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# FINITE MASS SELF-SIMILAR BLOWING-UP SOLUTIONS OF A CHEMOTAXIS SYSTEM WITH NON-LINEAR DIFFUSION

ADRIEN BLANCHET<sup>1</sup> AND PHILIPPE LAURENÇOT<sup>2</sup>

ABSTRACT. For a specific choice of the diffusion, the parabolic-elliptic Patlak-Keller-Segel system with non-linear diffusion (also referred to as the quasi-linear Smoluchowski-Poisson equation) exhibits an interesting threshold phenomenon: there is a critical mass  $M_c > 0$  such that all the solutions with initial data of mass smaller or equal to  $M_c$  exist globally while the solution blows up in finite time for a large class of initial data with mass greater than  $M_c$ . Unlike in space dimension 2, finite mass self-similar blowing-up solutions are shown to exist in space dimension  $d \geq 3$ .

## 1. INTRODUCTION

In space dimension  $d = 2$ , the parabolic-elliptic Patlak-Keller-Segel (PKS) system is a simplified model which describes the collective motion of cells in the following situation: cells diffuse in space and emit a chemical signal, the chemo-attractant, which results in the cells attracting each other. If  $\rho$  denotes the density of cells and  $c$  the concentration of the chemo-attractant, the PKS system reads [13, 19]

$$(1) \quad \begin{cases} \partial_t \rho(t, x) = \operatorname{div} [\nabla \rho(t, x) - \rho(t, x) \nabla c(t, x)] , \\ c(t, x) = (E_2 \star \rho)(t, x) , \quad E_2(x) = -\frac{1}{2\pi} \ln |x| , \end{cases} \quad (t, x) \in [0, \infty) \times \mathbb{R}^2 .$$

This model may be seen as an elementary brick to understand the aggregation of cells in mathematical biology as it exhibits the following interesting and biologically relevant feature: there is a critical mass above which the density of cells is expected to concentrate near isolated points after a finite time, a property which is related to the formation of fruiting bodies in the slime mold *Dictyostelium discoideum*. Such a phenomenon does not take place if the density of cells is too low. More precisely, given a non-negative integrable initial condition  $\rho_0$  with finite second moment, the system (1) has a unique maximal classical solution  $(\rho, c)$  defined on some maximal time interval  $[0, T)$ ,  $T \in (0, \infty]$ . Its first component  $\rho$  is non-negative and the mass of  $\rho$  (that is, its  $L^1$ -norm) remains constant through time evolution

$$\|\rho(t)\|_1 = M := \|\rho_0\|_1 , \quad t \in [0, T) .$$

It is well-known that, if  $M < 8\pi$ , the solution to (1) exists globally in time while it blows up in finite time if  $M > 8\pi$ , see [3, 6, 11, 12] and the references therein.

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More recently, it was shown that there is global existence as well for the critical mass  $M = 8\pi$ , the blowup occurring in infinite time with a profile being a Dirac mass of mass  $8\pi$  [1]. When the mass  $M$  is above  $8\pi$ , the shape of the finite time blowup is not self-similar according to asymptotic expansions computed in [5, 15] (see also [10] for a related problem in a bounded domain). In addition, there is no integrable and radially symmetric blowing-up self-similar solution to (1) [18, Theorem 8].

In space dimension  $d \geq 3$ , the system (1) seems to be less relevant from the biological point of view as blowup may occur whatever the value of  $M$  [9, 17]. This means that the diffusion is too weak to balance the aggregation resulting from the chemotactic term. It is however well-known that one can enhance the effect of diffusion to prevent crowding by considering a diffusion of porous medium type which increases the diffusion of the cells when their density  $\rho$  is large. This is the generalised version of the Patlak-Keller-Segel model considered in, e.g., [2, 4, 22, 23, 24]:

$$(2) \quad \begin{cases} \partial_t \rho(t, x) = \operatorname{div} (\nabla [\rho^m(t, x)] - \rho(t, x) \nabla c(t, x)) , \\ c(t, x) = (E_d \star \rho)(t, x), \quad E_d(x) = c_d |x|^{2-d}, \end{cases} \quad (t, x) \in [0, \infty) \times \mathbb{R}^d,$$

where  $m > 1$ ,  $c_d := 1/((d-2)\sigma_d)$ , and  $\sigma_d := 2\pi^{d/2}/\Gamma(d/2)$  denotes the surface area of the sphere  $\mathbb{S}^{d-1}$  of  $\mathbb{R}^d$ . The system (2) also arises in astrophysics [4] (being then referred to as the generalised Smoluchowski-Poisson equation), and  $\rho$  and  $c$  denote the density of particles and the gravitational potential, respectively.

For (2), it turns out that there is only one critical exponent of the non-linear diffusion, namely  $m_d := 2(d-1)/d$ , such that the mass plays a similar role to that in (1). Indeed, if  $m > m_d$  the diffusion enhancement is too strong and the solutions always exist globally in time whereas if  $m < m_d$  the diffusion is not strong enough to compensate the aggregation term and there are solutions blowing up in finite time whatever the value of the mass [22, 23]. The relevant diffusion is thus achieved in the case when  $m = m_d$ . In this case, it was proved in [2] that there is a unique threshold mass  $M_c > 0$  with the following properties: if the mass  $M = \|\rho_0\|_1$  of the initial condition  $\rho_0$  is less or equal to  $M_c$ , then the corresponding solution to (2) exists globally in time, whereas given any  $M > M_c$  there are initial data  $\rho_0$  with mass  $M$  such that the corresponding solution blows up in finite time. Thus, for the peculiar choice  $m = m_d$  and  $d \geq 3$ , the system (2) exhibits the same qualitative behaviour as the PKS system (1) in space dimension 2. Still, there is a fundamental difference as the latter has no fast-decaying stationary solution with mass  $8\pi$  while the former has a two-parameter family of non-negative, integrable, and compactly supported stationary solutions with mass  $M_c$  for each  $d \geq 3$  [2, Section 3].

It is then tempting to figure out whether this striking difference extends above the critical mass  $M_c$  and this leads us to investigate the existence of blowing-up (or backward) self-similar solutions with finite mass. More precisely, since mass remains unchanged throughout time evolution, we look for solutions  $(\rho, c)$  to (2) with  $m = m_d$  and  $d \geq 3$  of the form

$$(3) \quad \rho(t, x) = \frac{1}{s(t)^d} \Phi \left( \frac{x}{s(t)} \right) \quad \text{and} \quad c(t, x) = \frac{1}{s(t)^{d-2}} \Psi \left( \frac{x}{s(t)} \right)$$

for  $(t, x) \in [0, T) \times \mathbb{R}^d$  with  $s(t) := [d(T - t)]^{1/d}$ , the time  $T$  being an arbitrary positive real number. Note that  $s(t)$  converges to zero as  $t$  increases to the blowup time  $T$ .

Our main result is then the following:

**Theorem 1** (Existence of finite mass self-similar blowing-up solutions). *There exists  $M_2 \in (M_c, \infty)$  such that, for any  $M$  in  $(M_c, M_2]$ , there exists at least a non-negative self-similar blowing-up solution  $(\rho_M, c_M)$  to (2) of the form (3) with a radially symmetric, compactly supported, and non-increasing profile  $\Phi_M$  satisfying  $\|\rho_M(t)\|_1 = \|\Phi_M\|_1 = M$  for  $t \in [0, T)$  and  $\|\rho_M(t)\|_\infty \rightarrow \infty$  as  $t \rightarrow T$ .*

As a consequence of Theorem 1, we realize that non-negative, integrable, and radially symmetric self-similar blowing-up solutions to (2) with a non-increasing profile only exist below a threshold mass. Another by-product of our analysis is the existence of non-negative and non-integrable self-similar blowing-up solutions to (2), see Proposition 8 below.

## 2. BLOWING-UP SELF-SIMILAR PROFILES

From now on,

$$d \geq 3 \quad \text{and} \quad m = m_d = \frac{2(d-1)}{d},$$

and we look for a solution  $(\rho, c)$  to (2) of the form

$$(4) \quad \rho(t, x) = \frac{1}{s(t)^d} \Phi\left(\frac{x}{s(t)}\right) \quad \text{and} \quad c(t, x) = \frac{1}{s(t)^{d-2}} \Psi\left(\frac{x}{s(t)}\right)$$

with  $s(t) = [d(T - t)]^{1/d}$  and  $(t, x) \in [0, T) \times \mathbb{R}^d$  for some given  $T > 0$ . We further assume that  $\Phi$  enjoys the following properties:

$$\begin{cases} \Phi \in \mathcal{C}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \text{ is radially symmetric and non-negative,} \\ \Phi^{m-1} \in W^{1,\infty}(\mathbb{R}^d). \end{cases}$$

Inserting the ansatz (4) in (2) gives that  $(\Phi, \Psi)$  solves

$$\begin{cases} \operatorname{div}(\nabla[\Phi^m(y)] - \Phi(y)\nabla\Psi(y) - \Phi(y)y) = 0, \\ \Psi(y) = (E_d \star \Phi)(y), \end{cases}$$

for  $y \in \mathbb{R}^d$ . Since  $\Psi = E_d \star \Phi$ , the radial symmetry of  $\Phi$  ensures that of  $\Psi$  and, introducing the profiles  $(\varphi, \psi)$  of  $(\Phi, \Psi)$

$$(5) \quad \Phi(y) = \varphi(|y|), \quad \Psi(y) = \psi(|y|), \quad y \in \mathbb{R}^d.$$

By [14, Theorem 9.7, Formula (5)], we have

$$(6) \quad \psi(r) = \frac{1}{(d-2)r^{d-2}} \int_0^r \varphi(s) s^{d-1} ds + \frac{1}{d-2} \int_r^\infty \varphi(s) s ds$$

for  $r \geq 0$ . We can also write the equation for  $\varphi$  as

$$(7) \quad \partial_r (r^{d-1} \varphi(r) \partial_r J(r)) = 0 \quad \text{with} \quad J(r) := \frac{2(d-1)}{d-2} \varphi^{(d-2)/d}(r) - \psi(r) - \frac{r^2}{2}$$

for  $r \in (0, \infty)$ . Since we are looking for an integrable profile, we formally conclude that

$$(8) \quad \partial_r J(r) = 0 \quad \text{for } r \in \mathcal{P}_\varphi := \{s \in (0, \infty) : \varphi(s) > 0\}.$$

In particular,  $J$  is constant on any connected component of  $\mathcal{P}_\varphi$ . But, if  $\mathcal{C}$  is a connected component of  $\mathcal{P}_\varphi$ , we have either

$$(9) \quad \mathcal{C} = (0, R_s) \quad \text{for some } R_s \in (0, \infty],$$

or

$$\mathcal{C} = (R_i, R_s) \quad \text{for some } R_i \in (0, \infty) \quad \text{and } R_s \in (0, \infty].$$

**Remark 2.** *If we additionally assume that the profile  $\varphi$  is non-increasing then  $\mathcal{P}_\varphi$  has only one connected component which is necessarily of the form (9).*

Now, take a connected component  $\mathcal{C}$  of  $\mathcal{P}_\varphi$ . It follows from (8) that there is  $\mu \in \mathbb{R}$  such that

$$(10) \quad J(r) = \frac{2(d-1)}{d-2} \varphi^{(d-2)/d}(r) - \psi(r) - \frac{r^2}{2} = -\mu \quad \text{for } r \in \mathcal{C}.$$

Owing to the assumed integrability of  $\Phi$ , the function  $r \mapsto r^{d-1}\varphi(r)$  belongs to  $L^1(0, \infty)$  and it follows from (6) that the function  $r \mapsto r^{d-2}\psi(r)$  is bounded in  $\mathcal{C}$ . Therefore (10) only complies with the integrability of  $\Phi$  if  $R_s < \infty$  which implies the boundedness of  $\mathcal{C}$ . Introducing

$$\Xi := \varphi^{(d-2)/d}$$

and taking the Laplacian of both sides of (10) yield that  $\Xi$  is a positive solution to

$$(11) \quad -\frac{d^2\Xi}{dr^2}(r) - \frac{d-1}{r} \frac{d\Xi}{dr}(r) = \frac{d-2}{2(d-1)} (\Xi(r)^{d/(d-2)} - d) \quad \text{in } \mathcal{C},$$

with either

$$(12) \quad \partial_r \Xi(0) = \Xi(R_s) = 0 \quad \text{if } \mathcal{C} = (0, R_s)$$

or

$$(13) \quad \Xi(R_i) = \Xi(R_s) = 0 \quad \text{if } \mathcal{C} = (R_i, R_s).$$

A final change of scale, namely

$$\eta(r) := \frac{1}{\lambda_d} \Xi\left(\frac{r}{\mu_d}\right), \quad \lambda_d := d^{(d-2)/d}, \quad \mu_d := d^{1/d} \left(\frac{d-2}{2(d-1)}\right)^{1/2},$$

leads us to the following boundary-value problem for  $\eta$ : either

$$(14) \quad \begin{cases} \frac{d^2\eta}{dr^2}(r) + \frac{d-1}{r} \frac{d\eta}{dr}(r) + \eta(r)^{d/(d-2)} - 1 = 0, & r \in (0, \mu_d R_s), \\ \frac{d\eta}{dr}(0) = 0, & \eta(\mu_d R_s) = 0, \end{cases}$$

or

$$(15) \quad \begin{cases} \frac{d^2\eta}{dr^2}(r) + \frac{d-1}{r} \frac{d\eta}{dr}(r) + \eta(r)^{d/(d-2)} - 1 = 0, & r \in (\mu_d R_i, \mu_d R_s), \\ \eta(\mu_d R_i) = 0, & \eta(\mu_d R_s) = 0. \end{cases}$$

We have thus reduced our study to one or several boundary-value problems (depending on the number of connected components of  $\mathcal{P}_\varphi$ ) for a nonlinear second order differential equation. The purpose of the next section is then a precise study of this ordinary differential equation.

However, before going on, let us point out that (11) is not equivalent to (10). Indeed, since

$$\partial_r J(r) = \frac{2(d-1)}{d-2} \partial_r \Xi(r) + \frac{1}{r^{d-1}} \int_0^r \Xi(s)^{d/(d-2)} s^{d-1} ds - r, \quad r \in \mathcal{C},$$

by (7), the fact that  $\Xi$  is a solution to (11) only guarantees that  $\partial_r(r^{d-1}\partial_r J(r)) = 0$  for  $r \in \mathcal{C}$ . Consequently, there are constants  $C_1$  and  $C_2$  such that

$$\partial_r J(r) = -\frac{(d-2)C_1}{r^{d-1}}, \quad J(r) = \frac{C_1}{r^{d-2}} + C_2, \quad r \in \mathcal{C},$$

from which (10) follows only if  $C_1 = 0$ . On the one hand, if  $\mathcal{C} = (R_i, R_s)$  with  $0 < R_i < R_s$ , it is yet unclear whether the boundary conditions (13) might imply this property. On the other hand, if  $\mathcal{C} = (0, R_s)$ , the boundary conditions (12) ensure that  $\partial_r J(0) = 0$  and thus  $C_1 = 0$ . We shall only deal with this case in the remaining of this paper and thus focus on the non-increasing profiles  $\varphi$ .

### 3. AN AUXILIARY ORDINARY DIFFERENTIAL EQUATION

For  $a \in \mathbb{R}$ , let  $u(\cdot, a) \in \mathcal{C}^1([0, r_{\max}(a)])$  denote the maximal solution to the Cauchy problem

$$(16) \quad \begin{cases} u''(r, a) + \frac{d-1}{r} u'(r, a) + |u(r, a)|^{p-1} u(r, a) - 1 = 0, & r \in [0, r_{\max}(a)], \\ u(0, a) = a, \quad u'(0, a) = 0, \end{cases}$$

with  $r_{\max}(a) \in (0, \infty]$  and  $p = d/(d-2)$ .

Clearly, if  $a = 1$  then  $u(\cdot, 1) \equiv 1$  is a stationary solution and  $r_{\max}(1) = \infty$ . We first show that  $u(\cdot, a)$  is global for all  $a \in \mathbb{R}$  and oscillates around the value 1 if  $a \neq 1$ .

**Lemma 3.** *For each  $a \in \mathbb{R} \setminus \{1\}$ ,  $r_{\max}(a) = \infty$ , and the solution  $u(\cdot, a)$  to (16) is an oscillatory function in  $(0, \infty)$ . More precisely,*

- if  $a > 1$ , there is an increasing sequence  $(r_i(a))_{i \geq 0}$  of real numbers such that  $r_0(a) = 0$ ,

$$\begin{cases} u'(r_i(a), a) = 0, & (-1)^i u'(r, a) < 0 \quad \text{for } r \in (r_i(a), r_{i+1}(a)), \\ u(r_{2i}(a), a) > u(r_{2i+2}(a), a) > 1 > u(r_{2i+3}(a), a) > u(r_{2i+1}(a), a) \end{cases}$$

for  $i \geq 0$ ,

- if  $a < 1$ , there is an increasing sequence  $(r_i(a))_{i \geq 1}$  of real numbers such that  $r_1(a) = 0$

$$\begin{cases} u'(r_i(a), a) = 0, & (-1)^i u'(r, a) < 0 \quad \text{for } r \in (r_i(a), r_{i+1}(a)), \\ u(r_{2i}(a), a) > u(r_{2i+2}(a), a) > 1 > u(r_{2i+1}(a), a) > u(r_{2i-1}(a), a) \end{cases}$$

for  $i \geq 1$ .

These properties are illustrated in Figure 1. Notice that, for  $a = 7$ ,  $u(., 7)$  vanishes at a finite  $r$  and thus provides a solution to (14).

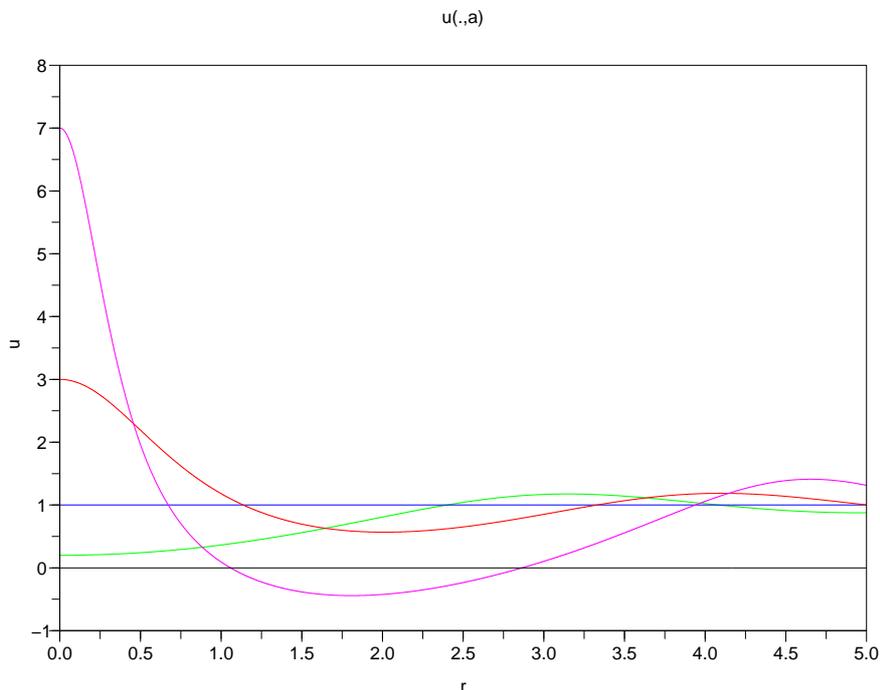


FIGURE 1. Various oscillating behaviours of  $u(., a)$  for  $a \in \{0.2, 1, 3, 7\}$ .

*Proof of Lemma 3.* For any  $r \in [0, r_{\max}(a))$  consider the functional

$$(17) \quad E(r, a) := \frac{|u'(r, a)|^2}{2} + \frac{|u(r, a)|^{p+1}}{p+1} - u(r, a).$$

By (16), for all  $r \in [0, r_{\max}(a))$

$$(18) \quad \frac{dE}{dr}(r, a) = -\frac{d-1}{r} |u'(r, a)|^2 \leq 0,$$

Obviously  $E(r, a) \geq -p/(p+1)$ . Owing to (18),  $E(r, a) \in [-p/(p+1), E(0, a)]$  for  $r \in [0, r_{\max}(a))$  which prevents  $u(., a)$  of becoming unbounded at a finite value of  $r$ , thereby implying that  $r_{\max}(a) = \infty$ . We next argue using Sturm's oscillations theorem as in [16, Lemma 9], to establish the oscillatory behaviour of  $u(., a)$  for  $a \neq 1$ .  $\square$

According to (14), we are interested in finding solutions to the initial value problem (16) which are positive and vanish at a finite value of  $r$ . We thus focus on the case  $a > 0$  and investigate the positivity properties of  $u(., a)$ .

**Lemma 4.** *There is a constant  $a_c > 1$  such that*

- if  $a \in (0, a_c)$ , then  $u(r, a) > 0$  for all  $r \geq 0$ ,

- if  $a = a_c$ , then there is  $R(a_c) > 0$  such that

$$\begin{cases} u(R(a_c), a_c) = 0 \\ u'(R(a_c), a_c) = 0 \\ u(r, a_c) > 0 \quad \text{for } r \in [0, R(a_c)), \end{cases}$$

- if  $a \in (a_c, \infty)$ , then there is  $R(a) > 0$  such that

$$\begin{cases} u(R(a), a) = 0 \\ u'(R(a), a) < 0 \\ u(r, a) > 0 \quad \text{for } r \in [0, R(a)). \end{cases}$$

These three possibilities are drawn in Figure 2.

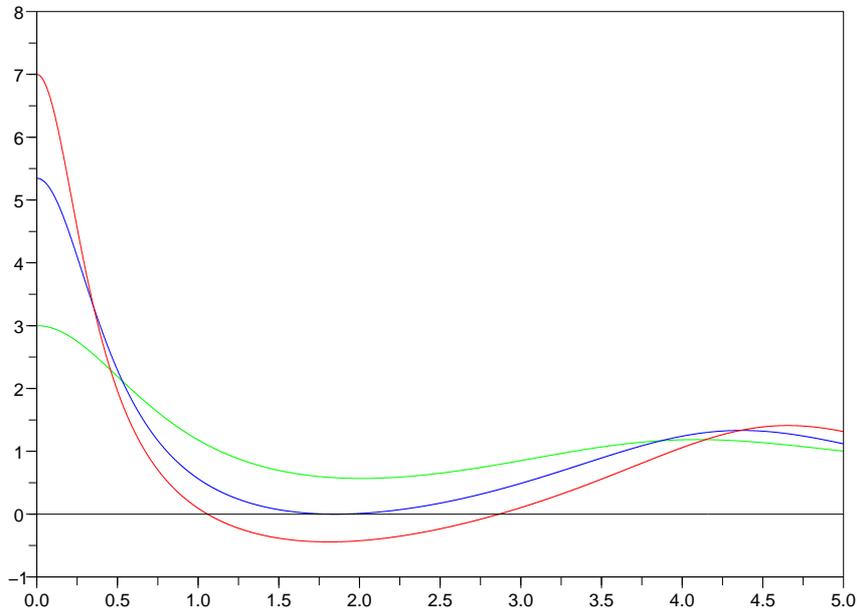


FIGURE 2. Behaviour of  $u(., a)$  for  $a > a_c$ ,  $a = a_c$  and  $a < a_c$ .

*Proof of Lemma 4.* For  $a > 0$ , we define

$$R(a) := \inf\{R > 0 : u(r, a) > 0 \quad \text{for } r \in [0, R)\}.$$

Notice that the positivity of  $a$  and the continuity of  $u(., a)$  guarantee that  $R(a) > 0$ . We consider the sets

$$\begin{aligned} \mathcal{P} &:= \{a > 0 : R(a) = \infty\}, \\ \mathcal{N} &:= \{a > 0 : R(a) < \infty \text{ and } u'(R(a), a) < 0\}, \\ \mathcal{N}_0 &:= \{a > 0 : R(a) < \infty \text{ and } u'(R(a), a) = 0\}. \end{aligned}$$

Clearly,  $\mathcal{P} \cup \mathcal{N} \cup \mathcal{N}_0 = (0, \infty)$  and  $1 \in \mathcal{P}$ . Actually, if  $a \in (0, (p+1)^{1/p})$ , then  $E(0, a) < 0$  and the monotonicity (18) of  $E$  entails that  $E(r, a) < 0$  for all  $r \geq 0$ . But, if  $R(a) < \infty$ , it readily follows from the definition (17) of the functional  $E$  that  $E(R(a), a) \geq 0$  whence a contradiction. Therefore,  $R(a) = \infty$  for any  $a \in (0, (p+1)^{1/p})$  so that

$$(19) \quad (0, (p+1)^{1/p}) \subset \mathcal{P}.$$

Consider now  $a \in \mathcal{N}_0$ . Then  $U(x) := u(|x|, a)$  is a radial positive solution to the homogeneous Dirichlet-Neumann free boundary problem  $\Delta U + U^p - 1 = 0$  in  $B(0, R(a))$  with  $U = \partial_\nu U = 0$  on  $\partial B(0, R(a))$ . According to [20, Theorem 3 (iii)], there is only one value of  $a$  for which this solution has a positive radial solution and it is unique. Consequently, there is a unique  $a_c > 0$  such that  $\mathcal{N}_0 = \{a_c\}$ .

Consider next  $a \in \mathcal{N} \cup \mathcal{N}_0$  and recall that  $a > 1$  by (19). Following [16, Lemma 11], let us assume for contradiction that there is  $\varrho \in (0, R(a))$  such that  $u'(\varrho, a) = 0$ . Either  $\varrho \leq 1$  and we infer from the definition, the monotonicity of  $E$ , see (17)-(18), and the definition of  $R(a)$  that  $0 > E(\varrho, a) \geq E(R(a), a) \geq 0$  which is a contradiction. Or  $\varrho > 1$  and the oscillating behaviour of the solutions implies, using the notation of Lemma 3, that  $\varrho \geq r_2(a)$ . This implies that  $r_1(a) < R(a)$ . Then  $u(r_1(a), a) \in (0, 1)$  and using again (17), (18), and the definition of  $R(a)$ , we conclude that  $0 > E(r_1(a), a) \geq E(R(a), a) \geq 0$ , hence a contradiction. Therefore,

$$(20) \quad u'(r, a) < 0 \quad \text{for } r \in (0, R(a)) \quad \text{if } a \in \mathcal{N} \cup \mathcal{N}_0.$$

Let us now prove that  $\mathcal{P}$  and  $\mathcal{N}$  are open subsets of  $(0, \infty)$ . We first consider  $a \in \mathcal{N}$ : by (20) there are  $\varrho > R(a)$  and  $\varepsilon > 0$  such that  $u(\varrho, a) < 0$  and  $u'(r, a) < -2\varepsilon$  for  $r \in (0, \varrho)$ . By continuous dependence, there is  $\delta \in (0, a)$  such that  $u(\varrho, b) < 0$  and  $u'(r, b) < -\varepsilon$  for  $r \in (0, \varrho)$  and  $b \in (a - \delta, a + \delta)$ . Since  $u(0, b) = b > 0$ , we readily deduce that, for each  $b \in (a - \delta, a + \delta)$ , we have  $R(b) \in (0, \varrho)$  with  $u'(R(b), b) < -\varepsilon < 0$ . Consequently,  $(a - \delta, a + \delta) \cap \mathcal{N}$  is open in  $(0, \infty)$ . Consider next  $a \in \mathcal{P}$ ,  $a > 1$ . By Lemma 3 and (17), we have  $u(r, a) \geq u(r_1(a), a) \in (0, 1)$  for  $r \in [0, r_1(a)]$  and  $E(r_1(a), a) < 0$ . By continuous dependence, there is  $\delta > a$  such that  $u(r, b) \geq u(r_1(a), a)/2 > 0$  for  $r \in [0, r_1(a)]$ ,  $u(r_1(a), b) \in (0, 1)$ , and  $E(r_1(a), b) < 0$  for  $b \in (a - \delta, a + \delta)$ . Assume now for contradiction that there is  $b \in (a - \delta, a + \delta)$  such that  $R(b) < \infty$ . Owing to (17), (18), and the definition of  $R(b)$ , we obtain  $0 > E(r_1(a), b) > E(R(b), b) \geq 0$  and a contradiction. Consequently,  $(a - \delta, a + \delta) \subset \mathcal{P}$  and  $\mathcal{P}$  is also open in  $(0, \infty)$ .

We finally argue as in [16, Lemma 15] to show that there is  $A > 0$  such that  $(A, \infty) \subset \mathcal{N}$ .

Since  $\mathcal{P}$  and  $\mathcal{N}$  are open subsets of  $(0, \infty)$ ,  $\mathcal{N}_0 = \{a_c\}$ ,  $(0, (p+1)^{1/p}) \subset \mathcal{P}$ , and  $(A, \infty) \subset \mathcal{N}$ , we readily conclude that  $\mathcal{P} = (0, a_c)$  and  $\mathcal{N} = (a_c, \infty)$ .  $\square$

We next study the properties of the map  $a \mapsto R(a)$ . An efficient tool for that purpose is the variation of  $u(\cdot, a)$  with respect to  $a$  defined by

$$\vartheta(r, a) := \frac{\partial u}{\partial a}(r, a), \quad (r, a) \in [0, \infty) \times (0, \infty),$$

which solves the second order linear differential equation

$$(21) \quad \begin{aligned} \vartheta''(r, a) + \frac{d-1}{r} \vartheta'(r, a) + p u(r, a)^{p-1} \vartheta(r, a) &= 0, \quad r \in [0, \infty), \\ \vartheta(0, a) &= 1, \quad \vartheta'(0, a) = 0, \end{aligned}$$

We argue as in [7, 25] to prove the following lemma.

**Lemma 5.** *If  $a > a_c$ , there is a unique  $z(a) \in (0, R(a))$  such that*

$$\begin{cases} \vartheta(r, a) > 0 & \text{for } r \in [0, z(a)), \\ \vartheta(z(a), a) = 0 \\ \vartheta(r, a) < 0 & \text{for } r \in (z(a), R(a)]. \end{cases}$$

*In addition,  $u(z(a), a) > 1$  and the ratio  $\vartheta(\cdot, a)/u(\cdot, a)$  is a decreasing function of  $r$  on  $(0, R(a))$ .*

*Proof of Lemma 5.* Since the proof follows rather closely that of [25] and [7, Lemma 2.1], we sketch it briefly for the sake of completeness. Fix  $a > a_c$  and set  $u = u(\cdot, a)$  and  $\vartheta = \vartheta(\cdot, a)$  to simplify notations. We first argue as in [16, Lemma 17] to show that  $\vartheta$  vanishes at least once in the interval  $(0, z_1(a))$ , where  $z_1(a)$  denotes the unique zero in  $(0, R(a))$  of  $u - 1$ . Indeed, (16) also reads

$$(u(r) - 1)'' + \frac{d-1}{r} (u(r) - 1)' + \frac{u(r)^p - 1}{u(r) - 1} (u(r) - 1) = 0, \quad r \in [0, \infty)$$

and  $(u(r)^p - 1)/(u(r) - 1) \leq p u(r)^{p-1}$  for  $r \in [0, z_1(a))$ . It then follows from Sturm's comparison theorem that  $\vartheta$  vanishes at least once in the interval  $(0, z_1(a))$ . Let  $z \in (0, z_1(a))$  denote the first zero of  $\vartheta$ .

We now aim at showing that  $\vartheta$  cannot vanish once more in the interval  $(z, R(a))$ . To this end, we define

$$\xi(r) := r^{d-1} [u'(r) \vartheta(r) - u(r) \vartheta'(r)] = -r^{d-1} u(r)^2 \left( \frac{\vartheta}{u} \right)'(r), \quad r \in [0, R(a)],$$

which encodes the monotonicity of  $\vartheta/u$ . It follows from (16) and (21) that

$$(22) \quad \xi'(r) = r^{d-1} ((p-1) u^p(r) + 1) \vartheta(r), \quad r \in [0, R(a)].$$

Clearly,  $\xi'(r) > 0$  for  $r \in (0, z)$  and  $\xi(0) = 0$ , so that  $\xi(r) > 0$  for  $r \in (0, z]$ . Assume now for contradiction that there is  $\varrho \in (z, R(a))$  such that

$$\xi(r) > 0 \text{ for } r \in (0, \varrho) \text{ and } \xi(\varrho) = 0.$$

Observing that  $\vartheta'(z) < 0$ , we realize that, if  $\vartheta(\varrho) \geq 0$ , there is  $\sigma \in (z, \varrho]$  such that  $\vartheta(r) < 0$  for  $r \in (z, \sigma)$  and  $\vartheta(\sigma) = 0$ . In that case,  $\vartheta'(\sigma) \geq 0$  and thus  $\xi(\sigma) = -\sigma^{d-1} u(\sigma) \vartheta'(\sigma) \leq 0$ , leading us to a contradiction. Consequently,

$$(23) \quad \vartheta(\varrho) < 0.$$

We next introduce the functions

$$\begin{aligned} T(r) &:= \frac{2(u(r)^p - 1)}{(p-1)u(r)^p + 1} \xi(r) - \zeta(r), \\ \zeta(r) &:= r^d [u'(r) \vartheta'(r) + (u(r)^p - 1) \vartheta(r)] + (d-2) r^{d-1} u'(r) \vartheta(r), \end{aligned}$$

for  $r \in [0, R(a))$  and use (16), (21), and (22) to obtain

$$\begin{aligned} \zeta'(r) &= 2 r^{d-1} (u(r)^p - 1) \vartheta(r), \\ (24) \quad T'(r) &= 2p^2 \frac{u(r)^{p-1}}{[(p-1)u(r)^p + 1]^2} u'(r) \xi(r), \end{aligned}$$

for  $r \in [0, R(a))$ . Integrating (24) over  $(0, \varrho)$  and using the negativity of  $u'$  and the positivity of  $\xi$  on this interval give

$$(25) \quad \zeta(\varrho) = -T(\varrho) > 0.$$

Since  $\xi(\varrho) = 0$ , we have  $u(\varrho) \vartheta'(\varrho) = u'(\varrho) \vartheta(\varrho)$  and we have

$$\zeta(\varrho) = Q(\varrho) \frac{\vartheta(\varrho)}{u(\varrho)},$$

where

$$Q(r) := r^d [u'(r)^2 + u(r)^{p+1} - u(r)] + (d-2) r^{d-1} u(r) u'(r), \quad r \in [0, R(a)).$$

It then follows from (23), (25), and the positivity of  $u$  that

$$(26) \quad Q(\varrho) < 0.$$

Finally, define

$$P(r) := r^d \left( u'(r)^2 + 2 \frac{u(r)^{p+1}}{p+1} - 2u(r) \right) + (d-2) r^{d-1} u(r) u'(r)$$

for  $r \in [0, R(a))$ . On the one hand, we notice that

$$(27) \quad P(r) = Q(r) - u(r) - \frac{p-1}{p+1} u(r)^{p+1} < Q(r), \quad r \in [0, R(a)).$$

On the other hand, we deduce from (16) and (18) that

$$P'(r) = r^{d-1} u(r) \left( \frac{d-2}{d-1} u(r)^p - (d+2) \right), \quad r \in [0, R(a)).$$

At this point, we realize that we have necessarily  $a > (d+2)(d-1)/(d-2)$  and that there is  $s \in (0, R(a))$  such that  $P'(r) > 0$  if  $r \in (0, s)$  and  $P'(r) < 0$  if  $r \in (s, R(a))$ . Since  $P(0) = 0$  and  $P(R(a)) > 0$ , we conclude that  $P(\varrho) > 0$  and then  $Q(\varrho) > 0$  by (27). But this contradicts (26). We have thus established that  $\xi$  is positive in  $(0, R(a))$  from which Lemma 5 follows.  $\square$

We are now in a position to state and prove some properties of the map  $a \mapsto R(a)$ .

**Proposition 6.** *The map  $a \mapsto R(a)$  is a decreasing function on  $(a_c, \infty)$  and there is  $z_1 > 0$  such that*

$$(28) \quad \lim_{a \searrow a_c} R(a) = R(a_c) \quad \text{and} \quad \lim_{a \rightarrow \infty} a^{(p-1)/2} R(a) = z_1.$$

The monotonicity of  $a \mapsto R(a)$  is shown in Figure 3. According to numerical simulations, the function  $a \mapsto a^{(p-1)/2} R(a)$  also seems to be a decreasing function of  $a \in [a_c, \infty)$ , see Figure 3.

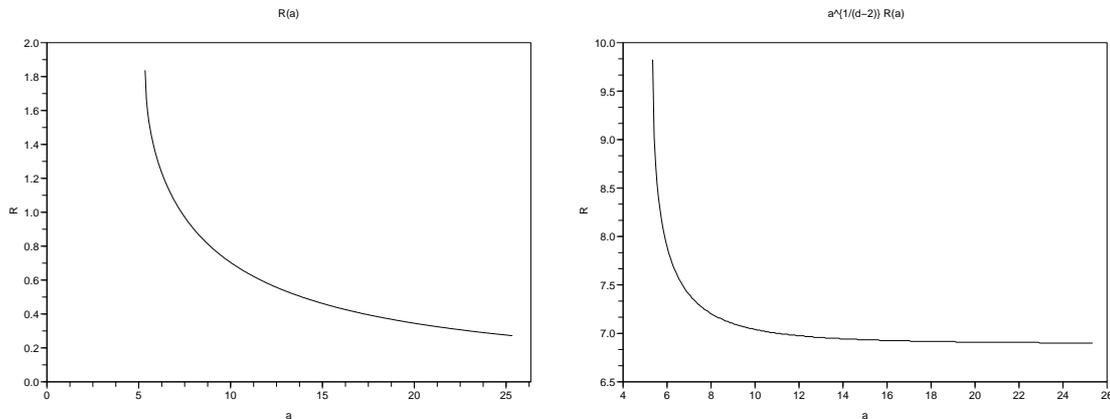


FIGURE 3. Monotonicity of the radius  $R$  and  $a \mapsto a^{(p-1)/2} R(a)$  ( $d = 3$ ).

*Proof of Proposition 6.* By Lemma 4,  $u'(R(a), a) < 0$  for all  $a \in (a_c, \infty)$  and the implicit function theorem warrants that  $R \in \mathcal{C}^1((a_c, \infty))$  with

$$\frac{dR}{da}(a) = -\frac{\vartheta(R(a), a)}{u'(R(a), a)}.$$

Since  $\vartheta(R(a), a) < 0$  by Lemma 5, the previous formula implies the strict monotonicity of  $a \mapsto R(a)$ . We next define

$$R_l := \sup_{a \in (a_c, \infty)} R(a) \in (0, \infty].$$

If  $R_l > R(a_c)$ , there is  $\varrho \in (R(a_c), R_l)$  such that  $u(\varrho, a_c) > 0$  by Lemmata 3 and 4. Then, there is  $\delta > 0$  such that  $R(a) > \varrho$  for  $a \in (a_c, a_c + \delta)$ . It then follows from the continuous dependence of  $u(\cdot, a)$  with respect to  $a$  and the monotonicity of  $u(\cdot, a)$  with respect to  $r$  that

$$0 = u(R(a_c), a_c) = \lim_{a \searrow a_c} u(R(a_c), a) \geq \lim_{a \searrow a_c} u(\varrho, a) = u(\varrho, a_c) > 0,$$

and a contradiction. Therefore,  $R_l \leq R(a_c)$  is finite and we have

$$u(R_l, a_c) = \lim_{a \searrow a_c} u(R(a), a) = 0,$$

from which we conclude that  $R_l = R(a_c)$ .

Finally, define

$$(29) \quad v(r, a) := \frac{1}{a} u\left(\frac{r}{a^{(p-1)/2}}, a\right), \quad (r, a) \in [0, \infty) \times (0, \infty).$$

Owing to (16),  $v(\cdot, a)$  solves

$$\begin{cases} v''(r, a) + \frac{d-1}{r} v'(r, a) + |v(r, a)|^{p-1} v(r, a) - a^{-p} = 0, & r \in [0, \infty), \\ v(0, a) = 1, \quad v'(0, a) = 0, \end{cases}$$

In addition,

$$(30) \quad v(r, a) > 0 \quad \text{for } r \in (0, a^{(p-1)/2} R(a))$$

for  $a > a_c$  by Lemma 4. Since  $a^{-p} \rightarrow 0$  as  $a \rightarrow \infty$ , we have

$$(31) \quad \lim_{a \rightarrow \infty} \sup_{r \in [0, \varrho]} |v(r, a) - w(r)| = 0 \quad \text{for all } \varrho > 0,$$

where  $w$  denotes the unique solution to

$$(32) \quad \begin{cases} w''(r) + \frac{d-1}{r} w'(r) + |w(r)|^{p-1} w(r) = 0, & r \in [0, \infty), \\ w(0) = 1, \quad w'(0) = 0. \end{cases}$$

By [8], there is  $z_1 > 0$  such that

$$(33) \quad w(r) > 0 \quad \text{and} \quad w'(r) < 0 \quad \text{for } r \in [0, z_1), \quad w(z_1) = 0, \quad w'(z_1) < 0.$$

Owing to (33), there is  $\delta > 0$  such that  $w(r) < 0$  for  $r \in (z_1, z_1 + \delta)$ . It then follows from (31) that, given  $r \in (z_1, z_1 + \delta)$ ,  $v(r, a) < 0$  for  $a$  large enough (depending on  $r$ ), whence  $a^{(p-1)/2} R(a) \leq r$  for  $a$  large enough by (30). Letting  $r \rightarrow z_1$  guarantees that

$$\limsup_{a \rightarrow \infty} a^{(p-1)/2} R(a) \leq z_1.$$

Next, if  $\varrho \in (0, \gamma)$ , we have  $w(r) > w(\varrho) > 0$  for  $r \in [0, \varrho]$  and we infer from (31) that  $v(r, a) > w(\varrho)/2 > 0$  for  $r \in [0, \varrho]$  and  $a$  large enough. Consequently,  $\varrho < a^{(p-1)/2} R(a)$  for  $a$  large enough, from which we conclude that

$$\liminf_{a \rightarrow \infty} a^{(p-1)/2} R(a) \geq z_1.$$

Combining the above two inequalities completes the proof of Proposition 6.  $\square$

The above information allow us to estimate from above and from below a specific integral of  $u(\cdot, a)$ .

**Proposition 7.** *For  $a \in [a_c, \infty)$ , we define*

$$\mathcal{M}(a) := d |B(0, 1)| \int_0^{R(a)} u(r, a)^p r^{d-1} dr.$$

*Recalling that  $w$  is the solution to (32) and  $z_1$  is its first positive zero, we have*

$$(34) \quad \lim_{a \rightarrow \infty} \mathcal{M}(a) = \mathcal{M}_c := d |B(0, 1)| \int_0^{z_1} w(r)^p r^{d-1} dr,$$

$$(35) \quad \mathcal{M}_2 := \sup_{a \in [a_c, \infty)} \mathcal{M}(a) < \infty.$$

*Proof of Proposition 7.* Let  $a \geq a_c$ . Since  $u(0, a) = a$ , it follows from the monotonicity of  $u(\cdot, a)$  that

$$\mathcal{M}(a) \leq d |B(0, 1)| \int_0^{R(a)} a^p r^{d-1} dr = |B(0, 1)| (a^{(p-1)/2} R(a))^d.$$

The upper bound (35) is then a straightforward consequence of (28) and the above inequality.

Next, recalling that  $v(\cdot, a)$  is defined by (29), we have

$$\mathcal{M}(a) = d |B(0, 1)| \int_0^{a^{(p-1)/2} R(a)} v(r, a)^p r^{d-1} dr,$$

and we infer from (28) and (31) that (34) holds true.  $\square$

#### 4. PROOF OF THEOREM 1

Thanks to the analysis done in the previous sections, we are now in a position to construct self-similar blowing-up solutions to (2) having either finite or infinite mass.

**Proposition 8.** *Given  $a > 0$  and  $T > 0$ , define*

$$\varphi(r) := \lambda_d^{d/(d-2)} u(\mu_d r, a)^{d/(d-2)} \quad \text{for } r \in [0, \infty) \quad \text{if } a \in (0, a_c),$$

and

$$\varphi(r) := \begin{cases} \lambda_d^{d/(d-2)} u(\mu_d r, a)^{d/(d-2)} & \text{for } r \in [0, R(a)/\mu_d] \\ 0 & \text{for } r \geq R(a)/\mu_d, \end{cases} \quad \text{if } a \in [a_c, \infty).$$

Define next  $\psi$ ,  $\Phi$ , and  $\Psi$  by (6) and (5), respectively. Then the functions  $(\rho, c)$  defined by (4) in  $(0, T) \times \mathbb{R}^d$  with  $s(t) = [d(T - t)]^{1/d}$  is a non-negative self-similar blowing-up solution to (2) with finite mass if  $a \geq a_c$  and infinite mass if  $a \in (0, a_c)$ .

The proof of Proposition 8 readily follows from the analysis performed in Sections 2 and 3. As for Theorem 1, it is a straightforward consequence of Proposition 8, the threshold values  $M_c$  and  $M_2$  being given by

$$M_c := d^{1/d} \left( \frac{2(d-1)}{d-2} \right)^{(d-1)/2} \mathcal{M}_c \quad \text{and} \quad M_2 := d^{1/d} \left( \frac{2(d-1)}{d-2} \right)^{(d-1)/2} \mathcal{M}_2.$$

#### 5. DISCUSSION

We have proved the existence of non-negative, integrable, and radially symmetric self-similar blowing-up solutions for (2). The profile  $\varphi$  of these self-similar solutions is compactly supported and non-increasing, and the mass of the corresponding self-similar solution ranges in the bounded interval  $(M_c, M_2]$ , the threshold mass  $M_c$  corresponding to the onset of blowup found in [2]. Our analysis thus reveals the existence of a second threshold value  $M_2 > M_c$  of the mass above which no radially symmetric and non-increasing self-similar blowing-up solution exist. The meaning of this second threshold value for the mass is yet unclear. It is worth mentioning at this point that a related situation was uncovered for the critical unstable thin-film equation

$$\partial_t u = -\partial_x (u^n \partial_x^3 u + u^{n+2} \partial_x u), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

in [21] for  $n \in (0, 3/2)$ . It is likely that, given  $M \in (M_c, M_2]$ , there is only a unique radially symmetric and non-increasing self-similar blowing-up solution with mass  $M$  and Figure 4 provides some numerical evidence of this fact. Besides this uniqueness question, the question of stability of these blowing-up solutions is also of interest.

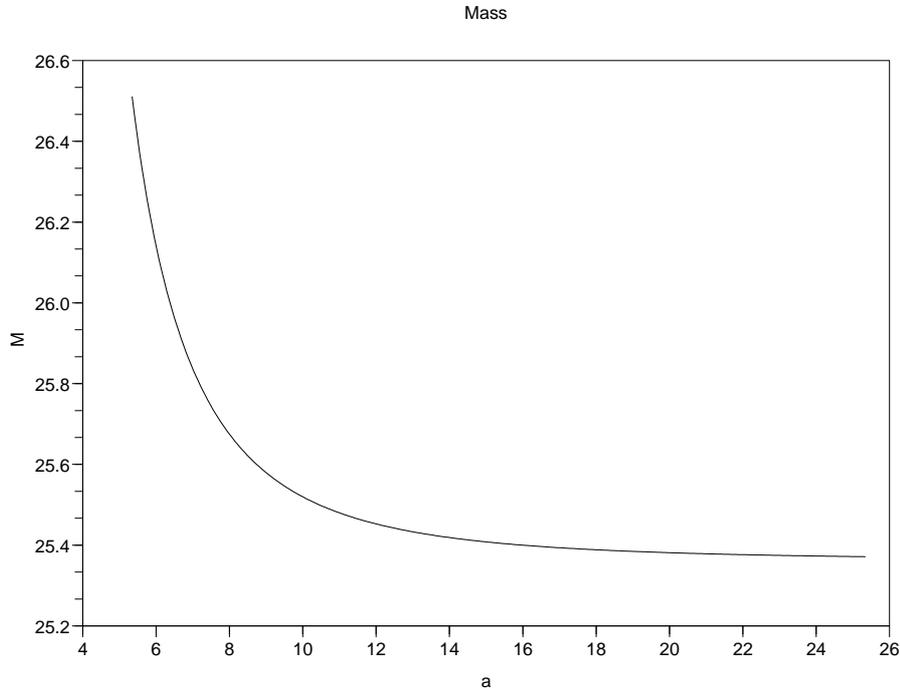


FIGURE 4. Monotonicity of the mass  $a \mapsto \mathcal{M}(a)$ .

Another challenging question is the existence (or non-existence) of integrable profiles  $\varphi$  with a non-connected positivity set as discussed in Section 2. Figure 5 provides numerical evidence that, if  $a > a_c$  is large enough,  $u(\cdot, a)$  may have several zeroes and each positive “hump” actually corresponds to a solution of (15) for suitable values of  $R_i$  and  $R_s$ . Whether the additional constraint (10) may be satisfied does not seem to be clear.

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#### REFERENCES

- [1] A. BLANCHET, J.A. CARRILLO, AND N. MASMOUDI, *Infinite time aggregation for the critical two-dimensional Patlak-Keller-Segel model*, Comm. Pure Appl. Math. **61** (2008), 1449–1481

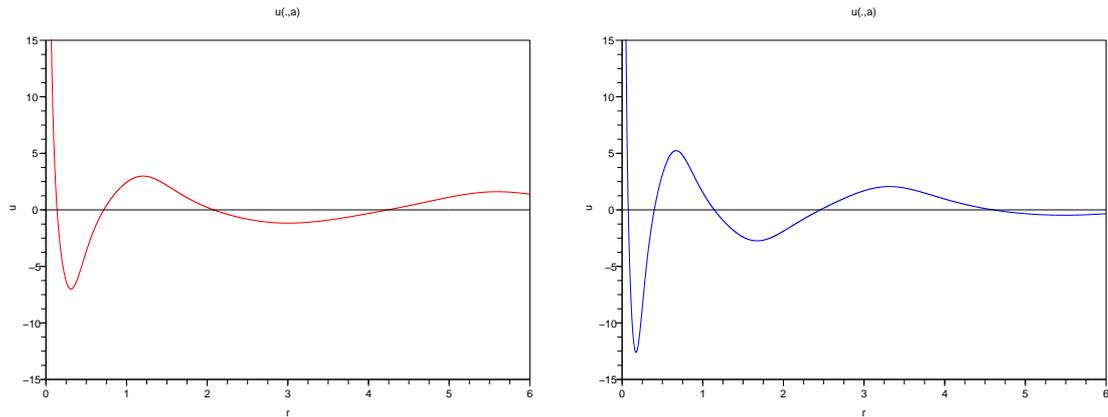


FIGURE 5. Positivity set of  $u(., a)$  with two ( $a = 50$ , left) and three ( $a = 90$ , right) connected components ( $d = 3$ ).

- [2] A. BLANCHET, J.A. CARRILLO, AND PH. LAURENÇOT, *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions*, Calc. Var. Partial Differential Equations **35** (2009), 133–168.
- [3] A. BLANCHET, J. DOLBEAULT, AND B. PERTHAME, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations **44** (2006), 32 pp. (electronic).
- [4] P.-H. CHAVANIS AND C. SIRE, *Anomalous diffusion and collapse of self-gravitating Langevin particles in  $D$  dimensions*, Phys. Rev. E **69** (2004), 016116.
- [5] P.-H. CHAVANIS AND C. SIRE, *Virial theorem and dynamical evolution of self-gravitating Brownian particles in an unbounded domain. I. Overdamped models*, Phys. Rev. E **73** (2006), 066103.
- [6] J. DOLBEAULT AND B. PERTHAME, *Optimal critical mass in the two-dimensional Keller-Segel model in  $\mathbb{R}^2$* , C. R. Math. Acad. Sci. Paris **339** (2004), 611–616.
- [7] P.L. FELMER, A. QUAAS, M. TANG, AND J. YU, *Monotonicity properties for ground states of the scalar field equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 105–119.
- [8] R.H. FOWLER, *Further studies of Emden's and similar differential equations*, Quart. J. Math. **2** (1931), 259–288.
- [9] M.A. HERRERO, E. MEDINA, AND J.J.L. VELÁZQUEZ, *Finite-time aggregation into a single point in a reaction-diffusion system*, Nonlinearity **10** (1997), 1739–1754.
- [10] M.A. HERRERO AND J.J.L. VELÁZQUEZ, *Singularity patterns in a chemotaxis model*, Math. Ann. **306** (1996), 583–623.
- [11] D. HORSTMANN, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein. **105** (2003), 103–165.
- [12] W. JÄGER AND S. LUCKHAUS, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), 819–824.
- [13] E.F. KELLER AND L.A. SEGEL, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399–415.
- [14] E.H. LIEB AND M. LOSS, *Analysis*, Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, second ed., 2001.
- [15] P.M. LUSHNIKOV, *Critical chemotactic collapse*, preprint, 2009.
- [16] M.K. KWONG, *Uniqueness of positive solutions of  $\Delta u - u + u^p = 0$  in  $R^N$* , Arch. Rational Mech. Anal. **105** (1989), 243–266.
- [17] T. NAGAI, *Behavior of solutions to a parabolic-elliptic system modelling chemotaxis*, J. Korean Math. Soc. **37** (2000), 721–733.

- [18] Y. NAITO AND T. SUZUKI, *Self-similarity in chemotaxis systems*, Colloq. Math. **111** (2008), 11–34.
- [19] C.S. PATLAK, *Random walk with persistence and external bias*, Bull. Math. Biophys. **15** (1953), 311–338.
- [20] J. SERRIN AND M. TANG, *Uniqueness of ground states for quasilinear elliptic equations*, Indiana Univ. Math. J. **49** (2000), 897–923.
- [21] D. SLEPČEV AND M.C. PUGH, *Selfsimilar blowup of unstable thin-film equations*, Indiana Univ. Math. J. **54** (2005), 1697–1738.
- [22] Y. SUGIYAMA, *Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems*, Differential Integral Equations **19** (2006), 841–876.
- [23] Y. SUGIYAMA, *Application of the best constant of the Sobolev inequality to degenerate Keller-Segel models*, Adv. Differential Equations **12** (2007), 121–144.
- [24] T. SUZUKI AND R. TAKAHASHI, *Degenerate parabolic equation with critical exponent derived from the kinetic theory, I, generation of the weak solution*, Adv. Differential Equations **14** (2009), 433–476.
- [25] M. TANG, *Uniqueness of positive radial solutions for  $\Delta u - u + u^p = 0$  on an annulus*, J. Differential Equations **189** (2003), 148–160.

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