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SEMI-LINEAR PARABOLIC EQUATIONS ON THE HEISENBERG GROUP WITH A SINGULAR POTENTIAL

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ABSTRACT. In this work, we discuss the asymptotic behavior of solutions for semi-linear parabolic equations on the Heisenberg group with a singular potential. The singularity is controlled by Hardy's inequality, and the nonlinearity is controlled by Sobolev's inequality. We also establish the existence of a global branch of the corresponding steady states via the classical Rabinowitz theorem.

Key words: Semi-linear parabolic equations, Heisenberg group, Hardy's inequality, Sobolev's inequality, Singular potential, Global bifurcation, Blow-up

A.M.S. Classification

1. INTRODUCTION

In this work, we study a class of parabolic equations on the Heisenberg group \mathbb{H}^d . Let us recall that the Heisenberg group is the space \mathbb{R}^{2d+1} with the (non commutative) law of product

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 2((y|x') - (y'|x))).$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + 2y_j \partial_s, \quad Y_j = \partial_{y_j} - 2x_j \partial_s, \quad j = 1, \dots, d \quad \text{and} \quad S = \partial_s = \frac{1}{4}[Y_j, X_j].$$

In the sequel, we shall denote $Z_j = X_j$ and $Z_{j+d} = Y_j$ for $j \in \{1, \dots, d\}$. We fix here some notations :

$$z = (x, y) \in \mathbb{R}^{2d}, \quad w = (z, s) \in \mathbb{H}^d, \quad \rho(z, s) = (|z|^4 + |s|^2)^{1/4}$$

where ρ is the Heisenberg distance. Moreover, the Laplacian-Kohn operator on \mathbb{H}^d and Heisenberg gradient are given by

$$\Delta_{\mathbb{H}^d} = \sum_{j=1}^n X_j^2 + Y_j^2; \quad \nabla_{\mathbb{H}^d} = (Z_1, \dots, Z_{2d}).$$

Let Ω be an open and bounded domain of \mathbb{H}^d , we define thus the associated Sobolev space by

$$H^1(\Omega, \mathbb{H}^d) = \left\{ f \in L^2(\Omega); \nabla_{\mathbb{H}^d} f \in L^2(\Omega) \right\}$$

and $H_0^1(\Omega, \mathbb{H}^d)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega, \mathbb{H}^d)$.

We are concerned in the following semi-linear parabolic problem

$$(1.1) \quad \begin{cases} \partial_t u - \Delta_{\mathbb{H}^d} u - \mu \frac{|z|^2}{\rho^4} u = \lambda u + |u|^{p-2} u, & w \in \Omega, t > 0, \\ u(0, w) = u_0(w), & w \in \Omega, \\ u|_{\partial\Omega} = 0, & t > 0, \end{cases}$$

where λ is a real constant and $2 < p < 2^*$; the index $2^* = 2 + \frac{2}{d}$ is the critical index of Sobolev's inequality on the Heisenberg group [6, 9, 10, 18]:

$$(1.2) \quad \|u\|_{L^{2^*}(\Omega)} \leq C_\Omega \|u\|_{H^1(\Omega, \mathbb{H}^d)},$$

for all $u \in H_0^1(\Omega, \mathbb{H}^d)$.

The following Hardy inequality is first proved in [11, 7]:

$$(1.3) \quad \bar{\mu} \int_{\Omega} \frac{|z|^2}{\rho(w)^4} |u(w)|^2 dw \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2$$

for all $u \in H_0^1(\Omega, \mathbb{H}^d)$. By the work of Kombe [19], we have the following improved Hardy inequality, for all $u \in C_0^\infty(\mathbb{H}^d \setminus \{0\})$:

$$(1.4) \quad \frac{1}{C^2 r^2(B)} \int_B u(w)^2 dw + \bar{\mu} \int_{\Omega} \frac{|z|^2}{\rho(w)^4} |u(w)|^2 dw \leq \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2,$$

where $\bar{\mu} = (\frac{Q-2}{2})^2$, C is a positive constant and $r(B)$ is the radius of the ball B . Moreover $\bar{\mu}$ is optimal and it is not attained in $H_0^1(\Omega, \mathbb{H}^d)$.

We recall the following compact embedding result:

Lemma 1.1. *Let $\Omega \in \mathbb{H}^d$ be a bounded open domain. Then $H_0^1(\Omega, \mathbb{H}^d)$ is compactly embedded in to $L^p(\Omega)$, $2 \leq p < 2^*$.*

In a remarkable paper, J. A. Goldstein and Q. S. Zhang [14] considered the following particular case

$$(1.5) \quad \begin{cases} \partial_t u - \Delta_{\mathbb{H}^d} u = \mu \frac{|z|^2}{\rho^4} u & t \in (0, T], T > 0, \\ u(w, 0) = u_0(w), & w \in \mathbb{H}^d. \end{cases}$$

They found that if $\mu > \bar{\mu}$, then the problem (1.5) has no negative solutions except $u_0 = 0$, and if $\mu \leq \bar{\mu}$, then the problem (1.5) has a positive solution for some $u_0 > 0$.

On the Eucliden space \mathbb{R}^d , problem (1.5) has been studied first by P. Barras and Goldstein [3] for the potential $V(x) = \frac{1}{|x|^2}$. Cabrel and Martel [5, Theorem 1, 2], extend this result to

some potential $V(x) = \frac{1}{\delta(x)^2}$, where $\delta(x) = \text{dist}(x, \partial\Omega)$, $\Omega \subset \mathbb{R}^d$ is of class C^2 . They show that the behavior of the solutions depends heavily on the critical value of the parameter μ which is the best constant of the classical Hardy inequality.

The work [3] generated a lot of activity on this topic and various questions have been investigated as, for example: general positive singular potentials, the asymptotic behavior of the solutions, semilinear equations, etc. See, for example, [15, 14, 27, 29].

Stimulated by the recent paper in the Eucliden space \mathbb{R}^d of Karachalios and Zographopoulos [20] which studied the global bifurcation of nontrivial equilibrium solutions

on the bounded domain case for a reaction term $f(s) = \lambda s - |s|^2 s$, where λ is a bifurcation parameter; our focus here is devoted to some results concerning the existence of a global attractor for the equation (1.1) and the existence of a global branch of the corresponding steady states

$$(1.6) \quad \begin{cases} -\Delta_{\mathbb{H}^d} u - \mu \frac{|z|^2}{\rho(w)^4} u = \lambda u + |u|^{p-2} u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

with respect λ . Let us recall some definitions on semiflows :

Definition 1.2. *Let E be a complete metric space, a semiflow is a family of continuous maps $\mathcal{S}(t) : E \rightarrow E$, $t \geq 0$, satisfying the semigroup identities*

$$\mathcal{S}(0) = I, \quad \mathcal{S}(t+t') = \mathcal{S}(t)\mathcal{S}(t').$$

For $\mathcal{B} \subset E$ and $t \geq 0$, let

$$\mathcal{S}(t)\mathcal{B} := \{u(t) = \mathcal{S}(t)u_0; u_0 \in \mathcal{B}\}.$$

The positive orbit of u through u_0 is the set

$$\gamma^+(u_0) = \{u(t) = \mathcal{S}(t)u_0, t \geq 0\},$$

and the positive orbit of \mathcal{B} is the set $\gamma^+(\mathcal{B}) = \cup_{t \geq 0} \mathcal{S}(t)\mathcal{B}$. The \mathcal{W} -limit set of u_0 is

$$\mathcal{W}(u_0) = \{\phi \in E : u(t_j) = \mathcal{S}(t_j)u_0 \rightarrow \phi, t_j \rightarrow +\infty\}.$$

The α -limit set of u_0 is

$$\alpha(u_0) = \{\phi \in E : u(t_j) \rightarrow \phi, t_j \rightarrow -\infty\}.$$

The subset \mathcal{A} attracts a set \mathcal{B} if $\text{dist}(\mathcal{S}(t)\mathcal{B}, \mathcal{A}) \rightarrow 0$, $t \rightarrow +\infty$.

\mathcal{A} is invariant if $\mathcal{S}(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$.

The functional $\mathcal{J} : E \rightarrow \mathbb{R}$ is a Lyapunov functional for the semiflow $\mathcal{S}(t)$ if

i) \mathcal{J} is continuous,

ii) $\mathcal{J}(\mathcal{S}(t)u_0) \leq \mathcal{J}(\mathcal{S}(t')u_0)$ for $0 \leq t' \leq t$.

iii) $\mathcal{J}(\mathcal{S}(t))$ is constant for some orbit u and for all $t \in \mathbb{R}$.

And we have the following theorem from the papers of Ball [1, 2] :

Theorem 1.3. *Let $\mathcal{S}(t)$ be an asymptotically compact semiflow and suppose that there exists a Lyapunov functional \mathcal{J} . Suppose further that the set \mathcal{E} is bounded. Then $\mathcal{S}(t)$ is dissipative, so there exists a global attractor $\mathcal{A}(t)$.*

For each complete orbit u containing u_0 lying in $\mathcal{A}(t)$, the limit sets $\alpha(u_0)$ and $\mathcal{W}(u_0)$ are connected subsets of \mathcal{E} on which \mathcal{J} is constant.

If \mathcal{E} is totally disconnected (in particular if it is countable), the limits

$$(1.7) \quad \phi_- = \lim_{t \rightarrow -\infty} u(t), \quad \phi_+ = \lim_{t \rightarrow +\infty} u(t)$$

exist and are equilibrium points. Furthermore, any solution $\mathcal{S}(t)u_0$ tends to an equilibrium point as $t \rightarrow \pm\infty$

The existence of a global branch of nonnegative solutions will be proved via the classical Rabinowitz theorem [25]:

Theorem 1.4. *Assume that X is a Banach space with norm $\|\cdot\|$ and let $G(\lambda, \cdot) = \lambda L + H(\lambda, \cdot)$, where L is a compact linear map on X and $H(\lambda, \cdot)$ is compact on X and satisfies*

$$(1.8) \quad \lim_{\|u\| \rightarrow 0} \frac{\|H(\lambda, u)\|}{\|u\|} = 0.$$

If λ is a simple eigenvalue of L , then the closure of the set

$$C = \{(\lambda, u) \in \mathbb{R} \times X : (\lambda, u) \text{ solves } u = G(\lambda, u), u \neq 0\},$$

possesses a maximal continuum (connected branch) of solutions C_λ , such that $(\lambda, 0) \in C_\lambda$ and C_λ either

- (i) meets infinity in $\mathbb{R} \times X$, or*
- (ii) meets $(\lambda^*, 0)$, where $\lambda^* \neq \lambda$ is also an eigenvalue of L .*

The outline of the paper is as follows : In Section 2, we study the existence of global branch of nonnegative solutions of (1.6) with respect to the parameter λ . In Section 3, we describe the asymptotic behavior of solutions of (1.1) when u_0 has low energy smaller than the mountain pass level.

2. EXISTENCE OF A GLOBAL BRANCH OF THE CORRESPONDING STEADY STATES

From the study of spectral decomposition of $H_0^1(\Omega, \mathbb{H}^d)$ with respect to the operator $-\Delta_{\mathbb{H}^d} - \mu \frac{|z|^2}{\rho(w)^4}$ where the singular potential V satisfies Hardy's inequality (1.3), we have:

Proposition 2.1. *Let $0 < \mu \leq \bar{\mu}$. Then there exist $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty$, such that for each $k \geq 1$, the following Dirichlet problem*

$$(2.9) \quad \begin{cases} -\Delta_{\mathbb{H}^d} \phi_k - \mu \frac{|z|^2}{\rho(w)^4} \phi_k = \lambda_k \phi_k, & \text{in } \Omega \\ \phi_k|_{\partial\Omega} = 0 \end{cases}$$

admits a nontrivial solution in $H_0^1(\Omega, \mathbb{H}^d)$. Moreover, $\{\phi_k\}_{k \geq 1}$ constitutes an orthonormal basis of Hilbert space $H_0^1(\Omega, \mathbb{H}^d)$.

Remark that the first eigenvalue $\lambda_{1,\mu}$ characterized by

$$(2.10) \quad \lambda_{1,\mu} = \inf_{u \in H_0^1(\Omega, \mathbb{H}^d) \setminus \{0\}} \frac{\int_{\Omega} (|\nabla_{\mathbb{H}^d} u|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u|^2) dw}{\|u\|_{L^2(\Omega)}^2},$$

is simple with a positive associated eigenfunction $\phi_{1,\mu}$.

For the proof of this proposition, we refer to [21].

We discuss the behavior of $\lambda_{1,\mu}$ when $0 < \mu < \bar{\mu}$ and $\mu \uparrow \bar{\mu}$:

Proposition 2.2. *Let $0 < \mu < \bar{\mu}$ and $\mu \uparrow \bar{\mu}$. Then,*

- (i) $(\lambda_{1,\mu})_\mu$ is a decreasing sequence, and there exist $\lambda_* > 0$ such that $\lambda_{1,\mu} \rightarrow \lambda_*$.*
- (ii) The corresponding normalized eigenfunction $\phi_{1,\mu}$ convergis weakly to 0 in $H_0^1(\Omega, \mathbb{H}^d)$.*

Proof:

- Let $\mu_1 < \mu_2$. The characterization (2.10) of $\lambda_{1,\mu}$ implies that $\lambda_{1,\mu_1} > \lambda_{1,\mu_2}$. The improved Hardy inequality (1.4) implies that $\lambda_{1,\mu}$ is bounded from below by $\frac{1}{C^2 r^2(B)}$. So, there exist $\lambda_* > 0$ such that $\lambda_{1,\mu} \rightarrow \lambda_*$.

- The eigenfunction $\phi_{1,\mu}$ satisfies, for any $v \in C_0^\infty(\Omega)$:

$$(2.11) \quad \int_{\Omega} \nabla_{\mathbb{H}^d} \phi_{1,\mu} \overline{\nabla_{\mathbb{H}^d} v} \, dw - \mu \int_{\Omega} \frac{|z|^2}{\rho(w)^4} \phi_{1,\mu} \bar{v} \, dw = \lambda_{1,\mu} \int_{\Omega} \phi_{1,\mu} \bar{v} \, dw.$$

We still denote by $\phi_{1,\mu}$ the sequence of normalized eigenfunction, forming a bounded sequence in $H_0^1(\Omega, \mathbb{H}^d)$. Then there exists $u \in H_0^1(\Omega, \mathbb{H}^d)$ such that

$$\begin{aligned} \phi_{1,\mu} &\rightharpoonup u \text{ in } H_0^1(\Omega, \mathbb{H}^d), \\ \phi_{1,\mu} &\rightarrow u \text{ in } L^q(\Omega), \text{ for any } 2 \leq q < 2^*. \end{aligned}$$

For some fixed small enough $\varepsilon > 0$ and any for $v \in C_0^\infty(\Omega)$, we have

$$\int_{\Omega} \frac{|z|^2}{\rho(w)^4} (\phi_{1,\mu} - u) \bar{v} \, dw \leq \|v\|_{L^\infty(\Omega)} \left(\int_{\Omega} |\phi_{1,\mu} - u|^{\frac{Q-\varepsilon}{Q-2-\varepsilon}} \, dw \right)^{\frac{Q-2-\varepsilon}{Q-\varepsilon}} \left(\int_{\Omega} \left(\frac{|z|}{\rho(w)^2} \right)^{Q-\varepsilon} \, dw \right)^{\frac{2}{Q-\varepsilon}}.$$

Thus,

$$\int_{\Omega} \frac{|z|^2}{\rho(w)^4} \phi_{1,\mu} \bar{v} \, dw \rightarrow \int_{\Omega} \frac{|z|^2}{\rho(w)^4} u \bar{v} \, dw, \text{ as } \mu \uparrow \bar{\mu}.$$

We assume that $u \neq 0$, so passing to the limit in (2.11), we get that u is a nontrivial solution of the problem

$$-\Delta_{\mathbb{H}^d} u - \bar{\mu} \frac{|z|^2}{\rho(w)^4} u = \bar{\mu} u, \quad u \in H_0^1(\Omega, \mathbb{H}^d).$$

However, $\bar{\mu}$ is not attained in $H_0^1(\Omega, \mathbb{H}^d)$, so $u = 0$.

Thanks to Hardy inequality (1.3) and Poincaré inequality,

$$(2.12) \quad \|u\|_{\mu} = \left(\int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u(z, s)|^2 - \mu \frac{|z|^2}{\rho(z, s)^4} |u(z, s)|^2 \right] dz ds \right)^{\frac{1}{2}}$$

is equivalent to the norm on $H_0^1(\Omega, \mathbb{H}^d)$ for all $0 \leq \mu < \bar{\mu}$, so that we will use $\|\cdot\|_{\mu}$ as the norm of $H_0^1(\Omega, \mathbb{H}^d)$.

Theorem 2.3. *Let $\Omega \in \mathbb{H}^d$ be a bounded domain and assume that $0 < \mu < \bar{\mu}$. Then, the principal eigenvalue $\lambda_{1,\mu}$ considered in $H_0^1(\Omega, \mathbb{H}^d)$ with the norm $\|\cdot\|_{\mu}$, is a bifurcating point of the problem (1.6) and $C_{\lambda_{1,\mu}}$ is a global branch of nonnegative solutions of (1.6).*

Proof: First we prove the existence of $C_{\lambda_{1,\mu}}$:

We define the space X as a completion of $C_0^\infty(\Omega)$ with respect to the norm induced by

$$(2.13) \quad \langle u, v \rangle_X \equiv \int_{\Omega} \left[\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu \frac{|z|^2}{\rho(z, s)^4} u \bar{v} \right] dz ds - \frac{\lambda_{1,\mu}}{2} \int_{\Omega} u \bar{v} \, dz ds.$$

We have

$$\|u\|_X = \|u\|_{\mu}^2 - \frac{\lambda_{1,\mu}}{2} \|u\|_{L^2(\Omega)}^2 \leq \|u\|_{\mu}^2,$$

and from the characterization of $\lambda_{1,\mu}$, we have

$$\|u\|_X \geq \|u\|_{\mu}^2 - \frac{\lambda_{1,\mu}}{2} \|u\|_{L^2(\Omega)}^2 \geq \|u\|_{\mu}^2 - \frac{1}{2} \|u\|_{\mu}^2 \geq \frac{1}{2} \|u\|_{\mu}^2.$$

Since $C^\infty(\Omega)$ is dense both in X and $H_0^1(\Omega, \mathbb{H}^d)$, it follows that $X = H_0^1(\Omega, \mathbb{H}^d)$, and the inner product in X is given by $\langle u, v \rangle_X = \langle u, v \rangle_{\mu}$.

Let

$$a(u, v) = \int_{\Omega} uv \, dz ds, \text{ for all } u, v \in X.$$

The bilinear form $a(u, v)$ is continuous in X , so the Riesz representation theorem implies that there exists a bounded linear operator L such that

$$(2.14) \quad a(u, v) = \langle Lu, v \rangle, \text{ for all } u, v \in X.$$

The operator L is self adjoint and compact and its largest eigenvalue ν_1 is characterized by

$$(2.15) \quad \nu_1 = \sup_{u \in X} \frac{\langle Lu, u \rangle}{\langle u, u \rangle_X} = \sup_{u \in X} \frac{\|u\|_{L^2(\Omega)}}{\int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u|^2 - \mu \frac{|z|^2}{\rho(z, s)^4} |u|^2 \right] dz ds} = \frac{1}{\lambda_{1, \mu}}.$$

We define energy functional $I_{\mu, \lambda}$ on $H_0^1(\Omega, \mathbb{H}^d)$ by

$$(2.16) \quad I_{\mu, \lambda}(u) = \frac{1}{2} \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u|^2 - \mu \frac{|z|^2}{\rho(z, s)^4} |u|^2 \right] dz ds - \frac{1}{p} \int_{\Omega} |u|^p dz ds - \frac{\lambda}{2} \int_{\Omega} |u|^2 dz ds.$$

Similarly to the classical case, $I_{\mu, \lambda}(\cdot)$ is well-defined on $H_0^1(\Omega, \mathbb{H}^d)$ and belongs to $C^1(H_0^1(\Omega, \mathbb{H}^d); \mathbb{R})$ and we have

$$\langle I'_{\mu, \lambda}(u), v \rangle = \int_{\Omega} \left[\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu \frac{|z|^2}{\rho(z, s)^4} u \bar{v} - |u|^{p-2} u \bar{v} - \lambda u \bar{v} \right] dz ds$$

for any $v \in H_0^1(\Omega, \mathbb{H}^d)$. Let $N(\lambda, \cdot) : \mathbb{R} \times X \rightarrow X^*$ where X^* is the dual space of X be defined as by

$$(2.17) \quad \langle N(\lambda, u), v \rangle = \int_{\Omega} \left[\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu \frac{|z|^2}{\rho(z, s)^4} u \bar{v} - |u|^{p-2} u \bar{v} - \lambda u \bar{v} \right] dz ds$$

for all $v \in X$. Since $I'_{\mu, \lambda}(u)$ is a bounded linear functional, $N(\lambda, \cdot)$ is well defined, and $N(\lambda, \cdot) = u - G(\lambda, u)$ where $G(\lambda, u) = \lambda Lu + H(u)$,

$$(2.18) \quad \langle H(u), v \rangle = \int_{\Omega} |u|^{p-2} u \bar{v} dz ds \forall v \in X.$$

Thanks to the compact embedding (1.1), the map H is compact. On the other hand, we have

$$|\langle H(u), v \rangle| \leq \|u\|_{L^p(\Omega)}^{p-1} \|v\|_{L^p(\Omega)},$$

Since $X = H_0^1(\Omega, \mathbb{H}^d)$ and thanks to the compact embedding (1.1), we have

$$(2.19) \quad \frac{1}{\|u\|_X} |\langle H(u), v \rangle| \leq \|u\|_X^{p-2} \|v\|_X.$$

Thus

$$(2.20) \quad \lim_{\|u\|_X \rightarrow 0} \frac{\|H(u)\|_{X^*}}{\|u\|_X} = \lim_{\|u\|_X \rightarrow 0} \sup_{\|v\|_X \leq 1} \frac{1}{\|u\|_X} |\langle H(u), v \rangle| = 0.$$

It remains to prove that $C_{\lambda_{1, \mu}}$ is a global branch for nonnegative solutions of (1.6) : First, we prove that there exist $\varepsilon_0 > 0$ such that $u > 0$ for any $(\lambda, u) \in C_{\lambda_{1, \mu}} \cap B_{\varepsilon_0}(\lambda_{1, \mu}, 0)$ where $B_{\varepsilon_0}(\lambda_{1, \mu}, 0)$ is the open ball of $C_{\lambda_{1, \mu}}$ with center $(\lambda_{1, \mu}, 0)$ and radius ε_0 . By contradiction, we assume that there exists $(\lambda_n, u_n) \in C_{\lambda_{1, \mu}}$ a sequence of solutions of (1.6), such that $\lambda_n \rightarrow \lambda_{1, \mu}$, $u_n \rightarrow 0$ in $H_0^1(\Omega, \mathbb{H}^d)$ and that $(u_n)_n$ are changing sign in Ω .

Let $u_n^- \equiv \min\{0, u_n\}$ and $\mathcal{U}_n^- \equiv \{x \in \Omega : u_n(x) < 0\}$. Since $u_n = u_n^+ - u_n^-$ is a weak solution of (1.6), u_n^- satisfies

$$(2.21) \quad \begin{cases} -\Delta_{\mathbb{H}^d} u_n^- - \mu \frac{|z|^2}{\rho^4} u_n^- = \lambda u_n^- + |u_n|^{p-2} u_n^- & \text{in } \Omega, \\ u_n^- \Big|_{\partial\Omega} = 0. \end{cases}$$

We thus have

$$(2.22) \quad \int_{\mathcal{U}_n^-} \left[|\nabla_{\mathbb{H}^d} u_n^-|^2 - \mu \frac{|z|^2}{\rho(z,s)^4} |u_n^-|^2 \right] dz ds = \lambda_n \int_{\mathcal{U}_n^-} |u_n^-|^2 dz ds + \int_{\mathcal{U}_n^-} |u_n|^{p-2} |u_n^-|^2 dz ds.$$

But λ_n is bounded, so we get by Hölder inequality, Sobolev inequality and Sobolev embedding

$$(2.23) \quad \|u_n^-\|_{H_0^1(\mathcal{U}_n^-)}^2 \leq \lambda_n |\mathcal{U}_n^-|^{\frac{2}{Q}} \left(\int_{\mathcal{U}_n^-} |u_n^-|^{2^*} dz ds \right)^{\frac{2}{2^*}} + \|u_n^-\|_{L^p(\mathcal{U}_n^-)}^p$$

$$(2.24) \quad \leq C_1 |\mathcal{U}_n^-|^{\frac{2}{Q}} \|u_n^-\|_{H_0^1(\mathcal{U}_n^-)}^2 + C_2 \|u_n^-\|_{H_0^1(\mathcal{U}_n^-)}^p,$$

thus

$$(2.25) \quad 1 \leq C_1 |\mathcal{U}_n^-|^{\frac{2}{Q}} + C_2 \|u_n^-\|_{H_0^1(\mathcal{U}_n^-)}^{p-2}.$$

Since $\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)} \rightarrow 0$ and $p > 2$, we derive that

$$(2.26) \quad |\mathcal{U}_n^-| \geq C_3, \forall n,$$

where the constant $C_3 > 0$ depends neither on λ_n nor u_n .

Next we denote by $v_n = \frac{u_n}{\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)}}$, then there exists a subsequence of v_n , which we denote again by v_n , such that

$$\begin{aligned} v_n &\rightharpoonup v_0 \text{ in } H_0^1(\Omega, \mathbb{H}^d), \\ v_n &\rightarrow v_0 \text{ in } L^2(\Omega). \end{aligned}$$

Since $u_n = G(\lambda_n, u_n) = \lambda_n L u_n + H(u_n)$,

$$v_n = \lambda_n L v_n + \frac{H(u_n)}{\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)}}.$$

As L is a compact linear operator and $H(u_n) = 0(\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)})$, so $v_0 = \lambda_{1,\mu} L v_0$ and then $v_0 = \phi_{1,\mu} > 0$. Hence, by applying Egorov's Theorem [4, Theorem IV.28] or [17], v_n converges uniformly to $\phi_{1,\mu}$ in the exterior of a set of arbitrarily small measure. Then, there exists Σ a piece of Ω of arbitrarily small measure in which v_n is positive outside Σ for n large enough, obtaining a contradiction with (2.26) and we conclude that the functions u_n are nonnegative, for n large enough. It then follows that $u > 0$ for any $(\lambda, u) \in C_{\lambda_1, \mu} \cap B_{\varepsilon_0}(\lambda_{1,\mu}, 0)$ with $\varepsilon_0 > 0$ small enough. Assume now that there exists $(\lambda, u) \in C_{\lambda_1, \mu}$ such that $u(w_0) \leq 0$ at some point $w_0 \in \Omega$. From the previous part, we have $u(w) > 0$ for all $w \in \Omega$ whenever $(\lambda, u) \in C_{\lambda_1, \mu}$ is close to $(\lambda_{1,\mu}, 0)$. Since $C_{\lambda_1, \mu}$ is connected, there exists $(\lambda^*, u^*) \in C_{\lambda_1, \mu}$, such that $u^*(w) \geq 0$ for all $w \in \Omega$, except possibly some point $w_0 \in \Omega$ where $u^*(w_0) = 0$, and in any neighbourhood of (λ^*, u^*) , we can find a point $(\bar{\lambda}, \bar{u}) \in C_{\lambda_1, \mu}$ with $\bar{u}(w) < 0$ for some $w \in \Omega$. Then, the maximum principle implies that $u^* = 0$ on Ω . Thus we can construct a sequence $(\lambda_n, u_n) \in C_{\lambda_1, \mu}$ such that $u_n > 0$

for all n , $u_n \rightarrow 0$ in $H_0^1(\Omega, \mathbb{H}^d)$ and $\lambda_n \rightarrow \lambda^*$.

Let $v_n = \frac{u_n}{\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)}}$, then

$$v_n = \lambda_n L v_n + \frac{H(u_n)}{\|u_n\|_{H_0^1(\Omega, \mathbb{H}^d)}}.$$

So, the subsequence $(v_n)_n$ converges to $v_0 = \lambda^* L v_0$. Since $v_n > 0$, for all n and $\|v_0\|_{H_0^1(\Omega, \mathbb{H}^d)} = 1$, we have $v_0 > 0$. Thus λ^* is an eigenvalue of (1.6) corresponding to a positive eigenfunction. But $\lambda_{1,\mu}$ is the only positive eigenvalue of (1.6) corresponding to a positive eigenfunction, so we deduce that $\lambda^* = \lambda_{1,\mu}$, and that $(\lambda^*, u^*) = (\lambda_{1,\mu}, 0)$. This contradicts the fact that every neighbourhood of (λ^*, u^*) must contain a point $(\bar{\lambda}, \bar{u}) \in C_{\lambda_{1,\mu}}$ with $\bar{u}(w) < 0$ for some $w \in \Omega$. Hence $u(w) > 0$ for all $w \in \Omega$ whenever $(\lambda, u) \in C_{\lambda_{1,\mu}}$, and $C_{\lambda_{1,\mu}}$ cannot cross points of the form $(\lambda, 0)$, where $\lambda \neq \lambda_{1,\mu}$.

3. ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR PROBLEM (1.1)

Similarly [22, 23], we are interested here in the description of the behavior of solutions of (1.1) when u_0 has low energy smaller than the mountain pass level

$$c_{\mu,\lambda} = \inf_{h \in \Gamma} \max_{t \in [0,1]} I_{\mu,\lambda}(h(t)), \text{ where}$$

$$(3.27) \quad \Gamma = \{ h \in C([0, 1]; H_0^1(\Omega, \mathbb{H}^d)); h(0) = 0 \text{ and } h(1) = e \}.$$

In view of [21], since $2 < p < 2^*$, the functional $I_{\mu,\lambda}$ satisfies the Palais-Smale condition and admits at least a positive solution (called *mountain pass solution*).

Lemma 3.1. *For $\lambda > 0$, $0 < \mu < \bar{\mu}$ and $2 < p < 2^*$, the function $f(t) = \lambda t + |t|^{p-2} t$, $t \in \mathbb{R}$ defines a locally Lipschitz map $f : H_0^1(\Omega, \mathbb{H}^d) \rightarrow H^{-1}(\Omega, \mathbb{H}^d)$.*

Proof: The function $f_1(u) = \lambda u$, defines a locally Lipschitz map $f_1 : L^2(\Omega) \rightarrow L^2(\Omega)$, so $f_1 : H_0^1(\Omega, \mathbb{H}^d) \rightarrow H^{-1}(\Omega, \mathbb{H}^d)$ is locally Lipschitz. Let $u \in L^p(\Omega)$ and $f_2(u) = |u|^{p-2} u$. The function $f_2 : L^p(\Omega) \rightarrow L^{p'}(\Omega)$ is locally Lipschitz, thanks to the following estimate :

$$(3.28) \quad \|f_2(u) - f_2(v)\|_{L^{p'}(\Omega)} \leq (p-1) \left(\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \right)^p \|u - v\|_{L^p(\Omega)},$$

for all $u, v \in L^p(\Omega)$. So thanks to compact embedding (1.1) and from $L^{p'}(\Omega) \subset H^{-1}(\Omega; \mathbb{H}^d)$, the function $f_2 : H_0^1(\Omega, \mathbb{H}^d) \rightarrow H^{-1}(\Omega, \mathbb{H}^d)$ is locally Lipschitz.

Proposition 3.2. *Let $u_0 \in H_0^1(\Omega, \mathbb{H}^d)$, $\lambda > 0$ and $0 < \mu < \bar{\mu}$, the problem (1.1) has a unique weak solution u such that*

$$u \in C([0, T]; H_0^1(\Omega, \mathbb{H}^d)) \cap C^1([0, T]; H^{-1}(\Omega, \mathbb{H}^d)),$$

and we have

$$(3.29) \quad \frac{d}{dt} I_{\mu,\lambda}(u(t)) = - \| \partial_t u \|_{L^2(\Omega)}^2.$$

Proof: By means of the Hille-Yosida theorem, $\mathcal{T}(t) = \{e^{-tL_\mu}\}_{t \geq 0}$ is the semigroup generated by the operator $L_\mu = -\Delta_{\mathbb{H}^d} - \mu \frac{|z|^2}{\rho(z, s)^4}$. Since $f : H_0^1(\Omega, \mathbb{H}^d) \rightarrow H^{-1}(\Omega)$ is locally Lipschitz, so by Pazy [24, Theorem 1.4] or Haraux [16, Theorem 6.2.2] or Goldstein [12, Theorem 2.4]; there exists a unique solution of (1.1) defined on a maximal interval $[0, T_{max})$, where $0 < T_{max} \leq +\infty$ and

$$u \in C([0, T]; H_0^1(\Omega, \mathbb{H}^d)) \cap C^1([0, T]; H^{-1}(\Omega)),$$

satisfying the variation of constants formula

$$(3.30) \quad u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-\tau) f(u(\tau)) d\tau.$$

Moreover, if $T_{max} < +\infty$, we say that T_{max} is a blow-up time, whereas if $T_{max} = +\infty$, we say that u is global solution.

We will show that u satisfies (3.29) : Let $u \in D(L_\mu)$ ($D(L_\mu)$ be the domain of definition of L_μ) and $t \in [0, T], T < T_{max}$. Since $I_{\mu,\lambda} \in C^1(H_0^1(\Omega, \mathbb{H}^d); \mathbb{R})$, we have

$$(3.31) \quad \begin{aligned} \langle I'_{\mu,\lambda}(u), \Delta_{\mathbb{H}^d} u + \mu \frac{|z|^2}{\rho(w)^4} u + f(u) \rangle &= - \int_{\Omega} | \Delta_{\mathbb{H}^d} u + \mu \frac{|z|^2}{\rho(w)^4} u + f(u) |^2 dw \\ &= - \int_{\Omega} | \partial_t u |^2 dw. \end{aligned}$$

Set $g(t) = f(u(t))$ and let $g_n \in C^1([0, T]; H_0^1(\Omega, \mathbb{H}^d))$, $u_{0n} \in D(L_\mu)$ such that

$$\begin{aligned} g_n &\rightarrow g \text{ in } C^1([0, T]; H_0^1(\Omega, \mathbb{H}^d)), \\ u_{0n} &\rightarrow u_0 \text{ in } H_0^1(\Omega, \mathbb{H}^d). \end{aligned}$$

Define $u_n(t) = \mathcal{T}(t)u_{0n} + \int_0^t \mathcal{T}(t-\tau) g_n(\tau) d\tau$, then $u_n \in C^1([0, T]; H_0^1(\Omega, \mathbb{H}^d))$ satisfies

$$\partial_t u_n - \Delta_{\mathbb{H}^d} u_n - \mu V u_n = g_n$$

and

$$u_n \rightarrow u \text{ in } H_0^1(\Omega, \mathbb{H}^d).$$

Thus, from (3.31),

$$\begin{aligned} I_{\mu,\lambda}(u_n(t)) - I_{\mu,\lambda}(u_{0n}) &= \int_0^t \langle I'_{\mu,\lambda}(u_n(\tau)), \Delta_{\mathbb{H}^d} u_n + \mu \frac{|z|^2}{\rho(w)^4} u_n + g_n(\tau) \rangle d\tau \\ &= - \int_0^t \|\partial_\tau u_n(\tau)\|_{L^2(\Omega)}^2 d\tau + \int_0^t \langle I'_{\mu,\lambda}(u_n(\tau)), g_n(\tau) - f(u_n(\tau)) \rangle d\tau. \end{aligned}$$

Passing to the limit, we deduce (3.29).

Next, we introduce the following sets :

$$(3.32) \quad \begin{aligned} \mathcal{O}^+ &\equiv \{u \in H_0^1(\Omega, \mathbb{H}^d) : I_{\mu,\lambda}(u) < c_{\mu,\lambda}; \langle I'_{\mu,\lambda}(u), u \rangle > 0\}, \\ \mathcal{O}^- &\equiv \{u \in H_0^1(\Omega, \mathbb{H}^d) : I_{\mu,\lambda}(u) < c_{\mu,\lambda}; \langle I'_{\mu,\lambda}(u), u \rangle < 0\}, \\ \mathcal{N} &\equiv \{u \in H_0^1(\Omega, \mathbb{H}^d) : \langle I'_{\mu,\lambda}(u), u \rangle = 0\}. \end{aligned}$$

\mathcal{N} is named the *Nehari manifold* relative to $I_{\mu,\lambda}$. The mountain-pass level $c_{\mu,\lambda}$ defined in (3.27) may also be characterized as

$$(3.33) \quad c_{\mu,\lambda} = \inf_{u \in \mathcal{N}} I_{\mu,\lambda}(u).$$

Theorem 3.3. *If there exist $t_0 \geq 0$ such that $I_{\mu,\lambda}(u(t_0)) \leq 0$, then $u(t)$ blows-up in finite time.*

Proof: Let $t_0 \geq 0$ such that $I_{\mu,\lambda}(u(t_0)) \leq 0$ and we suppose that $u(t)$ is a global solution to the problem (1.1). Since $u(t)$ satisfies (3.29), we have

$$I_{\mu,\lambda}(u(t_0)) = I_{\mu,\lambda}(u(t)) + \int_{t_0}^t \|\partial_\tau u(\tau)\|_{L^2(\Omega)}^2 d\tau.$$

Set $g(t) \equiv \int_{\Omega} |u(t)|^2 dw$, then

$$\begin{aligned}
\frac{d}{dt}g(t) &= 2 \int_{\Omega} u(t) \partial_t u(t) dw \\
&= -2 \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u(t)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u(t)|^2 \right] dw \\
&\quad + 2\lambda \int_{\Omega} |u(t)|^2 dw + 2 \int_{\Omega} |u(t)|^p dw \\
&= 4 \int_{t_0}^t \|\partial_{\tau} u(\tau)\|_{L^2(\Omega)}^2 d\tau - 4I_{\mu,\lambda}(u(t_0)) + 2\left(1 - \frac{2}{p}\right) \int_{\Omega} |u(t)|^p dw \\
(3.34) \quad &\geq 2\left(1 - \frac{2}{p}\right) \int_{\Omega} |u(t)|^p dw > 0.
\end{aligned}$$

Hence we get for any $t \geq t_0$, $g(t) \geq g(t_0) = \int_{\Omega} |u(t_0)|^2 dw$. Let $\epsilon \in (1, \frac{p}{2})$, so we deduce by (3.34), that for any $t \geq t_0$:

$$\begin{aligned}
-\frac{1}{\epsilon-1} \frac{d}{dt} g^{1-\epsilon}(t) &= g^{-\epsilon}(t) \frac{d}{dt} g(t) \\
&\geq 2\left(1 - \frac{2}{p}\right) g^{-\epsilon}(t) \int_{\Omega} |u(t)|^p dw \\
&\geq C g^{-\epsilon}(t) \left(\int_{\Omega} |u(t)|^2 dw \right)^{\frac{p}{2}} \\
&\geq C \left(\int_{\Omega} |u(t_0)|^2 dw \right)^{\frac{p}{2}-\epsilon}.
\end{aligned}$$

Hence for $t \geq t_0$ sufficiently large, we have

$$\begin{aligned}
0 &< \left(\int_{\Omega} |u(t)|^2 dw \right)^{1-\epsilon} = g^{1-\epsilon}(t) \\
&\leq g^{1-\epsilon}(t_0) + C(\epsilon-1) g^{\frac{p}{2}-\epsilon}(t_0) (t_0 - t).
\end{aligned}$$

Then

$$-1 < C(\epsilon-1) g^{\frac{p}{2}-1}(t_0) (t_0 - t)$$

and so $t < t_0 + \left[C(\epsilon-1) g^{\frac{p}{2}-1}(t_0) \right]^{-1}$, which is a contradiction.

Theorem 3.4. *Assume that $u_0 \in \mathcal{O}^+$ and $\lambda < \lambda_{1,\mu}$, then the problem (1.1) admits a global solution $u(t)$. Moreover, there exists a positive number α such that*

$$(3.35) \quad \|u(t)\|_{\mu} = O(e^{-\alpha t}), \text{ as } t \rightarrow +\infty.$$

Proof: Let $u_0 \in \mathcal{O}^+$ and $u(t) = u(t, w)$ be the unique solution, the existence of which has been proved in Proposition 3.2. From (3.29), we have that $t \mapsto I_{\mu,\lambda}(u(t))$ is strictly decreasing, so

$$(3.36) \quad I_{\mu,\lambda}(u(t)) \leq I_{\mu,\lambda}(u_0) \leq c_{\mu,\lambda}.$$

Suppose there exists $t^* \in (0, T_{max})$ such that $u(t^*) \notin \mathcal{O}^+$. Then

$$\langle I'_{\mu,\lambda}(u(t^*)), u(t^*) \rangle \leq 0.$$

Moreover since $t \mapsto \langle I'_{\mu,\lambda}(u(t)), u(t) \rangle$ is continuous, there exists $t_0 \in (0, t^*]$ such that

$$\langle I'_{\mu,\lambda}(u(t_0)), u(t_0) \rangle = 0.$$

Hence $u(t_0) = 0$ in Ω , or $u(t_0) \in \mathcal{N}$. If $u(t_0) = 0$ in Ω , then by the uniqueness of $u(t)$, we conclude that $u(t) = 0$ for any $t \in [t_0, T_{max})$. Thus $u(t)$ is global by extending to 0 for all $t \geq T_{max}$, and so $I_{\mu,\lambda}(u(t)) > 0$ for any $t \geq 0$ by Theorem 3.3. But $I_{\mu,\lambda}(u(t_0)) = 0$, which is a contradiction, and so $u(t_0) \in \mathcal{N}$. It is well known that $c_{\mu,\lambda} = \inf_{u \in \mathcal{N}} I_{\mu,\lambda}(u)$ [28, Theorem 4.2], thus $c_{\mu,\lambda} \leq I_{\mu,\lambda}(u(t_0))$, which is a contradiction with (3.36). So, we conclude that $u(t) \in \mathcal{O}^+$ for all $t \in [t_0, T_{max})$.

On other hand, we can write

$$\begin{aligned}
(3.37) \quad I_{\mu,\lambda}(u(t)) &= \frac{1}{p} \langle I'_{\mu,\lambda}(u(t)), u(t) \rangle \\
&+ \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u(t, w)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u(t, w)|^2 \right] dw - \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \int_{\Omega} |u(t, w)|^2 dw \\
&> \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u(t, w)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u(t, w)|^2 \right] dw - \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \int_{\Omega} |u(t, w)|^2 dw \\
&\geq \left(\frac{1}{2} - \frac{1}{p}\right) \left(1 - \frac{\lambda}{\lambda_{1,\mu}}\right) \|u(t, \cdot)\|_{\mu}^2 > 0.
\end{aligned}$$

Since $u(t)$ satisfies (3.29), we have

$$(3.38) \quad \int_{t_0}^t \|\partial_t u(\tau, \cdot)\|_{L^2(\Omega)}^2 d\tau + \left(\frac{1}{2} - \frac{1}{p}\right) \left(1 - \frac{\lambda}{\lambda_{1,\mu}}\right) \|u(t, \cdot)\|_{\mu}^2 \leq I_{\mu,\lambda}(u(t_0, \cdot)) < c_{\mu,\lambda}.$$

Then we have

$$(3.39) \quad \int_{t_0}^t \|\partial_t u(\tau)\|_{L^2(\Omega)}^2 d\tau < c_{\mu,\lambda}, \text{ and } \|u(t)\|_{\mu}^2 < \left[\left(\frac{1}{2} - \frac{1}{p}\right) \left(1 - \frac{\lambda}{\lambda_{1,\mu}}\right)\right]^{-1} c_{\mu,\lambda},$$

which implies that $u(t)$ is a global solution of the problem (1.1) and \mathcal{O}^+ is invariant set.

Letting $t \rightarrow +\infty$ in (3.39), the integral $\int_{t_0}^{+\infty} \|\partial_t u(\tau)\|_{L^2(\Omega)}^2 d\tau$ is finitely determined.

Therefore, there exists a sequence $(t_n)_{n \geq 0}$ with $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$, such that

$$(3.40) \quad \int_{\Omega} |\partial_t u(t_n)|^2 dw \rightarrow 0, \text{ and } u(t_n) \rightharpoonup v \text{ in } H_0^1(\Omega, \mathbb{H}^d).$$

Letting $t_n \rightarrow +\infty$, we obtain that $v \in H_0^1(\Omega, \mathbb{H}^d)$ is a solution of problem (1.6). So

$$(3.41) \quad \langle I'_{\mu,\lambda}(v), v \rangle = 0.$$

If $v \neq 0$, then $v \in \mathcal{N}$, and so

$$(3.42) \quad I_{\mu,\lambda}(v) \geq c_{\mu,\lambda}.$$

Since $u(t_n)$ satisfies (3.29), it follows by Hölder inequality and from (3.40), that

$$\begin{aligned}
(3.43) \quad |\langle I'_{\mu,\lambda}(u(t_n, \cdot)), u(t_n, \cdot) \rangle| &\leq \left| \int_{\Omega} u(t_n, w) \partial_t u(t_n, w) dw \right| \\
&\leq \|u(t_n, \cdot)\|_{L^2(\Omega)} \|\partial_t u(t_n, \cdot)\|_{L^2(\Omega)} \\
&\leq \sqrt{\lambda_{1,\mu}} \|u(t_n, \cdot)\|_{\mu} \|\partial_t u(t_n, \cdot)\|_{L^2(\Omega)} \\
&\leq C \|\partial_t u(t_n, \cdot)\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore,

$$(3.44) \quad \lim_{n \rightarrow +\infty} \langle I'_{\mu,\lambda}(u(t_n)), u(t_n) \rangle = 0.$$

We deduce by (3.37), (3.41) and (3.44) that

$$\begin{aligned}
I_{\mu,\lambda}(v) &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} v(w)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |v(w)|^2 \right] dw - \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \int_{\Omega} |v(w)|^2 dw \\
&\leq \lim_{n \rightarrow +\infty} \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u(t_n, w)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u(t_n, w)|^2 \right] dw \\
&\quad - \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \int_{\Omega} |u(t_n, w)|^2 dw + \lim_{n \rightarrow +\infty} \langle I'_{\mu,\lambda}(u(t_n)), u(t_n) \rangle \\
&\leq \lim_{n \rightarrow +\infty} I_{\mu,\lambda}(u(t_n)) \\
&\leq I_{\mu,\lambda}(u_0) < c_{\mu,\lambda},
\end{aligned}$$

which contradicts (3.42), and so $v = 0$ in Ω .

Hence by (3.40), we have

$$u(t_n, \cdot) \rightarrow 0 \text{ in } L^q(\Omega), \quad 2 \leq q < 2^*.$$

Since

$$\begin{aligned}
\|u(t_n, \cdot)\|_{\mu}^2 &= \langle I'_{\mu,\lambda}(u(t_n)), u(t_n) \rangle + \lambda \int_{\Omega} |u(t_n, w)|^2 dw + \int_{\Omega} |u(t_n, w)|^p dw \\
&\rightarrow 0, \text{ as } n \rightarrow +\infty,
\end{aligned}$$

we have

$$(3.45) \quad u(t_n, \cdot) \rightarrow 0 \text{ in } H_0^1(\Omega, \mathbb{H}^d), \text{ as } n \rightarrow +\infty.$$

For simplicity, let us denote by t the divergent sequence and by $u(t) = u(t_n, w)$. We have from (3.44) that

$$\begin{aligned}
I_{\mu,\lambda}(u(t)) &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u(t)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u(t)|^2 \right] dw - \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \int_{\Omega} |u(t)|^2 dw \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \|u(t)\|_{\mu}^2 - \left(\frac{1}{2} - \frac{1}{p}\right) \|u(t)\|_{L^2(\Omega)}^2.
\end{aligned}$$

So, due to (3.29) we have

$$\begin{aligned}
\|u(t)\|_{\mu}^2 &= \frac{2p}{p-2} I_{\mu,\lambda}(u(t)) + \|u(t)\|_{L^2(\Omega)}^2 \\
&\leq \frac{2p}{p-2} I_{\mu,\lambda}(u_0) + \|u(t)\|_{L^2(\Omega)}^2 \\
(3.46) \quad &< \frac{2p}{p-2} c_{\mu,\lambda} + o(1).
\end{aligned}$$

Therefore there exists t_0 such that for all $t \geq t_0$,

$$(3.47) \quad \|u(t)\|_{\mu}^2 \leq \frac{2p}{p-2} c_{\mu,\lambda}.$$

On the other hand,

$$\begin{aligned}
\int_{\Omega} |u(t)|^p dw &\leq C_{\Omega}^p \left(\frac{\bar{\mu}}{\bar{\mu} - \mu} \right)^{\frac{p}{2}} \|u(t)\|_{\mu}^p \\
&\leq C_{\Omega}^p \left(\frac{\bar{\mu}}{\bar{\mu} - \mu} \right)^{\frac{p}{2}} \left[\frac{2p}{p-2} c_{\mu,\lambda} \right]^{\frac{p-2}{2}} \|u(t)\|_{\mu}^2.
\end{aligned}$$

Let $C_1 = C_\Omega^p \left(\frac{\bar{\mu}}{\bar{\mu} - \mu} \right)^{\frac{p}{2}} \left[\frac{2p}{p-2} c_{\mu,\lambda} \right]^{\frac{p-2}{2}}$, we have

$$(3.48) \quad \begin{aligned} (1 - C_1) \|u(t)\|_\mu^2 &\leq \|u(t)\|_\mu^2 - \|u(t)\|_{L^p(\Omega)}^p \\ &\leq \langle I'_{\mu,\lambda}(u(t)), u(t) \rangle + \lambda \|u(t)\|_{L^2(\Omega)}^2. \end{aligned}$$

Let us recall that if we set $g(t) \equiv \int_\Omega |u(t)|^2 dw$, then

$$\begin{aligned} \frac{d}{dt} g(t) &= 2 \int_\Omega u(t) \partial_t u(t) dw \\ &= -2 \int_\Omega \left[|\nabla_{\mathbb{H}^d} u(t)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u(t)|^2 \right] dw \\ &\quad + 2\lambda \int_\Omega |u(t)|^2 dw + 2 \int_\Omega |u(t)|^p dw \\ &= -2 \langle I'_{\mu,\lambda}(u(t)), u(t) \rangle. \end{aligned}$$

So we get from (3.37) that for any $t \geq t_0$, we have

$$(3.49) \quad \begin{aligned} \int_t^\infty \langle I'_{\mu,\lambda}(u(\tau)), u(\tau) \rangle d\tau &= \frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2 \leq \frac{1}{2\lambda_{1,\mu}} \|u(t)\|_\mu^2 \\ &\leq \frac{1}{\lambda_{1,\mu}} \frac{p}{p-2} I_{\mu,\lambda}(u(t)). \end{aligned}$$

So from (3.48) and (3.49), we have for any $t \geq t_0$ that

$$(3.50) \quad \begin{aligned} \int_t^\infty I_{\mu,\lambda}(u(\tau)) d\tau &\leq \frac{1}{2\lambda_{1,\mu}} \int_t^\infty \|u(\tau)\|_\mu^2 \\ &\leq (1 - C_1)^{-1} \frac{1}{2\lambda_{1,\mu}} \left[\int_t^\infty \langle I'_{\mu,\lambda}(u(\tau)), u(\tau) \rangle d\tau + \lambda \int_t^\infty \|u(\tau)\|_{L^2(\Omega)}^2 d\tau \right] \\ &\leq \frac{(1 - C_1)^{-1}}{2\lambda_{1,\mu}^2} \frac{p}{p-2} I_{\mu,\lambda}(u(t)) + (1 - C_1)^{-1} \frac{\lambda}{2\lambda_{1,\mu}} \int_t^\infty \|u(\tau)\|_{L^2(\Omega)}^2 d\tau. \end{aligned}$$

Since $\lim_{t \rightarrow +\infty} \|u(t)\|_{L^2(\Omega)} = 0$, there exists $t_1 > t_0$ such that for any $t \geq t_1$, we have

$$(3.51) \quad \int_t^\infty I_{\mu,\lambda}(u(\tau)) d\tau \leq \frac{(1 - C_1)^{-1}}{2\lambda_{1,\mu}^2} \frac{p}{p-2} I_{\mu,\lambda}(u(t)).$$

Thus

$$(3.52) \quad \int_t^\infty I_{\mu,\lambda}(u(\tau)) d\tau \leq C(t_1) e^{-\alpha t},$$

with $\alpha = \frac{(1 - C_1)^{-1}}{2\lambda_{1,\mu}^2} \frac{p}{p-2}$. But we remark that

$$(3.53) \quad I_{\mu,\lambda}(u(t+1)) \leq \int_t^{t+1} I_{\mu,\lambda}(u(\tau)) d\tau < \int_t^\infty I_{\mu,\lambda}(u(\tau)) d\tau,$$

hence we deduce that for any $t \geq t_1$, we have

$$(3.54) \quad I_{\mu,\lambda}(u(t+1)) < C(t_1) e^{-\alpha t},$$

and we can conclude that for any $t \geq t_1$, we have

$$(3.55) \quad \|u(t)\|_\mu = O(e^{-\alpha t}).$$

Theorem 3.5. *Assume that $u_0 \in \mathcal{O}^-$. Then the solution $u(t)$ of the problem (1.1) blows up in finite time.*

Proof: Let $u_0 \in \mathcal{O}^-$ and $u(t) = u(w, t, u_0)$ be the unique solution, the existence of which has been proved in Proposition 3.2. From the inequality (3.29), we have that $t \mapsto I_{\mu,\lambda}(u(t))$ is strictly decreasing, so

$$(3.56) \quad I_{\mu,\lambda}(u(t)) \leq I_{\mu,\lambda}(u_0) \leq c_{\mu,\lambda}.$$

Suppose there exists $\tilde{t} \in (0, T_{max})$ such that $u(\tilde{t}) \notin \mathcal{O}^-$. Then

$$\langle I'_{\mu,\lambda}(u(\tilde{t})), u(\tilde{t}) \rangle \geq 0.$$

And since the application $t \mapsto \langle I'_{\mu,\lambda}(u(t)), u(t) \rangle$ is continuous, there exists $\tilde{t}_0 \in (0, \tilde{t}]$ such that

$$\langle I'_{\mu,\lambda}(u(\tilde{t}_0)), u(\tilde{t}_0) \rangle = 0.$$

Hence $u(\tilde{t}_0) = 0$ in Ω , or $u(\tilde{t}_0) \in \mathcal{N}$. If $u(\tilde{t}_0) = 0$ in Ω , then by the uniqueness of $u(t)$, we conclude that $u(t) = 0$ for any $t \in [\tilde{t}_0, T_{max})$. Thus $u(t)$ is global by extending to 0 for all $t \geq T_{max}$, and thanks to Theorem 3.3, $I_{\mu,\lambda}(u(t)) > 0$ for any $t \geq 0$. But $I_{\mu,\lambda}(u(\tilde{t}_0)) = 0$, which is a contradiction, and so $u(\tilde{t}_0) \in \mathcal{N}$. But by [28],

$$c_{\mu,\lambda} = \inf_{u \in \mathcal{N}} I_{\mu,\lambda}(u),$$

then $c_{\mu,\lambda} \leq I_{\mu,\lambda}(u(\tilde{t}_0))$, which contradicts (3.56). So, we conclude that $u(t) \in \mathcal{O}^-$ for all $t \in [\tilde{t}_0, T_{max})$. We suppose by contradiction that $T_{max} = +\infty$, i.e. $u(t) = u(t, \cdot)$ exists for all $t \geq 0$. For $u \in \mathcal{O}^-$, we have

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2(\Omega)}^2 = -2 \langle I'_{\mu,\lambda}(u(t)), u(t) \rangle > 0.$$

Then $t \mapsto \|u(t, \cdot)\|_{L^2(\Omega)}$ is strictly increasing and so

$$(3.57) \quad \lim_{t \rightarrow +\infty} \|u(t, \cdot)\|_{L^2(\Omega)} = c \in (0, +\infty].$$

We suppose that $c < +\infty$. Following the same reasoning as in the proof of Theorem 3.4, we deduce that we can select a divergent subsequence, still denoted by t , such that when $t \rightarrow +\infty$,

$$u(t, \cdot) \rightarrow 0 \quad \text{in } H_0^1(\Omega, \mathbb{H}^d).$$

Letting $t \rightarrow +\infty$ in the inequality

$$\sqrt{\lambda_{1,\mu}} \|u(t, \cdot)\|_{L^2(\Omega)} \leq \|u(t, \cdot)\|_{\mu},$$

we get that $0 < c \leq 0$, which is a contradiction. So we conclude that

$$(3.58) \quad \lim_{t \rightarrow +\infty} \|u(t, \cdot)\|_{L^2(\Omega)} = +\infty.$$

Set $g(t) = \|u(t, \cdot)\|_{L^2(\Omega)}^2$, so

$$(3.59) \quad \begin{aligned} -\frac{2}{p-2} \frac{d}{dt} g^{1-\frac{p}{2}}(t) &= g'(t) g^{-\frac{p}{2}}(t) \\ &= -2 \|u(t, \cdot)\|_{L^2(\Omega)}^{-p} \left(\|u(t, \cdot)\|_{\mu}^2 - \lambda \|u(t, \cdot)\|_{L^2(\Omega)}^2 \|u(t, \cdot)\|_{L^p(\Omega)}^2 \right) \\ &\geq -2 \|u(t, \cdot)\|_{L^2(\Omega)}^{-p} \|u(t, \cdot)\|_{\mu}^2 + 2 \|u(t, \cdot)\|_{L^2(\Omega)}^{-p} \|u(t, \cdot)\|_{L^p(\Omega)}^2. \end{aligned}$$

By Hölder inequality, we have

$$\|u(t, \cdot)\|_{L^p(\Omega)}^p \geq |\Omega|^{1-\frac{p}{2}} \|u(t, \cdot)\|_{L^2(\Omega)}^p,$$

and by (3.58), there exist $t_1 > 0$ and a constant $C_1 > 0$, such that for $t \geq t_1$, we have

$$(3.60) \quad \|u(t, \cdot)\|_{L^2(\Omega)} \geq C_1.$$

Then, there exist $t_1 > 0$ and a constant $C_2 > 0$, such that for $t \geq t_1$, we have

$$(3.61) \quad -\frac{2}{p-2} \frac{d}{dt} g^{1-\frac{p}{2}}(t) \geq -2\lambda_{1,\mu} C_1^{2-p} + 2|\Omega|^{1-\frac{p}{2}} \geq C_2.$$

Hence, we have from (3.61), that for any $t \geq t_1$,

$$0 < g(t) \leq g(t_1) + \frac{p-2}{2} C_2 (t - t_1),$$

which is a contradiction if t is sufficiently large. So we conclude that $T_{max} < +\infty$.

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