Far field modeling of electromagnetic time reversal and application to selective focusing on small scatterers

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FAR FIELD MODELLING OF ELECTROMAGNETIC TIME-REVERSAL AND APPLICATION TO SELECTIVE FOCUSING ON SMALL SCATTERERS.

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Abstract. A time harmonic far field model for closed electromagnetic time reversal mirrors is proposed. Then, a limit model corresponding to small perfectly conducting scatterers is derived. This asymptotic model is used to prove the selective focusing properties of the time reversal operator. In particular, a mathematical justification of the DORT method (Decomposition of the Time Reversal Operator method) is given for axially symmetric scatterers.

Key words. electromagnetic scattering, time reversal, far field, small obstacles

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1. Introduction. In the last decade, acoustic time reversal has definitely demonstrated its efficiency in target characterization by wave focusing in complex media (see the review papers [13, 15]). In particular, it has been shown that selective focusing can be achieved using the eigenvectors (resp. eigenfunctions) of the so-called time reversal matrix (resp. operator). Known as the DORT method (french acronym for Diagonalization of the Time Reversal Operator, cf. [14, 32, 26, 31, 16, 25, 18]), this technique involves three steps. First, an incident wave is emitted in the medium containing the scatterers by the time reversal mirror (TRM). The scattered field is then measured by the mirror and time-reversed (or phase-conjugated in the time harmonic case). Finally, the obtained signal is then reemitted in the medium. By definition, the time reversal operator $\mathbf{T}$ is the operator describing two successive cycles Emission/Reception/Time-Reversal. If the propagation medium is non dissipative, the operator $\mathbf{T}$ is hermitian, since $\mathbf{T} = \mathbf{F}^*\mathbf{F}$, where $\mathbf{F}$ denotes the far field operator. The DORT method can thus be seen as a singular value decomposition of $\mathbf{F}$. Moreover, in a particular range of frequencies (for which the scatterers can be considered as point-like scatterers), $\mathbf{T}$ has as many significant eigenvalues as there are scatterers in the medium, and the corresponding eigenfunctions generate incident waves that selectively focus on the scatterers. From the mathematical point of view, a detailed analysis of this problem has been proposed for the acoustic scattering problem by small scatterers in the free space in [19] and in a two-dimensional straight waveguide in [29]. Let us emphasize that time reversal has also been intensively studied in the context of random media (cf. [17] and the references therein).

Recently, electromagnetic focusing using time reversal has been demonstrated experimentally [23] and used for imaging applications [24]. One of the first works dealing with mathematical and numerical aspects of electromagnetic time reversal is the paper [34]. The authors analyze therein the DORT method in the case of a homogeneous medium containing perfectly conducting or dielectric objects of particular shapes (circular rods and spheres). Their method is based on a low frequency approximation of a multipole expansion of the scattered field (*i.e.* a Fourier-Bessel series involving Hankel functions for circular rods and vector spherical functions for spheres). In [8], the authors proposed an iterative process based on time reversal to determine optimal

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electromagnetic measurements (*i.e.* to determine the incident waves maximizing the scattered field). More recently, the DORT method has been used for targets localization, especially in the context of imaging [6, 7, 1]. The analysis followed in these works is based on the singular value decomposition of the multistatic response matrix, which corresponds to the case where the mirror is described by a discrete array of transducers (emitters and receivers). In this paper, we propose a time harmonic far field model of electromagnetic time reversal in the case of a continuous distribution of transducers. Only *closed mirrors* (i.e. completely surrounding the scatterers) are considered in this work and the limited aperture case is not studied. Except this difference, the present work can be seen as the extension of the results obtained for acoustic time reversal in the free space [19] and in straight waveguides [29]. We pay a very careful attention to the derivation of the limit scattering model for small perfectly conducting scatterers. The functional framework used hereafter for the far field and the time reversal operators is the one commonly used in inverse electromagnetic scattering theory (cf. [11, 5, 20]).

We start the paper with a short description in Section 2 of the mathematical model of time reversal. In particular, we define the incident field emitted by the TRM (electromagnetic Herglotz waves), the measured fields (the far field pattern) and the time reversal operator. In Section 3, we restrict our analysis to the case of small scatterers (of typical size δ). We show that the small scatterers asymptotics can be deduced from the classical low frequency scattering asymptotics (the Rayleigh approximation) involving the polarization tensors of the scatterers. More precisely, our analysis corresponds to the case where $k\delta$ and δ/d tend simultaneously to 0, where k denotes the wavenumber and d the minimum separation distance between the scatterers. Finally, we study in Section 4 the spectral focusing properties of the eigenfunctions of the limit far field operator obtained in Section 3. We show that each small scatterer gives rise to at most 6 distinct eigenvalues (recovering the results obtained in [7, 1] for the case of a discrete TRM). Furthermore, if the polarizability tensors of the scatterers are diagonal (e.g. for axially symmetric scatterers) and under the additional assumption that $kd \to \infty$, we prove that each associated eigenfunction generates an incident wave that selectively focuses on the corresponding scatterer.

2. A far field model for electromagnetic time-reversal. In order to obtain an expression of the time reversal operator, we begin this paper by recalling the far field model of electromagnetic scattering. Consider the scattering problem of an incident electromagnetic plane wave by a perfectly conducting bounded obstacle contained in an homogeneous medium. Without loss of generality, we assume that the electric permittivity ε and the magnetic permeability μ are both equal to 1. Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^3$ with smooth boundary Γ, and outward unit normal $\boldsymbol{\nu}$ and let $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$ be the propagation domain. Let $L^2_t(S^2)$ be the space of tangential vector fields of the unit sphere S^2:

$$L^2_t(S^2) = \left\{ \boldsymbol{f} \in \left(L^2(S^2)\right)^3 \mid \forall \boldsymbol{\alpha} \in S^2,\ \boldsymbol{f}(\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha} = 0 \right\}$$

and consider the incident plane wave $(\boldsymbol{E}_I^{\boldsymbol{\alpha},\boldsymbol{f}}, \boldsymbol{H}_I^{\boldsymbol{\alpha},\boldsymbol{f}})$ of direction $\boldsymbol{\alpha} \in S^2$ and electric polarization $\boldsymbol{f} \in L^2_t(S^2)$:

$$\left\{ \begin{array}{l} \boldsymbol{E}_I^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{\alpha})\, \mathrm{e}^{ik\boldsymbol{\alpha}\cdot\boldsymbol{x}}, \\ \boldsymbol{H}_I^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x}) = (\boldsymbol{\alpha} \times \boldsymbol{f}(\boldsymbol{\alpha}))\, \mathrm{e}^{ik\boldsymbol{\alpha}\cdot\boldsymbol{x}}, \end{array} \right. \tag{2.1}$$

Throughout the paper, the time dependence is assumed to be of the form $e^{-i\omega t}$ and will always be implicit. Introducing the wave number $k = \omega\sqrt{\varepsilon\mu} = \omega$, the scattered field $(\boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}}, \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}})$ solves the following exterior boundary value problem:

$$\begin{cases} \mathbf{curl}\, \boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}} = ik\, \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}} & (\Omega) \\ \mathbf{curl}\, \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}} = -ik\, \boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}} & (\Omega) \\ \boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}} \times \boldsymbol{\nu} = -\boldsymbol{E}_I^{\boldsymbol{\alpha},\boldsymbol{f}} \times \boldsymbol{\nu} & (\Gamma) \\ \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}} \cdot \boldsymbol{\nu} = -\boldsymbol{H}_I^{\boldsymbol{\alpha},\boldsymbol{f}} \cdot \boldsymbol{\nu} & (\Gamma) \\ \boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}}, \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}} \text{ are outgoing.} \end{cases} \tag{2.2}$$

Classically, the outgoing behavior of the scattered field is imposed by one of the two Silver-Müller radiation conditions:

$$\begin{cases} \lim\limits_{|\boldsymbol{x}|\to\infty} \Big(\boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x}) \times \boldsymbol{x} + |\boldsymbol{x}|\, \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x}) \Big) = 0 \\ \lim\limits_{|\boldsymbol{x}|\to\infty} \Big(\boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x}) \times \boldsymbol{x} - |\boldsymbol{x}|\, \boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x}) \Big) = 0 \end{cases}$$

uniformly in every direction $\boldsymbol{x}/|\boldsymbol{x}| \in S^2$, where $|\,.\,|$ is the euclidian norm in $\mathbb{R}^3$.

We are now in position to introduce the far field pattern of the electromagnetic field $(\boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}}, \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}})$. Its main properties are collected in the next proposition (see [11] for the proofs).

Proposition 2.1. *The scattered field $(\boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}}, \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}})$ has the asymptotic behavior in the direction $\boldsymbol{\beta} \in S^2$ as $|\boldsymbol{x}| \to \infty$:*

$$\begin{cases} \boldsymbol{E}^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{\beta}|\boldsymbol{x}|) = \dfrac{e^{ik|\boldsymbol{x}|}}{ik|\boldsymbol{x}|} \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) + O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right), \\ \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{\beta}|\boldsymbol{x}|) = \dfrac{e^{ik|\boldsymbol{x}|}}{ik|\boldsymbol{x}|} \left(\boldsymbol{\beta} \times \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha}))\right) + O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right). \end{cases}$$

The scattering amplitude $\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha}))$ is given for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S^2$ and all $\boldsymbol{f} \in L^2_t(S^2)$ by the formula

$$\boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) = -\frac{k^2}{4\pi} \boldsymbol{\beta} \times \int_\Gamma \Big[\boldsymbol{\nu}(\boldsymbol{y}) \times \boldsymbol{H}_T^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{y}) \Big] \times \boldsymbol{\beta}\, e^{-ik\boldsymbol{\beta}\cdot\boldsymbol{y}}\, \mathrm{d}\boldsymbol{y} \tag{2.3}$$

where $\boldsymbol{H}_T^{\boldsymbol{\alpha},\boldsymbol{f}} = \boldsymbol{H}_I^{\boldsymbol{\alpha},\boldsymbol{f}} + \boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{f}}$ is the total magnetic field. Moreover, $\boldsymbol{A}(\cdot, \cdot; \cdot)$ satisfies the following reciprocity relation

$$\boldsymbol{g}(\boldsymbol{\beta}) \cdot \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) = \boldsymbol{f}(\boldsymbol{\alpha}) \cdot \boldsymbol{A}(-\boldsymbol{\beta}, -\boldsymbol{\alpha}; \boldsymbol{g}(\boldsymbol{\beta})) \tag{2.4}$$

for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S^2$ and all $\boldsymbol{f}, \boldsymbol{g} \in L^2_t(S^2)$.

Assume now that the TRM emits a Herglotz wave, *i.e.* a superposition of plane waves of the form (2.1). More precisely, denote by $(\boldsymbol{E}_I^{\boldsymbol{f}}, \boldsymbol{H}_I^{\boldsymbol{f}})$ the incident Herglotz wave of polarization $\boldsymbol{f} \in L^2_t(S^2)$, defined by

$$\begin{cases} \boldsymbol{E}_I^{\boldsymbol{f}}(\boldsymbol{x}) = \displaystyle\int_{S^2} \boldsymbol{E}_I^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{\alpha} = \int_{S^2} \boldsymbol{f}(\boldsymbol{\alpha}) e^{ik\boldsymbol{\alpha}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{\alpha} \\ \boldsymbol{H}_I^{\boldsymbol{f}}(\boldsymbol{x}) = \displaystyle\int_{S^2} \boldsymbol{H}_I^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{\alpha} = \int_{S^2} (\boldsymbol{\alpha} \times \boldsymbol{f})(\boldsymbol{\alpha}) e^{ik\boldsymbol{\alpha}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{\alpha} \end{cases} \tag{2.5}$$

By linearity, Proposition 2.1 yields the following result.

Corollary 2.2. *When illuminated by the Herglotz wave* $(\boldsymbol{E}_I^{\boldsymbol{f}}, \boldsymbol{H}_I^{\boldsymbol{f}})$, *the scattering obstacle generates the diffracted field* $(\boldsymbol{E}^{\boldsymbol{f}}, \boldsymbol{H}^{\boldsymbol{f}})$ *which admits in the direction* $\boldsymbol{\beta} \in S^2$ *the far field asymptotics*

$$\begin{cases} \boldsymbol{E}^{\boldsymbol{f}}(\boldsymbol{\beta}|\boldsymbol{x}|) = \dfrac{e^{i|\boldsymbol{x}|}}{ik|\boldsymbol{x}|}\mathbf{F}\boldsymbol{f}(\boldsymbol{\beta}) + O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right), \\ \boldsymbol{H}^{\boldsymbol{f}}(\boldsymbol{\beta}|\boldsymbol{x}|) = \dfrac{e^{i|\boldsymbol{x}|}}{ik|\boldsymbol{x}|}\boldsymbol{\beta} \times \mathbf{F}\boldsymbol{f}(\boldsymbol{\beta}) + O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right). \end{cases}$$

where $\mathbf{F}\boldsymbol{f}(\boldsymbol{\beta})$ *is given by*

$$\mathbf{F}\boldsymbol{f}(\boldsymbol{\beta}) = \int_{S^2} \boldsymbol{A}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha}))\, \mathrm{d}\boldsymbol{\alpha}. \tag{2.6}$$

Using the expression (2.3) of the scattering amplitude, one can show that the far field operator $\mathbf{F} : \boldsymbol{f} \longmapsto \mathbf{F}\boldsymbol{f}$ defined by (2.6) is continuous from $L_t^2(S^2)$ onto itself. Moreover, using the reciprocity relation (2.4), one can show the following result (see [9] for the proof).

Proposition 2.3. *The far field operator* $\mathbf{F} : L_t^2(S^2) \longrightarrow L_t^2(S^2)$ *defined by (2.6) is a compact and normal operator. As in the acoustic case, its adjoint is the operator* $\mathbf{F}^* : L_t^2(S^2) \to L_t^2(S^2)$ *defined by*

$$\forall \boldsymbol{f} \in L_t^2(S^2), \qquad \mathbf{F}^*\boldsymbol{f} = \overline{\mathbf{R}\mathbf{F}\overline{\mathbf{R}\boldsymbol{f}}} \tag{2.7}$$

where $\mathbf{R}$ *is the symmetry operator defined by* $\mathbf{R}\boldsymbol{f}(\boldsymbol{\alpha}) = \boldsymbol{f}(-\boldsymbol{\alpha})$ *for all* $\boldsymbol{\alpha} \in S^2$ *and* $\boldsymbol{f} \in L_t^2(S^2)$.

We are now able to define the time reversal operator $\mathbf{T}$. During the time-reversal process, the TRM first emits an incident electromagnetic Herglotz wave $(\boldsymbol{E}_I^{\boldsymbol{f}}, \boldsymbol{H}_I^{\boldsymbol{f}})$ of polarization $\boldsymbol{f}$. Then the scattering obstacle generates a scattered field $(\boldsymbol{E}^{\boldsymbol{f}}, \boldsymbol{H}^{\boldsymbol{f}})$. The TRM measures and conjugates the corresponding electric far field $\mathbf{F}\boldsymbol{f}$. The resulting field is then used as a polarization $\boldsymbol{g}$ of a new incident Herglotz wave. Therefore, we have

$$\boldsymbol{g} = \overline{\mathbf{R}\mathbf{F}\boldsymbol{f}}, \tag{2.8}$$

where the presence of the symmetry operator is due to the fact that the far field measured in a direction $\boldsymbol{\beta}$ is reemitted in the opposite direction $-\boldsymbol{\beta}$. The time reversal operator $\mathbf{T}$ is then obtained by iterating this cycle twice:

$$\mathbf{T}\boldsymbol{f} = \overline{\mathbf{R}\mathbf{F}\boldsymbol{g}} = \overline{\mathbf{R}\mathbf{F}\overline{\mathbf{R}\mathbf{F}\boldsymbol{f}}}. \tag{2.9}$$

Thanks to Proposition 2.3, we have shown the following result.

Proposition 2.4. *The time reversal operator* $\mathbf{T}$ *is the compact, selfadjoint and positive operator given by*

$$\begin{array}{rlcl} \mathbf{T}: & L_t^2(S^2) & \longrightarrow & L_t^2(S^2) \\ & \boldsymbol{f} & \longmapsto & \mathbf{T}\boldsymbol{f} = \mathbf{F}\mathbf{F}^*\boldsymbol{f} = \mathbf{F}^*\mathbf{F}\boldsymbol{f}. \end{array}$$

The nonzero eigenvalues of $\mathbf{T}$ *are exactly the positive numbers*

$$|\lambda_1|^2 \geq |\lambda_2|^2 \geq \cdots > 0,$$

where the sequence $(\lambda_p)_{p\geq 1}$ *denotes the nonzero complex eigenvalues of the normal compact far field operator* $\mathbf{F}$. *Moreover, the corresponding eigenfunctions* $(\boldsymbol{f}_p)_{p\geq 1}$ *of* $\mathbf{F}$ *are exactly the eigenfunctions of* $\mathbf{T}$.

3. Scattering by perfectly conducting small scatterers. In this Section, we show that the asymptotics of the electromagnetic scattering problem by small scatterers is closely connected to the classical low frequency scattering (the Rayleigh approximation, cf. [21, 12]). In particular, this asymptotics involves the electromagnetic polarizability tensors of the scatterers [30, 2, 3]. The fact that the two limit models are similar is straightforward when the scattering obstacle has only one connected component. As it is shown in Subsection 3.1, this follows from a scaling argument. The proof is less obvious when the obstacle is multiply connected (one cannot anymore use a unique change of variables to work in a reference domain of fixed size). We study this question using an integral equation approach in Subsection 3.2.

3.1. The case of one scatterer. Let us assume that the perfectly conducting scatterer is of small size δ and that it is obtained from a reference obstacle after a dilation. More precisely, let us set:

$$\mathcal{O}^\delta = \{\boldsymbol{x} = \boldsymbol{s} + \delta\boldsymbol{\xi}\ ;\ \boldsymbol{\xi} \in \mathcal{O}\}\,.$$

Its boundary is denoted by Γ^δ and its exterior by $\Omega^\delta := \mathbb{R}^3 \setminus \overline{\mathcal{O}^\delta}$. Given an incident plane wave $(\boldsymbol{E}_I^{\alpha,\boldsymbol{f}}, \boldsymbol{H}_I^{\alpha,\boldsymbol{f}})$, let $(\boldsymbol{E}^\delta, \boldsymbol{H}^\delta)$ be the solution of the scattering problem by the perfectly conducting obstacle (for the sake of clarity, we drop here the reference to the angle of incidence and to the polarization in the scattered field):

$$\begin{cases} \mathbf{curl}\,\boldsymbol{E}^\delta = ik\boldsymbol{H}^\delta & (\Omega^\delta) \\ \mathbf{curl}\,\boldsymbol{H}^\delta = -ik\boldsymbol{E}^\delta & (\Omega^\delta) \\ \mathbf{div}\,\boldsymbol{E}^\delta = 0 & (\Omega^\delta) \\ \mathbf{div}\,\boldsymbol{H}^\delta = 0 & (\Omega^\delta) \\ \boldsymbol{E}^\delta \times \boldsymbol{\nu} = -\boldsymbol{E}_I^{\alpha,\boldsymbol{f}} \times \boldsymbol{\nu} & (\Gamma^\delta) \\ \boldsymbol{H}^\delta \cdot \boldsymbol{\nu} = -\boldsymbol{H}_I^{\alpha,\boldsymbol{f}} \cdot \boldsymbol{\nu} & (\Gamma^\delta) \\ \boldsymbol{E}^\delta,\ \boldsymbol{H}^\delta \text{ outgoing.} \end{cases} \tag{3.1}$$

Introducing the scaled fields

$$\begin{cases} \boldsymbol{e}^\delta(\boldsymbol{\xi}) = \boldsymbol{E}^\delta(\boldsymbol{s} + \delta\boldsymbol{\xi}), \\ \boldsymbol{h}^\delta(\boldsymbol{\xi}) = \boldsymbol{H}^\delta(\boldsymbol{s} + \delta\boldsymbol{\xi}), \end{cases} \qquad \boldsymbol{\xi} \in \Omega := \mathbb{R}^3 \setminus \overline{\mathcal{O}}$$

we obtain that

$$\begin{cases} \mathbf{curl}\,\boldsymbol{e}^\delta = i\,(k\delta)\,\boldsymbol{h}^\delta & (\Omega) \\ \mathbf{curl}\,\boldsymbol{h}^\delta = -i\,(k\delta)\,\boldsymbol{e}^\delta & (\Omega) \\ \mathbf{div}\,\boldsymbol{e}^\delta = 0 & (\Omega) \\ \mathbf{div}\,\boldsymbol{h}^\delta = 0 & (\Omega) \\ \boldsymbol{e}^\delta \times \boldsymbol{\nu} = -\boldsymbol{e}_I^{\alpha,\boldsymbol{f}} \times \boldsymbol{\nu} & (\Gamma) \\ \boldsymbol{h}^\delta \cdot \boldsymbol{\nu} = -\boldsymbol{h}_I^{\alpha,\boldsymbol{f}} \cdot \boldsymbol{\nu} & (\Gamma) \\ \boldsymbol{e}^\delta,\ \boldsymbol{h}^\delta \text{ outgoing,} \end{cases} \tag{3.2}$$

where $\Gamma = \partial\Omega$ and

$$\begin{cases} \boldsymbol{e}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{\xi}) = \boldsymbol{E}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s} + \delta\boldsymbol{\xi}) = \boldsymbol{E}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s}) + O(k\delta), \\ \boldsymbol{h}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{\xi}) = \boldsymbol{H}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s} + \delta\boldsymbol{\xi}) = \boldsymbol{H}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s}) + O(k\delta). \end{cases}$$

When $\delta \to 0$, problem (3.2) appears as a low frequency electromagnetic scattering problem ($k\delta \to 0$) associated with an incident wave that behaves like the constant field $(\boldsymbol{E}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s}), \boldsymbol{H}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s}))$ asymptotically. The electromagnetic scattering problem for small frequencies has been studied for a long time (cf. [35, 36, 22, 28]) and the asymptotic behavior of its solution is by now well known (see the reference book [12] for a detailed presentation and [4] for convergence results of higher order terms). In particular, the first order approximation $(\boldsymbol{e}^0, \boldsymbol{h}^0)$ of $(\boldsymbol{e}^\delta, \boldsymbol{h}^\delta)$ (the so-called Rayleigh approximation) is given by the next result, which follows from [12, Chap. 5]).

THEOREM 3.1. *Let* $\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$ *and* $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$ *be the vector potentials defined by*

$$\begin{cases} \Delta \boldsymbol{\Phi} = 0, & (\Omega) \\ \boldsymbol{\Phi} = \boldsymbol{x} + \boldsymbol{c}, & (\Gamma) \\ \boldsymbol{\Phi} = O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right) & |\boldsymbol{x}| \to \infty \end{cases} \tag{3.3}$$

and

$$\begin{cases} \Delta \boldsymbol{\Psi} = 0, & (\Omega) \\ \dfrac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{\nu}} = \boldsymbol{\nu}, & (\Gamma) \\ \boldsymbol{\Psi} = O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right) & |\boldsymbol{x}| \to \infty \end{cases} \tag{3.4}$$

where the constant vector $\boldsymbol{c} \in \mathbb{R}^3$ *is chosen such that* $\displaystyle\int_\Gamma \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{\nu}} = 0$.

Then, as $\delta \longrightarrow 0$, *we have*

$$\begin{cases} \boldsymbol{e}^\delta \longrightarrow \boldsymbol{e}^0 := -\nabla \boldsymbol{\Phi}\, \boldsymbol{f}(\boldsymbol{\alpha}) \\ \boldsymbol{h}^\delta \longrightarrow \boldsymbol{h}^0 := -\nabla \boldsymbol{\Psi}\, (\boldsymbol{\alpha} \times \boldsymbol{f}(\boldsymbol{\alpha})) \end{cases}$$

locally in $H_{\mathbf{curl}}(\Omega)$.

Using the above result, one can easily obtain the asymptotics of the far field associated to $\boldsymbol{E}^\delta$.

COROLLARY 3.2. *Let* $(\boldsymbol{E}^\delta, \boldsymbol{H}^\delta)$ *be the solution of the scattering problem* (3.1). *Let* $\boldsymbol{\mathcal{P}}$ *and* $\boldsymbol{\mathcal{M}}$ *be respectively the electric polarizability and magnetic polarizability tensors defined by (I denotes the identity)*

$$\boldsymbol{\mathcal{P}} = |\mathcal{O}|\, I - \int_\Gamma \boldsymbol{x} \left(\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{\nu}}\right)^T \, \mathrm{d}\gamma_{\boldsymbol{x}} \qquad \boldsymbol{\mathcal{M}} = |\mathcal{O}|\, I - \int_\Gamma \boldsymbol{\nu} \boldsymbol{\Psi}^T \, \mathrm{d}\gamma_{\boldsymbol{x}}$$

where the vector potentials $\boldsymbol{\Phi}$ *and* $\boldsymbol{\Psi}$ *are respectively defined by* (3.3) *and* (3.4).
Then, the far field $\boldsymbol{A}^\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha}))$ *of* $\boldsymbol{E}^\delta$, *defined by*

$$\boldsymbol{E}^\delta(\boldsymbol{\beta}|\boldsymbol{x}|) = \boldsymbol{A}^\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) \frac{\mathrm{e}^{ik|\boldsymbol{x}|}}{ik|\boldsymbol{x}|} + O\left(\frac{1}{|\boldsymbol{x}|^2}\right),$$

admits as $\delta \to 0$ *the following asymptotics :*

$$\boldsymbol{A}^\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) = \frac{(ik\delta)^3}{4\pi}\, \boldsymbol{\beta} \times \Big[\boldsymbol{\beta} \times (\boldsymbol{\mathcal{P}} \boldsymbol{f}(\boldsymbol{\alpha})) - \boldsymbol{\mathcal{M}}(\boldsymbol{\alpha} \times \boldsymbol{f}(\boldsymbol{\alpha}))\Big]\, \mathrm{e}^{ik\,(\boldsymbol{\alpha} - \boldsymbol{\beta}) \cdot \boldsymbol{s}} + O(\delta^4). \tag{3.5}$$

Proof. Following [12], we have

$$\boldsymbol{A}^\delta(\boldsymbol{\alpha},\boldsymbol{\beta};\boldsymbol{f}(\boldsymbol{\alpha})) = \frac{k^2}{4\pi}\,\boldsymbol{\beta}\times\left\{\boldsymbol{\beta}\times\int_{\Gamma^\delta}\left[\boldsymbol{\nu}_{\boldsymbol{x}}\times\left(\boldsymbol{H}^\delta(\boldsymbol{x})+\boldsymbol{H}_I^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{x})\right)\right]\,\mathrm{e}^{-ik\,\boldsymbol{\beta}\cdot\boldsymbol{x}}\,\mathrm{d}\gamma_{\boldsymbol{x}}\right\} \tag{3.6}$$

The change of variables $\boldsymbol{\xi} = (\boldsymbol{x}-\boldsymbol{s})/\delta$ in the above integral shows that

$$\boldsymbol{A}^\delta(\boldsymbol{\alpha},\boldsymbol{\beta};\boldsymbol{f}(\boldsymbol{\alpha})) = \left(\frac{(k\delta)^2}{4\pi}\,\boldsymbol{\beta}\times\left\{\boldsymbol{\beta}\times\int_{\Gamma}\left[\boldsymbol{\nu}_{\boldsymbol{\xi}}\times\left(\boldsymbol{h}^\delta(\boldsymbol{\xi})+\boldsymbol{h}_I^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{\xi})\right)\right]\,\mathrm{d}\gamma_{\boldsymbol{\xi}}\right\}\right)\mathrm{e}^{-ik\,\boldsymbol{\beta}\cdot\boldsymbol{s}} + O(\delta^3). \tag{3.7}$$

Comparing (3.6) with the term between parenthesis in the above expression, we see that this term is nothing but the electric far field associated to the solution $(\boldsymbol{e}^\delta,\boldsymbol{h}^\delta)$ of the low frequency scattering problem (3.2). Consequently, this term can be expressed using the polarizability tensors (cf. equation (5.158) in [12]):

$$\frac{(k\delta)^2}{4\pi}\,\boldsymbol{\beta}\times\left\{\boldsymbol{\beta}\times\int_{\Gamma}\left[\boldsymbol{\nu}_{\boldsymbol{\xi}}\times\left(\boldsymbol{h}^\delta(\boldsymbol{\xi})+\boldsymbol{h}_I^{\boldsymbol{\alpha},\boldsymbol{f}}(\boldsymbol{\xi})\right)\right]\,\mathrm{d}\gamma_{\boldsymbol{\xi}}\right\} = \frac{(ik\delta)^3}{4\pi}\boldsymbol{\beta}\times\Big[\boldsymbol{\beta}\times(\boldsymbol{\mathcal{P}}\boldsymbol{f}(\boldsymbol{\alpha}))-\boldsymbol{\mathcal{M}}(\boldsymbol{\alpha}\times\boldsymbol{f}(\boldsymbol{\alpha}))\Big]\mathrm{e}^{ik\,\boldsymbol{\alpha}\cdot\boldsymbol{s}} + O\left(\delta^4\right),$$

where we have used the fact that the incident electromagnetic field $(\boldsymbol{e}_I,\boldsymbol{h}_I)$ converges to the constants $\left(\boldsymbol{E}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s}),\boldsymbol{H}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{s})\right) = (\boldsymbol{f}(\boldsymbol{\alpha}),\boldsymbol{\alpha}\times\boldsymbol{f}(\boldsymbol{\alpha}))\,\mathrm{e}^{ik\,\boldsymbol{\alpha}\cdot\boldsymbol{s}}$ as δ tends to 0. Plugging the last relation in (3.7) yields (3.5).□

3.2. Multiply connected scatterer. We consider now the case where the scatterer has M connected components:

$$\mathcal{O}^\delta = \bigcup_{p=1}^{M}\mathcal{O}_p^\delta$$

where each component $\mathcal{O}_p^\delta$ is obtained from a reference domain $\mathcal{O}_p$ by a dilation and a translation:

$$\mathcal{O}_p^\delta = \{\boldsymbol{x} = \boldsymbol{s}_p + \delta\boldsymbol{\xi}\ ;\ \boldsymbol{\xi}\in\mathcal{O}_p\}.$$

Finally we denote once again by $\Omega^\delta = \mathbb{R}^3\setminus\overline{\mathcal{O}^\delta}$ the exterior domain and by $\Gamma^\delta = \bigcup_{p=1}^{M}\Gamma_p^\delta$ its boundary.

In order to study the asymptotics $\delta\to 0$, we seek an integral representation of the solution $(\boldsymbol{E}^\delta,\boldsymbol{H}^\delta)$ of (3.1) in the form

$$\left\{\begin{aligned} \boldsymbol{E}^\delta(\boldsymbol{y}) &= \delta\,\mathbf{curl}\,\mathbf{curl}\int_{\Gamma^\delta}G_k(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{J}^\delta(\boldsymbol{x})\,\mathrm{d}\gamma_{\boldsymbol{x}},\\ \boldsymbol{H}^\delta(\boldsymbol{y}) &= -\delta\,(ik)\,\mathbf{curl}\int_{\Gamma^\delta}G_k(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{J}^\delta(\boldsymbol{x})\,\mathrm{d}\gamma_{\boldsymbol{x}},\end{aligned}\right. \qquad \boldsymbol{y}\in\Omega^\delta, \tag{3.8}$$

where $\boldsymbol{J}^\delta$ is the (unknown) electric surface current and

$$G_k(\boldsymbol{x},\boldsymbol{y}) = \frac{\mathrm{e}^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

denotes the Green function of $-\Delta - k^2$ in $\mathbb{R}^3$.

Using the identity $\mathbf{curl}\,\mathbf{curl} = \nabla\,\mathbf{div}\, - \Delta$ and the fact that for $\boldsymbol{x} \neq \boldsymbol{y}$, we have $\Delta G_k(\boldsymbol{x},\boldsymbol{y}) = -k^2\, G_k(\boldsymbol{x},\boldsymbol{y})$, one can show that the electric field can also be written in the form (cf. [10, p. 64])

$$\boldsymbol{E}^\delta(\boldsymbol{y}) = \delta\left(k^2 \int_{\Gamma^\delta} G_k(\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{J}^\delta(\boldsymbol{x})\;\mathrm{d}\gamma_{\boldsymbol{x}} + \nabla \int_{\Gamma^\delta} G_k(\boldsymbol{x},\boldsymbol{y})\,\mathbf{div}_{\Gamma^\delta}\,\boldsymbol{J}^\delta(\boldsymbol{x})\;\mathrm{d}\gamma_{\boldsymbol{x}}\right), \quad (3.9)$$

where $\mathbf{div}_{\Gamma^\delta}$ denotes the surface divergence operator on Γ^δ.

The unknown current $\boldsymbol{J}^\delta = (\boldsymbol{J}^\delta_1, \ldots, \boldsymbol{J}^\delta_M)$ is uniquely determined by writing the perfectly conducting boundary condition on each scatterer:

$$(\boldsymbol{E}^\delta \times \boldsymbol{\nu})_{|\Gamma^\delta_p} = -(\boldsymbol{E}_I^{\alpha,\boldsymbol{f}} \times \boldsymbol{\nu})_{|\Gamma^\delta_p} \qquad \forall\, p = 1, \ldots, M. \quad (3.10)$$

It is well known (see for instance [27, Theorem 5.5.1]) that the trace of a potential of the form (3.9) is given by

$$(\boldsymbol{E}^\delta \times \boldsymbol{\nu})_{|\Gamma^\delta_p} = \sum_{q=1}^{M} \delta\,\left(k^2 S^{k,\delta}_{pq} + T^{k,\delta}_{pq}\right)\boldsymbol{J}^\delta_q,$$

where the integral operators $S^{k,\delta}_{pq} : TH^s(\Gamma_q) \to TH^{s+1}(\Gamma_p)$ and $T^{k,\delta}_{pq} : TH^s(\Gamma_q) \to TH^{s-1}(\Gamma_p)$, ($TH^s(\Gamma_q)$ denotes the Sobolev space of tangent vector fields [27]) are defined for $\boldsymbol{y} \in \Gamma_p$ by

$$\left\{\begin{array}{lcl} \left(S^{k,\delta}_{pq}\,\boldsymbol{J}^\delta_q\right)(\boldsymbol{y}) & = & \displaystyle\int_{\Gamma^\delta_q} G_k(\boldsymbol{x},\boldsymbol{y})\,\left(\boldsymbol{J}^\delta_q(\boldsymbol{x}) \times \boldsymbol{\nu}_{\boldsymbol{y}}\right)\;\mathrm{d}\gamma_{\boldsymbol{x}} \\ \left(T^{k,\delta}_{pq}\,\boldsymbol{J}^\delta_q\right)(\boldsymbol{y}) & = & \displaystyle\left(\nabla_{\boldsymbol{y}} \int_{\Gamma^\delta_q} G_k(\boldsymbol{x},\boldsymbol{y})\,\mathbf{div}_{\Gamma^\delta_q}\boldsymbol{J}^\delta_q(\boldsymbol{x})\;\mathrm{d}\gamma_{\boldsymbol{x}}\right)_{|\Gamma^\delta_p} \times \boldsymbol{\nu}_{\boldsymbol{y}}. \end{array}\right.$$

For $q \neq p$, the kernels of the above integral operators are infinitely differentiable. The operator S^δ_{pp} is the classical single layer potential, and has a singular but integrable kernel. The operator T^δ_{pp} can also be written using a formula involving only integrable kernels (see [27, p. 242]):

$$\left(T^{k,\delta}_{pp}\,\boldsymbol{J}^\delta_p\right)(\boldsymbol{y}) = \int_{\Gamma^\delta_p} \Big[\left(\nabla_{\boldsymbol{y}} G_k(\boldsymbol{x},\boldsymbol{y}) \times (\boldsymbol{\nu}_{\boldsymbol{y}} - \boldsymbol{\nu}_{\boldsymbol{x}})\right)\,\mathbf{div}_{\Gamma^\delta_p}\boldsymbol{J}^\delta_p(\boldsymbol{x}) \\ - G_k(\boldsymbol{x},\boldsymbol{y})\,\mathbf{curl}_{\Gamma^\delta_p}\mathbf{div}_{\Gamma^\delta_p}\boldsymbol{J}^\delta_p(\boldsymbol{x})\Big]\,\mathrm{d}\gamma_{\boldsymbol{x}}.$$

In the above relation, $\mathbf{div}_{\Gamma^\delta_p}$ and $\mathbf{curl}_{\Gamma^\delta_p}$ denote respectively the surface divergence operator and tangential rotational operator on Γ^δ_p. Then, the integral equation (3.10) reads

$$\sum_{q=1}^{M} \delta\left(k^2 S^{k,\delta}_{pq} + T^{k,\delta}_{pq}\right)\boldsymbol{J}^\delta_q = -(\boldsymbol{E}^{inc} \times \boldsymbol{\nu})_{|\Gamma^\delta_p} \qquad \forall\, p = 1, \ldots, M. \quad (3.11)$$

In order to work in a functional framework independent of δ, we introduce the new variables

$$\left\{\begin{array}{l} \boldsymbol{\xi} = \dfrac{\boldsymbol{x} - \boldsymbol{s}_q}{\delta} \in \mathcal{O}_q, \\ \boldsymbol{\eta} = \dfrac{\boldsymbol{y} - \boldsymbol{s}_p}{\delta} \in \mathcal{O}_p, \end{array}\right.$$

and the scaled fields

$$\begin{cases} \boldsymbol{j}_q^\delta(\boldsymbol{\xi}) = \boldsymbol{J}_q^\delta(\boldsymbol{x}), \\ \mathcal{G}_{pq}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta}) = G_k(\boldsymbol{x},\boldsymbol{y}). \end{cases}$$

With the above notation, we have

$$\left(S_{pq}^{k,\delta}\,\boldsymbol{J}_q^\delta\right)(\boldsymbol{y}) := \delta^2\left(\mathcal{S}_{pq}^{k,\delta}\,\boldsymbol{j}_q^\delta\right)(\boldsymbol{\eta}) = \delta^2\int_{\Gamma_q}\mathcal{G}_{pq}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta})\,\left(\boldsymbol{j}_q^\delta(\boldsymbol{\xi})\times\boldsymbol{\nu}_{\boldsymbol{\eta}}\right)\,\mathrm{d}\gamma_{\boldsymbol{\xi}}$$

and

$$\begin{aligned}\left(T_{pq}^{k,\delta}\,\boldsymbol{J}_p^\delta\right)(\boldsymbol{y}) \quad &= \begin{cases} \left(\nabla_{\boldsymbol{\eta}}\displaystyle\int_{\Gamma_q}\mathcal{G}_{pq}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta})\,\mathbf{div}_{\Gamma_q}\boldsymbol{j}_q^\delta(\boldsymbol{\xi})\,\mathrm{d}\gamma_{\boldsymbol{\xi}}\right)_{|\Gamma_p}\times\boldsymbol{\nu}_{\boldsymbol{\eta}} & \text{for } q\neq p, \\ \displaystyle\int_{\Gamma_q}\Big[\left(\nabla_{\boldsymbol{\eta}}\mathcal{G}_{pp}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta})\times(\boldsymbol{\nu}_{\boldsymbol{\eta}}-\boldsymbol{\nu}_{\boldsymbol{\xi}})\right)\,\mathbf{div}_{\Gamma_q}\,\boldsymbol{j}_p^\delta(\boldsymbol{\xi}) & \\ \qquad -\mathcal{G}_{pp}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta})\,\mathbf{curl}_{\Gamma_p}\mathbf{div}_{\Gamma_p}\,\boldsymbol{j}_p^\delta(\boldsymbol{\xi})\Big]\,\mathrm{d}\gamma_{\boldsymbol{\xi}}, & \text{for } q=p \end{cases} \\ &:= \quad \left(\mathcal{T}_{pq}^{k,\delta}\,\boldsymbol{j}_q^\delta\right)(\boldsymbol{\eta}).\end{aligned}$$

Consequently, equation (3.11) can be written

$$\mathcal{B}_{pp}^{k,\delta}\,\boldsymbol{j}_p^\delta + \sum_{q\neq p}\mathcal{B}_{pq}^{k,\delta}\,\boldsymbol{j}_q^\delta = -(\boldsymbol{e}_I^{\alpha,\boldsymbol{f}}\times\boldsymbol{\nu})_{|\Gamma_p} \qquad \forall\, p=1,\ldots,M, \tag{3.12}$$

with

$$\mathcal{B}_{pq}^{k,\delta} = (k\delta)^2\;\delta\,\mathcal{S}_{pq}^{k,\delta} + \delta\,\mathcal{T}_{pq}^{k,\delta} \tag{3.13}$$

and

$$\boldsymbol{e}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{\eta}) = \boldsymbol{E}_I^{\alpha,\boldsymbol{f}}(\boldsymbol{y}).$$

Let us consider first the diagonal terms in (3.12), by investigating the behavior of the kernels involved in the expression of $\mathcal{B}_{pp}^{k,\delta}$ as $\delta\to 0$. Since

$$\begin{cases} \mathcal{G}_{pp}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta}) = \dfrac{1}{\delta}G_{k\delta}(\boldsymbol{\xi},\boldsymbol{\eta}), \\ \nabla_{\boldsymbol{\eta}}\mathcal{G}_{pp}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta}) = \dfrac{1}{\delta}\nabla_{\boldsymbol{\eta}}G_{k\delta}(\boldsymbol{\xi},\boldsymbol{\eta}), \end{cases}$$

we see that

$$\mathcal{B}_{pp}^{k,\delta} = (k\delta)^2\,\widetilde{\mathcal{S}}_{pp}^{k\delta} + \widetilde{\mathcal{T}}_{pp}^{k\delta} := \widetilde{\mathcal{B}}_{pp}^{k\delta}$$

where

$$\begin{cases} \left(\widetilde{\mathcal{S}}_{pp}^{k\delta}\,\boldsymbol{j}_p\right)(\boldsymbol{y}) = \displaystyle\int_{\Gamma_q}G_{k\delta}(\boldsymbol{\xi},\boldsymbol{\eta})\,\left(\boldsymbol{j}_p(\boldsymbol{\xi})\times\boldsymbol{\nu}_{\boldsymbol{\eta}}\right)\,\mathrm{d}\gamma_{\boldsymbol{\xi}}, \\ \left(\widetilde{\mathcal{T}}_{pp}^{k\delta}\,\boldsymbol{j}_q\right)(\boldsymbol{y}) = \displaystyle\int_{\Gamma_q}\Big[(\nabla_{\boldsymbol{\eta}}G_{k\delta}(\boldsymbol{\xi},\boldsymbol{\eta})(\boldsymbol{\xi},\boldsymbol{\eta})\times(\boldsymbol{\nu}_{\boldsymbol{\eta}}-\boldsymbol{\nu}_{\boldsymbol{\xi}}))\,\mathbf{div}_{\Gamma_p}\,\boldsymbol{j}_p(\boldsymbol{\xi}) \\ \qquad\qquad -G_{k\delta}(\boldsymbol{\xi},\boldsymbol{\eta})(\boldsymbol{\xi},\boldsymbol{\eta})\,\mathbf{curl}_{\Gamma_p}\mathbf{div}_{\Gamma_p}\,\boldsymbol{j}_p(\boldsymbol{\xi})\Big]\,\mathrm{d}\gamma_{\boldsymbol{\xi}} \end{cases}$$

The crucial point here is that $\widetilde{\mathcal{B}}_{pp}^{k\delta}$ is exactly the operator involved in the integral equation formulation of the simple scattering problem associated with the reference scatterer $\mathcal{O}_p$ at low frequency $k\delta \to 0$. Moreover, since the zero frequency limit exists, $\widetilde{\mathcal{B}}_{pp}^{k\delta} = \mathcal{B}_{pp}^{k,\delta}$ admits a limit $\mathcal{B}_{pp}^{0}$.

Let us consider now the off diagonal terms $\mathcal{B}_{pq}^{k,\delta}$, $q \neq p$. Denote by

$$d = \min_{1 \le p < q \le N} |\boldsymbol{s}_p - \boldsymbol{s}_q| \tag{3.14}$$

the minimal distance between the centers of the obstacles. Using the relation

$$|s_p - s_q + \delta(\boldsymbol{\xi} - \boldsymbol{\eta})| = |s_p - s_q| \left(1 + O\left(\frac{\delta}{d}\right)\right),$$

one can easily check that

$$\begin{cases} \mathcal{G}_{pq}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta}) = G_k(\boldsymbol{s}_q,\boldsymbol{s}_p)\left[1 + O(k\delta) + O\left(\dfrac{\delta}{d}\right)\right] \\ \nabla_{\boldsymbol{\eta}}\, \mathcal{G}_{pq}^{k,\delta}(\boldsymbol{\xi},\boldsymbol{\eta}) = G_k(\boldsymbol{s}_q,\boldsymbol{s}_p)\left[O(k\delta) + O\left(\dfrac{\delta}{d}\right)\right], \end{cases} \qquad \forall q \neq p.$$

Inserting the above asymptotics in (3.13) shows that

$$\mathcal{B}_{pq}^{k,\delta} = O\left(\frac{\delta}{d}\right)\left[O(k\delta) + O\left(\frac{\delta}{d}\right)\right] \qquad \text{for } q \neq p.$$

Summing up, the behavior of the solution $(\boldsymbol{E}^\delta, \boldsymbol{H}^\delta)$ of (3.1) for small scatterers (namely for $k\delta \to 0$ and $\delta/d \to 0$) is given by the low frequency limit of the simple scattering problem. Therefore, the multiple scattering effects can be neglected when $k\delta \to 0$ and $\delta/d \to 0$, and the electric far field can be obtained simply by superposition of the far fields given in Corollary 3.2. We have thus proved the following result.

Theorem 3.3. *Assume that the scatterer has M connected components*

$$\mathcal{O}^\delta = \bigcup_{p=1}^{M} \mathcal{O}_p^\delta,$$

where each component $\mathcal{O}_p^\delta$ is obtained from a reference scatterer $\mathcal{O}_p$ (centered at the origin) of smooth boundary Γ_p by a dilation of ratio δ centered at a given point $\boldsymbol{s}_p \in \mathbb{R}^3$:

$$\mathcal{O}_p^\delta = \{\boldsymbol{x} = \boldsymbol{s}_p + \delta\boldsymbol{\xi}\ ;\ \boldsymbol{\xi} \in \mathcal{O}_p\}.$$

For all $p = 1, \ldots, M$, let $\boldsymbol{\Phi}_p$ and $\boldsymbol{\Psi}_p$ be the vector potentials defined by

$$\begin{cases} \Delta\boldsymbol{\Phi}_p = 0, & (\mathbb{R}^3 \setminus \overline{\mathcal{O}_p}) \\ \boldsymbol{\Phi}_p = \boldsymbol{x} + \boldsymbol{c}_p, & (\Gamma_p) \\ \boldsymbol{\Phi}_p = O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right) & |\boldsymbol{x}| \to \infty \end{cases} \tag{3.15}$$

and

$$\begin{cases} \Delta\boldsymbol{\Psi}_p = 0, & (\mathbb{R}^3 \setminus \overline{\mathcal{O}_p}) \\ \dfrac{\partial\boldsymbol{\Psi}_p}{\partial\boldsymbol{\nu}} = \boldsymbol{\nu}, & (\Gamma_p) \\ \boldsymbol{\Psi}_p = O\left(\dfrac{1}{|\boldsymbol{x}|^2}\right) & |\boldsymbol{x}| \to \infty \end{cases} \tag{3.16}$$

where the constant vector $\boldsymbol{c}_p \in \mathbb{R}^3$ *is chosen such that* $\displaystyle\int_{\Gamma_p} \frac{\partial \boldsymbol{\Phi}_p}{\partial \boldsymbol{\nu}} = 0$.

Let $\boldsymbol{\mathcal{P}}_p$ *and* $\boldsymbol{\mathcal{M}}_p$ *be respectively the electric polarizability and magnetic polarizability tensors of the reference scatterer* $\mathcal{O}_p$ *(*I *denotes the identity):*

$$\begin{cases} \boldsymbol{\mathcal{P}}_p = |\mathcal{O}_p|\, I - \displaystyle\int_{\Gamma_p} \boldsymbol{x} \left(\frac{\partial \boldsymbol{\Phi}_p}{\partial \boldsymbol{\nu}} \right)^T \, \mathrm{d}\gamma_{\boldsymbol{x}}, \\ \\ \boldsymbol{\mathcal{M}}_p = |\mathcal{O}_p|\, I - \displaystyle\int_{\Gamma_p} \boldsymbol{\nu} \boldsymbol{\Psi}_p^T \, \mathrm{d}\gamma_{\boldsymbol{x}}. \end{cases} \tag{3.17}$$

Finally, let $(\boldsymbol{E}^\delta, \boldsymbol{H}^\delta)$ *be the solution of the scattering problem* (3.1) *and* $\boldsymbol{A}^\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha}))$ *the far field of* $\boldsymbol{E}^\delta$*:*

$$\boldsymbol{E}^\delta(\boldsymbol{\beta}|\boldsymbol{x}|) = \boldsymbol{A}^\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) \frac{\mathrm{e}^{ik|\boldsymbol{x}|}}{ik|\boldsymbol{x}|} + O\left(\frac{1}{|\boldsymbol{x}|^2}\right).$$

Then, as $\delta \to 0$*, we have*

$$\frac{4\pi}{(ik\delta)^3} \boldsymbol{A}^\delta(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) \longrightarrow \boldsymbol{A}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) \tag{3.18}$$

where

$$\boldsymbol{A}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha})) = \sum_{p=1}^{M} \boldsymbol{\beta} \times \Big[\boldsymbol{\beta} \times (\boldsymbol{\mathcal{P}}_p \boldsymbol{f}(\boldsymbol{\alpha})) - \boldsymbol{\mathcal{M}}_p(\boldsymbol{\alpha} \times \boldsymbol{f}(\boldsymbol{\alpha})) \Big] \mathrm{e}^{ik\,(\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot \boldsymbol{s}_p}. \tag{3.19}$$

The convergence (3.18) *holds uniformly for all* $\alpha, \beta \in S^2$ *and for all wavenumber* k *and minimal distance* d *(defined by* (3.14)*) satisfying* $k\delta \to 0$ *and* $\delta/d \to 0$.

4. Selective focusing using time-reversal. From now on, we assume that $k\delta \to 0$ and $\delta/d \to 0$. According to Theorem 3.3, the eigenfunctions of the far field operator $\mathbf{F}^\delta$ can be approximated by those of the operator $\mathbf{F}^0 : L^2_t(S^2) \to L^2_t(S^2)$ defined by

$$(\mathbf{F}^0 \boldsymbol{f})(\boldsymbol{\beta}) = \int_{S^2} \boldsymbol{A}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha}))\, \mathrm{d}\boldsymbol{\alpha} \qquad \forall\, \boldsymbol{f} \in L^2_t(S^2). \tag{4.1}$$

Substituting the expression (3.19) of $\boldsymbol{A}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{f}(\boldsymbol{\alpha}))$ in (4.1), we obtain that

$$(\mathbf{F}^0 \boldsymbol{f})(\boldsymbol{\beta}) = \sum_{p=1}^{M} \boldsymbol{\beta} \times \Big[\boldsymbol{\beta} \times \Big(\boldsymbol{\mathcal{P}}_p \boldsymbol{E}_I^{\boldsymbol{f}}(\boldsymbol{s}_p) \Big) - \boldsymbol{\mathcal{M}}_p \boldsymbol{H}_I^{\boldsymbol{f}}(\boldsymbol{s}_p) \Big] \mathrm{e}^{-ik\,\boldsymbol{\beta}\cdot\boldsymbol{s}_p}, \tag{4.2}$$

where $(\boldsymbol{E}_I^{\boldsymbol{f}}, \boldsymbol{H}_I^{\boldsymbol{f}})$ denote the electromagnetic Herglotz wave associated to $\boldsymbol{f}$ defined by (2.5). Finally, let us notice that

$$(\mathbf{F}^0 \boldsymbol{f})(\boldsymbol{\beta}) = -\sum_{p=1}^{M} \Big[\Delta(\boldsymbol{\beta}) \boldsymbol{\mathcal{P}}_p \boldsymbol{E}_I^{\boldsymbol{f}}(\boldsymbol{s}_p) + \boldsymbol{\beta} \times (\boldsymbol{\mathcal{M}}_p \boldsymbol{H}_I^{\boldsymbol{f}}(\boldsymbol{s}_p)) \Big] \mathrm{e}^{-ik\,\boldsymbol{\beta}\cdot\boldsymbol{s}_p}, \tag{4.3}$$

where we have set for every $\boldsymbol{\alpha} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \in S^2$:

$$\Delta(\boldsymbol{\alpha}) = \mathbf{I} - \boldsymbol{\alpha}\boldsymbol{\alpha}^T. \tag{4.4}$$

Remark 4.1. *Note that formula* (4.2) *shows that, the electric far field radiated by the scatterers as their size tends to* 0 *corresponds to the superposition of* M *(uncoupled) electric and magnetic dipoles located at the points* $\boldsymbol{s}_p$ *and associated with the electric and magnetic moments* $\boldsymbol{p}_p = \boldsymbol{\mathcal{P}}_p\, \boldsymbol{E}_I^{\boldsymbol{f}}(\boldsymbol{s}_p)$ *and* $\boldsymbol{m}_p = \boldsymbol{\mathcal{M}}_p\, \boldsymbol{H}_I^{\boldsymbol{f}}(\boldsymbol{s}_p)$.

Remark 4.2. *Formula* (4.3) *shows that* $\mathbf{F}^0$ *has at most* $6M$ *nonzero eigenvalues, since its range satisfies* $\operatorname{Ran}\mathbf{F}^0 \subset \bigoplus_{1\le p\le M} \Big\{ \big(\Delta(\boldsymbol{\beta})\operatorname{Ran}\boldsymbol{\mathcal{P}}_p\big) \oplus \big(\boldsymbol{\beta}\times \operatorname{Ran}\boldsymbol{\mathcal{M}}_p\big)\Big\}$.

The aim of this section is twofold: first, to compute approximate eigenfunctions of $\mathbf{F}^0$, and then to prove that these eigenfunctions selectively focus on the scatterers. As we will see, this can be achieved provided the two following assumptions are satisfied:

1. The polarizability tensors $\boldsymbol{\mathcal{P}}_p$ and $\boldsymbol{\mathcal{M}}_p$ are diagonal (in the same basis). This is in particular true for axially symmetric scatterers (cf. [12, p. 167]).
2. The scatterers are distant enough (well separated scatterers). More precisely, we assume that $kd \to \infty$, where $d = \min\limits_{1\le p<q\le N} |\boldsymbol{s}_p - \boldsymbol{s}_q|$ is the minimal distance between the obstacles.

From now on, we will assume that these two conditions are satisfied.

Theorem 4.3. *For* $p \in \{1,\dots,M\}$, *let* $(\boldsymbol{e}_{p,1}, \boldsymbol{e}_{p,2}, \boldsymbol{e}_{p,3})$ *be an orthonormal basis of* $\mathbb{R}^3$ *such that the polarizability tensors* $\boldsymbol{\mathcal{P}}_p, \boldsymbol{\mathcal{M}}_p$ *of the reference scatterer* $\mathcal{O}_p$ *are diagonal:*

$$\boldsymbol{\mathcal{P}}_p = \begin{bmatrix} \lambda_{p,1} & 0 & 0 \\ 0 & \lambda_{p,2} & 0 \\ 0 & 0 & \lambda_{p,3} \end{bmatrix} \qquad \boldsymbol{\mathcal{M}}_p = \begin{bmatrix} \lambda'_{p,1} & 0 & 0 \\ 0 & \lambda'_{p,2} & 0 \\ 0 & 0 & \lambda'_{p,3} \end{bmatrix} \tag{4.5}$$

Given $\ell \in \{1,2,3\}$, *define the following elements of* $L_t^2(S^2)$ *(recall that* $\Delta(\boldsymbol{\alpha})$ *is defined by* (4.4)*):*

$$\begin{cases} \boldsymbol{f}_{p,\ell}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}\times(\boldsymbol{\alpha}\times \boldsymbol{e}_{p,\ell})\, \mathrm{e}^{-ik\,\boldsymbol{\alpha}\cdot\boldsymbol{s}_p} = -\Delta(\boldsymbol{\alpha})\boldsymbol{e}_{p,\ell}\, \mathrm{e}^{-ik\,\boldsymbol{\alpha}\cdot\boldsymbol{s}_p}, \\ \boldsymbol{g}_{p,\ell}(\boldsymbol{\alpha}) = (\boldsymbol{\alpha}\times \boldsymbol{e}_{p,\ell})\, \mathrm{e}^{-ik\,\boldsymbol{\alpha}\cdot\boldsymbol{s}_p}, \end{cases} \qquad \boldsymbol{\alpha}\in S^2. \tag{4.6}$$

Then, the family of functions $\{\boldsymbol{f}_{p,\ell},\ \boldsymbol{g}_{p,\ell}\ ;\ 1\le \ell\le 3\,, 1\le p\le M\}$ *is linearly independent in* $L_t^2(S^2)$. *Moreover, the functions* $\boldsymbol{f}_{p,\ell}$ *and* $\boldsymbol{g}_{p,\ell}$ *constitute approximate eigenfunctions of the limit far field operator* $\mathbf{F}^0$ *defined by* (3.19)-(4.1) *as* $kd\to\infty$*:*

$$\begin{cases} \mathbf{F}^0\boldsymbol{f}_{p,\ell} = -\dfrac{8\pi}{3}\lambda_{p,\ell}\ \boldsymbol{f}_{p,\ell} + O\left((kd)^{-N}\right), \\ \mathbf{F}^0\boldsymbol{g}_{p,\ell} = -\dfrac{8\pi}{3}\lambda'_{p,\ell}\ \boldsymbol{g}_{p,\ell} + O\left((kd)^{-N}\right), \end{cases} \qquad \textit{for all } N\in\mathbb{N}. \tag{4.7}$$

Proof. To see that the functions $\boldsymbol{f}_{p,\ell}$ and $\boldsymbol{g}_{p,\ell}$, for $\ell = 1,2,3$ and $p = 1,\dots,M$, are linearly independent, it suffices to note that these functions are exactly the far field patterns of electric and magnetic dipoles located at the points $\boldsymbol{s}_p$ and associated with electric or magnetic dipole moment $\boldsymbol{e}_{p,\ell}$. Consequently, by uniqueness of the far field pattern (which follows from Rellich's lemma, cf. [11]), the condition

$$\sum_{p=1}^{M}\sum_{\ell=1}^{3}(z_{p,\ell}\,\boldsymbol{f}_{p,\ell} + z'_{p,\ell}\,\boldsymbol{g}_{p,\ell}) = 0 \qquad z_{p,\ell},\, z'_{p,\ell}\in\mathbb{C},$$

implies that $z_{p,\ell} = z'_{p,\ell} = 0$ for all $p = 1, \dots, M$ and $\ell = 1, 2, 3$.
Fix now $q \in \{1, \dots, M\}$ and $\ell \in \{1, 2, 3\}$ and let us compute $\mathbf{F}^0 \boldsymbol{f}_{q,\ell}$. We have

$$\begin{cases} \boldsymbol{E}_I^{\boldsymbol{f}_{q,\ell}}(\boldsymbol{s}_p) = -\left(\displaystyle\int_{S^2} \Delta(\boldsymbol{\alpha})\, \mathrm{e}^{ik\,\boldsymbol{\alpha}\cdot(\boldsymbol{s}_p - \boldsymbol{s}_q)}\, \mathrm{d}\boldsymbol{\alpha}\right) \boldsymbol{e}_{q,\ell} := \boldsymbol{D}_{pq}\, \boldsymbol{e}_{q,\ell} \\ \boldsymbol{H}_I^{\boldsymbol{f}_{q,\ell}}(\boldsymbol{s}_p) = \displaystyle\int_{S^2} \boldsymbol{\alpha} \times [\boldsymbol{\alpha} \times (\boldsymbol{\alpha} \times \boldsymbol{e}_{q,\ell})]\, \mathrm{e}^{ik\,\boldsymbol{\alpha}\cdot(\boldsymbol{s}_p - \boldsymbol{s}_q)}\, \mathrm{d}\boldsymbol{\alpha} := \boldsymbol{D}'_{pq}\, \boldsymbol{e}_{q,\ell} \end{cases}$$

A straightforward computation shows that

$$\boldsymbol{D}_{qq} = -\int_{S^2} \Delta(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} = -\frac{8\pi}{3}\mathbf{I}$$

while by symmetry

$$\boldsymbol{D}'_{qq} = \int_{S^2} \boldsymbol{\alpha} \times \left[\boldsymbol{\alpha} \times (\boldsymbol{\alpha} \times \boldsymbol{e}_{q,\ell})\right] \mathrm{d}\boldsymbol{\alpha} = 0.$$

On the other hand, let us note that the elements of the 3×3 matrices $\boldsymbol{D}_{pq}$ and $\boldsymbol{D}'_{pq}$ for $p \neq q$ are oscillatory integrals of the form $\displaystyle\int_{S^2} \psi(\boldsymbol{\alpha})\, \mathrm{e}^{ik\,\boldsymbol{\alpha}\cdot(\boldsymbol{s}_p - \boldsymbol{s}_q)}\, \mathrm{d}\boldsymbol{\alpha}$, where ψ is a smooth function. It follows then from the stationary phase theorem (see for instance [33, Chap. VIII]) that

$$\boldsymbol{D}_{pq} = \boldsymbol{D}'_{pq} = O\left((kd)^{-N}\right) \qquad \forall\, p \neq q,\ \forall N \in \mathbb{N}.$$

Consequently, formula (4.3) simplifies into

$$\begin{aligned} (\mathbf{F}^0 \boldsymbol{f}_{q,\ell})(\boldsymbol{\beta}) &= -\frac{8\pi}{3}\Delta(\boldsymbol{\beta})\boldsymbol{\mathcal{P}}_q \boldsymbol{e}_{q,\ell}\, \mathrm{e}^{-ik\,\boldsymbol{\beta}\cdot\boldsymbol{s}_q} + O\left((kd)^{-N}\right), \\ &= -\frac{8\pi}{3}\lambda_{q,\ell}\, \Delta(\boldsymbol{\beta})\boldsymbol{e}_{q,\ell}\, \mathrm{e}^{-ik\,\boldsymbol{\beta}\cdot\boldsymbol{s}_q} + O\left((kd)^{-N}\right), \end{aligned}$$

which proves the first relation of (4.7). The second relation of (4.7) follows using the same arguments, since

$$\begin{cases} \boldsymbol{E}_I^{\boldsymbol{g}_{q,\ell}}(\boldsymbol{s}_p) = -\boldsymbol{H}_I^{\boldsymbol{f}_{q,\ell}}(\boldsymbol{s}_p) = -\boldsymbol{D}'_{pq}\, \boldsymbol{e}_{q,\ell} \\ \boldsymbol{H}_I^{\boldsymbol{g}_{q,\ell}}(\boldsymbol{s}_p) = \boldsymbol{E}_I^{\boldsymbol{f}_{q,\ell}}(\boldsymbol{s}_p) = \boldsymbol{D}_{pq}\, \boldsymbol{e}_{q,\ell} \end{cases}$$

and the proof is thus complete.□

Remark 4.4. *In the special case of scattering by small triaxial ellipsoids (see [12, Chap. 8]), with semi-axes* $a_{p,1} > a_{p,2} > a_{p,3}$*, the electric and magnetic polarizability tensors admit in the basis constituted by the axis of each ellipsoid the diagonal form* (4.5)*, with*

$$\begin{cases} \lambda_{p,\ell} = \dfrac{4\pi}{3I_{p,\ell}} \\ \lambda'_{p,\ell} = \dfrac{4\pi}{3}\dfrac{a_{p,1}a_{p,2}a_{p,3}}{1 - a_{p,1}a_{p,2}a_{p,3}I_{p,\ell}} \end{cases} \qquad \ell = 1, 2, 3,$$

with

$$I_{p,\ell} = \frac{2\pi}{3}\int_0^\infty \frac{\mathrm{d}x}{(x + a_{p,\ell}^2)\sqrt{x^2 + a_{p,1}^2}\sqrt{x^2 + a_{p,2}^2}\sqrt{x^2 + a_{p,3}^2}}.$$

In the special case of spheres of radii a_p*, we have* $\boldsymbol{\mathcal{P}}_p = 2\boldsymbol{\mathcal{M}}_p = 4\pi a_p^3 I$.

The next result provides the expected selective focusing properties of the eigenfunctions of the far field operator $\mathbf{F}^0$ (and thus of time reversal operator $\mathbf{T}^0 = (\mathbf{F}^0)^*\mathbf{F}^0$).

THEOREM 4.5. *For* $p \in \{1, \ldots, M\}$*, the approximate eigenfunctions* $(\boldsymbol{f}_{p,\ell}, \boldsymbol{g}_{p,\ell})_{1\le\ell\le3}$ *defined by* (4.6) *generate electromagnetic Herglotz waves that focus selectively on the scatterer* p*.*

Proof. Plugging the expression (4.6) of $\boldsymbol{f}_{p,\ell}$ and $\boldsymbol{g}_{p,\ell}$ in (2.5), we obtain that

$$\begin{cases} \boldsymbol{E}_I^{\boldsymbol{f}_{p,\ell}}(\boldsymbol{x}) = \boldsymbol{H}_I^{\boldsymbol{g}_{p,\ell}}(\boldsymbol{x}) = \displaystyle\int_{S^2} (\boldsymbol{\alpha} \times (\boldsymbol{\alpha} \times \boldsymbol{e}_{p,\ell}))\, \mathrm{e}^{ik\,\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{s}_p)}\, \mathrm{d}\boldsymbol{\alpha}, \\ \boldsymbol{H}_I^{\boldsymbol{f}_{p,\ell}}(\boldsymbol{x}) = -\boldsymbol{E}_I^{\boldsymbol{g}_{p,\ell}}(\boldsymbol{x}) = -\displaystyle\int_{S^2} (\boldsymbol{\alpha} \times \boldsymbol{e}_{p,\ell})\, \mathrm{e}^{ik\,\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{s}_p)}\, \mathrm{d}\boldsymbol{\alpha}. \end{cases}$$

The conclusion follows once again from the stationary phase theorem, since for $\boldsymbol{x} \neq \boldsymbol{s}_p$ the above integrals behave like $O\left((k|x-s_p|)^{-N}\right)$ for all $N \in \mathbb{N}$, and are independent of k for $\boldsymbol{x} = \boldsymbol{s}_p$. □

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