L^p weighted theory for Navier-Stokes equations in exterior domains

Chérif Amrouche, Huy Hoang Nguyen

L^p weighted theory for Navier-Stokes equations in exterior domains

Chérif AMROUCHE⋆ and Huy Hoang NGUYEN†

Laboratoire de Mathématiques Appliquées

CNRS UMR 5142

Université de Pau et des Pays de l'Adour

IPRA - Avenue de l'Université 64013 Pau, France

⋆ *cherif.amrouche@univ-pau.fr*

† *huy-hoang.nguyen@etud.univ-pau.fr*

In honor of Professor Peter D. LAX

Abstract - This paper is devoted to some mathematical questions related to the stationary Navier-Stokes problem in three-dimensional exterior domains. Our approach is based on a combination of properties of Oseen problems in $\mathbb{R}^3$ and in exterior domains of $\mathbb{R}^3$.

Keywords: Navier-Stokes equations; Oseen equations; weighted Sobolev spaces; fluid mechanics.

AMS class: 35Q30, 76D03, 76D05, 76D07

1 Introduction and preliminary results

This paper continues our previous studies in [7] related to the three-dimensional stationary Navier-Stokes equations. Let Ω' be a bounded open region of $\mathbb{R}^3$, not necessarily connected, with a Lipschitz-continuous boundary and let Ω be the complement of $\overline{\Omega'}$. We suppose that Ω' has a finite number of connected components and each connected component has a connected boundary, so that Ω is connected. In this paper, we study the following exterior Navier-Stokes problem:

$$(\mathcal{NS})\quad \begin{cases} -\nu\Delta \boldsymbol{u} + \boldsymbol{u}.\nabla\boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{in } \Omega, \\ \text{div } \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \Gamma, \\ \boldsymbol{u} \to \boldsymbol{u}_\infty & \text{at infinity,} \end{cases}$$

where $\nu > 0$, $\boldsymbol{f}$ and $\boldsymbol{u}_\infty \in \mathbb{R}^3$ are respectively the viscosity of the fluid, the external force field acting on the fluid and a given constant vector of $\mathbb{R}^3$. The problem consists in looking for the velocity field $\boldsymbol{u} = (u_1, u_2, u_3)$ of the fluid and the pressure function π. We shall assume that the origin of the coordinate frame is attached to Ω'. The third equation of the system states that the fluid adheres at the surface of the body, which is the common no-slip condition. Since the domain Ω is unbounded, the last equation is really necessary. In this equation, we have two different cases concerning the behavior of $\boldsymbol{u}$ at infinity. If $\boldsymbol{u}_\infty = 0$, the flow is at rest at infinity and in the remaining case, if $\boldsymbol{u}_\infty \neq 0$, the flow is past at infinity.

In this paper, we are interested in considering the case Ω being an exterior domain in $\mathbb{R}^3$ and $\boldsymbol{u}_\infty \neq 0$. We note that the case $\Omega = \mathbb{R}^3$ was considered in our previous paper [7]. Our purpose is to study some regularity properties of the weak solutions to the problem $(\mathcal{NS})$.

To our knowledge, in the three-dimensional situation, following Farwig [12] and Galdi [13], they consider the problem $(\mathcal{NS})$ in the case $\boldsymbol{u}_\infty = 0$ or $\boldsymbol{u}_\infty \neq 0$. In the case $\boldsymbol{u}_\infty \neq 0$, they consider the external force field $\boldsymbol{f}$ belonging to the classical spaces $\mathbf{L}^p(\Omega)$, and in [12] with the weight $(1 + |\boldsymbol{x}\,|)^\alpha$ for some p and $\alpha \in]0, 1[$. The solutions are obtained in the homogeneous Sobolev spaces with or without the weight. In this paper, we are interested in the case in which the external force field belongs to the weighted Sobolev spaces $\mathbf{W}_0^{-1,p}$, that permits us to obtain generalized solutions in the weighted Sobolev spaces $\mathbf{W}_0^{1,p}$. We consider also the case in which the external force field belongs to $\mathbf{L}^q$ or $\mathbf{L}^q \cap \mathbf{W}_0^{-1,p}$ and some regularity properties. Our main interest is directed at L^p-regularity of weak solutions, under suitable assumptions on the right-hand side $\boldsymbol{f}$. This point is improved in this paper. We assume different levels of regularity of $\boldsymbol{f}$, and then describe the corresponding level of smoothness of the weak solutions associated to $\boldsymbol{f}$. We refine a regularity theory which may be found in [13]. Galdi assumes that $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ for all $p \in (1, p_0]$, with some $p_0 > 3$ (see Section IX.7 [13]). More precisely, in Theorem 5.9, we recover Galdi's regularity results.

This paper is organised as follows: In this section, the problem will be introduced and we recall well-known results about weighted Sobolev spaces. In Section 2, a result about existence of weak solutions for the problem $(\mathcal{NS})$ will be presented. In next sections, we shall obtain some regularity properties of the weak solution $\boldsymbol{u}$ and the associated pressure π. In Section 4, the exterior Oseen problem is considered. The identity energy will be given in the last section.

In this paper, we use bold type characters to denote vector distributions or spaces of vector distributions with 3 components and $C > 0$ usually denotes a generic constant the value of which may change from line to line. We shall also denote by B_R the open ball of radius $R > 0$ centered at the origin and $B^R = \mathbb{R}^3 - \overline{B_R}$. In particular, since Ω' is bounded, we can find some R_0 such that $\Omega' \subset B_{R_0}$ and we introduce, for any $R \geq R_0$, the set

$$\Omega_R = \Omega \cap B_R \text{ and } \Omega^R = \Omega - \overline{\Omega_R}.$$

We now recall the main notations and results , concerning the weighted Sobolev spaces, which we shall use later on.
We define $\mathcal{D}(\Omega)$ to be the linear space of infinite differentiable functions with compact support on Ω. Now, let $\mathcal{D}'(\Omega)$ denote the dual space of $\mathcal{D}(\Omega)$, often called the space of distributions on Ω. We denote by $\langle .,. \rangle$ the duality pairing between $\mathcal{D}(\Omega)'$ and $\mathcal{D}(\Omega)$. Remark that when $\boldsymbol{f}$ is a locally integrable function,

then $\boldsymbol{f}$ can be identified with a distribution by

$$\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle = \int_\Omega \boldsymbol{f}\,(\boldsymbol{x})\,.\,\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}.$$

Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ (or more simply $X^\perp$, if there is no ambiguity as to the duality product) the subspace of B' orthogonal to X, *i.e.*

$$B' \perp X = X^\perp = \{f \in B' | \forall\ v \in X, < f, v >= 0\} = (B/X)'.$$

The space $X^\perp$ is also called the polar space of X in B'. A typical point in $\mathbb{R}^3$ is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$ and its norm is given by $r = |\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$. We define the weight function $\rho(\boldsymbol{x}) = 1 + r$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. We now define the weighted Sobolev space

$$W_0^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_1} \in L^p(\Omega), \nabla u \in \mathbf{L}^p(\Omega)\},$$

where

$$w_1 = \begin{cases} (1+r) & \text{if } p \neq 3, \\ (1+r)\,\ln(2+r) & \text{if } p = 3. \end{cases}$$

This space is a reflexive Banach space when endowed with the norm:

$$||\,u||_{W_0^{1,p}(\Omega)} = (||\frac{u}{w_1}||^p_{L^p(\Omega)} + ||\nabla u\,||^p_{\mathbf{L}^p(\Omega)})^{1/p}.$$

We also introduce the space

$$W_0^{2,p}(\Omega) = \{u \in \mathcal{D}'(\Omega), \frac{u}{w_2} \in L^p(\Omega), \frac{\nabla u}{w_1} \in \mathbf{L}^p(\Omega), D^2 u \in \mathbf{L}^p(\Omega)\},$$

where

$$w_2 = \begin{cases} (1+r)^2 & \text{if } p \notin \{\frac{3}{2}, 3\}, \\ (1+r)^2\,\ln(2+r), & \text{otherwise}, \end{cases}$$

which is a Banach space equipped with its natural norm given by

$$||\,u||_{W_0^{2,p}(\Omega)} = (||\frac{u}{w_2}||^p_{L^p(\Omega)} + ||\frac{\nabla u}{w_1}||^p_{\mathbf{L}^p(\Omega)} + ||D^2 u||^p_{\mathbf{L}^p(\Omega)})^{1/p}.$$

We note that the logarithmic weight only appears if $p = 3$ or $p = \frac{3}{2}$ and all the local properties of $W_0^{1,p}(\Omega)$ (respectively, $W_0^{2,p}(\Omega)$) coincide with those of the corresponding classical Sobolev space $W^{1,p}(\Omega)$ (respectively, $W^{2,p}(\Omega)$). For $m = 1$ or $m = 2$, we set $\overset{\circ}{W}{}_0^{m,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W_0^{m,p}(\Omega)}$ and we denote the dual space of $\overset{\circ}{W}{}_0^{m,p}(\Omega)$ by $W_0^{-m,p'}(\Omega)$, which is the space of distributions. When $\Omega = \mathbb{R}^3$, we have $W_0^{m,p}(\mathbb{R}^3) = \overset{\circ}{W}{}_0^{m,p}(\mathbb{R}^3)$. If Ω is a Lipschitz exterior domain, then

$$\overset{\circ}{W}{}_0^{1,p}(\Omega) = \{\, v \in W_0^{1,p}(\Omega);\ v = 0 \text{ on } \Gamma\,\}.$$

If Ω is a $C^{1,1}$ exterior domain, then

$$\overset{\circ}{W}{}_0^{2,p}(\Omega) \;=\; \{\, v \in W_0^{2,p}(\Omega);\; v = \partial_n v = 0 \text{ on } \Gamma \,\},$$

where $\partial_n v$ is the normal derivative of v. For all $\lambda \in \mathbb{N}^3$ where $0 \leq |\lambda| \leq 2m$ with $m = 1$ or $m = 2$, the mapping

$$u \in W_0^{m,p}(\Omega) \to \partial^\lambda u \in W_0^{m-|\lambda|,p}(\Omega)$$

is continuous. Also recall the following Sobolev embeddings (see [1]):

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \;\text{ where }\; p^* = \frac{3p}{3-p} \;\text{ and }\; 1 < p < 3. \tag{1.1}$$

Consequently, by duality, we have

$$L^q(\Omega) \hookrightarrow W_0^{-1,p'}(\Omega) \;\text{ where }\; q = \frac{3p'}{3+p'} \;\text{ and }\; p' > 3/2. \tag{1.2}$$

Note also that if $\nabla u \in \mathbf{L}^p(\Omega)$ with $p > 3$ (respectively, $p = 3$) and $u \in L^r(\Omega)$ for some $r \geq 1$, then we have $u \in L^\infty(\Omega)$ (respectively, $u \in L^q(\Omega)$ for any real $q \geq r$). Moreover,

- For all $u \in W_0^{1,3}(\Omega) \cap L^r(\Omega)$, we have

$$||\, u \,||_{L^q} \leq C\,(\; ||\, \nabla u \,||_{\mathbf{L}^3} + ||\, u \,||_{L^r}) \;\text{ for all } q \geq r; \tag{1.3}$$

- For all $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$ with $p > 3$, we have

$$||\, u \,||_{L^q} \leq C\,(\; ||\, \nabla u \,||_{\mathbf{L}^p} + ||\, u \,||_{L^r}) \;\text{ for all } q \in [\, r, \infty\,]. \tag{1.4}$$

We introduce the space

$$X_0^{1,p}(\Omega) = \left\{ u \in W_0^{1,p}(\Omega);\; \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\Omega) \right\}$$

which is a Banach space equipped with the following norm

$$||u||_{X_0^{1,p}(\Omega)} = ||\frac{u}{w_1}||_{L^p(\Omega)} + \sum_{i=1}^{3} ||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)} + ||\frac{\partial u}{\partial x_1}||_{W_0^{-1,p}(\Omega)}.$$

We also introduce the space

$$\overset{\circ}{X}{}_0^{1,p}(\Omega) \;=\; \{\, u \in X_0^{1,p}(\Omega);\; u = 0 \text{ on } \Gamma \,\},$$

and we know that $\mathcal{D}(\Omega)$ is dense in $\overset{\circ}{X}{}_0^{1,p}(\Omega)$ (cf. [10]). Now we introduce the following Lemma.

Lemma 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain. Assume that $u \in \overset{\circ}{W}{}_0^{1,p}(\Omega)$ such that $\dfrac{\partial u}{\partial x_1} \in L^q(\Omega)$ with $1 < \frac{2}{p} + \frac{1}{q}$. Then $u \in L^r(\Omega)$ with $\frac{1}{r} = \frac{1}{3}(\frac{2}{p} + \frac{1}{q} - 1)$ and we have the estimate as follows*

$$||u||_{L^r(\Omega)} \leq C(||u||_{W_0^{1,p}(\Omega)} + ||\frac{\partial u}{\partial x_1}||_{L^q(\Omega)}). \tag{1.5}$$

Proof. We extend u by zero outside Ω and denote $\widetilde{u}$ the extended function. Then $\widetilde{u} \in W_0^{1,p}(\mathbb{R}^3)$ and $\dfrac{\partial \widetilde{u}}{\partial x_1} \in L^q(\mathbb{R}^3)$. We set

$$X_{p,q}(\mathbb{R}^3) = \{v \in W_0^{1,p}(\mathbb{R}^3);\ \frac{\partial v}{\partial x_1} \in L^q(\mathbb{R}^3)\}.$$

It is easy to prove that $\mathcal{D}(\mathbb{R}^3)$ is dense in $X_{p,q}(\mathbb{R}^3)$, *i.e*, there exists $\varphi_k \in \mathcal{D}(\mathbb{R}^3)$ such that $\varphi_k \to \widetilde{u}$ in $X_{p,q}(\mathbb{R}^3)$. Thanks to Babenko [11], we have the following inequality

$$\begin{aligned} ||\varphi_k||_{L^r(\mathbb{R}^3)} &\leq C||\frac{\partial \varphi_k}{\partial x_2}||^{1/3}_{L^p(\mathbb{R}^3)}||\frac{\partial \varphi_k}{\partial x_3}||^{1/3}_{L^p(\mathbb{R}^3)}||\frac{\partial \varphi_k}{\partial x_1}||^{1/3}_{L^q(\mathbb{R}^3)} \\ &\leq C(||\nabla \varphi_k||_{L^p(\mathbb{R}^3)} + ||\frac{\partial \varphi_k}{\partial x_1}||_{L^q(\mathbb{R}^3)}) \end{aligned}$$

with $\frac{1}{r} = \frac{1}{3}(\frac{2}{p} + \frac{1}{q} - 1)$. Since (φ_k) is bounded in $L^r(\mathbb{R}^3)$, then $\widetilde{u} \in L^r(\mathbb{R}^3)$ and we obtain (1.5). $\square$

We introduce the

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain and $u \in \overset{\circ}{X}{}_0^{1,\,p}(\Omega)$.*
i) If $1 < p < 3$, then $u \in L^{\frac{4p}{4-p}}(\Omega) \cap L^{\frac{3p}{3-p}}(\Omega)$ and the following estimate holds

$$||\, u\,||_{L^{\frac{4p}{4-p}}(\Omega)} + ||\, u\,||_{L^{\frac{3p}{3-p}}(\Omega)} \;\leq\; C\,||\, u\,||_{X_0^{1,\,p}(\Omega)}. \tag{1.6}$$

ii) If $p = 3$, then there exists a unique constant $k(u)$ such that $u + k(u) \in \bigcap_{r\geq 12} L^r(\Omega)$ and the following estimate holds

$$||\, u + k(u)\,||_{L^r(\Omega)} \;\leq\; C\,||\, u\,||_{X_0^{1,\,p}(\Omega)} \quad \textit{for any} \quad r \geq 12. \tag{1.7}$$

iii) If $3 < p < 4$, then there exists a unique constant $k(u)$ such that $u + k(u) \in L^{4p/(4-p)}(\Omega) \cap L^\infty(\Omega)$ and the following estimate holds

$$||\, u + k(u)\,||_{L^{\frac{4p}{4-p}}(\Omega)} + ||\, u + k(u)\,||_{L^\infty(\Omega)} \;\leq\; C\,||\, u\,||_{X_0^{1,\,p}(\Omega)}. \tag{1.8}$$

Proof. Let $u \in \overset{\circ}{X}{}_0^{1,\,p}(\Omega)$ with $1 < p < 4$. Extend u by zero outside Ω and denote $\widetilde{u}$ by the extended function. It is clear that $\widetilde{u}$ belongs to $W_0^{1,p}(\mathbb{R}^3)$. It remains to prove that $\dfrac{\partial \widetilde{u}}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$. Let $R_0 > 0$ be a real and sufficient large such that $\overline{\Omega'}$ is contained in B_{R_0} and R_1, R_2 be reals such that $R_2 > R_1 > R_0$. Choose now some functions ψ_1 and ψ_2 satisfying

$$\psi_1 \in C^\infty(\mathbb{R}^3),\ \psi_1(\boldsymbol{x}) = 0 \text{ if } |\boldsymbol{x}| \leq R_1,\ \psi_1(\boldsymbol{x}) = 1 \text{ if } |\boldsymbol{x}| \geq R_2,$$

$$\forall \boldsymbol{x} \in \mathbb{R}^3, \psi_1(\boldsymbol{x}) + \psi_2(\boldsymbol{x}) = 1.$$

We then can write $\widetilde{u} = \widetilde{u}\psi_1 + \widetilde{u}\psi_2 = \widetilde{u}_1 + \widetilde{u}_2$. It is easy to prove that $\dfrac{\partial \widetilde{u}_1}{\partial x_1}$ and $\dfrac{\partial \widetilde{u}_2}{\partial x_1}$ belong to $W_0^{-1,p}(\mathbb{R}^3)$, then $\dfrac{\partial \widetilde{u}}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$ and we can deduce $\widetilde{u} \in X_0^{1,p}(\mathbb{R}^3)$. Moreover,

$$||\,\widetilde{u}\,||_{X_0^{1,p}(\mathbb{R}^3)} \;\leq\; C\,||\,u\,||_{X_0^{1,p}(\Omega)}.$$

Since $-\Delta\widetilde{u} + \dfrac{\partial \widetilde{u}}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^3)$ and

$$< -\Delta\widetilde{u} + \frac{\partial \widetilde{u}}{\partial x_1}, 1 >_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)} = 0 \quad \text{if} \quad p < 3/2,$$

we know from Theorem 4.4 [9] there exists a unique $v \in X_0^{1,p}(\mathbb{R}^3) \cap L^{4p/(4-p)}(\mathbb{R}^3)$ such that

$$-\Delta v + \frac{\partial v}{\partial x_1} \; = \; -\Delta\widetilde{u} + \frac{\partial \widetilde{u}}{\partial x_1}$$

and satisfying the following estimate

$$\begin{aligned} \|\, v \,\|_{X_0^{1,p}(\mathbb{R}^3)} + \|\, v \,\|_{L^{4p/(4-p)}(\mathbb{R}^3)} \quad &\leq \; C \,\|\, -\Delta\widetilde{u} + \frac{\partial \widetilde{u}}{\partial x_1} \,\|_{W_0^{-1,p}(\mathbb{R}^3)} \\ &\leq \; C \,\|\, u \,\|_{X_0^{1,p}(\Omega)}. \end{aligned} \tag{1.9}$$

The function $z = \widetilde{u} - v \in X_0^{1,p}(\mathbb{R}^3)$ verifying the equation

$$-\Delta z + \frac{\partial z}{\partial x_1} = 0 \quad \text{in} \quad \mathbb{R}^3,$$

then z is a polynomial that belongs to $W_0^{1,p}(\mathbb{R}^3)$. Therefore, there exists a constant k such that $z = k$, with $k = 0$ if $1 < p < 3$. It means that $u - k = v$ in Ω. The estimate (1.6) is immediately deduced from (1.9). The estimates (1.7) and (1.8) are consequences of (1.3) and (1.4). $\square$

Remark 1.3. The above result is available for all $u \in X_0^{1,p}(\Omega)$ because we know that u can be extended by $Pu \in X_0^{1,p}(\mathbb{R}^3)$.

Defining now

$$X_0^{2,p}(\Omega) \;\; = \;\; \{\, v \in W_0^{2,p}(\Omega); \;\; \frac{\partial v}{\partial x_1} \in L^p(\Omega) \,\}.$$

Note that

$$W_0^{2,p}(\Omega) \hookrightarrow L^{p*}(\Omega) \quad \text{where} \quad p* = \frac{3p}{3-2p} \quad \text{and} \quad 1 < p < 3/2.$$

By duality, we have

$$L^q \hookrightarrow W_0^{-2,p'}(\Omega) \quad \text{where} \quad q = \frac{3p'}{2p'+3} \quad \text{and} \quad p' > 3.$$

Note also that if $v \in W_0^{2,p}(\Omega)$ with $\frac{3}{2} \leq p < 3$ and $\nabla v \in \mathbf{L}^r(\Omega)$ for some r, then $\nabla v \in \mathbf{L}^q(\Omega)$ for all $q \geq r$ if $p = 3/2$ and $\nabla v \in \mathbf{L}^r(\Omega) \cap \mathbf{L}^\infty(\Omega)$ if $3/2 < p < 3$.

We now introduce a lemma concerning the extension of $X_0^{2,p}(\Omega)$ in $\mathbb{R}^3$.

Lemma 1.4. *Assume $v \in X_0^{2,p}(\Omega)$. Then there exists $\widetilde{v} \in X_0^{2,p}(\mathbb{R}^3)$ such that $\widetilde{v} = v$ in Ω and*

$$\|\, \widetilde{v} \,\|_{X_0^{2,p}(\mathbb{R}^3)} \;\leq\; C \,\|\, v \,\|_{X_0^{2,p}(\Omega)}. \tag{1.10}$$

Proof. We know that there exists an linear and continuous extended operator P of $W_0^{2,p}(\Omega)$ in $W_0^{2,p}(\mathbb{R}^3)$. Setting $\widetilde{v} = Pv$ and using again the partition of unity

$$\widetilde{v} = \widetilde{v}\psi_1 + \widetilde{v}\psi_2,$$

then it is easy to prove that $\widetilde{v} \in W_0^{2,p}(\mathbb{R}^3)$ and $\widetilde{v}$ satisfies the estimate (1.10). $\square$

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain and $u \in X_0^{2,p}(\Omega)$.*
i) If $1 < p < 3/2$, then $u \in L^{2p/(2-p)}(\Omega) \cap L^{3p/(3-2p)}(\Omega)$.
ii) If $3/2 \leq p < 2$, then there exists a unique constant k such that $u+k \in L^q(\Omega)$ for all $q \geq 2p/(2-p)$.

Proof. The proof is similar as in the one of Lemma 1.2 by using once again the partition of unity and Proposition 4.3 [9]. $\square$

Proposition 1.6. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain and $u \in X_0^{2,p}(\Omega)$.*
i) If $1 < p < 3$, then $\nabla u \in \mathbf{L}^{4p/(4-p)}(\Omega) \cap \mathbf{L}^{3p/(3-p)}(\Omega)$.
ii) If $p = 3$, then there exists a unique $k \in \mathscr{P}_1$, independent on x_1, such that $\nabla(u+k) \in \bigcap_{r \geq 12} \mathbf{L}^r(\Omega)$.
iii) If $3 < p < 4$, then there exists a unique $k \in \mathscr{P}_1$, independent on x_1, such that $\nabla(u+k) \in \mathbf{L}^{4p/(4-p)}(\Omega) \cap \mathbf{L}^{\infty}(\Omega)$.

Proof. This proposition is a consequence of Lemma 1.2 and Remark 1.3. $\square$

2 Existence of weak solutions in weighted Sobolev spaces

First of all, we shall study the existence of weak solutions of Navier-Stokes problem in weighted Sobolev spaces in this chapter. Without loss of generality, we can set $\boldsymbol{u}_\infty = \lambda \boldsymbol{e}_1$ with $\boldsymbol{e}_1 = (1,0,0)$ and $\lambda \geq 0$. From now on, we consider the case of a fixed $\lambda > 0$.

In 1933, Jean Leray [15] who introduced the concept of the weak solution:

Definition 2.1. A weak solution to the problem $(\mathcal{NS})$ is a field $\boldsymbol{u} \in \mathbf{H}^1_{loc}(\overline{\Omega})$ vanishing on $\partial\Omega$, with $\nabla \boldsymbol{u} \in \mathbf{L}^2(\Omega)$, div $\boldsymbol{u} = 0$ in Ω and $\lim_{|\boldsymbol{x}| \to \infty} \int_{S_2} |\boldsymbol{u}(\sigma|\boldsymbol{x}|) - \boldsymbol{u}_\infty| d\sigma = 0$ where S_2 is the unit sphere of $\mathbb{R}^3$ such that for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega), \text{div } \boldsymbol{v} = 0\}$:

$$\nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} \, d\boldsymbol{x} + \int_\Omega (\boldsymbol{u}.\nabla \boldsymbol{u}) . \boldsymbol{\varphi} \, d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle .$$

As in [2], it is easy to prove the following theorem.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain. Given a force $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega)$, the problem $(\mathcal{NS})$ has a weak solution $\boldsymbol{u}$ satisfying $\boldsymbol{u} - \boldsymbol{u}_\infty \in \mathbf{W}_0^{1,2}(\Omega)$ and there exists a function $\pi \in L^2_{loc}(\Omega)$, unique up to a constant, such that $(\boldsymbol{u}, \pi)$ solves the problem $(\mathcal{NS})$ in the sense of distributions and we have the following estimation*

$$||\boldsymbol{u} - \boldsymbol{u}_\infty||_{\mathbf{W}_0^{1,2}(\Omega)} \leq \frac{C}{\nu} ||\boldsymbol{f}||_{\mathbf{W}_0^{-1,2}(\Omega)} + C(\nu)|\boldsymbol{u}_\infty|(1 + |\boldsymbol{u}_\infty|). \tag{2.1}$$

In Theorem 2.2, we see that a pressure π locally belongs to $L^2(\Omega)$. At the beginning, we shall establish, without additional assumption, of the properties of integrability at infinity of the pressure.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and let $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega)$. The pressure π obtained in Theorem 2.2 has a representative such that*

$$\pi = \tau^1 + \tau^2 \textit{ with } \tau^1 \in L^2(\Omega) \textit{ and } \tau^2 \in W_0^{1,3/2}(\Omega).$$

Proof. Let R_1 and R_2 be reals such that $R_2 > R_1 > R_0$ and choose some functions ψ_1 and ψ_2 such that

$$\psi_1 \in C^\infty(\mathbb{R}^3),\ \psi_1(\boldsymbol{x}) = 0 \text{ if } |\boldsymbol{x}| \leq R_1,\ \psi_1(\boldsymbol{x}) = 1 \text{ if } |\boldsymbol{x}| \geq R_2,$$

$$\forall \boldsymbol{x} \in \mathbb{R}^3, \psi_1(\boldsymbol{x}) + \psi_2(\boldsymbol{x}) = 1.$$

Let $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_\infty$ where $\boldsymbol{u}$ is a solution given by Theorem 2.2 and let $\pi \in L^2_{loc}(\Omega)$ be the associated pressure. We define $(\boldsymbol{v}^1, \pi^1)$ as follows

$$(\boldsymbol{v}^1, \pi^1) = (\boldsymbol{v}\psi_1, \pi\psi_1) \text{ in } \Omega, \quad (\boldsymbol{v}^1, \pi^1) = (\mathbf{0}, 0) \text{ in } \overline{\Omega'},$$

and set $(\boldsymbol{v}^2, \pi^2) = (\boldsymbol{v}\psi_2, \pi\psi_2)$ in Ω. It is easy to check that $(\boldsymbol{v}^1, \pi^1) \in \mathbf{W}_0^{1,2}(\mathbb{R}^3) \times L^2_{loc}(\mathbb{R}^3)$ and $(\boldsymbol{v}^2, \pi^2) \in \mathbf{H}^1(\Omega_2) \times L^2(\Omega_2)$. Moreover, we can establish the equalities in the sense of distributions (respectively in $\boldsymbol{\mathcal{D}}'(\mathbb{R}^3)$ if $i = 1$ and in $\boldsymbol{\mathcal{D}}'(\Omega_2)$ if $i = 2$):

$$-\nu\Delta\boldsymbol{v}^i + \lambda\frac{\partial \boldsymbol{v}^i}{\partial x_1} + \nabla\pi^i = \boldsymbol{f}^{\,i} \text{ and } \operatorname{div} \boldsymbol{v}^i = g^i, \tag{2.2}$$

where

$$\begin{gathered}\boldsymbol{f}^{\,i} = [\boldsymbol{f}\psi_i - \nu\boldsymbol{v}\Delta\psi_i - 2\nu\nabla\boldsymbol{v}\nabla\psi_i + \pi\nabla\psi_i] + [\lambda\boldsymbol{v}\frac{\partial\psi_i}{\partial x_1} - (\boldsymbol{v}.\nabla\boldsymbol{v})\psi_i] := k_i + h_i,\\ g^i = -\boldsymbol{v}.\nabla\psi_i.\end{gathered} \tag{2.3}$$

Since ψ_1 is C^∞ on $\mathbb{R}^3$ with supp $\psi_1 \subset \Omega$, we have naturally denoted by $\boldsymbol{f}\psi_1$ the distributions on $\mathbb{R}^3$ given by:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3),\ <\boldsymbol{f}\psi_1, \boldsymbol{\varphi}>_{\mathbb{R}^3} = <\boldsymbol{f}, \boldsymbol{\varphi}\psi_1>_\Omega .$$

This notation also applies to each other term in the definition (2.3) with $i = 1$. Considering now with $i = 2$, the regularity of $\boldsymbol{v}$ and π near the boundary depends on the regularity of $(\boldsymbol{f}^{\,2}, g^2)$ and on the properties of the Oseen problem in the bounded domain Ω_2. Similarly, the regularity of $\boldsymbol{v}$ and π near the infinity depends on the regularity of $(\boldsymbol{f}^{\,1}, g^1)$ and on the properties of the Oseen problem in the bounded domain $\mathbb{R}^3$. We have $\pi = \pi^1 + \pi^2$ and from Theorem 2.2, we obtain $\pi^2 \in L^2(\Omega)$. Thus, the main of the proof deals with the properties of π^1 and therefore of $(\boldsymbol{f}^{\,1}, g^1)$. We consider

$$-\nu\Delta\boldsymbol{a}^1 + \lambda\frac{\partial\boldsymbol{a}^1}{\partial x_1} + \nabla b^1 = k_1 \text{ and } \operatorname{div} \boldsymbol{a}^1 = -\boldsymbol{v}\nabla\psi_1 \text{ in } \mathbb{R}^3. \tag{2.4}$$

Since ψ_1 is bounded and has bounded derivatives with compact support, it is easy to check that the term $\boldsymbol{f}\psi_1$, $\boldsymbol{v}\Delta\psi_1$, $\nabla\boldsymbol{v}\nabla\psi_1$ and $\pi\nabla\psi_1$ belong to $\mathbf{W}_0^{-1,2}(\mathbb{R}^3)$ and because $\mathbf{W}_0^{1,2}(\mathbb{R}^3) \subset \mathbf{L}^6(\mathbb{R}^3)$ then we have $\boldsymbol{v}.\dfrac{\partial\psi_1}{\partial x_1} \in \mathbf{L}^q(\mathbb{R}^3)$ for all $q \in [1, 6]$.

Even simple is to prove that $g^1 = -\boldsymbol{v}.\nabla\psi_1 \in L^2(\mathbb{R}^3) \cap W_0^{-1,2}(\mathbb{R}^3)$ and therefore $\dfrac{\partial g^1}{\partial x_1} \in W_0^{-2,2}(\mathbb{R}^3)$ satisfying the following compatibility condition

$$\left\langle \frac{\partial g^1}{\partial x_1}, 1 \right\rangle_{W_0^{-2,2}(\mathbb{R}^3)\times W_0^{2,2}(\mathbb{R}^3)} = 0.$$

Applying Theorem 1.10 [7], there exists a unique solution $(\boldsymbol{a}^1, b^1) \in (\mathbf{X}_0^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$ of (2.4) such that $\boldsymbol{a}^1 \in \mathbf{L}^{r_1}(\mathbb{R}^3)$ where $4 \leq r_1 \leq 6$. Thanks to Hölder inequality, we deduce that $(\boldsymbol{v}.\nabla\boldsymbol{v})\psi_1 \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ and, in particular, we have $\boldsymbol{v}.\dfrac{\partial\psi_1}{\partial x_1} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$. Therefore, from Theorem 1.9 (see [7]), the system as follows

$$-\nu\Delta\boldsymbol{a}^2 + \lambda\frac{\partial\boldsymbol{a}^2}{\partial x_1} + \nabla b^2 = h_1 \quad \text{and} \quad \text{div } \boldsymbol{a}^2 = 0 \text{ in } \mathbb{R}^3, \tag{2.5}$$

has a unique solution $(\boldsymbol{a}^2, b^2) \in \mathbf{L}^{s_1}(\mathbb{R}^3) \times W_0^{1,3/2}(\mathbb{R}^3)$ such that $\nabla\boldsymbol{a}^2 \in \mathbf{L}^{r_2}(\mathbb{R}^3)$, $\nabla^2\boldsymbol{a}^2 \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ and $\dfrac{\partial\boldsymbol{a}^2}{\partial x_1} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ for all $s_1 \in [6, \infty)$ and $r_2 \in [12/5, 3]$.
We set $\boldsymbol{z} = \boldsymbol{v}^1 - \boldsymbol{a}^1 - \boldsymbol{a}^2$ and $\theta = \pi^1 - b^1 - b^2$. Subtracting (2.2) to (2.4) and (2.5), we get

$$-\nu\Delta\boldsymbol{z} + \lambda\frac{\partial\boldsymbol{z}}{\partial x_1} + \nabla\theta = \mathbf{0} \quad \text{and} \quad \text{div } \boldsymbol{z} = 0 \text{ in } \mathbb{R}^3. \tag{2.6}$$

Therefore, we have

$$-\nu\Delta\mathbf{curl}\ \boldsymbol{z} + \lambda\frac{\partial(\mathbf{curl}\ \boldsymbol{z})}{\partial x_1} = \mathbf{0} \quad \text{in } \mathbb{R}^3,$$

and we get $\boldsymbol{\Psi} = \mathbf{curl}\ \boldsymbol{z}$, then for $i = 1, 2, 3$,

$$-\nu\Delta\Psi_i + \lambda\frac{\partial\psi_1}{\partial x_1} = 0 \quad \text{in } \mathbb{R}^3,$$

where $\Psi_i \in L^2(\mathbb{R}^3) + L^{r_2}(\mathbb{R}^3) \hookrightarrow S'(\mathbb{R}^3)$. Then, from Lemma 4.1 [9], $\boldsymbol{\Psi}$ is a polynomial which belongs to $\mathbf{L}^2(\mathbb{R}^3) + \mathbf{L}^{r_2}(\mathbb{R}^3)$. Consequently, $\boldsymbol{\Psi} = \mathbf{0} = \mathbf{curl}\ \boldsymbol{z}$ and div $\boldsymbol{z} = 0$. Therefore,

$$-\Delta\boldsymbol{z} = \mathbf{curl}\ \mathbf{curl}\ \boldsymbol{z} + \nabla \text{ div } \boldsymbol{z} = \mathbf{0} \quad \text{in } \mathbb{R}^3.$$

Similarly, it is easy to prove that $\boldsymbol{z}$ is a constant, then we can deduce from (2.6) that $\nabla\theta = 0$ and by the way the existence of a constant c such that $\pi^1 = b^1 + b^2 + c$. Therefore, the proposition is proved setting $\tau^1 = \pi^2 + b^1$, $\tau^2 = b^2$. □

3 Regularity of weak solutions

Let $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_\infty$ where $\boldsymbol{u}$ is the weak solution of the Navier-Stokes problem $(\mathcal{NS})$ given by Theorem 2.2. Then we rewrite the Navier-Stokes problem $(\mathcal{NS})$

as follows:

$$(\mathcal{NS})\quad\begin{cases} -\nu\Delta \boldsymbol{v} + \lambda\dfrac{\partial \boldsymbol{v}}{\partial x_1} + \nabla\pi = \boldsymbol{f} - \boldsymbol{v}.\nabla\boldsymbol{v} & \text{in } \Omega,\\ \operatorname{div}\, \boldsymbol{v} = 0 & \text{in } \Omega,\\ \boldsymbol{v} = -\boldsymbol{u}_\infty & \text{on } \Gamma,\\ \boldsymbol{v} \longrightarrow 0 & \text{if } |x| \to \infty. \end{cases} \tag{3.1}$$

We start our studies by adding assumptions on the force field $\boldsymbol{f}$. First, we assume additionally that $\boldsymbol{f} \in \mathbf{W}_0^{-1,3}(\Omega)$, and then, we will consider the case more generally $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega) \cap \mathbf{W}_0^{-1,p}(\Omega)$ with $p \geq 3$. Following this idea, we state and prove the

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with a $C^{1,1}$ boundary. Given $p \geq 3$ and $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega) \cap \mathbf{W}_0^{-1,p}(\Omega)$. Then, each weak solution $\boldsymbol{u}$ to the problem $(\mathcal{NS})$ satisfies*

$$\boldsymbol{v} \in \mathbf{W}_0^{1,2}(\Omega) \cap \mathbf{W}_0^{1,p}(\Omega) \cap \mathbf{L}^{r_1}(\Omega) \quad \text{and} \quad \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{W}_0^{-1,r_2}(\Omega) \tag{3.2}$$

for any $r_1 \geq 6$ and any $r_2 \geq 3$. Besides, the associated pressure has a representative

$$\pi \in L^3(\Omega) \cap L^p(\Omega), \tag{3.3}$$

and if $p > 3$, then we have $\boldsymbol{v} \in \mathbf{L}^\infty(\Omega)$.

Proof. We use once again the partition of unit introduced in Proposition 2.3. We first prove the case $p = 3$ and then consider the case $p > 3$.

a) The case $p = 3$: $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega) \cap \mathbf{W}_0^{-1,3}(\Omega)$. Let $\boldsymbol{u}$ be a weak solution of $(\mathcal{NS})$ given by Theorem 2.2 and $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_\infty$. Since $\boldsymbol{v} \in \mathbf{L}^6(\Omega)$ and $\boldsymbol{v}.\nabla\boldsymbol{v} = \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})$, we have that $\boldsymbol{v}.\nabla\boldsymbol{v} \in \mathbf{W}_0^{-1,3}(\Omega)$, $\boldsymbol{v}\dfrac{\partial \psi_1}{\partial x_1} \in \mathbf{L}^{3/2}(\mathbb{R}^3) \hookrightarrow \mathbf{W}_0^{-1,3}(\mathbb{R}^3)$ and $\boldsymbol{f}\psi_1 - (\boldsymbol{v}.\nabla\boldsymbol{v})\psi_1 \in \mathbf{W}_0^{-1,3}(\mathbb{R}^3)$. Moreover, since $\boldsymbol{v} \in \mathbf{H}^1_{loc}(\Omega)$ and $\pi \in L^2(\Omega_2)$, we deduce easily from Sobolev imbedding theorem that

$$-2\nu\nabla\boldsymbol{v}\nabla\psi_1 - \nu\boldsymbol{v}\Delta\psi_1 + \pi\nabla\psi_1 \in \mathbf{W}_0^{-1,3}(\mathbb{R}^3), \quad -\boldsymbol{v}.\nabla\psi_1 \in L^3(\mathbb{R}^3).$$

Hence, the pair $(\boldsymbol{f}^{\,1}, g^1)$ belongs to $\mathbf{W}_0^{-1,3}(\mathbb{R}^3) \times L^3(\mathbb{R}^3)$. Otherwise, we can easily see that $g^1 \in L^{3/2}(\mathbb{R}^3) \subset W_0^{-1,3}(\mathbb{R}^3)$ and therefore $\dfrac{\partial g^1}{\partial x_1} \in W_0^{-2,3}(\mathbb{R}^3)$ satisfying the following compatibility condition

$$\left\langle \frac{\partial g^1}{\partial x_1}, 1 \right\rangle_{W_0^{-2,3}(\mathbb{R}^3) \times W_0^{2,3/2}(\mathbb{R}^3)} = 0.$$

Then, applying Theorem 1.10 [7], the following Oseen system

$$-\nu\Delta\boldsymbol{w} + \lambda\frac{\partial \boldsymbol{w}}{\partial x_1} + \nabla q = \boldsymbol{f}^{\,1} \quad \text{and} \quad \operatorname{div}\, \boldsymbol{w} = g^1 \text{ in } \mathbb{R}^3 \tag{3.4}$$

has a unique solution $(\boldsymbol{w}, q) \in (\mathbf{X}_0^{1,3}(\mathbb{R}^3) \times L^3(\mathbb{R}^3))$ such that $\boldsymbol{w} \in \mathbf{L}^r(\mathbb{R}^3)$ for any $r \geq 12$. We set $\boldsymbol{z} = \boldsymbol{v}^1 - \boldsymbol{w}$ and $\theta = \pi^1 - q$. Subtracting (2.2) to (3.4), we get

$$-\nu\Delta\boldsymbol{z} + \lambda\frac{\partial \boldsymbol{z}}{\partial x_1} + \nabla\theta = \mathbf{0} \quad \text{and} \quad \operatorname{div}\, \boldsymbol{z} = 0 \text{ in } \mathbb{R}^3.$$

Proceeding analogously as in the proof of Proposition 2.3, we can deduce that $\nabla \boldsymbol{z} = \mathbf{0}$ in $\mathbb{R}^3$. Since $\boldsymbol{z}$ belongs to $\mathbf{W}_0^{1,2}(\mathbb{R}^3) + \mathbf{W}_0^{1,3}(\mathbb{R}^3)$, then $\boldsymbol{z}$ must be a constant $\boldsymbol{c}$ and $\nabla \boldsymbol{v}^1 = \nabla \boldsymbol{w}$. As $\boldsymbol{z} \in \mathbf{L}^6(\mathbb{R}^3) + \mathbf{L}^{12}(\mathbb{R}^3)$, then $\boldsymbol{c} = \mathbf{0}$, *i.e.* $\boldsymbol{v}^1 = \boldsymbol{w}$ and $\boldsymbol{v}^1 \in \mathbf{W}_0^{1,2}(\mathbb{R}^3) \cap \mathbf{W}_0^{1,3}(\mathbb{R}^3)$. Moreover, we have $\boldsymbol{v}^1 \in \mathbf{L}^{r_1}(\mathbb{R}^3)$ and $\dfrac{\partial \boldsymbol{v}^1}{\partial x_1} \in \mathbf{W}_0^{-1,r_2}(\mathbb{R}^3)$ for any $r_1 \geq 6$ and any $r_2 \geq 3$. Since $\boldsymbol{z} = 0$, we deduce that $\nabla \theta = 0$, then θ must be a constant, *i.e*, there exists a constant a such that $\pi^1 = q + a$ with $q \in L^3(\mathbb{R}^3)$. Let us now come to the regularity near the boundary. Recall that $(\boldsymbol{v}^2, \pi^2) \in \mathbf{H}^1(\Omega_2) \times L^2(\Omega_2)$ satisfies (2.2) with $i = 2$. Moreover, we can prove-like we proved- that $(\boldsymbol{f}^{\,2}, g^2) \in \mathbf{W}^{-1,3}(\Omega_2) \times L^3(\Omega_2)$. Thanks to Green's formula and div $\boldsymbol{v} = 0$, we have

$$\int_{\Omega_2} g^2(\boldsymbol{x}) d\boldsymbol{x} = -\int_{\Gamma} \psi_2 \boldsymbol{u}_\infty . \boldsymbol{n} d\sigma. \tag{3.5}$$

With such data, and since Ω_2 has $C^{1,1}$ boundary, we can deduce from Proposition 4.2 [10] that $(\boldsymbol{v}^2, \pi^2) \in \mathbf{W}^{1,3}(\Omega_2) \times L^3(\Omega_2)$ which immediately imply that $(\boldsymbol{v}^2, \pi^2) \in \mathbf{W}_0^{1,3}(\Omega) \times L^3(\Omega)$. This ends the proof of the case $p = 3$.

b) The case $p > 3$: Let $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega) \cap \mathbf{W}_0^{-1,p}(\Omega)$. It is clear that $\boldsymbol{f} \in \mathbf{W}_0^{-1,3}(\Omega)$ and since we have proved the theorem for $p = 3$, we know that $\boldsymbol{v} \in \mathbf{W}_0^{1,2}(\Omega) \cap \mathbf{W}_0^{1,3}(\Omega) \cap \mathbf{L}^{r_1}(\Omega)$ for any $r_1 \geq 6$ and $\pi \in L^3(\Omega)$. Then

$$(\boldsymbol{f}^{\,1}, g^1) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3) \text{ and } (\boldsymbol{f}^{\,2}, g^2) \in \mathbf{W}^{-1,p}(\Omega_2) \times L^p(\Omega_2).$$

As in the case *a)*, we prove that $(\boldsymbol{v}^1, \pi^1) \in \mathbf{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and $(\boldsymbol{v}^2, \pi^2) \in \mathbf{W}_0^{1,p}(\Omega) \times L^p(\Omega)$, *i.e*, $\boldsymbol{v} \in \mathbf{W}_0^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$. Moreover $\boldsymbol{v} \in \mathbf{L}^\infty(\Omega)$. The proof is complete. $\square$

From Sobolev embedding theorem and the properties of the duality, we know that $\mathbf{L}^{3/2}(\Omega) \hookrightarrow \mathbf{W}_0^{-1,3}(\Omega)$. If we now reinforce the assumptions of Theorem 3.1, $\boldsymbol{f}$ belongs to $\mathbf{L}^{3/2}(\Omega)$ instead of $\mathbf{W}_0^{-1,3}(\Omega)$, we can prove the following.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $C^{1,1}$ boundary.*
i) Assume that $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega) \cap \mathbf{L}^{3/2}(\Omega)$. Then each weak solution $\boldsymbol{u}$ to the problem $(\mathcal{NS})$ satisfies

$$\boldsymbol{v} \in \mathbf{W}_0^{1,2}(\Omega) \cap \mathbf{W}_0^{1,3}(\Omega) \cap \mathbf{L}^{r_1}(\Omega), \tag{3.6}$$

$$\frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{3/2}(\Omega) \cap \mathbf{L}^3(\Omega) \cap \mathbf{W}_0^{-1,r_2}(\Omega) \quad \textit{and} \quad \nabla^2 \boldsymbol{v} \in \mathbf{L}^{3/2}(\Omega) \tag{3.7}$$

for any $r_1 \geq \frac{9}{2}$, $r_2 \geq 3$. Besides, the associated pressure π belongs to $W_0^{1,3/2}(\Omega)$.
ii) Let $\dfrac{3}{2} < p < 3$. Assume that $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega) \cap \mathbf{L}^p(\Omega)$. Then each solution $\boldsymbol{u}$ to the problem $(\mathcal{NS})$ satisfies

$$\boldsymbol{v} \in \mathbf{W}_0^{1,2}(\Omega) \cap \mathbf{W}_0^{1,p*}(\Omega) \cap \mathbf{L}^{r_1}(\Omega) \text{ and } \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{W}_0^{-1,r_2}(\Omega) \tag{3.8}$$

for any $r_1 \in [3p, \infty]$ if $\frac{3}{2} < p < 2$, for any $r_1 \in [6, \infty]$ if $2 \leq p < 3$ and for any $r_2 \geq 3$. Besides, the associated pressure satisfies

$$\pi \in L^3(\Omega) \cap L^{p*}(\Omega) \tag{3.9}$$

where $p* = \dfrac{3p}{3-p}$ *. Moreover, we have*

$$\nabla^2 \boldsymbol{v} \in \mathbf{L}^p(\Omega),\ \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^p(\Omega) \text{ and } \pi \in W_0^{1,p}(\Omega). \tag{3.10}$$

Proof. i) Let $\boldsymbol{u}$ be a weak solution of $(\mathcal{NS})$. Since $\mathbf{L}^{3/2}(\Omega) \hookrightarrow \mathbf{W}_0^{-1,3}(\Omega)$, from Theorem 3.1, we know that $\boldsymbol{u}$ and π satisfy (3.2) and (3.3)with $p = 3$. Now it remains to prove that $\boldsymbol{v}$ belongs to $\mathbf{L}^{9/2}(\Omega)$ and $\dfrac{\partial \boldsymbol{v}}{\partial x_1}$, $\nabla^2 \boldsymbol{v}$, $\nabla\pi$ belong to $\mathbf{L}^{3/2}(\Omega)$. It is then clear that $\boldsymbol{f}^{\,1} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$and $g^1 \in X_0^{1,3/2}(\mathbb{R}^3)$. Then, by applying Theorem 1.9 [7], the following Oseen system

$$-\nu\Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \nabla\mu = \boldsymbol{f}^{\,1} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = g^1 \text{ in } \mathbb{R}^3, \tag{3.11}$$

has a unique solution $(\boldsymbol{w}, \mu)$ such that $\boldsymbol{w} \in \mathbf{L}^s(\mathbb{R}^3)$, $\nabla \boldsymbol{w} \in \mathbf{L}^r(\mathbb{R}^3)$, $\nabla^2 \boldsymbol{w} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$, $\dfrac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ and the pressure $\mu \in W_0^{1,3/2}(\mathbb{R}^3)$ for all $s \in [6, \infty)$ and $r \in [12/5, 3]$. We set $\boldsymbol{z} = \boldsymbol{v}^1 - \boldsymbol{w}$ and $\theta = \pi^1 - \mu$. Subtracting (3.1) to (3.11), we get

$$-\nu\Delta \boldsymbol{z} + \lambda \frac{\partial \boldsymbol{z}}{\partial x_1} + \nabla\theta = \mathbf{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \text{ in } \mathbb{R}^3.$$

By the analogous techniques as in the proof of Theorem 3.1, we conclude $\boldsymbol{v}^1 = \boldsymbol{w}$, $\pi^1 = \mu \in W_0^{1,3/2}(\mathbb{R}^3)$, $\dfrac{\partial \boldsymbol{v}^1}{\partial x_1} \in \mathbf{L}^{3/2}(\mathbb{R}^3)$ and $\nabla^2 \boldsymbol{v}^1 \in \mathbf{L}^{3/2}(\mathbb{R}^3)$. Thanks to Lemma 1.4 [7] with $q = \dfrac{3}{2}$, we can deduce $\boldsymbol{v}^1 \in \mathbf{L}^{9/2}(\mathbb{R}^3)$. Let us now come to the regularity near the boundary. First, we verify easily that $(\boldsymbol{f}^{\,2}, g^2) \in \mathbf{L}^{3/2}(\Omega_2) \times W^{1,3/2}(\Omega_2)$. With such data, and since Ω_2 has $C^{1,1}$ boundary, we can deduce from Proposition 4.3 [10] that $(\boldsymbol{v}^2, \pi^2) \in \mathbf{W}^{2,3/2}(\Omega_2) \times W^{1,3/2}(\Omega_2)$ which immediately imply that $(\boldsymbol{v}^2, \pi^2) \in \mathbf{W}_0^{2,3/2}(\Omega) \times W^{1,3/2}(\Omega)$. Finally, since $\boldsymbol{v} = \boldsymbol{v}^1 + \boldsymbol{v}^2$ and $\pi = \pi^1 + \pi^2$, we obtain (3.6) and (3.7).

ii) Thanks to the Sobolev embedding theorem, since $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ where $\dfrac{3}{2} < p < 3$, we can deduce that $\boldsymbol{f} \in \mathbf{W}_0^{-1,p*}(\Omega)$ with $p* = \frac{3p}{3-p}$ and $p* > 3$. From Theorem 3.1, we have (3.2) and (3.3) but $p*$ plays a role as p in Theorem 3.1. From Hölder's inequality, we obtain $\boldsymbol{v}.\nabla \boldsymbol{v} \in \mathbf{L}^q(\Omega)$ for all $\dfrac{3}{2} \leq q_2 < 3$ and then $\boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v} \in \mathbf{L}^p(\Omega)$. Proceeding similarly as in the previous case, we prove (3.10). By applying Lemma 1.1, we have $\boldsymbol{v} \in \mathbf{L}^{3p}(\Omega)$ and we deduce (3.8). Finally, we obtain $\pi \in L^{p*}(\Omega)$ from $\pi \in W_0^{1,p}(\Omega)$. The theorem is completely proved. □

In Theorem 3.2 (i), we proved $\boldsymbol{v} \in \mathbf{L}^{r_1}(\Omega)$ for any $r_1 \geq 9/2$. To obtain $\boldsymbol{v} \in \mathbf{L}^{r_1}(\Omega)$ with $r_1 < 9/2$, we have to assume additionally a condition for $\boldsymbol{f}$. We can state the

Theorem 3.3. *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior domain with* $C^{1,1}$ *boundary. Assume that* $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega) \cap \mathbf{L}^{3/2}(\Omega) \cap \mathbf{L}^{4/3}(\Omega)$*. Then each weak solution* $\boldsymbol{u}$ *and the* *associate pressure π to the problem $(\mathcal{NS})$ satisfy the results in Theorem 3.2 i). Moreover,*

$$\nabla^2 \boldsymbol{v} \in \mathbf{L}^{4/3}(\Omega),\ \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{4/3}(\Omega),\ \pi \in\ W_0^{1,4/3}(\Omega) \quad and \quad \boldsymbol{v} \in \mathbf{L}^{r_1}(\Omega) \tag{3.12}$$

for any $r_1 \geq 4$.

Proof. From Theorem 3.2 and Theorem 5.26 [10], we have the following estimate

$$\begin{aligned} &\lambda|| \frac{\partial \boldsymbol{v}}{\partial x_1} ||_{\mathbf{L}^{3/2}(\Omega)} \leq C(|| \boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v} \,||_{\mathbf{L}^{3/2}(\Omega)} + ||\boldsymbol{u}_\infty||_{\mathbf{W}^{4/3,3/2}(\Gamma)}) \\ &\leq C(|| \boldsymbol{f} \,||_{\mathbf{L}^{3/2}(\Omega)} + ||\ \boldsymbol{v}\ ||_{\mathbf{L}^6(\Omega)} ||\nabla \boldsymbol{v}\, ||_{\mathbf{L}^2(\Omega)} + ||\boldsymbol{u}_\infty||_{\mathbf{W}^{4/3,3/2}(\Gamma)}) \\ &\leq C(|| \boldsymbol{f} \,||_{\mathbf{L}^{3/2}(\Omega)} + ||\ \boldsymbol{v}\ ||_{\mathbf{L}^6(\Omega)} ||\boldsymbol{f}\, ||_{\mathbf{W}_0^{-1,2}(\Omega)} + ||\boldsymbol{u}_\infty||_{\mathbf{W}^{4/3,3/2}(\Gamma)}). \end{aligned} \tag{3.13}$$

Applying Lemma 1.4 [7] with $q = 3/2$, we can deduce

$$||\boldsymbol{v}||_{L^{9/2}(\Omega)} + \lambda|| \frac{\partial \boldsymbol{v}}{\partial x_1} ||_{\mathbf{L}^{3/2}(\Omega)} \leq C(|| \boldsymbol{f} \,||_{\mathbf{L}^{3/2}(\Omega)} + ||\ \boldsymbol{v}\ ||_{\mathbf{L}^6(\Omega)} + 1).$$

We define the sequence $\{q_k\}$ as follows

$$\frac{2q_{k+1}}{2 - q_{k+1}} = 3q_k, \quad k \in \mathbb{N} \tag{3.14}$$

with $q_0 = 2$. Clearly, the sequence $\{q_k\}$ is strictly decreasing and converges to $4/3$. By induction, we can deduce for $4/3 \leq q_k \leq 2$ with $k \in \mathbb{N}$ that

$$\begin{aligned} ||\boldsymbol{v}||_{L^{3q_{k+1}}(\Omega)} + \lambda|| \frac{\partial \boldsymbol{v}}{\partial x_1} ||_{\mathbf{L}^{q_{k+1}}(\Omega)} &\leq C(|| \boldsymbol{f} \,||_{\mathbf{L}^{q_{k+1}}(\Omega)} + ||\ \boldsymbol{v}\ ||_{\mathbf{L}^{2q_{k+1}/(2-q_{k+1})}(\Omega)} + 1) \\ &\leq C(|| \boldsymbol{f} \,||_{\mathbf{L}^{q_{k+1}}(\Omega)} + ||\ \boldsymbol{v}\ ||_{\mathbf{L}^{3q_k}(\Omega)} + 1). \end{aligned} \tag{3.15}$$

Thanks to Babenko [11] and (3.15), we have the following estimate

$$\begin{aligned} ||\, \boldsymbol{v}\, ||_{\mathbf{L}^{3q_{k+1}}(\Omega)} \ &\leq\ C\, || \, \frac{\partial \boldsymbol{v}}{\partial x_1} \, ||^{1/3}_{\mathbf{L}^{q_{k+1}}(\Omega)} || \, \frac{\partial \boldsymbol{v}}{\partial x_2} \, ||^{1/3}_{\mathbf{L}^2(\Omega)} || \, \frac{\partial \boldsymbol{v}}{\partial x_3} \, ||^{1/3}_{\mathbf{L}^2(\Omega)} \\ &\leq\ C\, || \, \frac{\partial \boldsymbol{v}}{\partial x_1} \, ||^{1/3}_{\mathbf{L}^{q_{k+1}}(\Omega)} \\ &\leq\ C\,(1 + ||\, \boldsymbol{v}\, ||_{\mathbf{L}^{3q_k}(\Omega)})^{1/3}. \end{aligned} \tag{3.16}$$

Then we deduce

$$\begin{aligned} 1 + ||\, \boldsymbol{v}\, ||_{\mathbf{L}^{3q_{k+1}}(\Omega)} \ &\leq\ C\,(1 + ||\, \boldsymbol{v}\, ||_{\mathbf{L}^{3q_k}(\Omega)})^{1/3} \\ &\leq\ C^{1+\frac{1}{3}+\ldots+\frac{1}{3^k}}\,(1 + ||\, \boldsymbol{v}\, ||_{\mathbf{L}^{3q_0}(\Omega)})^{\frac{1}{3^k}}. \end{aligned}$$

When $k \to +\infty$, then $q_k \to 4/3$ and we can deduce $\boldsymbol{v} \in \mathbf{L}^4(\Omega)$ with the following estimate

$$1 + ||\, \boldsymbol{v}\, ||_{\mathbf{L}^4(\Omega)} \ \leq\ C\,(1 + ||\, \boldsymbol{v}\, ||_{\mathbf{L}^6(\Omega)}) \ \leq\ C.$$

Since $\boldsymbol{v} \in \mathbf{L}^4(\Omega)$ and $\nabla \boldsymbol{v} \in \mathbf{L}^2(\Omega)$, we obtain $\boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v} \in \mathbf{L}^{4/3}(\Omega)$ and we deduce (3.12). The Theorem is completely proved.

□

4 The exterior Oseen problem

For our studies, we shall introduce the following problem. Let

$$\text{a fixed } \boldsymbol{z} \in \mathbf{L}^3(\Omega) \text{ such that div } \boldsymbol{z} = 0 \text{ in } \Omega, \tag{4.1}$$

we search a solution $(\boldsymbol{w}, \theta)$ to the following Oseen problem

$$\begin{aligned} -\nu\Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \boldsymbol{z}.\nabla \boldsymbol{w} + \nabla\theta = \boldsymbol{f} \quad & \text{in } \Omega, \\ \text{div } \boldsymbol{w} = 0 \quad & \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{u}_* \quad & \text{on } \Gamma. \end{aligned} \tag{4.2}$$

We introduce the space

$$\mathbf{V}^p(\Omega) = \{\boldsymbol{v} \ \in \overset{\circ}{\mathbf{W}}{}_0^{1,\,p}(\Omega), \ \text{div } \boldsymbol{v} = 0\}.$$

As in [2], we can prove the following lemma

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain. Assume that $\boldsymbol{z}$ satisfies* (4.1)*, $\boldsymbol{u}_* = \mathbf{0}$ and let $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega)$. Then Problem* (4.2) *has a solution $(\boldsymbol{w}, \theta) \ \in \overset{\circ}{\mathbf{W}}{}_0^{1,\,2}(\Omega) \times L^2_{loc}(\Omega)$.*

We have the following corollary.

Corollary 4.2. *With the same hypothesis as in Lemma* 4.1*, we can deduce that $\theta \in L^2(\Omega)$. Moreover, we have $\boldsymbol{w} \in \mathbf{L}^4(\Omega)$, $\dfrac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{W}_0^{-1,2}(\Omega)$.*

Proof. We use once again the partition of unit with the role of $\boldsymbol{w}$ (θ, respectively) as $\boldsymbol{v}$ (π, respectively) introduced in Proposition 2.3. Proceeding analogously as in the proof of Theorem 3.1, we can deduce that θ belongs to $L^2(\Omega)$. Moreover, we have $\dfrac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{W}_0^{-1,2}(\Omega)$ because $\Delta \boldsymbol{w}$, $\boldsymbol{z}.\nabla \boldsymbol{w}$, $\nabla\theta$ and $\boldsymbol{f}$ belong to $\mathbf{W}_0^{-1,2}(\Omega)$. Thanks to Lemma 1.2, we deduce $\boldsymbol{w} \in \mathbf{L}^4(\Omega)$. □

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain. Assume that $\boldsymbol{z}$ satisfies* (4.1)*, $\boldsymbol{u}_* = \mathbf{0}$ and let $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega)$. Then Problem* (4.2) *has a unique solution $(\boldsymbol{w}, \theta) \in \ (\overset{\circ}{\mathbf{W}}{}_0^{1,\,2}(\Omega) \cap \mathbf{L}^4(\Omega)) \times L^2(\Omega)$ with $\dfrac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{W}_0^{-1,2}(\Omega)$ and $\boldsymbol{w}$ satisfies the energy equality*

$$\nu \int_\Omega |\nabla \boldsymbol{w}\,|^2 d\boldsymbol{x} = <\boldsymbol{f}\,, \boldsymbol{w}>_{\mathbf{W}_0^{-1,2}(\Omega) \times \overset{\circ}{\mathbf{W}}{}_0^{1,\,2}(\Omega)}\,. \tag{4.3}$$

Moreover, we have the estimate

$$\begin{aligned} &||\boldsymbol{w}\,||_{\mathbf{L}^4(\Omega)} + ||\nabla \boldsymbol{w}||_{\mathbf{L}^2(\Omega)} + ||\frac{\partial \boldsymbol{w}}{\partial x_1}||_{\mathbf{W}_0^{-1,2}(\Omega)} + ||\nabla\theta||_{\mathbf{L}^2(\Omega)} \\ &\leq C(||\boldsymbol{f}\,||_{\mathbf{W}_0^{-1,2}(\Omega)} + ||\boldsymbol{z}.\nabla \boldsymbol{w}\,||_{\mathbf{W}_0^{-1,2}(\Omega)}). \end{aligned} \tag{4.4}$$

Proof. The existence of $(\boldsymbol{w}, \theta) \in \ (\overset{\circ}{\mathbf{W}}{}_0^{1,\,2}(\Omega) \cap \mathbf{L}^4(\Omega)) \times L^2(\Omega)$ such that $\dfrac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{W}_0^{-1,2}(\Omega)$ is given by Lemma 4.1 and Corollary 4.2. Since the space $\boldsymbol{\mathcal{V}}(\Omega)$ is dense in $\mathbf{V}^2(\Omega)$, for any $\boldsymbol{\varphi} \in \mathbf{V}^2(\Omega)$, we have

$$\begin{aligned}&\nu \textstyle\int_\Omega \nabla \boldsymbol{w}.\nabla \boldsymbol{\varphi} d\boldsymbol{x} + \lambda < \dfrac{\partial \boldsymbol{w}}{\partial x_1}, \boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,2}(\Omega)\times \overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega)} \\ &+ < \boldsymbol{z}.\nabla \boldsymbol{w}, \boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,2}(\Omega)\times \overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega)} = < \boldsymbol{f}, \boldsymbol{\varphi} >_{\mathbf{W}_0^{-1,2}(\Omega)\times \overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega)} .\end{aligned} \tag{4.5}$$

Since $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\overset{\circ}{\mathbf{X}}{}_0^{1,2}(\Omega)$ (see [8]), for all $\boldsymbol{\psi}, \boldsymbol{\varphi} \in \overset{\circ}{\mathbf{X}}{}_0^{1,2}(\Omega)$, we obtain

$$\left\langle \frac{\partial \boldsymbol{\psi}}{\partial x_1}, \boldsymbol{\varphi} \right\rangle_{\mathbf{W}_0^{-1,2}(\Omega)\times \overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega)} = -\left\langle \boldsymbol{\psi}, \frac{\partial \boldsymbol{\varphi}}{\partial x_1} \right\rangle_{\overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega)\times \mathbf{W}_0^{-1,2}(\Omega)} .$$

Then, we deduce

$$\left\langle \frac{\partial \boldsymbol{\psi}}{\partial x_1}, \boldsymbol{\psi} \right\rangle_{\mathbf{W}_0^{-1,2}(\Omega)\times \overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega)} = 0 \tag{4.6}$$

and we have for any $\boldsymbol{\varphi} \in \mathbf{V}^2(\Omega)$,

$$\begin{aligned}\langle \mathrm{div}\ (\boldsymbol{z} \otimes \boldsymbol{\varphi}), \boldsymbol{\varphi} \rangle_{\mathbf{W}_0^{-1,2}(\Omega)\times \overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega)} &= -\int_\Omega z_i\, \varphi_j \frac{\partial \varphi_j}{\partial x_i} d\boldsymbol{x} \\ &= -\frac{1}{2}\int_\Omega z_i \frac{\partial \varphi_j^2}{\partial x_i} d\boldsymbol{x} = 0. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7) and (4.5), we have (4.3). The uniqueness of $(\boldsymbol{w}, \theta)$ is a immediate consequence of (4.3). □

We consider the following nonhomogeneous problem.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain. Assume that $\boldsymbol{z}$ satisfies* (4.1), *$\boldsymbol{u}_* \in \mathbf{H}^{1/2}(\Gamma)$ and let $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\Omega)$. Then Problem* (4.2) *has a unique solution $(\boldsymbol{w}, \theta) \in (\mathbf{X}_0^{1,2}(\Omega) \cap \mathbf{L}^4(\Omega)) \times L^2(\Omega)$ and we have the estimate*

$$\begin{aligned}&||\boldsymbol{w}\,||_{\mathbf{L}^4(\Omega)} + ||\nabla \boldsymbol{w}||_{\mathbf{L}^2(\Omega)} + ||\frac{\partial \boldsymbol{w}}{\partial x_1}||_{\mathbf{W}_0^{-1,2}(\Omega)} + ||\nabla \theta||_{\mathbf{L}^2(\Omega)} \\ &\leq C(||\boldsymbol{f}\,||_{\mathbf{W}_0^{-1,2}(\Omega)} + ||\boldsymbol{z}.\nabla \boldsymbol{w}\,||_{\mathbf{W}_0^{-1,2}(\Omega)} + ||\boldsymbol{u}_*||_{\mathbf{H}^{1/2}(\Gamma)}).\end{aligned} \tag{4.8}$$

Proof. It is easily to show Lemma 4.4 by applying Lemma 4.3 and Lemma 5.8 [10] with the case $p = 2$. □

Our objective is to consider the Navier-Stokes equations by using the properties of the Oseen equations. We now consider some properties of the Oseen equations. Beforehand, we introduce

$$Y_0^{1,p}(\Omega) \ = \begin{cases} X_0^{1,p}(\Omega) \cap L^{4p/(4-p)}(\Omega) & \text{if } \ 1 < p < 4, \\ X_0^{1,p}(\Omega) & \text{if } \ p \geq 4, \end{cases}$$

and

$$\overset{\circ}{Y}{}_0^{1,p}(\Omega) \ = \ \{ v \in Y_0^{1,p}(\Omega);\ v = 0 \ \text{ on } \ \Gamma \},$$

with the same definition when $\Omega = \mathbb{R}^3$. Now defining

$$\mathcal{N}_p^+(\Omega) = \{(\boldsymbol{u}, \pi) \in \overset{\circ}{\mathbf{Y}}{}_0^{1,p}(\Omega) \times L^p(\Omega), \mathbf{T}(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega\},$$

$$\mathcal{N}_p^-(\Omega) = \{(\boldsymbol{u}, \pi) \in \overset{\circ}{\mathbf{Y}}{}_0^{1,p}(\Omega) \times L^p(\Omega), \mathbf{T}^*(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega\},$$

with

$$\mathbf{T}(\boldsymbol{u},\pi) = (-\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi, -\text{div } u),$$

and its adjoint

$$\mathbf{T}^*(\boldsymbol{u},\pi) = (-\Delta \boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi, -\text{div } u).$$

Moreover, if $1 < p < 4$, $\boldsymbol{u}$ satisfies the properties i)-iii) of Lemma 1.2. We introduce the characterization of the kernel $\mathcal{N}_p^+(\Omega)$. (see [10]).

Lemma 4.5. *Let Ω be an exterior domain with a $C^{1,1}$ boundary.*
1) If $1 \leq p < 4$, then $\mathcal{N}_p^+(\Omega) = \{(\mathbf{0}, 0)\}$.
2) If $p \geq 4$, then $\mathcal{N}_p^+(\Omega) = \{(\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c);\ \boldsymbol{c} \in \mathbb{R}^3\}$ where

$$(\boldsymbol{\lambda}_c, \mu_c) \in \bigcap_{r>4/3} \mathbf{Y}_0^{1,r}(\Omega) \times \bigcap_{s>3/2} L^s(\Omega)$$

is the unique solution of the following system

$$-\Delta \boldsymbol{\lambda}_c + \frac{\partial \boldsymbol{\lambda}_c}{\partial x_1} + \nabla \mu_c = 0,\ \text{div}\, \boldsymbol{\lambda}_c = 0 \ \textit{in}\ \Omega,\ \ \boldsymbol{\lambda}_c = \boldsymbol{c}\ \textit{on}\ \Gamma. \tag{4.9}$$

Remark that we have the similar results for $\mathcal{N}_p^-(\Omega)$. We now introduce the

Theorem 4.6. [10] *Let Ω be an exterior domain with a $C^{1,1}$ boundary. Assume that $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\Omega)$ and $\boldsymbol{u}_* \in \mathbf{W}^{1/p',p}(\Gamma)$. Moreover, if $1 < p \leq 4/3$, assume that we have the compatibility condition*

$$\forall (\boldsymbol{v}, \eta) \in \mathcal{N}_{p'}^-(\Omega), \quad < \boldsymbol{f}, \boldsymbol{v} >_\Omega + < (\nabla \boldsymbol{v} - \eta I).\boldsymbol{n}, \boldsymbol{u}_* >_\Gamma = 0. \tag{4.10}$$

i) If $1 < p < 4$, then the following problem

$$-\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f},\ \text{div}\, \boldsymbol{u} = 0 \text{ in } \Omega,\ \ \boldsymbol{u} = \boldsymbol{u}_* \text{ on } \Gamma \tag{4.11}$$

has a unique solution $(\boldsymbol{u}, \pi) \in \mathbf{Y}_0^{1,p}(\Omega) \times L^p(\Omega)$ satisfying the estimate

$$\|\, \boldsymbol{u}\, \|_{\mathbf{Y}_0^{1,p}(\Omega)} + \|\, \pi\, \|_{L^p(\Omega)} \leq C\, (\, \|\, \boldsymbol{f}\, \|_{\mathbf{W}_0^{-1,p}(\Omega)} + \|\, \boldsymbol{u}_*\, \|_{\mathbf{W}^{1/p',p}(\Gamma)}\,).$$

ii) If $p \geq 4$, then problem (4.11) has a solution $(\boldsymbol{u}, \pi) \in \mathbf{Y}_0^{1,p}(\Omega) \times L^p(\Omega)$, unique up to an element of $\mathcal{N}_p^+(\Omega)$, satisfying the estimate

$$\inf_{(\boldsymbol{v},\eta)\in\mathcal{N}_p^+(\Omega)} (\, \|\, \boldsymbol{u} + \boldsymbol{v} \|_{\mathbf{Y}_0^{1,p}(\Omega)} + \|\, \pi + \eta\, \|_{L^p(\Omega)}) \leq C\, (\, \|\, \boldsymbol{f}\, \|_{\mathbf{W}_0^{-1,p}(\Omega)} + \|\, \boldsymbol{u}_*\, \|_{\mathbf{W}^{1/p',p}(\Gamma)}\,).$$

The next Lemma characterizes the kernel $\mathcal{N}^{p,q}(\Omega)$ of the exterior Oseen system:

$$\begin{aligned} \mathcal{N}^{p,q}(\Omega) = \quad & \{(\boldsymbol{u}, \pi) \in [\mathbf{Y}_0^{1,p}(\Omega) + \mathbf{Y}_0^{1,q}(\Omega)] \times [L^p(\Omega) + L^q(\Omega)], \\ & \mathbf{T}(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega\, , \boldsymbol{u} = 0 \text{ on } \Gamma\} \end{aligned}$$

with $1 < p < q < \infty$.

Lemma 4.7. *Let Ω be an exterior domain with a $C^{1,1}$ boundary and $1 < p < q < \infty$.*
i) If $q < 4$, then $\mathcal{N}^{p,q}(\Omega) = \{(\mathbf{0}, 0)\}$.
ii) If $q \geq 4$, then

$$\mathcal{N}^{p,q}(\Omega) = \{(\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c);\ \boldsymbol{c} \in \mathbb{R}^3\}$$

where

$$(\boldsymbol{\lambda}_c, \mu_c) \in \bigcap_{r>4/3} \mathbf{Y}_0^{1,r}(\Omega) \times \bigcap_{s>3/2} L^s(\Omega)$$

is the unique solution of the system (4.9). Moreover, we have $\boldsymbol{\lambda}_c \in \mathbf{L}^s(\Omega) \cap \mathbf{L}^\infty(\Omega)$ for all $s > 2$.

Proof. Let $(\boldsymbol{z}, \theta) \in \mathcal{N}^{p,q}(\Omega)$, then $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{v}$ with $\boldsymbol{u} \in \mathbf{Y}_0^{1,p}(\Omega)$, $\boldsymbol{v} \in \mathbf{Y}_0^{1,q}(\Omega)$ and $\boldsymbol{u} = \boldsymbol{v}$ on Γ. Let now $\widetilde{\boldsymbol{v}} \in \mathbf{Y}_0^{1,q}(\mathbb{R}^3)$ be an extended function of $\boldsymbol{v}$ outside Ω. We set $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$ in Ω, $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{v}}$ outside Ω and $\widetilde{\boldsymbol{z}} = \widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{v}}$. Then $\widetilde{\boldsymbol{u}} \in \mathbf{Y}_0^{1,p}(\mathbb{R}^3)$, $\widetilde{\boldsymbol{z}} = 0$ outside Ω and we can prove that $\operatorname{div} \widetilde{\boldsymbol{z}} = 0$ in $\mathbb{R}^3$. We now extend θ by 0 outside Ω and denote $\widetilde{\theta}$ its extended function. It is easy to see that $\widetilde{\theta} \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$. Now setting

$$\boldsymbol{h} = -\Delta \widetilde{\boldsymbol{z}} + \frac{\partial \widetilde{\boldsymbol{z}}}{\partial x_1} + \nabla \widetilde{\theta},$$

then $\operatorname{supp} \boldsymbol{h} \subset \Gamma$ and $\boldsymbol{h} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$.

1) *The case $p > \dfrac{4}{3}$:* Thanks to Theorem 1.10 [7], there exists $\boldsymbol{w} \in \mathbf{Y}_0^{1,p}(\mathbb{R}^3)$ and $\alpha \in L^p(\mathbb{R}^3)$ such that

$$-\Delta \boldsymbol{w} + \frac{\partial \boldsymbol{w}}{\partial x_1} + \nabla \alpha = \boldsymbol{h} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 \text{ in } \mathbb{R}^3.$$

We now set that $\boldsymbol{y} = \boldsymbol{w} - \widetilde{\boldsymbol{z}}$ and $k = \alpha - \widetilde{\theta}$. Hence, we have

$$-\Delta \boldsymbol{y} + \frac{\partial \boldsymbol{y}}{\partial x_1} + \nabla k = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{y} = 0 \text{ in } \mathbb{R}^3,$$

and we deduce

$$-\Delta \mathbf{curl}\, \boldsymbol{y} + \frac{\partial (\mathbf{curl}\, \boldsymbol{y})}{\partial x_1} = 0 \text{ in } \mathbb{R}^3.$$

We take $\boldsymbol{\Phi} = \mathbf{curl}\, \boldsymbol{y}$. Then, for $i = 1, 2, 3$, we have

$$-\Delta \Phi_i + \frac{\partial \Phi_i}{\partial x_1} = 0$$

where $\Phi_i \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3) \hookrightarrow S'(\mathbb{R}^3)$. It is deduced that $\boldsymbol{\Phi}$ is a polynomial which belongs to $\mathbf{L}^p(\mathbb{R}^3) + \mathbf{L}^q(\mathbb{R}^3)$. Consequently, $\boldsymbol{\Phi} = 0 = \mathbf{curl}\, \boldsymbol{y}$. Therefore,

$$-\Delta \boldsymbol{y} = \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{y} + \nabla \operatorname{div} \boldsymbol{y} = 0.$$

Since $\boldsymbol{y} \in \mathbf{Y}_0^{1,p}(\mathbb{R}^3) + \mathbf{Y}_0^{1,q}(\mathbb{R}^3)$, then $\boldsymbol{y}$ must be a constant $\boldsymbol{c}$ and $\nabla \boldsymbol{w} = \nabla \widetilde{\boldsymbol{z}}$. Moreover, we obtain $\nabla k = 0$ in $\mathbb{R}^3$. Then k is a constant belonging to $L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$, it means $\alpha = \widetilde{\theta}$ in $\mathbb{R}^3$.
a) The case $q < 4$: As $\boldsymbol{y} \in \mathbf{L}^{4p/(4-p)}(\mathbb{R}^3) + \mathbf{L}^{4q/(4-q)}(\mathbb{R}^3)$, then $\boldsymbol{c} = 0$. Therefore, $\boldsymbol{w} = \widetilde{\boldsymbol{z}}$ in $\mathbb{R}^3$ and $\boldsymbol{w} = 0$ on Γ. Since $p < 4$, from Theorem 4.6, then $\boldsymbol{w} = 0$

in Ω, *i.e.*, $\boldsymbol{z} = 0$ in Ω. Therefore, $\nabla\theta = 0$ in Ω and we can deduce that θ is a constant which belongs to $L^p(\Omega) + L^q(\Omega)$. Hence, $\theta = 0$ in Ω.
b) The case $q \geq 4$: There exists a constant $\boldsymbol{c} = (c_1, c_2, c_3)$ such that $\boldsymbol{w} - \widetilde{\boldsymbol{z}} = \boldsymbol{c}$ and $\boldsymbol{w} = \boldsymbol{c}$ on Γ. Consider now the following problem

$$-\Delta\boldsymbol{\lambda}_i + \frac{\partial\boldsymbol{\lambda}_i}{\partial x_1} + \nabla\mu_i = 0,\ \operatorname{div}\boldsymbol{\lambda}_i = 0 \text{ in } \Omega,\ \boldsymbol{\lambda}_i = \boldsymbol{e}_i \text{ on } \Gamma, \tag{4.12}$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the canonical basis of $\mathbb{R}^3$. We know that the system (4.12) has a unique solution $(\boldsymbol{\lambda}_i, \mu_i)$ such that $\boldsymbol{\lambda}_i \in \bigcap_{r>4/3} \mathbf{Y}_0^{1,r}(\Omega)$ and $\mu_i \in \bigcap_{r>3/2} L^r(\Omega)$.
If $p < 4$, from Theorem 4.6, $\boldsymbol{w}$ is unique and then $w_i = \boldsymbol{c}.\boldsymbol{e}_i = \boldsymbol{c}.\boldsymbol{\lambda}_i$ on Γ, therefore $w_i = \boldsymbol{c}.\boldsymbol{\lambda}_i$ in Ω. Now we set $\boldsymbol{\lambda}_c = (\boldsymbol{c}.\boldsymbol{\lambda}_1, \boldsymbol{c}.\boldsymbol{\lambda}_2, \boldsymbol{c}.\boldsymbol{\lambda}_3)$ and $\mu_c = \boldsymbol{c}.\boldsymbol{\mu}$ with $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$. By construction of $\boldsymbol{\lambda}_c$ and μ_c, we deduce that $(\boldsymbol{\lambda}_c, \mu_c)$ is the unique solution of the following system

$$-\Delta\boldsymbol{\lambda}_c + \frac{\partial\boldsymbol{\lambda}_c}{\partial x_1} + \nabla\mu_c = 0,\ \operatorname{div}\boldsymbol{\lambda}_c = 0 \text{ in } \Omega,\ \boldsymbol{\lambda}_c = \boldsymbol{c} \text{ on } \Gamma,$$

such that $\boldsymbol{\lambda}_c \in \bigcap_{r>4/3} \mathbf{Y}_0^{1,r}(\Omega)$ and $\mu_c \in \bigcap_{r>3/2} L^r(\Omega)$. It is easy to see that $(\boldsymbol{w}, \alpha) = (\boldsymbol{w}, \theta) = (\boldsymbol{\lambda}_c, \mu_c)$ in Ω. Then we obtain $(\boldsymbol{z}, \theta) = (\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c)$ in Ω. If $p \geq 4$, we obtain again $(\boldsymbol{z}, \theta) = (\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c)$ in Ω by proceeding similarly as in the case $p < 4$.

2) *The case $1 < p < \frac{4}{3}$:* We set $\boldsymbol{m} = (m_j)_j$ with $m_j = < h_j, 1 >_{W_0^{-1,p}(\mathbb{R}^3)\times W_0^{1,p'}(\mathbb{R}^3)}$ and $H_i = h_i - \delta\delta_{ij}m_j$ where δ is the Dirac distribution and δ_{ij} denotes the Kronecker symbol. From Theorem 1.10 [7], there exists a unique solution $(\boldsymbol{w}_0, \alpha_0) \in \mathbf{Y}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that

$$-\Delta\boldsymbol{w}_0 + \frac{\partial\boldsymbol{w}_0}{\partial x_1} + \nabla\alpha_0 = \boldsymbol{H},\ \operatorname{div}\boldsymbol{w}_0 = 0 \text{ in } \mathbb{R}^3.$$

We now set that

$$\boldsymbol{w} = \boldsymbol{w}_0 - \boldsymbol{\mathcal{O}}.\boldsymbol{m} \text{ and } \alpha = \alpha_0 - \boldsymbol{\mathcal{P}}.\boldsymbol{m}$$

where $(\boldsymbol{\mathcal{O}}, \boldsymbol{\mathcal{P}})$ is the fundamental solution of Oseen equations. Then we have

$$-\Delta\boldsymbol{w} + \frac{\partial\boldsymbol{w}}{\partial x_1} + \nabla\alpha = \boldsymbol{h},\ \operatorname{div}\boldsymbol{w} = 0 \text{ in } \mathbb{R}^3.$$

Moreover, proceeding as in the case 1 of this Lemma, we obtain $\nabla\boldsymbol{w} = \nabla\widetilde{\boldsymbol{z}}$ and $\alpha = \widetilde{\theta}$ in $\mathbb{R}^3$. Note now that the pair $(\boldsymbol{a}, b) \in \mathcal{N}_{p'}^{-}(\Omega)$ satisfies the Green's formula: for all $(\boldsymbol{\psi}, \xi) \in \boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$,

$$\int_\Omega \{(-\Delta\boldsymbol{\psi} + \frac{\partial\boldsymbol{\psi}}{\partial x_1} + \nabla\xi)\boldsymbol{a} - b\operatorname{div}\boldsymbol{\psi}\}d\boldsymbol{x} = < (\nabla\boldsymbol{a} - bI).\boldsymbol{n}, \boldsymbol{\psi} >_\Gamma, \tag{4.13}$$

where $<, >_\Gamma$ denotes the duality pairing between $W^{-1/p',p'}(\Gamma)$ and $W^{1/p',p}(\Gamma)$. Thanks to the density of $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ in $\mathbf{X}_0^{1,p}(\Omega)$ and $\mathcal{D}(\overline{\Omega})$ in $L^p(\Omega)$, applying (4.13) with $(\boldsymbol{\psi}, \xi) = (\boldsymbol{w}_0, \alpha_0) \in \mathbf{X}_0^{1,p}(\Omega) \times L^p(\Omega)$ and $(\boldsymbol{a}, b) = (\boldsymbol{v}_\beta - \boldsymbol{\beta}, \theta_\beta) \in \mathcal{N}_{p'}^{-}(\Omega)$ $(\boldsymbol{\beta} \in \mathbb{R}^3)$, we obtain

$$< (\nabla\boldsymbol{v}_\beta - \theta_\beta I).\boldsymbol{n}, \boldsymbol{w}_0 >_{\mathbf{W}^{-1/p',p'}(\Gamma)\times\mathbf{W}^{1/p',p}(\Gamma)} = 0. \tag{4.14}$$

a) The case $q < 4$: Then we have $\boldsymbol{w} - \widetilde{\boldsymbol{z}} = 0$ in $\mathbb{R}^3$ and $\boldsymbol{w} = 0$ on Γ. Therefore, we deduce $\boldsymbol{w}_0 = \boldsymbol{\mathcal{O}}.\boldsymbol{m}$ on Γ. From (4.14), we have

$$\boldsymbol{m} < (\nabla \boldsymbol{v}_\beta - \theta_\beta I).\boldsymbol{n}, \boldsymbol{\mathcal{O}} >_\Gamma = 0. \tag{4.15}$$

By some calculs, we can obtain

$$\begin{aligned} 0 &= \int_\Omega \{(-\Delta \boldsymbol{\mathcal{O}} + \frac{\partial \boldsymbol{\mathcal{O}}}{\partial x_1} + \nabla \boldsymbol{\mathcal{P}})\boldsymbol{v}_\beta - \theta_\beta \operatorname{div} \boldsymbol{\mathcal{O}}\} d\boldsymbol{x} \\ &= < (\nabla \boldsymbol{v}_\beta - \theta_\beta I).\boldsymbol{n}, \boldsymbol{\mathcal{O}} >_\Gamma - \boldsymbol{\beta} \int_\Gamma \frac{\partial \boldsymbol{\mathcal{O}}}{\partial \boldsymbol{n}}, \end{aligned}$$

then we deduce $< (\nabla \boldsymbol{v}_\beta - \theta_\beta I).\boldsymbol{n}, \boldsymbol{\mathcal{O}} >_\Gamma \neq 0$ and from (4.15), $\boldsymbol{m} = 0$. Then $(\boldsymbol{w}_0, \alpha_0) = (0, 0)$ in Ω and we can deduce that $(\boldsymbol{w}, \alpha) = (0, 0)$ and $(\boldsymbol{z}, \theta) = (0, 0)$ in Ω.
b) The case $q \geq 4$: There exists a constant $\boldsymbol{c}$ such that $\boldsymbol{w} - \widetilde{\boldsymbol{z}} = \boldsymbol{c}$ in $\mathbb{R}^3$ and $\boldsymbol{w} = \boldsymbol{c}$ on Γ. Then we have $\boldsymbol{w}_0 = \boldsymbol{c} + \boldsymbol{\mathcal{O}}.\boldsymbol{m}$ on Γ. Applying (4.14), we deduce that

$$< (\nabla \boldsymbol{v}_\beta - \theta_\beta I).\boldsymbol{n}, \boldsymbol{c} + \boldsymbol{\mathcal{O}}.\boldsymbol{m} >_\Gamma = 0.$$

We set that $\boldsymbol{\mu} = \boldsymbol{c} + \boldsymbol{\mathcal{O}}.\boldsymbol{m}$. It is easy to prove that $\boldsymbol{\mu} \in \mathbf{W}^{1/r', r}(\Gamma)$ for all $\frac{4}{3} < r < 4$. Thanks to Theorem 4.6, the following system

$$-\Delta \boldsymbol{y}_0 + \frac{\partial \boldsymbol{y}_0}{\partial x_1} + \nabla \kappa = 0, \ \operatorname{div} \boldsymbol{y}_0 = 0 \text{ in } \Omega, \ \boldsymbol{y}_0 = \boldsymbol{\mu} \text{ on } \Gamma,$$

has a unique solution $(\boldsymbol{y}_0, \kappa) \in \mathbf{Y}_0^{1,r}(\Omega) \times L^r(\Omega)$. Then $(\boldsymbol{y}_0 - \boldsymbol{w}_0, \kappa - \alpha_0) \in \mathcal{N}^{p,r}(\Omega)$ and we deduce that $(\boldsymbol{y}_0, \kappa) = (\boldsymbol{w}_0, \alpha_0)$. Moreover, we can see that $\boldsymbol{\mu} \in \mathbf{W}^{1/p', p}(\Gamma)$. Then there exists $(\boldsymbol{s}, \omega) \in \mathbf{Y}_0^{1,q}(\Omega) \times L^q(\Omega)$ such that

$$-\Delta \boldsymbol{s} + \frac{\partial \boldsymbol{s}}{\partial x_1} + \nabla \omega = 0, \ \operatorname{div} \boldsymbol{s} = 0 \text{ in } \Omega, \ \boldsymbol{s} = \boldsymbol{\mu} \text{ on } \Gamma.$$

Then $(\boldsymbol{s} - \boldsymbol{w}_0, \omega - \alpha_0) \in \mathcal{N}^{r,q}(\Omega)$ and $(\boldsymbol{w}_0, \alpha_0) \in \mathbf{Y}_0^{1,q}(\Omega) \times L^q(\Omega)$. Therefore, we deduce that $(\boldsymbol{w}, \alpha) \in \mathbf{Y}_0^{1,q}(\Omega) \times L^q(\Omega)$. Since $\boldsymbol{w} = \boldsymbol{c}$ on Γ and thanks to the characterization of $\mathcal{N}^{q,q}(\Omega)$, we obtain that $(\boldsymbol{z}, \theta) = (\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c)$.
Finally, in the case $p = \frac{4}{3}$, let $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ satisfying $\int_{\mathbb{R}^3} \varphi_i = 1$. We set $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{O}} * \boldsymbol{\varphi}$ and $\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{P}} * \boldsymbol{\varphi}$. The reasonning can be applied by remplacing $\delta\delta_{ij}$ by φ_i, $\boldsymbol{\mathcal{U}}$ by $\boldsymbol{\mathcal{O}}$ and $\boldsymbol{\mathcal{K}}$ by $\boldsymbol{\mathcal{P}}$. □

Thanks to the above lemma, we immediately deduce the following corollary.

Corollary 4.8. *Let Ω be an exterior domain with a $C^{1,1}$ boundary. Assume $\boldsymbol{f} \in \mathbf{W}_0^{-1,p}(\Omega)$, $\boldsymbol{u}_* \in \mathbf{W}^{1/p', p}(\Gamma)$ with $1 < p < 4$ satisfying the compatibility condition* (4.10) *and $(\boldsymbol{u}, \pi) \in \mathbf{Y}_0^{1,p}(\Omega) \times L^p(\Omega)$ be the unique solution of the system* (4.11). *If in addition, $\boldsymbol{f} \in \mathbf{W}_0^{-1,q}(\Omega)$ and $\boldsymbol{u}_* \in \mathbf{W}^{1/q', q}(\Gamma)$ with $1 < q < 4$ satisfying the compatibility condition* (4.10) *by remplacing p by q, then we also have $(\boldsymbol{u}, \pi) \in \mathbf{Y}_0^{1,q}(\Omega) \times L^q(\Omega)$.*

We denote by $[q]$ the integer part of q. For any $k \in \mathbb{N}$, $\mathscr{P}_k$ (respectively, $\mathscr{P}_k^\Delta$) stands for the space of polynomials (respectively, harmonic polynomials) of degree $\leq k$. If k is strictly negative integer, we set by convention $\mathscr{P}_k = \{0\}$. We introduce the following space of polynomials:

$$\mathcal{N}_k \ = \ \{\, (\boldsymbol{\lambda}, \mu) \in \mathscr{P}_k \times \mathscr{P}^{\Delta}_{k-1}, \ -\Delta \boldsymbol{\lambda} + \frac{\partial \boldsymbol{\lambda}}{\partial x_1} + \nabla \mu = \mathbf{0}, \ \mathrm{div}\ \boldsymbol{\lambda} = 0 \ \text{ in } \mathbb{R}^3 \,\}.$$

Observe that $\mathcal{N}_0 = \mathbb{R}^3 \times \{0\}$ and $\mathcal{N}_1 = \mathscr{P}'_1 \times \mathbb{R}^3$ where $\mathscr{P}'_1$ is the space of polynomials of degree less than or equal to one not depending on x_1.

We now introduce the space $Z_p(\Omega)$ as follows:

$$Z_p(\Omega) \ = \ \begin{cases} X_0^{2,p}(\Omega) & \text{if } \ p \geq 4, \\ X_0^{2,p}(\Omega) \cap W_0^{1,\frac{4p}{4-p}}(\Omega) & \text{if } \ 2 \leq p < 4, \\ X_0^{2,p}(\Omega) \cap W_0^{1,\frac{4p}{4-p}}(\Omega) \cap L^{\frac{2p}{2-p}}(\Omega) & \text{if } \ 1 < p < 2. \end{cases}$$

Define that

$$\mathcal{A}_p^{+}(\Omega) \ = \ \{\, (\boldsymbol{u}, \pi) \in \mathbf{Z}_p(\Omega) \times W_0^{1,p}(\Omega), \ \mathbf{T}(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega, \ \boldsymbol{u} = \mathbf{0} \text{ on } \Gamma \,\}.$$

We can characterize the kernel $\mathcal{A}_p^{+}(\Omega)$ (see [10]), as follows:

Lemma 4.9. *Let Ω be an exterior domain with a $C^{1,1}$ boundary.*
i) If $1 < p < 2$, then $\mathcal{A}_p^{+}(\Omega) \ = \ \{\, (\mathbf{0}, 0) \,\}$.
ii) If $2 \leq p < 4$, then $\mathcal{A}_p^{+}(\Omega) \ = \ \{\, (\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c);\ \boldsymbol{c} \in \mathbb{R}^3 \,\}$, where

$$(\boldsymbol{\lambda}_c, \mu_c) \in \bigcap_{r > 4/3} \mathbf{Y}_0^{1,r}(\Omega) \times \bigcap_{s > 3/2} L^s(\Omega)$$

is the unique solution of the problem (4.9).
iii) If $p \geq 4$, then $\mathcal{A}_p^{+}(\Omega) \ = \ \{\, (\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c - \eta);\ (\boldsymbol{c}, \eta) \in \mathcal{N}_1 \,\}$, where

$$(\boldsymbol{\lambda}_c, \mu_c) \in \bigcap_{r > 4/3} \mathbf{Y}_0^{1,r}(\Omega) \times \bigcap_{s > 3/2} L^s(\Omega)$$

is the unique solution of the problem (4.9).

The next lemma characterizes the kernel $\mathcal{A}^{p,q}(\Omega)$ of the exterior Oseen system:

$$\begin{aligned} \mathcal{A}^{p,q}(\Omega) \ = \ & \{\, (\boldsymbol{u}, \pi) \in [\mathbf{Z}_p(\Omega) + \mathbf{Z}_q(\Omega)] \times [W_0^{1,p}(\Omega) + W_0^{1,q}(\Omega)], \\ & \mathbf{T}(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega, \ \boldsymbol{u} = \mathbf{0} \text{ on } \Gamma \,\}. \end{aligned}$$

Lemma 4.10. *Let Ω be an exterior domain with a $C^{1,1}$ boundary and $1 < p < q < \infty$.*
i) If $1 < p < 2$, then $\mathcal{A}^{p,q}(\Omega) \ = \ \{\, (\mathbf{0}, 0) \,\}$.
ii) If $2 \leq p < 4$, then $\mathcal{A}^{p,q}(\Omega) \ = \ \{\, (\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c);\ \boldsymbol{c} \in \mathbb{R}^3 \,\}$, where

$$(\boldsymbol{\lambda}_c, \mu_c) \in \bigcap_{r > 4/3} \mathbf{Y}_0^{1,r}(\Omega) \times \bigcap_{s > 3/2} L^s(\Omega)$$

is the unique solution of the problem (4.9).
iii) If $p \geq 4$, then $\mathcal{A}^{p,q}(\Omega) \ = \ \{\, (\boldsymbol{\lambda}_c - \boldsymbol{c}, \mu_c - \eta);\ (\boldsymbol{c}, \eta) \in \mathcal{N}_1 \,\}$, where

$$(\boldsymbol{\lambda}_c, \mu_c) \in \bigcap_{r > 4/3} \mathbf{Y}_0^{1,r}(\Omega) \times \bigcap_{s > 3/2} L^s(\Omega)$$

is the unique solution of the problem (4.9).

Proof. The proof can be obtained by proceeding similarly as in the one of Lemma 4.7. □

The following corollary is immediately deduced from the previous lemma.

Corollary 4.11. *Let Ω be an exterior domain with a $C^{1,1}$. Let $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, $\boldsymbol{u}_* \in \mathbf{W}^{1+1/p',p}(\Gamma)$ with $1 < p < 2$ and $(\boldsymbol{u}, \pi) \in \mathbf{Z}_p(\Omega) \times W_0^{1,p}(\Omega)$ be the unique solution of the system* (4.11)*. If in addition, $\boldsymbol{f} \in \mathbf{L}^q(\Omega)$, $\boldsymbol{u}_* \in \mathbf{W}^{1+1/q',q}(\Gamma)$ with $1 < q < 2$, then we also have $(\boldsymbol{u}, \pi) \in \mathbf{Z}_q(\Omega) \times W_0^{1,q}(\Omega)$.*

5 More regularity for the velocity field $\boldsymbol{u}$ and the pressure π of the Navier-Stokes system

We now introduce the following result which we shall need in this part. The proof of this lemma is similar as the one of Lemma 4.2 [7].

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz exterior domain and $\boldsymbol{z} \in \mathbf{L}^4(\Omega)$ such that* div $\boldsymbol{z} = 0$*. Then, for all $\varepsilon > 0$, there exist $\rho = \rho(\varepsilon, \boldsymbol{z}) > 0$ and a sequence $(\boldsymbol{z}_k)_{k\in\mathbb{N}} \in \mathbf{L}^3(\Omega) \cap \mathbf{L}^4(\Omega)$, such that* div $\boldsymbol{z}_k = 0$*, satisfying*

$$\boldsymbol{z}_k \to \boldsymbol{z} \text{ in } \mathbf{L}^4(\Omega). \tag{5.1}$$

Moreover, there exist sequences $(\boldsymbol{a}_k)$ and $(\boldsymbol{b}_k)$ in $\mathbf{L}^3(\Omega) \cap \mathbf{L}^4(\Omega)$ satisfying for each $k \in \mathbb{N}$

$$\boldsymbol{z}_k = \boldsymbol{a}_k + \boldsymbol{b}_k \text{ with } ||\boldsymbol{a}_k||_{\mathbf{L}^4(\Omega)} \leq \varepsilon \text{ and } \text{supp } \boldsymbol{b}_k \subset \overline{\Omega}_\rho. \tag{5.2}$$

From now on, Ω is an exterior domain with $C^{1,1}$ boundary in $\mathbb{R}^3$. Note that $\mathbf{L}^{6/5}(\Omega) \hookrightarrow \mathbf{W}_0^{-1,2}(\Omega)$ and $\mathbf{L}^{3/2}(\Omega) \hookrightarrow \mathbf{W}_0^{-1,3}(\Omega)$, and with the previous results in hand, we can now prove the following theorem.

Theorem 5.2. *Assume that $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega) \cap \mathbf{L}^{3/2}(\Omega)$. Then each weak solution $(\boldsymbol{u}, \pi)$ to the problem $(\mathcal{NS})$, satisfies*

$$\begin{gathered} \boldsymbol{v} \in \mathbf{L}^q(\Omega) \textit{ for all } q \in [3, \infty), \; \pi \in W_0^{1,6/5}(\Omega) \cap \; W_0^{1,3/2}(\Omega), \\ \nabla \boldsymbol{v} \in \mathbf{L}^{12/7}(\Omega) \cap \mathbf{L}^3(\Omega), \; \nabla^2 \boldsymbol{v} \in \mathbf{L}^{6/5}(\Omega) \cap \mathbf{L}^{3/2}(\Omega), \\ \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{6/5}(\Omega) \cap \mathbf{L}^3(\Omega). \end{gathered} \tag{5.3}$$

Proof. Let $\boldsymbol{u}$ be a weak solution of $(\mathcal{NS})$. As $\boldsymbol{f}$ satisfies the hypothesis of Theorem 3.3, then $(\boldsymbol{v}, \pi)$ verify (3.2), (3.3), (3.6), (3.7) and in particular, $\boldsymbol{v} \in \mathbf{L}^4(\Omega)$ and $\dfrac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{4/3}(\Omega)$. Let $\varepsilon > 0$, $\rho > 0$ and $\boldsymbol{v}_k = \boldsymbol{a}_k + \boldsymbol{b}_k$ be a sequence as $\boldsymbol{z}_k$ in Lemma 5.1. Since $\boldsymbol{v}_k \in \mathbf{L}^3(\Omega)$ and div $\boldsymbol{v}_k = 0$, from Lemma 4.4, there exists a unique solution $(\boldsymbol{w}_k, \theta_k) \in \mathbf{X}_0^{1,2}(\Omega) \times L^2(\Omega)$ satisfying

$$-\nu\Delta \boldsymbol{w}_k + \lambda \frac{\partial \boldsymbol{w}_k}{\partial x_1} + \boldsymbol{v}_k.\nabla \boldsymbol{w}_k + \nabla\theta_k = \boldsymbol{f} \quad \text{and} \quad \text{div } \boldsymbol{w}_k = 0 \text{ in } \Omega \tag{5.4}$$

with $\boldsymbol{w}_k = -\boldsymbol{u}_\infty$ on Γ. Thanks to Theorem 5.26 [10], we have

$$\begin{aligned} &||\, \boldsymbol{w}_k \,||_{\mathbf{L}^3(\Omega)} + ||\, \nabla \boldsymbol{w}_k \,||_{\mathbf{L}^{12/7}(\Omega)} + \\ &+ ||\, \frac{\partial \boldsymbol{w}_k}{\partial x_1} \,||_{\mathbf{L}^{6/5}(\Omega)} + ||\, \nabla^2 \boldsymbol{w}_k \,||_{\mathbf{L}^{6/5}(\Omega)} + ||\, \theta_k \,||_{W_0^{1,6/5}(\Omega)} \\ &\leq C(||\, \boldsymbol{f} \,||_{\mathbf{L}^{6/5}(\Omega)} + ||\, \boldsymbol{v}_k.\nabla \boldsymbol{w}_k \,||_{\mathbf{L}^{6/5}(\Omega)} + ||\, \boldsymbol{u}_\infty \,||_{\mathbf{W}^{7/6,5/6}(\Gamma)}), \end{aligned} \tag{5.5}$$

where $C \geq 0$ depends only on λ, ν and Ω. Note now that by construction of the sequence $\boldsymbol{b}_k$, we have $|\boldsymbol{b}_k| \leq \boldsymbol{v}_k$ almost everywhere in Ω, we have

$$\begin{aligned}
&\|\, \boldsymbol{v}_k.\nabla \boldsymbol{w}_k \,\|_{\mathbf{L}^{6/5}(\Omega)} \\
&\leq \|\, \boldsymbol{a}_k \,\|_{\mathbf{L}^4(\Omega)} \|\, \nabla \boldsymbol{w}_k \,\|_{\mathbf{L}^{12/7}(\Omega)} + \|\, \boldsymbol{b}_k \,\|_{\mathbf{L}^6(\overline{\Omega}_\rho)} \|\, \nabla \boldsymbol{w}_k \,\|_{\mathbf{L}^{3/2}(\overline{\Omega}_\rho)} \\
&\leq \varepsilon \|\, \nabla \boldsymbol{w}_k \,\|_{\mathbf{L}^{12/7}(\Omega)} + \|\, \boldsymbol{v} \,\|_{\mathbf{L}^6(\Omega)} \|\, \nabla \boldsymbol{w}_k \,\|_{\mathbf{L}^{3/2}(\overline{\Omega}_\rho)}
\end{aligned} \tag{5.6}$$

But there exists $C_1 \in \mathbb{R}$ such that

$$\forall k \in \mathbb{N}^*, \ \|\, \nabla \boldsymbol{w}_k \,\|_{\mathbf{L}^{3/2}(\overline{\Omega}_\rho)} \leq C_1 (\|\, \boldsymbol{f} \,\|_{\mathbf{L}^{6/5}(\Omega)} + \|\, \boldsymbol{u}_\infty \,\|_{\mathbf{W}^{7/6,5/6}(\Gamma)}). \tag{5.7}$$

Contradicting (5.7) means that there exists a sequence $(k_m)_{m \in \mathbb{N}^*}$ such that, for all $m \in \mathbb{N}^*$,

$$\begin{gathered}
\|\nabla \boldsymbol{w}_{k_m}\|_{\mathbf{L}^{3/2}(\overline{\Omega}_\rho)} = 1, \\
\| - \nu \Delta \boldsymbol{w}_{k_m} + \lambda \frac{\partial \boldsymbol{w}_{k_m}}{\partial x_1} + \boldsymbol{v}_{k_m}.\nabla \boldsymbol{w}_{k_m} + \nabla \theta_{k_m} \|_{\mathbf{L}^{6/5}(\Omega)} + \|\boldsymbol{w}_{k_m}\|_{\mathbf{W}^{7/6,5/6}(\Gamma)} \leq \frac{1}{m}.
\end{gathered} \tag{5.8}$$

Then we deduce from (5.5), (5.6) and (5.8) that

$$\begin{aligned}
&\|\, \boldsymbol{w}_{k_m} \,\|_{\mathbf{L}^3(\Omega)} + \|\, \nabla \boldsymbol{w}_{k_m} \,\|_{\mathbf{L}^{12/7}(\Omega)} + \|\, \nabla^2 \boldsymbol{w}_{k_m} \,\|_{\mathbf{L}^{6/5}(\Omega)} \\
&+ \|\, \frac{\partial \boldsymbol{w}_{k_m}}{\partial x_1} \,\|_{\mathbf{L}^{6/5}(\Omega)} + \|\, \theta_{k_m} \,\|_{W_0^{1,6/5}(\Omega)} \leq C.
\end{aligned}$$

Therefore $(\boldsymbol{w}_{k_m})_m$ is bounded in $\mathbf{W}_0^{2,6/5}(\Omega) \cap \mathbf{W}_0^{1,12/7}(\Omega)$, $\left(\frac{\partial \boldsymbol{w}_{k_m}}{\partial x_1} \right)_m$ is bounded in $\mathbf{L}^{6/5}(\Omega)$, $(\boldsymbol{w}_{k_m})_m$ is bounded in $\mathbf{L}^3(\Omega)$ and $(\theta_{k_m})_m$ is bounded in $W_0^{1,6/5}(\Omega)$. Thus, there exist subsequences, again denoted by $(\boldsymbol{w}_{k_m})_m$ and $(\theta_{k_m})_m$, such that $\boldsymbol{w}_{k_m} \rightharpoonup \boldsymbol{w}$ in $\mathbf{W}_0^{2,6/5}(\Omega) \cap \mathbf{W}_0^{1,12/7}(\Omega)$, $\frac{\partial \boldsymbol{w}_{k_m}}{\partial x_1} \rightharpoonup \frac{\partial \boldsymbol{w}}{\partial x_1}$ in $\mathbf{L}^{6/5}(\Omega)$, $\boldsymbol{w}_{k_m} \rightharpoonup \boldsymbol{w}$ in $\mathbf{L}^3(\Omega)$, and $\theta_{k_m} \rightharpoonup \theta$ in $W_0^{1,6/5}(\Omega)$. Moreover, since $\mathbf{W}^{2,6/5}(\overline{\Omega}_\rho) \hookrightarrow \mathbf{W}^{1,3/2}(\overline{\Omega}_\rho)$ with compact imbedding, we have $\boldsymbol{w}_{k_m} \to \boldsymbol{w}$ in $\mathbf{W}^{1,3/2}(\overline{\Omega}_\rho)$ with

$$\|\nabla \boldsymbol{w}\|_{\mathbf{L}^{3/2}(\overline{\Omega}_\rho)} = 1, \tag{5.9}$$

and

$$-\nu \Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \boldsymbol{v}.\nabla \boldsymbol{w} + \nabla \theta = 0 \ \text{ in } \ \Omega. \tag{5.10}$$

Since $\boldsymbol{w} \in \mathbf{W}_0^{1,2}(\Omega)$ and $\theta \in L^2(\Omega)$, then we have $\Delta \boldsymbol{w}$ and $\nabla \theta$ belonging to $\mathbf{W}_0^{-1,2}(\Omega)$. On the other hand, we deduce that $\boldsymbol{v}.\nabla \boldsymbol{w} = \text{ div } (\boldsymbol{v} \otimes \boldsymbol{w}) \in \mathbf{W}_0^{-1,2}(\Omega)$ because $\boldsymbol{v}$ and $\boldsymbol{w}$ belong to $\mathbf{L}^4(\Omega)$. Since $\mathbf{L}^{6/5}(\Omega) \hookrightarrow \mathbf{W}_0^{-1,2}(\Omega)$ we also have $\frac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{W}_0^{-1,2}(\Omega)$. Moreover, $\boldsymbol{w}$ is divergence free and, because of (5.8), it has also zero trace at the boundary. Then, we deduce $\boldsymbol{w} = 0$ in Ω which contradicts (5.9). Thanks to (5.5), (5.6) and (5.7), we have the following estimation

$$\begin{aligned}
&\|\, \boldsymbol{w}_k \,\|_{\mathbf{L}^3(\Omega)} + \|\, \nabla \boldsymbol{w}_k \,\|_{\mathbf{L}^{12/7}(\Omega)} \\
&\|\, \frac{\partial \boldsymbol{w}_k}{\partial x_1} \,\|_{\mathbf{L}^{6/5}(\Omega)} + \|\, \nabla^2 \boldsymbol{w}_k \,\|_{\mathbf{L}^{6/5}(\Omega)} + \|\, \theta_k \,\|_{W_0^{1,6/5}(\Omega)} \\
&\leq C(\|\, \boldsymbol{f} \,\|_{\mathbf{L}^{6/5}(\Omega)} + \|\, \boldsymbol{v} \,\|_{\mathbf{L}^6(\Omega)} \|\, \boldsymbol{f} \,\|_{\mathbf{L}^{6/5}(\Omega)} + \|\, \boldsymbol{u}_\infty \|_{\mathbf{W}^{1+1/6,6/5}(\Gamma)}).
\end{aligned}$$

We can show that there exist a subsequence of $(\boldsymbol{w}_k)_k$ which converges weakly towards $\boldsymbol{w}$ in $\mathbf{W}_0^{2,6/5}(\Omega) \cap \mathbf{W}_0^{1,12/7}(\Omega) \cap \mathbf{L}^3(\Omega)$ and a subsequence of $(\theta_k)_k$ which converges weakly towards θ in $W_0^{1,6/5}(\Omega)$ being a solution of the system as follows

$$-\nu\Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \boldsymbol{v}.\nabla \boldsymbol{w} + \nabla \theta = \boldsymbol{f} \quad \text{and} \quad \text{div } \boldsymbol{w} = 0 \text{ in } \Omega.$$

We set $\boldsymbol{y} = \boldsymbol{v} - \boldsymbol{w}$ and $\chi = \pi - \theta$. Then we deduce that $(\boldsymbol{y}, \chi)$ is a solution of the following system

$$-\nu\Delta \boldsymbol{y} + \lambda \frac{\partial \boldsymbol{y}}{\partial x_1} + \boldsymbol{v}.\nabla \boldsymbol{y} + \nabla \chi = 0 \quad \text{and} \quad \text{div } \boldsymbol{y} = 0 \text{ in } \Omega.$$

Since $\boldsymbol{y}$ satisfies the energy equality (4.3) with $\boldsymbol{f} = 0$, we deduce that $\boldsymbol{y} = \mathbf{0}$ then $\chi = 0$. Thanks to uniqueness arguments, we show that $\boldsymbol{w} = \boldsymbol{v}$ and $\theta = \pi$. Theorem is completely proved. □

Thanks to Theorem 3.2 (part ii), Theorem 5.2, Sobolev embedding theorem and by duality arguments, we can prove the following.

Corollary 5.3. *i) Assume that $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ for all $p \in [6/5, 2)$. Then the Navier-Stokes problem $(\mathcal{NS})$ has a solution $(\boldsymbol{u}, \pi)$ satisfying*

$$\begin{aligned} \boldsymbol{v} \in \mathbf{L}^q(\Omega),\ \nabla \boldsymbol{v} \in \mathbf{L}^{s_1}(\Omega),\ \pi \in W_0^{1,s_2}(\Omega), \\ \nabla^2 \boldsymbol{v} \in \mathbf{L}^{s_2}(\Omega),\ \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{s_3}(\Omega), \end{aligned} \tag{5.11}$$

for any $q \in [3, \infty]$, any $s_1 \in [12/7, 6)$, any $s_2 \in [6/5, 2)$ and any $s_3 \in [6/5, 6)$.

ii) Assume that $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ for all $p \in [6/5, 3)$. Then we have (5.11) for any $q \in [3, \infty]$, $s_1 \in [12/7, \infty)$, $s_2 \in [6/5, 3)$ and $s_3 \in [6/5, \infty)$.

We now prove the following Theorem.

Theorem 5.4. *Assume that $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ for all $p \in (1, 3/2]$. Then each weak solution $(\boldsymbol{u}, \pi)$ to the problem $(\mathcal{NS})$ satisfies*

$$\begin{aligned} \boldsymbol{v} \in \mathbf{L}^q(\Omega),\ \nabla \boldsymbol{v} \in \mathbf{L}^{s_1}(\Omega),\ \ \pi \in W_0^{1,s_2}(\Omega), \\ \nabla^2 \boldsymbol{v} \in \mathbf{L}^{s_2}(\Omega),\ \ \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{s_3}(\Omega), \end{aligned} \tag{5.12}$$

for any $q \in (2, \infty)$, any $s_1 \in (4/3, 3]$, any $s_2 \in (1, 3/2]$ and any $s_3 \in (1, 3]$.

Proof. Remark that from Theorem 5.2, as $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ and $\nabla \boldsymbol{v} \in \mathbf{L}^{12/7}(\Omega)$, we have $\boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v} \in \mathbf{L}^{12/11}(\Omega)$. By applying Theorem 5.26 [10] with $p = \frac{12}{11}$, the following system

$$-\nu\Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \nabla \theta = \boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v}, \ \ \text{div } \boldsymbol{w} = 0 \ \text{ in } \Omega; \ \ \boldsymbol{w} = -\boldsymbol{u}_\infty \ \text{ on } \Gamma,$$

has a unique solution $(\boldsymbol{w}, \theta)$ satisfying $\boldsymbol{w} \in \mathbf{L}^{2p/(2-p)}(\Omega) \cap \mathbf{L}^{3p/(3-2p)}(\Omega)$, $\nabla \boldsymbol{w} \in \mathbf{L}^{4p/(4-p)}(\Omega) \cap \mathbf{L}^{3p/(3-p)}(\Omega)$, $\nabla^2 \boldsymbol{w} \in \mathbf{L}^p(\Omega)$, $\frac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{L}^p(\Omega)$ and $\theta \in W_0^{1,p}(\Omega)$, *i.e.*, $(\boldsymbol{w}, \theta) \in \mathbf{Z}_{12/11}(\Omega) \times W_0^{1,12/11}(\Omega)$. On the other hand, from (5.3), we can show $(\boldsymbol{v}, \pi) \in \mathbf{Z}_{6/5}(\Omega) \times W_0^{1,6/5}(\Omega)$. Thanks to Corollary 4.11, we have $(\boldsymbol{w}, \theta) = (\boldsymbol{v}, \pi)$. Then, we obtain $\boldsymbol{v} \in \mathbf{L}^{12/5}(\Omega) \cap \mathbf{L}^{12/7}(\Omega)$, $\nabla \boldsymbol{v} \in \mathbf{L}^{4/3}(\Omega) \cap \mathbf{L}^{12/7}(\Omega)$, $\nabla^2 \boldsymbol{v}$, and $\dfrac{\partial \boldsymbol{v}}{\partial x_1}$ belong to $\mathbf{L}^{12/11}(\Omega)$, $\pi \in W_0^{1,12/11}(\Omega)$. Combining with the results in Theorem 5.2, we have $\boldsymbol{v} \in \mathbf{L}^q(\Omega)$ for all $q \in [12/5, \infty)$ and $\nabla \boldsymbol{v} \in \mathbf{L}^{4/3}(\Omega) \cap \mathbf{L}^3(\Omega)$. Hence, it is easy to prove that $\boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v}$ belongs to $\mathbf{L}^p(\Omega)$ for all $p \in]1, 3/2]$ and we can deduce that $\boldsymbol{v} \in \mathbf{L}^{\frac{2p}{2-p}}(\Omega) \cap \mathbf{L}^{\frac{3p}{3-2p}}(\Omega)$, $\nabla \boldsymbol{v} \in \mathbf{L}^{\frac{4p}{4-p}}(\Omega) \cap \mathbf{L}^{\frac{3p}{3-p}}(\Omega)$, $\nabla^2 \boldsymbol{v} \in \mathbf{L}^p(\Omega)$, $\dfrac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^p(\Omega)$ and $\pi \in W_0^{1,p}(\Omega)$. Clearly, we have (5.12) by combining with (5.3). □

Thanks to Corollary 5.3 and Theorem 5.4, we immediately obtain the following results.

Corollary 5.5. *i) Assume that* $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ *for all* $1 < p < 2$. *Then each weak solution* $(\boldsymbol{u}, \pi)$ *to* $(\mathcal{NS})$ *satisfies*

$$\begin{gathered} \boldsymbol{v} \in \mathbf{L}^q(\Omega), \ \nabla \boldsymbol{v} \in \mathbf{L}^{s_1}(\Omega), \ \pi \in W_0^{1,s_2}(\Omega), \\ \nabla^2 \boldsymbol{v} \in \mathbf{L}^{s_2}(\Omega), \ \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{s_3}(\Omega), \end{gathered} \tag{5.13}$$

for any $q \in (2, \infty]$, *any* $s_1 \in [4/3, 6)$, *any* $s_2 \in [1, 2)$ *and any* $s_3 \in [1, 6)$.

ii) Assume that $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ *for all* $1 < p < 3$. *Then we have* (5.13) *for any* $q \in (2, \infty]$, $s_1 \in (4/3, \infty)$, $s_2 \in (1, 3)$ *and* $s_3 \in (1, \infty)$.

We now search weak solutions of Navier-Stokes system $(\mathcal{NS})$ such that $\boldsymbol{v} \in \mathbf{L}^q(\Omega)$ and $\pi \in L^q(\Omega)$ for small values of q ($q \leq 2$) with similar properties for $\nabla \boldsymbol{v}$. The following theorem allow us to improve the results in Theorem 5.2 by taking an additional assumption for $\boldsymbol{f}$.

Theorem 5.6. *Let* $\dfrac{4}{3} < p < 2$ *and* $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega) \cap \mathbf{L}^{3/2}(\Omega) \cap \mathbf{W}_0^{-1,p}(\Omega)$. *Then each weak solution* $(\boldsymbol{u}, \pi)$ *to the problem* $(\mathcal{NS})$ *satisfies* (5.3). *Besides, we have*

$$\pi \in L^p(\Omega) \ \textit{and} \ \ \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{W}_0^{-1,s}(\Omega) \ \ \textit{for any } s \geq p. \tag{5.14}$$

In particular, if $\dfrac{4}{3} < p < \dfrac{12}{7}$, *we obtain additionally*

$$\boldsymbol{v} \in \mathbf{L}^q(\Omega) \ \ \textit{for any } q \geq \frac{4p}{4-p} \ \ \textit{and} \ \ \nabla \boldsymbol{v} \in \mathbf{L}^p(\Omega). \tag{5.15}$$

Proof. From Theorem 5.2, if $\boldsymbol{u}$ is a solution of $(\mathcal{NS})$, we have $\boldsymbol{v}$ satisfies (5.3). In particular, $\boldsymbol{v} \in \mathbf{L}^3(\Omega) \cap \mathbf{L}^4(\Omega)$ and div $(\boldsymbol{v} \otimes \boldsymbol{v}) \in \mathbf{W}_0^{-1,3/2}(\Omega) \cap \mathbf{W}_0^{-1,2}(\Omega)$.

1) *The case* $3/2 \leq p < 2$: We have $\boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v} \in \mathbf{W}_0^{-1,p}(\Omega)$. As p>4/3, then the compatibility condition (4.10) is automatically satisfied. Thanks to Theorem 4.6, the following system

$$-\nu \Delta \boldsymbol{w} + \lambda \frac{\partial \boldsymbol{w}}{\partial x_1} + \nabla \theta = \boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v}, \ \ \text{div } \boldsymbol{w} = 0 \ \text{ in } \Omega; \ \ \boldsymbol{w} = -\boldsymbol{u}_\infty \ \text{ on } \Gamma,$$

has a unique solution $(\boldsymbol{w}, \theta)$ satisfying $\boldsymbol{w} \in \mathbf{L}^{4p/(4-p)}(\Omega) \cap \mathbf{L}^{3p/(3-p)}(\Omega)$, $\theta \in L^p(\Omega)$, $\nabla \boldsymbol{w} \in \mathbf{L}^p(\Omega)$ and $\dfrac{\partial \boldsymbol{w}}{\partial x_1} \in \mathbf{W}_0^{-1,p}(\Omega)$. It is easy to see that $(\boldsymbol{w}, \theta) \in \mathbf{Y}_0^{1,p}(\Omega) \times L^p(\Omega)$ and $(\boldsymbol{v}, \pi) \in \mathbf{Y}_0^{1,3/2}(\Omega) \times L^{3/2}(\Omega)$. Applying Corollary 4.8, we have $(\boldsymbol{w}, \theta) = (\boldsymbol{v}, \pi)$, then we obtain (5.14).

2) *The case* $4/3 < p < 3/2$*:* Since $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega) \hookrightarrow \mathbf{W}_0^{-1,2}(\Omega)$, then in particular $\boldsymbol{f} \in \mathbf{W}_0^{-1,3/2}(\Omega)$. From the case 1) of this theorem, we have $\boldsymbol{v} \in \mathbf{L}^{4p/(4-p)}(\Omega) \cap \mathbf{L}^{3p/(3-p)}(\Omega)$. Applying with $p = 3/2$, we have $\boldsymbol{v} \in \mathbf{L}^{12/5}(\Omega) \cap \mathbf{L}^3(\Omega)$. Hence, we can show that $\boldsymbol{v}.\nabla \boldsymbol{v} = \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) \in \mathbf{W}_0^{-1,4/3}(\Omega) \cap \mathbf{W}_0^{-1,3/2}(\Omega)$ and $\boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v} \in \mathbf{W}_0^{-1,p}(\Omega)$. By applying Theorem 4.6 and Corollary 4.8, we have (5.14) and (5.15).

The proof is complete by combining the case 1) with the case 2). □

Remark 5.7. Because of the compatibility condition (4.10), the above problem is open for the case $1 < p \leq \dfrac{4}{3}$.

In Theorem 5.4, we know that if $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ for all $p \in (1, 3/2]$, then $\boldsymbol{v}$ satisfies (5.12). With additional assumption for $\boldsymbol{f}$, we shall prove that the weak solutions given in Theorem 5.4 satisfy better properties.

Proposition 5.8. *Given* $r > \dfrac{4}{3}$*. Assume that* $\boldsymbol{f} \in \mathbf{L}^p(\Omega) \cap \mathbf{W}_0^{-1,r}(\Omega)$ *for all* $p \in (1, 3/2]$*. Then each weak solution* $(\boldsymbol{u}, \pi)$ *to* $(\mathcal{NS})$ *satisfies* (5.12) *and* $\dfrac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{W}_0^{-1,s}(\Omega)$ *for any* $s \geq r$*. Moreover,*

$$\textit{if} \;\; \frac{4}{3} < r \leq \frac{3}{2}, \;\; \pi \in L^t(\Omega) \;\textit{for all}\;\; r \leq t \leq 3. \tag{5.16}$$

Proof. We know that $(\boldsymbol{u}, \pi)$ satisfies (5.12). In addition, thanks to Theorem 5.4, we have $\boldsymbol{v} \otimes \boldsymbol{v} \in \mathbf{L}^q(\mathbb{R}^3)$ for all $q > 1$. Then we deduce $\boldsymbol{f} - \boldsymbol{v}.\nabla \boldsymbol{v} \in \mathbf{W}_0^{-1,r}(\Omega)$. Proceeding as in Theorem 5.6, it is easy to prove that $\boldsymbol{v} \in \mathbf{L}^{\frac{4r}{4-r}}(\Omega) \cap \mathbf{L}^{\frac{3r}{3-r}}(\Omega)$, $\nabla \boldsymbol{v} \in \mathbf{L}^r(\Omega)$, $\dfrac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{W}_0^{-1,r}(\Omega)$ and $\pi \in L^r(\Omega)$. As $\boldsymbol{v} \in \mathbf{L}^q(\Omega)$ for any $q \geq 2$, we have $\dfrac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{W}_0^{-1,s}(\Omega)$ for any $s \geq r$. For the pressure, we note that thanks to (5.12), $\pi \in L^t(\Omega)$ for all $3/2 < t \leq 3$ and then, we have (5.16). The Theorem is completely proved. □

We now prove the following theorem.

Theorem 5.9. *Let* $4/3 < p < \infty$ *and* $q_0 \geq 3$*. Assume that* $\boldsymbol{f} \in \mathbf{L}^q(\Omega) \cap \mathbf{W}_0^{-1,p}(\Omega)$ *for all* $q \in (1, q_0]$*. Then the problem* $(\mathcal{NS})$ *has a solution* $(\boldsymbol{u}, \pi)$ *satisfying the properties of Corollary* 5.5 *part ii). Moreover, we have* $\pi \in W_0^{1,s_2}(\Omega)$ *and* $\nabla^2 \boldsymbol{v} \in \mathbf{L}^{s_2}(\Omega)$ *for all* $s_2 \in (1, q_0]$*. In particular, if* $4/3 < p \leq 3/2$*, we have additionally* $\pi \in L^{k_1}(\Omega)$ *for any* $k_1 \geq p$*.*

Proof. In particular, we have $\boldsymbol{f} \in \mathbf{L}^q(\Omega)$ for all $1 < q < 3$. From Corollary 5.5 part ii), we have

$$\begin{gathered} \boldsymbol{v} \in \mathbf{L}^{s_0}(\Omega), \;\; \nabla \boldsymbol{v} \in \mathbf{L}^{s_1}(\Omega), \;\; \pi \in W_0^{1,s_2}(\Omega), \\ \nabla^2 \boldsymbol{v} \in \mathbf{L}^{s_2}(\Omega), \;\; \frac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^{s_3}(\Omega), \end{gathered} \tag{5.17}$$

for any $s_0 \in (2,\infty]$, any $s_1 \in (4/3,\infty)$, any $s_2 \in (1,3)$ and any $s_3 \in (1,\infty)$. Using the partition of unity, we can deduce $(\boldsymbol{f}^{\,1}, g^1) \in \mathbf{L}^q(\mathbb{R}^3) \times X_0^{1,q}(\mathbb{R}^3)$ and $(\boldsymbol{f}^{\,2}, g^2) \in \mathbf{L}^q(\Omega_2) \times W_0^{1,q}(\Omega_2)$ for all $q \in (1, q_0]$ satisfying (3.5). Applying Theorem 1.9 [7], Proposition 4.3 [10] and proceeding as in Theorem 3.1, we can obtain that $\pi \in W_0^{1,q}(\Omega)$, $\nabla^2 \boldsymbol{v} \in \mathbf{L}^q(\Omega)$, $\dfrac{\partial \boldsymbol{v}}{\partial x_1} \in \mathbf{L}^q(\Omega)$. Combining with the previous results, we have (5.17) for all $s_2 \in (1, q_0]$, $s_3 \in (1,\infty)$. As $\boldsymbol{v}\otimes\boldsymbol{v} \in \mathbf{L}^r(\mathbb{R}^3)$ for any $r > 1$, then $\boldsymbol{f} - \boldsymbol{v}.\nabla\boldsymbol{v} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$. We use the same technique as in the proof of Theorem 5.6, we have $(\boldsymbol{v},\pi) \in (\mathbf{X}_0^{1,p}(\Omega) \times L^p(\Omega))$ such that $\boldsymbol{w} \in \mathbf{L}^s(\Omega)$ for all $\frac{4p}{4-p} \le s \le \frac{3p}{3-p}$. Note that $\pi \in L^{k_1}(\Omega)$ for any $k_1 \ge p$ if $4/3 < p \le 3/2$. The Theorem is completely proved. □

Now we introduce the stress tensor $\mathbf{T}$ and the related stretching tensor $\mathbf{D}$, $\mathbf{T}(\boldsymbol{u},\pi) = -\pi\mathbf{I} + 2\nu\mathbf{D}(\boldsymbol{u})$, where $\mathbf{I}$ is the identity matrix and $\mathbf{D}(\boldsymbol{u}) = \{D_{ij}\}(\boldsymbol{u})$ with

$$D_{ij}(\boldsymbol{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

We now consider the energy identity. The key idea to find the conditions to obtain the energy identity (5.18), is to test the Navier-Stokes problem with $\boldsymbol{v}$.

Theorem 5.10. *Let $\boldsymbol{f} \in \mathbf{L}^{6/5}(\Omega) \cap \mathbf{L}^{3/2}(\Omega)$ and $(\boldsymbol{u},\pi)$ be a weak solution of $(\mathcal{NS})$. Then we have the energy identity*

$$\nu \int_\Omega |\nabla \boldsymbol{v}|^2 d\boldsymbol{x} - \lambda \int_\Gamma \mathbf{T}.\boldsymbol{n}\, d\sigma = \int_\Omega \boldsymbol{f}.\boldsymbol{v}\, d\boldsymbol{x}. \tag{5.18}$$

Proof. Let $(\boldsymbol{u},\pi)$ be a weak solution of $(\mathcal{NS})$. From Theorem 5.2, we know that (5.3) takes place. Let $\boldsymbol{a}_0 \in \mathbf{H}^1(\Omega_{2R})$ where $R > R_0$ such that $\boldsymbol{a}_0 = \mathbf{0}$ on $\partial\Omega$, $\boldsymbol{a}_0 = \boldsymbol{u}_\infty$ on ∂B_{2R}, div $\boldsymbol{a}_0 = 0$ in Ω_{2R}. We set that $\boldsymbol{a} = \boldsymbol{u}_\infty$ in B^{2R} and $\boldsymbol{a} = \boldsymbol{a}_0$ in Ω_{2R}. Then, we have $\boldsymbol{a} - \boldsymbol{u}_\infty \in \mathbf{W}_0^{1,2}(\Omega)$ with compact support and div $\boldsymbol{a} = 0$. As $\boldsymbol{\mathcal{V}}(\Omega)$ is dense in $\mathbf{V}^2(\Omega)$ (cf. [2]), there exists a sequence $(\boldsymbol{\psi}_i) \in \boldsymbol{\mathcal{V}}(\Omega)$ with $(\boldsymbol{\psi}_i) \rightharpoonup \boldsymbol{v} - \boldsymbol{a} + \boldsymbol{u}_\infty$ in $\mathbf{V}^2(\Omega)$ with compact support. Since $\boldsymbol{v} \in \mathbf{L}^3(\Omega)$ then we deduce $\boldsymbol{v} - \boldsymbol{a} + \boldsymbol{u}_\infty \in \mathbf{L}^3(\Omega)$. Testing (3.1) with $(\boldsymbol{\psi}_i)$, we obtain

$$\begin{aligned}&\nu \textstyle\int_\Omega \nabla\boldsymbol{v}.\nabla\boldsymbol{\psi}_i d\boldsymbol{x} + \lambda < \dfrac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{\psi}_i >_{\mathbf{W}_0^{-1,2}(\Omega)\times \mathring{\mathbf{W}}_0^{1,\,2}(\Omega)} \\ &+ < \boldsymbol{v}.\nabla\boldsymbol{v}, \boldsymbol{\psi}_i >_{\mathbf{W}_0^{-1,2}(\Omega)\times \mathring{\mathbf{W}}_0^{1,\,2}(\Omega)} = < \boldsymbol{f}, \boldsymbol{\psi}_i >_{\mathbf{W}_0^{-1,2}(\Omega)\times \mathring{\mathbf{W}}_0^{1,\,2}(\Omega)}.\end{aligned}$$

When $i \to \infty$, we deduce that

$$\begin{aligned}&\nu \textstyle\int_\Omega |\nabla \boldsymbol{v}|^2 d\boldsymbol{x} - \nu \int_\Omega \nabla\boldsymbol{v}.\nabla\boldsymbol{a}\, d\boldsymbol{x} + \lambda < \dfrac{\partial \boldsymbol{v}}{\partial x_1}, \boldsymbol{v} + \boldsymbol{u}_\infty - \boldsymbol{a} >_{\mathbf{W}_0^{-1,2}\times \mathring{\mathbf{W}}_0^{1,\,2}} \\ &+ < \boldsymbol{v}.\nabla\boldsymbol{v}, \boldsymbol{v} + \boldsymbol{u}_\infty - \boldsymbol{a} >_{\mathbf{W}_0^{-1,2}\times \mathring{\mathbf{W}}_0^{1,\,2}} = < \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{a} + \boldsymbol{u}_\infty >_{\mathbf{W}_0^{-1,2}\times \mathring{\mathbf{W}}_0^{1,\,2}}.\end{aligned} \tag{5.19}$$

From (5.3), $\dfrac{\partial \boldsymbol{v}}{\partial x_1}$ and $\boldsymbol{v}.\nabla\boldsymbol{v}$ are in $\mathbf{L}^{3/2}(\Omega)$. Then, we can rewrite (5.19) as follows

$$\begin{aligned}&\nu \textstyle\int_\Omega |\nabla \boldsymbol{v}|^2 d\boldsymbol{x} - \nu \int_\Omega \nabla\boldsymbol{v}.\nabla\boldsymbol{a}\, d\boldsymbol{x} + \lambda \int_\Omega \dfrac{\partial \boldsymbol{v}}{\partial x_1}.(\boldsymbol{v} + \boldsymbol{u}_\infty - \boldsymbol{a})\, d\boldsymbol{x} \\ &+ \textstyle\int_\Omega \boldsymbol{v}.\nabla\boldsymbol{v}.(\boldsymbol{v} + \boldsymbol{u}_\infty - \boldsymbol{a})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}.(\boldsymbol{v} - \boldsymbol{a} + \boldsymbol{u}_\infty)\, d\boldsymbol{x}.\end{aligned} \tag{5.20}$$

Next, we multiply (3.1) with $\boldsymbol{u}_\infty - \boldsymbol{a} \in \mathbf{W}_0^{1,2}(\Omega)$ having compact support. Integrating on Ω and using integration by parts, we get

$$\begin{aligned} &-\nu \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{a}\, d\boldsymbol{x} - \lambda \int_\Gamma \mathbf{T}.\boldsymbol{n}\, d\sigma + \lambda \int_\Omega \frac{\partial \boldsymbol{v}}{\partial x_1}.(\boldsymbol{u}_\infty - \boldsymbol{a})\, d\boldsymbol{x} \\ &+ \int_\Omega \boldsymbol{v}.\nabla \boldsymbol{v}.(\boldsymbol{u}_\infty - \boldsymbol{a})\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f}.(\boldsymbol{u}_\infty - \boldsymbol{a})\, d\boldsymbol{x}. \end{aligned} \tag{5.21}$$

It is easy to see that $\int_\Omega \boldsymbol{v}.\nabla \boldsymbol{v}.\boldsymbol{v} = 0$ and $\int_\Omega \dfrac{\partial \boldsymbol{v}}{\partial x_1}.\boldsymbol{v} = 0$ (cf. [12]). From (5.20) and (5.21), we have (5.18). □

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, (2003).

[2] F. Alliot, C. Amrouche, *On the regularity and decay of the weak solutions to the steady-state Navier-Stokes equations in exterior domains*, Applied Nonlinear Analysis, Edited by A. Sequeira, H. Beirao da Veiga and J. H. Videman, Kluwer Academic, Dordrecht, (1999), pp. 1-18.

[3] F. Alliot, C. Amrouche, *The Stokes problem in $\mathbb{R}^3$: an approach in weighted Sobolev spaces*, Mathematical Models and Methods in Applied Sciences, **9**, (1999), pp. 723-754.

[4] C. Amrouche, V. Girault, *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension*, Czechoslovak Mathematical Journal, Praha, **44** (119), (1994), pp. 109-140.

[5] C. Amrouche, V. Girault, J. Giroire, *Weighted Sobolev spaces for the Laplace equation in $\mathbb{R}^n$*, Journal de Mathématiques Pures et Appliquées, **73** (6), (1994), pp. 579-606.

[6] C. Amrouche, V. Girault, J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: an approach in weighted Sobolev spaces*, Journal de Mathématiques Pures et Appliquées, **76** (1), (1997), pp. 55-81.

[7] C. Amrouche, HH. Nguyen, *The stationary three-dimensional Navier-Stokes Equations with a non-zero constant velocity at infinity*, Mathematical Methods in the Applied Sciences, (online: DOI 10.1002/mma. 1020), (2008).

[8] C. Amrouche, U. Razafison, *The stationary Oseen equations in $\mathbb{R}^3$. An approach in weighted Sobolev spaces*, Journal of Mathematical Fluids Mechanics, **9** (2), (2007), pp. 211-225.

[9] C. Amrouche, U. Razafison, *Weighted Sobolev spaces for a scalar model of the stationary Oseen equations in $\mathbb{R}^3$*, Journal of Mathematical Fluids Mechanics, **9** (2), (2007), pp. 181-210.

[10] C. Amrouche, U. Razafison, *On the Oseen problem in three-dimensional exterior domains*, Analysis and Applications, 4-2, (2006), pp. 133-162.

[11] K. I. Babenko, *On stationary solutions of the problem of flow past a body of a viscous incompressible fluid*, Mathematics of the USSR, Sbornik , **20**, (1973), pp. 1-25.

[12] R. Farwig, *The stationary Navier-Stokes equations in a 3D-exterior domain*, Recent topics on mathematical theory of viscous incompressible fluid, Tsukuba, (1996). Lecture Notes in Numerical and Applied Analysis, **16**, (1998), pp. 53-115.

[13] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, vol I and vol II, Springer tracts in natural philosophy, Springer, Berlin, (1994).

[14] V. Girault, P-A. Raviart, *Finite element methods for Navier-Stokes equations (Theory and Algorithms)*, Springer-Verlag Berlin Heidelberg, (1986).

[15] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Mathematica, **63**, (1934), pp. 193-248.

[16] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, (1969).