5	6	7	8	9	10
$\mathcal{K}_N$	1	1	2	3	7	12	24	45

The minimal  $b$  for a Chebyshev diagram  $\mathcal{C}(3, b)$  of  $K = S(\frac{\alpha}{\beta})$ ,  $\frac{\alpha}{\beta} > 1$ , is obtained with  $b = n + 1$  where  $n$  is the length of the continued fraction of  $\frac{\alpha}{\beta}$  or  $\frac{\alpha}{\alpha - \beta}$  (see [KP2]). This allows us, using Algorithm 3.2, to know the minimal  $b$  for which  $\mathcal{C}(3, b)$  is a projection of a given rational knot  $K$ .

In a similar manner, let  $K = S(\frac{\alpha}{\beta})$ ,  $\frac{\alpha}{\beta} > 1$ ,  $\beta$  even. The minimal integer  $b = 2n + 1$  for which there exists a continued fraction expansion  $r = [\varepsilon_1, 2\varepsilon_2, \dots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$ ,  $\varepsilon_i = \pm 1$ , such that  $S(r)$  is equivalent to  $K$ , is the smallest length of the continued fraction expansion  $[\pm 1, \pm 2, \dots, \pm 1, \pm 2]$  of either  $\frac{\alpha}{\beta}$ ,  $\frac{\alpha}{2\alpha - \beta}$ ,  $\frac{\alpha}{\beta'}$  or  $\frac{\alpha}{2\alpha - \beta'}$  where  $0 < \beta' < \alpha$ ,  $\beta'$  even and  $\beta\beta' = \pm 1 \pmod{\alpha}$ .

It happens that for some knots, there is a continued fraction expansion with smaller length including 0 instead of  $\pm 2$ . The list of these knots (up to crossing number 10) is  $8_{12}, 9_{13}, 9_{15}, 9_{26}, 10_8, 10_{12}, 10_{13}, 10_{25}, 10_{29}, 10_{38}, 10_{42}$ . For example for the knot  $8_{12} = S(\frac{29}{12})$ ,

we have  $\frac{29}{12} = [1, 0, 1, 2, 1, 0, 1, 2]$  while it is not possible to get a shorter continued fraction corresponding to  $8_{12}$ . We have enumerated all possible continued fraction expansions corresponding to diagrams  $\mathcal{C}(4, b)$  to determine the minimal  $b$  corresponding to a rational knot  $K$ .

In the next table we give the number of two-bridge knots of crossing number  $N$  that have a projection  $\mathcal{C}(3, n)$  with  $n \leq b$  and a projection  $\mathcal{C}(4, n')$  with  $n' \leq b$ .

Minimal $b$														
	$a = 3$								$a = 4$					
$b$	4	5	7	8	10	11	13	14	3	5	7	9	11	13
$N = 3$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$N = 4$		1	1	1	1	1	1	1		1	1	1	1	1
$N = 5$			2	2	2	2	2	2		2	2	2	2	2
$N = 6$			1	3	3	3	3	3		1	3	3	3	3
$N = 7$				1	7	7	7	7			5	7	7	7
$N = 8$					3	12	12	12			4	10	12	12
$N = 9$					1	5	24	24			1	14	24	24
$N = 10$						1	17	45				13	37	45

For example, looking at the 45 knots with crossing number 10 we see that: 1 of them has a projection  $\mathcal{C}(3, 11)$  (10 crossing points), 17 of them have a projection  $\mathcal{C}(3, 13)$  (12 crossing points) and 45 of them have a projection  $\mathcal{C}(3, 14)$  (13 crossing points). All of them have a projection  $\mathcal{C}(3, b)$  with  $b \leq 14$ . We can also observe that for  $b = 10$  we have all knots with crossing number not greater than 6, 7 knots with crossing number 7, 3 knots with crossing number 8 and 1 knot with crossing number 9. This last one is the Fibonacci knot  $\mathcal{C}(1, 1, 1, 1, 1, 1, 1, 1, 1) \sim \mathcal{C}(3, 10, 17, 0)$  (see [KP1]).

Looking at the 45 knots with crossing number 10 we see that: 13 of them have a projection  $\mathcal{C}(4, 9)$  (12 crossing points), 37 of them have a projection  $\mathcal{C}(4, 11)$  (15 crossing points), 45 of them have a projection  $\mathcal{C}(4, 13)$ .

Let  $K$  be a two-bridge knot. Once we know what we can expect as a diagram  $\mathcal{C}(a, b)$  for  $K$ , we look for it as a Chebyshev knot  $\mathcal{C}(a, b, c, \varphi)$ . We see from the previous table that every rational knot with crossing number  $N \leq 10$  is a Chebyshev knot  $\mathcal{C}(3, b, c, r)$  with  $b \leq 14$  and a Chebyshev knot  $\mathcal{C}(4, b, c, r)$  with  $b \leq 13$ . We proved in [KP2] that a two-bridge knot with crossing number  $N$  admits a plane Chebyshev projection  $\mathcal{C}(3, b)$  with  $b < \frac{3}{2}N$ . In comparison, the number of crossing points for Lissajous diagrams is far greater (see [BDHZ]).

We have limited ourselves to the bounds  $b \leq 21$ ,  $c \leq 300$  and  $(b-1)(c-1) \leq 13 \cdot 299 = 3887$ . The degrees of the polynomials  $R(\varphi)$  giving the critical values are bounded by 3887 when  $a = 3$  and 5382 when  $a = 4$ . A remarkable fact is that these polynomials have a large number of real roots (in average 58% when  $a = 3$  and 57% when  $a = 4$ , see Figure 9, first column). The proportion of non trivial knots  $\mathcal{C}(a, b, c, r)$  is approx. 25% when  $a = 3$  and



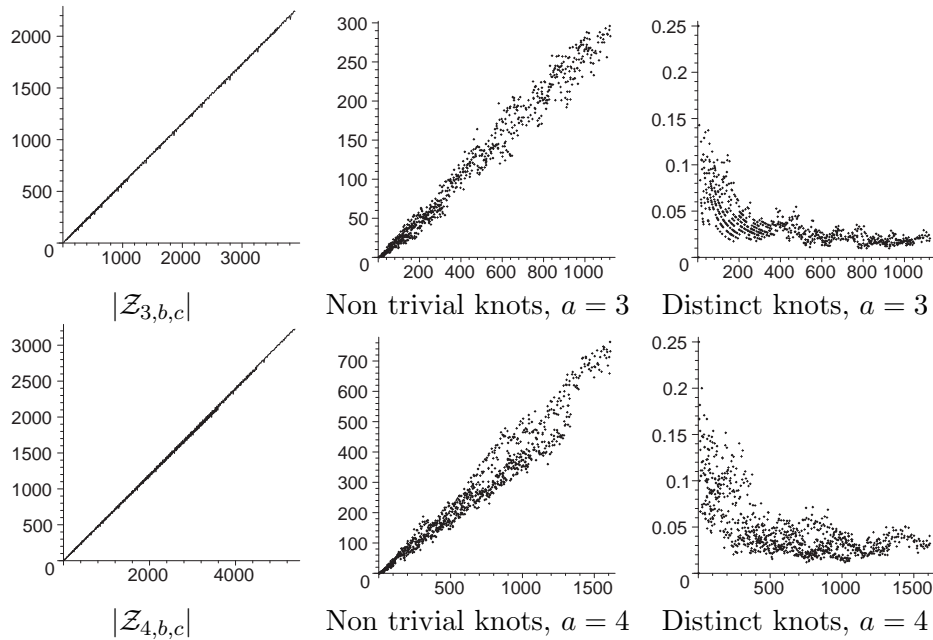


Figure 9: Number of real roots, number of non trivial knots, proportion of distinct knots

39% when  $a = 4$  (see column 2). The proportion of non trivial distinct knots  $\mathcal{C}(a, b, c, \varphi)$  is drawn in Figure 9, column 3.

We conclude our paper by a list of the first 95 two-bridge knots (up to crossing number 10). We give the Conway-Rolfsen numbering, their Schubert fraction (up to mirror symmetry) and their presentation as Chebyshev knots. Most of them have a parametrization with the minimal  $b$ . All of them have a parametrization with a minimal  $(b-1)(c-1)$ .

For example the knot  $9_5$  admits the Chebyshev parametrization  $\mathcal{C}(3, 17, 45, 1/364)$ . It is not minimal and we know that there is some other parametrization  $\mathcal{C}(3, 13, c, r)$  where  $c > 300$  and  $r$  is some rational number. This knot admits also the parametrization  $\mathcal{C}(4, 11, 152, 1/44)$  which is minimal with respect to  $b$  and  $\mathcal{C}(4, 19, 22, 1/20)$  that has minimal degree.

We get both  $10_{20}$  and  $10_{29}$  with minimal  $b = 14$  for the same value  $c = 292$ . We had to compute the polynomial  $R_{3,14,292}$  of degree 3783 and 2185 real roots (see 5.2). We obtain the knot  $10_{20} = S(\frac{35}{11})$  with  $\mathcal{C}(3, 14, 292, 1/94)$  and the knot  $10_{29} = S(\frac{63}{17})$  with  $\mathcal{C}(3, 14, 292, 1/93)$ .

## 7 Conclusion

We have shown that any two-bridge knot is a Chebyshev knot with  $a = 3$  and also with  $a = 4$ . For every  $a, b, c$  integers ( $a = 3, 4$  and  $a, b$  coprime), we have described an algorithm that gives all Chebyshev knots  $\mathcal{C}(a, b, c, \varphi)$ .

Our experiments fully justify the use of certified algorithms and exact computations

since numerical methods would have certainly failed in finding for example the knot  $10_{20} = S(\frac{35}{11})$  with  $\mathcal{C}(3, 14, 292, 1/94)$  and the knot  $10_{29} = S(\frac{63}{17})$  with  $\mathcal{C}(3, 14, 292, 1/93)$ . Also, an objective is now to consolidate and speed up our algorithms in order to increase its capabilities.

As the zero-dimensional systems we study have a triangular structure, we could try to get directly an exhaustive list of Chebyshev knots  $\mathcal{C}(a, b, c, \varphi)$  without computing additional resultants. In case when  $a$ ,  $b$  and  $c$  are relatively coprime, we can expect that the real variety  $V_{a,b,c}$  has only single points and try to get directly all possible signs. In that case, our three black-boxes could be implemented using exclusively univariate functions that compute recursively the roots of the systems to be solved without any additional rewriting.

## References

- [BDHZ] A. Booher, J. Daigle, J. Hoste, W. Zheng, *Sampling Lissajous and Fourier knots*, 2007. [arXiv:0707.4210](#)
- [BHJS] M. G. V. Bogle, J. E. Hearst, V. F. R. Jones, L. Stoilov, *Lissajous knots* Journal of Knot Theory and its Ramifications, 3(2): 121-140, 1994.
- [BPR] S. Basu, R. Pollack, M.-F. Roy, *Algorithms in real algebraic geometry*, Algorithms and Computations in Mathematics Vol. 10, Springer, 2003.
- [CLOS] D. Cox, J. Little, D. O’Shea, *Using Algebraic Geometry*, Graduate Texts in Mathematics, Springer, 1998.
- [Con] J. H. Conway, *An Enumeration of Knots and Links*, in Computational Problems in Abstract Algebra (Ed. J. Leech). Oxford, England: Pergamon Press, pp. 329-358, 1970.
- [DOS] A. Durfee, D. O’Shea, *Polynomial knots*, 2006. [arXiv:math/0612803v1](#)
- [ES] C. Ernst, D. W. Sumners, *The Growth of the Number of Prime Knots*, Math. Proc. Cambridge Philos. Soc. **102**, 303-315, 1987.
- [Fi] G. Fischer, *Plane Algebraic Curves*, A.M.S. Student Mathematical Library Vol 15, 2001.
- [FF] G. Freudenburg, J. Freudenburg, *Curves defined by Chebyshev polynomials*, 19 p., 2009. [arXiv:0902.3440](#)
- [GLS] M. Giusti, G. Lecerf, B. Salvy, *A Gröbner Free Alternative for Solving Polynomial Systems*, Journal of Complexity, 17(1):154-211, 2001.
- [HZ] J. Hoste, L. Zirbel, *Lissajous knots and knots with Lissajous projections*, Kobe Journal of mathematics, vol 24, n°2, 2007.
- [JP] V. F. R. Jones, J. Przytycki, *Lissajous knots and billiard knots*, Banach Center Publications, 42:145-163, 1998.
- [KP1] P. -V. Koseleff, D. Pecker, *Chebyshev knots*, 2008. [arXiv:0812.1089](#)
- [KP2] P. -V. Koseleff, D. Pecker, *Chebyshev diagrams for rational knots*, 2009. [arXiv:0906.4083](#)
- [La] C. Lamm, *There are infinitely many Lissajous knots*, Manuscripta Math., 93: 29-37, 1997.

- [Mu] K. Murasugi, *Knot Theory and its Applications*, Boston, Birkhäuser, 341p., 1996.
- [PRS] P. Pedersen, M.-F. Roy, A. Szpirglas, *Counting Real Zeros in the Multivariate Case*, Computational Algebraic Geometry, Birkhäuser, Progress in Mathematics, 109(6):61-76, 1993.
- [Rou] F. Rouillier, *Solving zero-dimensional systems through the rational univariate representation*, Journal of Applicable Algebra in Engineering, Communication and Computing, 9(5):433-461, 1999.
- [RZ] F. Rouillier, P. Zimmermann, *Efficient Isolation of Polynomial Real Roots*, J. of Computational and Applied Mathematics, 162(1):33-50, 2003.
- [Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its Applications, Advances Soviet Maths Vol 1, (1990).

## Table

Here is the list of the first 95 rational knots. We have given Chebyshev parametrizations for  $a = 3$  and  $a = 4$ . One corresponds to the minimal  $b$  and the other to the minimal total degree in  $b, c$ . For each parametrization we give the corresponding Schubert fraction  $(\alpha/\beta)$ , the number of double points (DP) in the corresponding diagram  $\mathcal{C}(a, b)$  so as the degree (deg) of  $V_{a,b,c}$ . Note that sometimes we have fewer double points with  $a = 4$  than with  $a = 3$ . For 6 knots ( $9_5, 10_{36}, 10_{39}, 10_3, 10_{30}, 10_{33}$ ), Chebyshev parametrizations with  $b$  minimal are not obtained with  $c \leq 300$ . Note that in [KP2], we have given an algorithm that determines for any two-bridge knot  $K$ , the minimal integer  $b$  and  $C(t) \in \mathbf{Q}[t]$ ,  $b + \deg C = 3N$ , such that  $x = T_3(t)$ ,  $y = T_b(t)$ ,  $z = C(t)$  is a parametrization of  $K$ .

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Chebyshev parametrizations of the first rational knots								
$K$	minimal $b$	$\alpha/\beta$	DP	deg	min. $(b-1)(c-1)$	$\alpha/\beta$	DP	deg
$3_1$	$\mathcal{C}(3, 4, 5, 0)$	$3/2$	3	12	$\mathcal{C}(3, 4, 5, 0)$	$3/2$	3	12
	$\mathcal{C}(4, 3, 5, 0)$	$3/2$	3	12	$\mathcal{C}(4, 5, 6, 1/5)$	$3/2$	6	30
$4_1$	$\mathcal{C}(3, 5, 7, 0)$	$5/3$	4	24	$\mathcal{C}(3, 5, 7, 0)$	$5/3$	4	24
	$\mathcal{C}(4, 5, 12, 1/23)$	$5/2$	6	66	$\mathcal{C}(4, 7, 8, 1/3)$	$5/2$	9	63
$5_1$	$\mathcal{C}(3, 7, 8, 0)$	$-5/4$	6	42	$\mathcal{C}(3, 7, 8, 0)$	$-5/4$	6	42
	$\mathcal{C}(4, 5, 8, 1/23)$	$-5/4$	6	42	$\mathcal{C}(4, 5, 8, 1/23)$	$-5/4$	6	42
$5_2$	$\mathcal{C}(3, 7, 17, 1/50)$	$-7/4$	6	96	$\mathcal{C}(3, 10, 11, 1/16)$	$7/4$	9	90
	$\mathcal{C}(4, 5, 7, 0)$	$-7/4$	6	36	$\mathcal{C}(4, 5, 7, 0)$	$-7/4$	6	36
$6_1$	$\mathcal{C}(3, 8, 10, 1/42)$	$-9/5$	7	63	$\mathcal{C}(3, 8, 10, 1/42)$	$-9/5$	7	63
	$\mathcal{C}(4, 7, 16, 1/39)$	$-9/4$	9	135	$\mathcal{C}(4, 7, 16, 1/39)$	$-9/4$	9	135
$6_2$	$\mathcal{C}(3, 8, 19, 1/46)$	$-11/7$	7	126	$\mathcal{C}(3, 8, 19, 1/46)$	$-11/7$	7	126
	$\mathcal{C}(4, 5, 11, 0)$	$11/8$	6	60	$\mathcal{C}(4, 5, 11, 0)$	$11/8$	6	60
$6_3$	$\mathcal{C}(3, 7, 11, 0)$	$13/8$	6	60	$\mathcal{C}(3, 7, 11, 0)$	$13/8$	6	60
	$\mathcal{C}(4, 7, 36, 1/42)$	$-13/8$	9	315	$\mathcal{C}(4, 9, 14, 1/29)$	$-13/8$	12	156
$7_1$	$\mathcal{C}(3, 10, 11, 0)$	$7/6$	9	90	$\mathcal{C}(3, 10, 11, 0)$	$7/6$	9	90
	$\mathcal{C}(4, 7, 27, 1/68)$	$7/6$	9	234	$\mathcal{C}(4, 9, 12, 1/18)$	$-7/6$	12	132
$7_2$	$\mathcal{C}(3, 10, 27, 1/50)$	$11/6$	9	234	$\mathcal{C}(3, 10, 27, 1/50)$	$11/6$	9	234
	$\mathcal{C}(4, 7, 9, 1/30)$	$11/6$	9	72	$\mathcal{C}(4, 7, 9, 1/30)$	$11/6$	9	72
$7_3$	$\mathcal{C}(3, 10, 28, 1/47)$	$13/4$	9	243	$\mathcal{C}(3, 10, 28, 1/47)$	$13/4$	9	243
	$\mathcal{C}(4, 7, 27, 1/80)$	$13/10$	9	234	$\mathcal{C}(4, 9, 15, 1/35)$	$13/10$	12	168
$7_4$	$\mathcal{C}(3, 10, 36, 1/306)$	$-15/4$	9	315	$\mathcal{C}(3, 11, 27, 1/238)$	$-15/11$	10	260
	$\mathcal{C}(4, 9, 64, 1/156)$	$-15/4$	12	756	$\mathcal{C}(4, 13, 21, 1/24)$	$-15/4$	18	360
$7_5$	$\mathcal{C}(3, 10, 35, 1/60)$	$17/12$	9	306	$\mathcal{C}(3, 13, 14, 1/24)$	$-17/10$	12	156
	$\mathcal{C}(4, 7, 9, 0)$	$17/10$	9	72	$\mathcal{C}(4, 7, 9, 0)$	$17/10$	9	72
$7_6$	$\mathcal{C}(3, 10, 33, 1/46)$	$19/8$	9	288	$\mathcal{C}(3, 14, 15, 1/26)$	$-19/7$	13	182
	$\mathcal{C}(4, 7, 40, 1/51)$	$19/8$	9	351	$\mathcal{C}(4, 9, 13, 5/44)$	$-19/8$	12	144
$7_7$	$\mathcal{C}(3, 8, 13, 0)$	$21/13$	7	84	$\mathcal{C}(3, 8, 13, 0)$	$21/13$	7	84
	$\mathcal{C}(4, 9, 61, 1/67)$	$-21/8$	12	720	$\mathcal{C}(4, 11, 16, 3/10)$	$-21/34$	15	225
$8_1$	$\mathcal{C}(3, 11, 13, 1/60)$	$13/7$	10	120	$\mathcal{C}(3, 11, 13, 1/60)$	$13/7$	10	120
	$\mathcal{C}(4, 9, 118, 1/67)$	$13/6$	12	1404	$\mathcal{C}(4, 15, 20, 2/15)$	$-13/24$	21	399
$8_2$	$\mathcal{C}(3, 11, 28, 1/80)$	$17/11$	10	270	$\mathcal{C}(3, 11, 28, 1/80)$	$17/11$	10	270
	$\mathcal{C}(4, 9, 35, 1/68)$	$17/14$	12	408	$\mathcal{C}(4, 13, 17, 1/6)$	$17/6$	18	288
$8_3$	$\mathcal{C}(3, 11, 13, 0)$	$17/13$	10	120	$\mathcal{C}(3, 11, 13, 0)$	$17/13$	10	120
	$\mathcal{C}(4, 11, 101, 1/85)$	$-17/4$	15	1500	$\mathcal{C}(4, 15, 17, 1/14)$	$-17/4$	21	336
$8_4$	$\mathcal{C}(3, 11, 46, 1/58)$	$19/15$	10	450	$\mathcal{C}(3, 17, 22, 1/144)$	$19/81$	16	336
	$\mathcal{C}(4, 7, 25, 1/75)$	$-19/14$	9	216	$\mathcal{C}(4, 11, 12, 3/14)$	$19/14$	15	165
$8_6$	$\mathcal{C}(3, 11, 87, 1/70)$	$23/13$	10	860	$\mathcal{C}(3, 17, 22, 1/44)$	$-23/13$	16	336
	$\mathcal{C}(4, 9, 91, 1/66)$	$23/16$	12	1080	$\mathcal{C}(4, 11, 21, 1/15)$	$-23/16$	15	300
$8_7$	$\mathcal{C}(3, 10, 70, 1/47)$	$-23/18$	9	621	$\mathcal{C}(3, 13, 18, 1/42)$	$23/18$	12	204
	$\mathcal{C}(4, 7, 13, 0)$	$-23/18$	9	108	$\mathcal{C}(4, 7, 13, 0)$	$-23/18$	9	108
$8_8$	$\mathcal{C}(3, 10, 14, 1/60)$	$-25/16$	9	117	$\mathcal{C}(3, 10, 14, 1/60)$	$-25/16$	9	117
	$\mathcal{C}(4, 7, 24, 1/236)$	$-25/14$	9	207	$\mathcal{C}(4, 7, 24, 1/236)$	$-25/14$	9	207
$8_9$	$\mathcal{C}(3, 11, 110, 1/168)$	$25/7$	10	1090	$\mathcal{C}(3, 17, 25, 1/96)$	$-25/57$	16	384
	$\mathcal{C}(4, 7, 16, 1/54)$	$-25/18$	9	135	$\mathcal{C}(4, 7, 16, 1/54)$	$-25/18$	9	135
$8_{11}$	$\mathcal{C}(3, 11, 100, 1/84)$	$27/17$	10	990	$\mathcal{C}(3, 14, 38, 1/66)$	$-27/17$	13	481
	$\mathcal{C}(4, 9, 30, 1/82)$	$27/8$	12	348	$\mathcal{C}(4, 11, 13, 2/25)$	$-27/8$	15	180
$8_{12}$	$\mathcal{C}(3, 11, 54, 1/152)$	$29/17$	10	530	$\mathcal{C}(3, 14, 41, 1/19)$	$29/17$	13	520
	$\mathcal{C}(4, 9, 103, 1/68)$	$29/12$	12	1224	$\mathcal{C}(4, 15, 18, 4/29)$	$-29/12$	21	357

Chebyshev parametrizations of the first rational knots								
$K$	minimal $b$	$\alpha/\beta$	DP	deg	min. $(b-1)(c-1)$	$\alpha/\beta$	DP	deg
$8_{13}$	$\mathcal{C}(3, 10, 17, 1/62)$	$-29/8$	9	144	$\mathcal{C}(3, 10, 17, 1/62)$	$-29/8$	9	144
	$\mathcal{C}(4, 9, 104, 1/66)$	$29/18$	12	1236	$\mathcal{C}(4, 13, 18, 17/45)$	$-29/8$	18	306
$8_{14}$	$\mathcal{C}(3, 11, 93, 1/86)$	$31/13$	10	920	$\mathcal{C}(3, 17, 22, 1/26)$	$-31/19$	16	336
	$\mathcal{C}(4, 9, 47, 1/200)$	$-31/44$	12	552	$\mathcal{C}(4, 13, 18, 1/20)$	$-31/18$	18	306
$9_1$	$\mathcal{C}(3, 13, 14, 0)$	$-9/8$	12	156	$\mathcal{C}(3, 13, 14, 0)$	$-9/8$	12	156
	$\mathcal{C}(4, 9, 87, 1/59)$	$-9/8$	12	1032	$\mathcal{C}(4, 13, 14, 3/43)$	$9/8$	18	234
$9_2$	$\mathcal{C}(3, 13, 37, 1/114)$	$-15/2$	12	432	$\mathcal{C}(3, 19, 24, 3/82)$	$15/8$	18	414
	$\mathcal{C}(4, 11, 33, 1/15)$	$-15/8$	15	480	$\mathcal{C}(4, 17, 20, 7/23)$	$-15/2$	24	456
$9_3$	$\mathcal{C}(3, 13, 143, 1/98)$	$-19/6$	12	1704	$\mathcal{C}(3, 16, 46, 1/54)$	$19/6$	15	675
	$\mathcal{C}(4, 9, 159, 1/108)$	$-19/16$	12	1896	$\mathcal{C}(4, 17, 18, 1/23)$	$19/44$	24	408
$9_4$	$\mathcal{C}(3, 13, 115, 1/164)$	$-21/4$	12	1368	$\mathcal{C}(3, 19, 24, 1/40)$	$21/16$	18	414
	$\mathcal{C}(4, 9, 106, 3/301)$	$-21/16$	12	1260	$\mathcal{C}(4, 17, 18, 3/17)$	$-21/4$	24	408
$9_5$	$b = 13$	$23/4$	12	$12c - 12$	$\mathcal{C}(3, 17, 45, 1/182)$	$23/29$	16	704
	$\mathcal{C}(4, 11, 152, 1/44)$	$23/6$	15	2265	$\mathcal{C}(4, 19, 22, 1/20)$	$23/6$	27	567
$9_6$	$\mathcal{C}(3, 13, 64, 1/102)$	$-27/22$	12	756	$\mathcal{C}(3, 16, 17, 1/34)$	$27/16$	15	240
	$\mathcal{C}(4, 9, 11, 1/55)$	$-27/16$	12	120	$\mathcal{C}(4, 9, 11, 1/55)$	$-27/16$	12	120
$9_7$	$\mathcal{C}(3, 13, 116, 1/80)$	$-29/20$	12	1380	$\mathcal{C}(3, 22, 24, 3/26)$	$-29/16$	21	483
	$\mathcal{C}(4, 9, 201, 1/131)$	$-29/16$	12	2400	$\mathcal{C}(4, 11, 17, 1/65)$	$-29/16$	15	240
$9_8$	$\mathcal{C}(3, 13, 121, 1/170)$	$-31/20$	12	1440	$\mathcal{C}(3, 16, 25, 3/97)$	$31/14$	15	360
	$\mathcal{C}(4, 11, 187, 1/147)$	$31/14$	15	2790	$\mathcal{C}(4, 17, 20, 6/19)$	$-31/14$	24	456
$9_9$	$\mathcal{C}(3, 13, 123, 1/80)$	$-31/22$	12	1464	$\mathcal{C}(3, 19, 52, 2/29)$	$-31/24$	18	918
	$\mathcal{C}(4, 9, 31, 1/66)$	$-31/22$	12	360	$\mathcal{C}(4, 9, 31, 1/66)$	$-31/22$	12	360
$9_{10}$	$\mathcal{C}(3, 13, 246, 1/110)$	$-33/10$	12	2940	$\mathcal{C}(3, 19, 53, 1/32)$	$33/10$	18	936
	$\mathcal{C}(4, 11, 29, 1/45)$	$33/10$	15	420	$\mathcal{C}(4, 11, 29, 1/45)$	$33/10$	15	420
$9_{11}$	$\mathcal{C}(3, 13, 114, 1/106)$	$-33/26$	12	1356	$\mathcal{C}(3, 23, 24, 1/10)$	$-33/19$	22	506
	$\mathcal{C}(4, 9, 33, 1/75)$	$-33/26$	12	384	$\mathcal{C}(4, 9, 33, 1/75)$	$-33/26$	12	384
$9_{12}$	$\mathcal{C}(3, 13, 36, 1/80)$	$-35/22$	12	420	$\mathcal{C}(3, 13, 36, 1/80)$	$-35/22$	12	420
	$\mathcal{C}(4, 11, 68, 1/77)$	$35/22$	15	1005	$\mathcal{C}(4, 15, 24, 1/17)$	$-35/22$	21	483
$9_{13}$	$\mathcal{C}(3, 13, 53, 1/78)$	$-37/26$	12	624	$\mathcal{C}(3, 16, 31, 1/42)$	$37/10$	15	450
	$\mathcal{C}(4, 9, 41, 1/91)$	$-37/26$	12	480	$\mathcal{C}(4, 17, 18, 1/6)$	$-37/26$	24	408
$9_{14}$	$\mathcal{C}(3, 11, 83, 1/74)$	$-37/23$	10	820	$\mathcal{C}(3, 17, 18, 1/9)$	$-37/29$	16	272
	$\mathcal{C}(4, 11, 176, 1/108)$	$37/14$	15	2625	$\mathcal{C}(4, 15, 22, 4/11)$	$37/8$	21	441
$9_{15}$	$\mathcal{C}(3, 13, 144, 1/310)$	$39/22$	12	1716	$\mathcal{C}(3, 17, 18, 1/34)$	$39/17$	16	272
	$\mathcal{C}(4, 9, 39, 1/85)$	$-39/22$	12	456	$\mathcal{C}(4, 13, 15, 7/45)$	$39/16$	18	252
$9_{17}$	$\mathcal{C}(3, 11, 16, 0)$	$-39/25$	10	150	$\mathcal{C}(3, 11, 16, 0)$	$-39/25$	10	150
	$\mathcal{C}(4, 9, 92, 1/92)$	$-39/14$	12	1092	$\mathcal{C}(4, 11, 24, 1/53)$	$39/14$	15	345
$9_{18}$	$\mathcal{C}(3, 13, 194, 1/144)$	$-41/12$	12	2316	$\mathcal{C}(3, 16, 43, 1/36)$	$-41/24$	15	630
	$\mathcal{C}(4, 9, 11, 0)$	$-41/24$	12	120	$\mathcal{C}(4, 9, 11, 0)$	$-41/24$	12	120
$9_{19}$	$\mathcal{C}(3, 11, 83, 1/82)$	$-41/25$	10	820	$\mathcal{C}(3, 14, 22, 1/110)$	$41/105$	13	273
	$\mathcal{C}(4, 9, 32, 1/416)$	$41/64$	12	372	$\mathcal{C}(4, 9, 32, 1/416)$	$41/64$	12	372
$9_{20}$	$\mathcal{C}(3, 13, 275, 1/86)$	$-41/26$	12	3288	$\mathcal{C}(3, 16, 46, 1/34)$	$41/26$	15	675
	$\mathcal{C}(4, 7, 17, 0)$	$41/30$	9	144	$\mathcal{C}(4, 7, 17, 0)$	$41/30$	9	144
$9_{21}$	$\mathcal{C}(3, 13, 179, 1/106)$	$-43/18$	12	2136	$\mathcal{C}(3, 16, 29, 9/172)$	$-43/18$	15	420
	$\mathcal{C}(4, 9, 94, 1/105)$	$43/68$	12	1116	$\mathcal{C}(4, 11, 23, 1/23)$	$43/18$	15	330
$9_{23}$	$\mathcal{C}(3, 13, 44, 1/98)$	$-45/26$	12	516	$\mathcal{C}(3, 13, 44, 1/98)$	$-45/26$	12	516
	$\mathcal{C}(4, 13, 16, 1/20)$	$-45/26$	18	270	$\mathcal{C}(4, 13, 16, 1/20)$	$-45/26$	18	270
$9_{26}$	$\mathcal{C}(3, 11, 25, 1/92)$	$-47/29$	10	240	$\mathcal{C}(3, 11, 25, 1/92)$	$-47/29$	10	240
	$\mathcal{C}(4, 9, 184, 1/79)$	$-47/34$	12	2196	$\mathcal{C}(4, 13, 17, 1/25)$	$47/18$	18	288

Chebyshev parametrizations of the first rational knots								
$K$	minimal $b$	$\alpha/\beta$	DP	deg	min. $(b-1)(c-1)$	$\alpha/\beta$	DP	deg
$9_{27}$	$\mathcal{C}(3, 13, 180, 1/84)$	$-49/18$	12	2148	$\mathcal{C}(3, 17, 51, 1/30)$	$49/31$	16	800
	$\mathcal{C}(4, 9, 39, 1/66)$	$-49/30$	12	456	$\mathcal{C}(4, 15, 20, 1/28)$	$49/80$	21	399
$9_{31}$	$\mathcal{C}(3, 10, 17, 0)$	$55/34$	9	144	$\mathcal{C}(3, 10, 17, 0)$	$55/34$	9	144
	$\mathcal{C}(4, 11, 68, 1/100)$	$55/34$	15	1005	$\mathcal{C}(4, 13, 24, 1/43)$	$-55/34$	18	414
$10_1$	$\mathcal{C}(3, 14, 38, 1/102)$	$-17/9$	13	481	$\mathcal{C}(3, 17, 19, 1/44)$	$17/9$	16	288
	$\mathcal{C}(4, 11, 141, 1/44)$	$-17/8$	15	2100	$\mathcal{C}(4, 19, 23, 1/33)$	$17/8$	27	594
$10_2$	$\mathcal{C}(3, 14, 43, 1/176)$	$-23/3$	13	546	$\mathcal{C}(3, 20, 25, 9/314)$	$23/15$	19	456
	$\mathcal{C}(4, 9, 278, 1/85)$	$23/20$	12	3324	$\mathcal{C}(4, 17, 20, 4/61)$	$-23/38$	24	456
$10_3$	$\mathcal{C}(3, 14, 16, 1/94)$	$-25/19$	13	195	$\mathcal{C}(3, 14, 16, 1/94)$	$-25/19$	13	195
	$b = 13$	$25/4$	18	$18c - 18$	$\mathcal{C}(4, 19, 20, 2/17)$	$25/6$	27	513
$10_4$	$\mathcal{C}(3, 14, 101, 1/130)$	$-27/23$	13	1300	$\mathcal{C}(3, 23, 26, 1/38)$	$27/23$	22	550
	$\mathcal{C}(4, 9, 257, 1/145)$	$27/20$	12	3072	$\mathcal{C}(4, 11, 34, 2/17)$	$27/20$	15	495
$10_5$	$\mathcal{C}(3, 13, 169, 9/1034)$	$33/28$	12	2016	$\mathcal{C}(3, 19, 24, 1/116)$	$33/28$	18	414
	$\mathcal{C}(4, 11, 194, 1/110)$	$-33/28$	15	2895	$\mathcal{C}(4, 13, 20, 2/35)$	$33/28$	18	342
$10_6$	$\mathcal{C}(3, 14, 128, 1/92)$	$-37/7$	13	1651	$\mathcal{C}(3, 20, 25, 1/42)$	$37/21$	19	456
	$\mathcal{C}(4, 13, 72, 1/47)$	$37/16$	18	1278	$\mathcal{C}(4, 17, 20, 1/15)$	$-37/58$	24	456
$10_7$	$\mathcal{C}(3, 14, 127, 1/128)$	$-43/27$	13	1638	$\mathcal{C}(3, 23, 33, 1/24)$	$43/27$	22	704
	$\mathcal{C}(4, 11, 229, 1/70)$	$-43/16$	15	3420	$\mathcal{C}(4, 19, 21, 1/17)$	$-43/8$	27	540
$10_8$	$\mathcal{C}(3, 14, 37, 1/144)$	$-29/23$	13	468	$\mathcal{C}(3, 14, 37, 1/144)$	$-29/23$	13	468
	$\mathcal{C}(4, 11, 77, 1/66)$	$-29/24$	15	1140	$\mathcal{C}(4, 13, 36, 2/45)$	$29/24$	18	630
$10_9$	$\mathcal{C}(3, 14, 281, 1/232)$	$-39/11$	13	3640	$\mathcal{C}(3, 17, 43, 1/186)$	$39/89$	16	672
	$\mathcal{C}(4, 9, 35, 1/133)$	$39/28$	12	408	$\mathcal{C}(4, 13, 17, 17/120)$	$39/28$	18	288
$10_{10}$	$\mathcal{C}(3, 13, 253, 1/250)$	$45/28$	12	3024	$\mathcal{C}(3, 19, 31, 1/31)$	$-45/118$	18	540
	$\mathcal{C}(4, 11, 102, 1/45)$	$-45/28$	15	1515	$\mathcal{C}(4, 15, 20, 1/10)$	$45/28$	21	399
$10_{11}$	$\mathcal{C}(3, 14, 101, 1/116)$	$-43/33$	13	1300	$\mathcal{C}(3, 20, 25, 1/196)$	$-43/185$	19	456
	$\mathcal{C}(4, 11, 126, 1/97)$	$-43/30$	15	1875	$\mathcal{C}(4, 21, 24, 2/31)$	$-43/76$	30	690
$10_{12}$	$\mathcal{C}(3, 13, 61, 1/178)$	$47/36$	12	720	$\mathcal{C}(3, 19, 24, 1/44)$	$47/36$	18	414
	$\mathcal{C}(4, 11, 115, 1/393)$	$-47/58$	15	1710	$\mathcal{C}(4, 13, 20, 1/18)$	$47/36$	18	342
$10_{13}$	$\mathcal{C}(3, 14, 211, 3/322)$	$-53/41$	13	2730	$\mathcal{C}(3, 23, 26, 1/26)$	$53/31$	22	550
	$\mathcal{C}(4, 11, 147, 1/84)$	$-53/22$	15	2190	$\mathcal{C}(4, 15, 18, 3/22)$	$-53/22$	21	357
$10_{14}$	$\mathcal{C}(3, 14, 139, 1/180)$	$-57/13$	13	1794	$\mathcal{C}(3, 17, 44, 2/43)$	$-57/13$	16	688
	$\mathcal{C}(4, 9, 35, 2/135)$	$57/44$	12	408	$\mathcal{C}(4, 9, 35, 2/135)$	$57/44$	12	408
$10_{15}$	$\mathcal{C}(3, 13, 17, 1/80)$	$43/24$	12	192	$\mathcal{C}(3, 13, 17, 1/80)$	$43/24$	12	192
	$\mathcal{C}(4, 9, 105, 1/40)$	$43/24$	12	1248	$\mathcal{C}(4, 17, 27, 2/35)$	$43/24$	24	624
$10_{16}$	$\mathcal{C}(3, 14, 127, 1/104)$	$-47/37$	13	1638	$\mathcal{C}(3, 23, 25, 1/9)$	$-47/37$	22	528
	$\mathcal{C}(4, 11, 37, 1/57)$	$-47/14$	15	540	$\mathcal{C}(4, 19, 21, 1/20)$	$-47/14$	27	540
$10_{17}$	$\mathcal{C}(3, 13, 194, 1/79)$	$41/32$	12	2316	$\mathcal{C}(3, 19, 24, 1/220)$	$41/32$	18	414
	$\mathcal{C}(4, 9, 16, 1/69)$	$41/32$	12	180	$\mathcal{C}(4, 9, 16, 1/69)$	$41/32$	12	180
$10_{18}$	$\mathcal{C}(3, 14, 37, 1/148)$	$-55/43$	13	468	$\mathcal{C}(3, 14, 37, 1/148)$	$-55/43$	13	468
	$\mathcal{C}(4, 11, 211, 1/52)$	$-55/32$	15	3150	$\mathcal{C}(4, 19, 21, 2/11)$	$55/32$	27	540
$10_{19}$	$\mathcal{C}(3, 13, 128, 1/158)$	$51/14$	12	1524	$\mathcal{C}(3, 22, 33, 1/48)$	$51/142$	21	672
	$\mathcal{C}(4, 9, 162, 1/181)$	$51/40$	12	1932	$\mathcal{C}(4, 13, 23, 1/148)$	$-51/40$	18	396
$10_{20}$	$\mathcal{C}(3, 14, 292, 1/94)$	$-35/11$	13	3783	$\mathcal{C}(3, 23, 29, 1/22)$	$-35/19$	22	616
	$\mathcal{C}(4, 11, 298, 1/133)$	$-35/16$	15	4455	$\mathcal{C}(4, 19, 28, 1/12)$	$35/54$	27	729
$10_{21}$	$\mathcal{C}(3, 14, 133, 1/108)$	$-45/31$	13	1716	$\mathcal{C}(3, 20, 46, 5/72)$	$-45/29$	19	855
	$\mathcal{C}(4, 11, 193, 1/60)$	$-45/16$	15	2880	$\mathcal{C}(4, 19, 23, 1/36)$	$45/16$	27	594
$10_{22}$	$\mathcal{C}(3, 14, 230, 1/554)$	$49/15$	13	2977	$\mathcal{C}(3, 20, 27, 1/126)$	$-49/15$	19	494
	$\mathcal{C}(4, 9, 96, 1/52)$	$49/36$	12	1140	$\mathcal{C}(4, 11, 25, 2/31)$	$49/36$	15	360

Chebyshev parametrizations of the first rational knots								
$K$	minimal $b$	$\alpha/\beta$	DP	deg	min. $(b-1)(c-1)$	$\alpha/\beta$	DP	deg
$10_{23}$	$\mathcal{C}(3, 13, 124, 1/362)$	$-59/18$	12	1476	$\mathcal{C}(3, 19, 26, 1/64)$	$59/18$	18	450
	$\mathcal{C}(4, 11, 38, 1/105)$	$-59/18$	15	555	$\mathcal{C}(4, 19, 20, 2/13)$	$59/36$	27	513
$10_{24}$	$\mathcal{C}(3, 14, 127, 3/319)$	$-55/31$	13	1638	$\mathcal{C}(3, 23, 36, 5/82)$	$55/31$	22	770
	$\mathcal{C}(4, 11, 247, 1/74)$	$-55/24$	15	3690	$\mathcal{C}(4, 19, 23, 1/37)$	$55/24$	27	594
$10_{25}$	$\mathcal{C}(3, 14, 148, 1/108)$	$-65/41$	13	1911	$\mathcal{C}(3, 17, 64, 1/46)$	$-65/19$	16	1008
	$\mathcal{C}(4, 9, 116, 2/135)$	$65/46$	12	1380	$\mathcal{C}(4, 15, 20, 1/43)$	$65/106$	21	399
$10_{26}$	$\mathcal{C}(3, 14, 110, 1/98)$	$-61/17$	13	1417	$\mathcal{C}(3, 17, 67, 1/86)$	$-61/17$	16	1056
	$\mathcal{C}(4, 9, 35, 1/77)$	$61/44$	12	408	$\mathcal{C}(4, 9, 35, 1/77)$	$61/44$	12	408
$10_{27}$	$\mathcal{C}(3, 13, 126, 1/218)$	$71/50$	12	1500	$\mathcal{C}(3, 22, 36, 1/50)$	$71/44$	21	735
	$\mathcal{C}(4, 11, 278, 1/115)$	$-71/44$	15	4155	$\mathcal{C}(4, 15, 26, 2/71)$	$71/44$	21	525
$10_{28}$	$\mathcal{C}(3, 13, 191, 1/112)$	$53/14$	12	2280	$\mathcal{C}(3, 19, 25, 5/138)$	$53/34$	18	432
	$\mathcal{C}(4, 11, 114, 1/139)$	$53/92$	15	1695	$\mathcal{C}(4, 15, 38, 1/38)$	$53/92$	21	777
$10_{29}$	$\mathcal{C}(3, 14, 292, 1/93)$	$-63/17$	13	3783	$\mathcal{C}(3, 17, 45, 1/42)$	$-63/37$	16	704
	$\mathcal{C}(4, 9, 168, 1/106)$	$63/46$	12	2004	$\mathcal{C}(4, 15, 23, 1/8)$	$63/46$	21	462
$10_{30}$	$\mathcal{C}(3, 14, 201, 1/96)$	$-67/41$	13	2600	$\mathcal{C}(3, 17, 39, 1/34)$	$-67/49$	16	608
	$b = 13$	$67/18$	18	$18c - 18$	$\mathcal{C}(4, 21, 25, 3/77)$	$67/18$	30	720
$10_{31}$	$\mathcal{C}(3, 13, 103, 1/80)$	$57/32$	12	1224	$\mathcal{C}(3, 19, 27, 1/32)$	$-57/32$	18	468
	$\mathcal{C}(4, 9, 111, 1/66)$	$57/32$	12	1320	$\mathcal{C}(4, 15, 20, 1/11)$	$57/32$	21	399
$10_{32}$	$\mathcal{C}(3, 14, 148, 1/172)$	$-69/19$	13	1911	$\mathcal{C}(3, 16, 56, 1/166)$	$-69/50$	15	825
	$\mathcal{C}(4, 11, 134, 1/103)$	$-69/50$	15	1995	$\mathcal{C}(4, 15, 22, 1/34)$	$-69/40$	21	441
$10_{33}$	$\mathcal{C}(3, 13, 182, 1/105)$	$65/18$	12	2172	$\mathcal{C}(3, 22, 40, 1/38)$	$65/148$	21	819
	$b = 13$	$65/18$	18	$18c - 18$	$\mathcal{C}(4, 25, 30, 5/17)$	$65/18$	36	1044
$10_{34}$	$\mathcal{C}(3, 13, 41, 1/90)$	$37/20$	12	480	$\mathcal{C}(3, 16, 20, 1/44)$	$-37/24$	15	285
	$\mathcal{C}(4, 11, 142, 1/122)$	$-37/20$	15	2115	$\mathcal{C}(4, 13, 19, 1/11)$	$-37/20$	18	324
$10_{35}$	$\mathcal{C}(3, 14, 38, 1/108)$	$-49/27$	13	481	$\mathcal{C}(3, 14, 38, 1/108)$	$-49/27$	13	481
	$\mathcal{C}(4, 13, 273, 3/697)$	$49/20$	18	4896	$\mathcal{C}(4, 19, 24, 1/9)$	$-49/78$	27	621
$10_{36}$	$b = 14$	$51/20$	13	$13c - 13$	$\mathcal{C}(3, 20, 32, 1/44)$	$51/31$	19	589
	$\mathcal{C}(4, 9, 179, 1/222)$	$51/28$	12	2136	$\mathcal{C}(4, 11, 17, 1/77)$	$-51/28$	15	240
$10_{37}$	$\mathcal{C}(3, 13, 17, 0)$	$53/30$	12	192	$\mathcal{C}(3, 13, 17, 0)$	$53/30$	12	192
	$\mathcal{C}(4, 13, 19, 5/59)$	$-53/30$	18	324	$\mathcal{C}(4, 13, 19, 5/59)$	$-53/30$	18	324
$10_{38}$	$\mathcal{C}(3, 14, 120, 1/134)$	$-59/25$	13	1547	$\mathcal{C}(3, 17, 23, 5/122)$	$-59/33$	16	352
	$\mathcal{C}(4, 13, 72, 1/43)$	$59/26$	18	1278	$\mathcal{C}(4, 23, 27, 2/19)$	$-59/34$	33	858
$10_{39}$	$b = 14$	$61/22$	13	$13c - 13$	$\mathcal{C}(3, 20, 25, 1/34)$	$61/39$	19	456
	$\mathcal{C}(4, 9, 277, 1/119)$	$61/36$	12	3312	$\mathcal{C}(4, 11, 39, 1/18)$	$-61/36$	15	570
$10_{40}$	$\mathcal{C}(3, 13, 190, 1/112)$	$75/44$	12	2268	$\mathcal{C}(3, 16, 34, 1/21)$	$75/46$	15	495
	$\mathcal{C}(4, 11, 57, 1/291)$	$75/106$	15	840	$\mathcal{C}(4, 17, 23, 2/25)$	$-75/44$	24	528
$10_{41}$	$\mathcal{C}(3, 14, 208, 1/110)$	$-71/41$	13	2691	$\mathcal{C}(3, 17, 55, 1/23)$	$-71/41$	16	864
	$\mathcal{C}(4, 9, 165, 2/141)$	$-71/112$	12	1968	$\mathcal{C}(4, 15, 25, 6/47)$	$71/30$	21	504
$10_{42}$	$\mathcal{C}(3, 13, 134, 1/166)$	$81/50$	12	1596	$\mathcal{C}(3, 16, 25, 1/22)$	$81/50$	15	360
	$\mathcal{C}(4, 11, 131, 1/132)$	$-81/34$	15	1950	$\mathcal{C}(4, 13, 32, 5/47)$	$81/34$	18	558
$10_{43}$	$\mathcal{C}(3, 13, 174, 1/114)$	$73/46$	12	2076	$\mathcal{C}(3, 16, 51, 15/976)$	$-73/46$	15	750
	$\mathcal{C}(4, 11, 277, 1/134)$	$-73/46$	15	4140	$\mathcal{C}(4, 13, 36, 1/48)$	$73/100$	18	630
$10_{44}$	$\mathcal{C}(3, 14, 132, 1/114)$	$-79/49$	13	1703	$\mathcal{C}(3, 20, 53, 5/148)$	$79/29$	19	988
	$\mathcal{C}(4, 11, 64, 1/119)$	$-79/30$	15	945	$\mathcal{C}(4, 15, 24, 3/13)$	$-79/50$	21	483
$10_{45}$	$\mathcal{C}(3, 11, 19, 0)$	$89/55$	10	180	$\mathcal{C}(3, 11, 19, 0)$	$89/55$	10	180
	$\mathcal{C}(4, 13, 132, 1/238)$	$89/34$	18	2358	$\mathcal{C}(4, 15, 26, 2/21)$	$89/144$	21	525