Reconstructing shapes with guarantees by unions of convex sets

Dominique Attali, André Lieutier

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Abstract

A simple way to reconstruct a shape $A \subset \mathbb{R}^N$ from a sample P is to output an r-offset $P + rB$, where $B = \{x \in \mathbb{R}^N \mid \|x\| \leq 1\}$ designates the unit Euclidean ball centered at the origin. Recently, it has been proved that the output $P + rB$ is homotopy equivalent to the shape A, for a dense enough sample P of A and for a suitable value of the parameter r [12, 23]. In this paper, we extend this result and find convex sets $C \subset \mathbb{R}^N$, besides the unit Euclidean ball B, for which $P + rC$ reconstructs the topology of A. This class of convex sets includes in particular N-dimensional cubes in $\mathbb{R}^N$. We proceed in two steps. First, we establish the result when P is an ε-offset of A. Building on this first result, we then consider the case when P is a finite noisy sample of A.

1 Introduction

In this paper, we study the Minkowski sum between a convex set and a point set that samples a shape, generalizing previous results that establish the Minkowski sum captures the topology of the shape when the convex set is an Euclidean ball.

Prior work and problem. Motivated by surface reconstruction from 3D scan data and manifold learning from point clouds, several authors have formulated precise conditions under which a reconstruction algorithm outputs a topologically correct approximation of a shape, given as input a possibly noisy sample of it [3, 14, 7, 23, 12]. Maybe one of the simplest algorithm consists in outputting an Euclidean r-offset of the sample, that is the union of Euclidean balls with radius r centered on the sample points. Assuming the reach of the shape is positive and the data points form a sufficiently dense and accurate sample of the shape, authors in [23, 12] have established that r-offsets of the data points are homotopy equivalent to the shape for suitable values of the offset parameter r (see Figure 1, left). The aim of this work is to understand how this result generalizes when, instead of unions of Euclidean balls, we consider for the reconstruction unions of translated and scaled copies of a convex set C centered on the sample. In other words, writing B for the unit Euclidean ball centered at the origin and letting P be a sample of a shape A, we would like to understand what happens if we replace the Euclidean r-offset $P + rB$ by the Minkowski sum $P + rC$. Do we keep the topology of the shape A as in Figure 1, middle or do we lose it as in Figure 1, right? We are particularly interested in the case where C is a polytope.

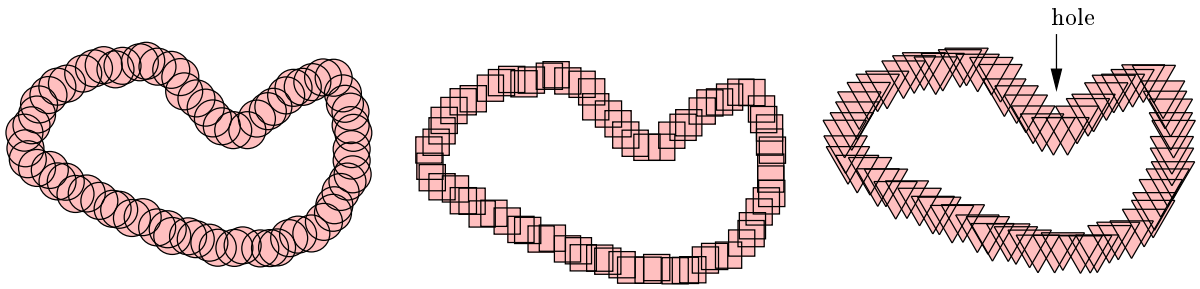


Figure 1: The union of disks and squares capture the topology of the sampled curve unlike the union of triangles.

Motivation. Our motivation to study this question is two-fold. First, in many practical applications such as stereo vision or image analysis, the accuracy of measures varies in magnitude according to the direction of measurements. In this context, it seems reasonable to recover the topology of the shape, using a convex set which takes into account the anisotropy of the measurement device. Second, we believe that unions of cubes could present some advantages over unions of Euclidean balls for topological computations in high dimension. In practice, the reconstruction represented by an α-offset is replaced by the more convenient corresponding α-shape which shares the same homotopy type [16]. Indeed, the α-shape has a simpler geometry and, being a simplicial complex, can benefit from existing theorems and algorithms dedicated to topological computations. However, if the ambient dimension is large, the α-shape may have a high complexity and its computation may be rather expensive and requires sophisticated data structures [5]. Our idea to circumvent this problem is the following. Given $\varepsilon > 0$, we define the cubical grid $G_\varepsilon = (\varepsilon\mathbb{Z})^N \subset \mathbb{R}^N$ and replace the sample $P \subset \mathbb{R}^N$ by a nearby sample $P_\varepsilon \subset G_\varepsilon$. Applying our result to the unit N-dimensional cube $C = [-1, 1]^N \in \mathbb{R}^N$, we shall see that the set $P_\varepsilon + k\varepsilon C$ captures the homotopy type of the shape, for some well-chosen integer k. Hence, our result allows us to reconstruct with the right homotopy type a shape by a union of voxels with vertices the cubical grid. Such a "cubical complex" has a simpler structure than the α-shape and may be more convenient for topological computations in high dimension. Following this idea, our work contributes to build a bridge between the point of view of distance functions in computational topology and the world of voxels in digital image processing.

Chosen approach and contributions. A first idea to tackle the problem mentioned above is to use the framework of semi-concave functions. Specifically, one can associate to any symmetric convex set C with a non-empty interior a norm $\|.\|_C$ defined by $x \mapsto \|x\|_C = \inf\{\alpha > 0 \mid x \in \alpha C\}$. The balls of the associated distance d_C are translated and scaled copies of C. The metric d_C is invariant by translation but is not isotropic unless C is the Euclidean ball. Suppose the boundary of C is smooth with a bounded curvature. Given a subset $Y \subset \mathbb{R}^N$, the squared distance-to-Y function $x \mapsto \min_{y \in Y} d_C(x, y)^2$ is semi-concave [6] and has therefore a generalized gradient which induces a continuous flow. Hence, a theory similar to what has been done in the Euclidean case [7, 11, 12] can be developed. However, we are interested in convex sets, such as polytopes, whose boundary are not necessarily smooth nor has a bounded curvature. It follows that the semi-concavity property is lost and no generalized gradient of the squared distance-to-Y function can induce a flow. The proof technique used in this paper presents interest in itself since it overcomes the limitation of

flow-based methods and applies to convex sets with non-smooth boundary. Taking inspiration in [23] where a deformation retract of an Euclidean offset of the sample onto the shape is constructed explicitly, we move away from this approach and introduce a new proof scheme based a sandwich lemma (Lemma 1). Results in this paper are positive as well as negative. We carefully identify a class of convex sets to which the above result can be extended. We also give examples of convex sets outside this class which won't provide a correct reconstruction. We proceed in two steps. First, we state a reconstruction theorem, when the sample P is an arbitrarily small Euclidean offset of the shape and secondly when P is a finite sample of the shape.

Outline. Section 2 presents definitions and the formal statements of our two reconstruction theorems. Section 3 proves the first reconstruction theorem and Section 4 proves the second. Section 5 concludes the paper.

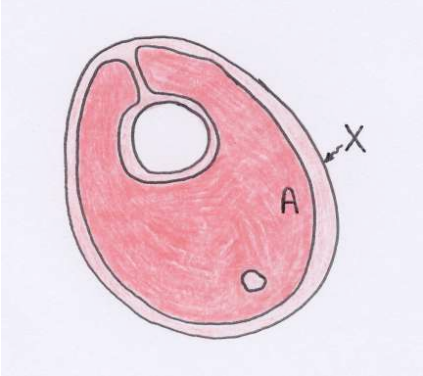
2 Statement of Results

Before we state our results in Section 2.3, we first introduce the necessary background in topology in Section 2.1 and identify in Section 2.2 classes of convex sets to which our results will apply.

2.1 Homotopy equivalences

First, we review some classical definitions in topology that can be found for instance in [19, 22]. Two continuous maps $h, k : X \to Y$ are *homotopic* and we write $h \simeq k$ if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = h(x)$ and $F(x,1) = k(x)$ for all $x \in X$. Let $f : X \to Y$ and $g : Y \to X$ be two continuous maps. Suppose that $f \circ g : Y \to Y$ is homotopic to the identity map of Y and $g \circ f : X \to X$ is homotopic to the identity map of X, *i.e.* suppose we have $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Then, the maps f and g are called *homotopy equivalences*, and each is said to be a *homotopy inverse* of the other. Furthermore, the spaces X and Y are said to have the same *homotopy type*, which we denote by $X \simeq Y$. We say that a subspace A of X is a *deformation retract* of X if there exists a continuous map $H : X \times [0,1] \to X$ such that $H(x,0) = x$, $H(x,1) \in A$ for all $x \in X$ and $H(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$. Such a function H is called a *deformation retraction* of X onto A. Let $r : X \to A$ be the retraction defined by $r(x) = H(x,1)$ and let $i : A \to X$ the inclusion map. We have $i \circ r \simeq 1_X$ and $r \circ i = 1_A$. Thus, if A is a deformation retract of X, the inclusion $i : A \to X$ is a homotopy equivalence. Note that the converse is not true: if the inclusion $A \hookrightarrow X$ is a homotopy equivalence, A is not necessarily a deformation retract of X. To see this, take for instance A to be an open disk and X a closed disk containing A. Hence, assuming X deformation retracts to A is stronger than assuming the inclusion map $A \hookrightarrow X$ is a homotopy equivalence, which in turn is stronger than assuming $A \simeq X$, as illustrated in Figure 2, left. We now state a technical lemma that will provide us key tools in establishing that two shapes have the same topology.

Lemma 1 (Sandwich Lemma)**.** *Consider four nested spaces $A_0 \subset X_0 \subset A_1 \subset X_1$. If A_1 deformation retracts to A_0 and X_1 deformation retracts to X_0, then X_0 deformation retracts to A_0. If the inclusions $A_0 \hookrightarrow A_1$ and $X_0 \hookrightarrow X_1$ are homotopy equivalences, then the inclusion $A_0 \hookrightarrow X_1$ is a homotopy equivalence.*



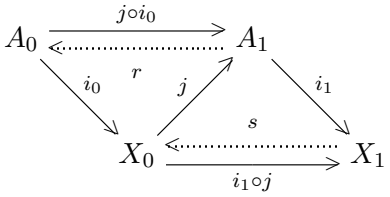


Figure 2: Left: two nested shapes $A \subset X$ which are close in Hausdorff distance and have the same homotopy type but for which the inclusion $A \hookrightarrow X$ is not a homotopy equivalence. Right: diagram for the proof of Lemma 1. All arrows but the dotted ones are inclusions.

Proof. To prove the first part of the lemma, suppose F is a deformation retraction of A_1 onto A_0 and G is a deformation retraction of X_1 onto X_0. Then, one can check that the map $H : X_0 \times [0,1] \to X_0$ defined by $H(x,t) = G(F(x,t),1)$ is a deformation retraction of X_0 onto A_0.

To prove the second part of the lemma, let $i_0 : A_0 \to X_0$, $j : X_0 \to A_1$ and $i_1 : A_1 \to X_1$ denote inclusions (see Figure 2, right). Suppose r is a homotopy inverse of $j \circ i_0$ and s is a homotopy inverse of $i_1 \circ j$. We prove that the inclusion $k = i_1 \circ j \circ i_0$ from A_0 to X_1 is a homotopy equivalence with homotopy inverse $r \circ j \circ s$. Indeed, using the fact that composition preserves the relation $\simeq$, we have $k \circ (r \circ j \circ s) = i_1 \circ (j \circ i_0 \circ r) \circ j \circ s \simeq i_1 \circ 1_{A_1} \circ j \circ s \simeq 1_{X_1}$ and similarly $(r \circ j \circ s) \circ k = r \circ j \circ (s \circ i_1 \circ j) \circ i_0 \simeq r \circ j \circ 1_{X_0} \circ i_0 \simeq 1_{A_0}$. □

2.2 Properties on convex bodies

A *convex body* designates a non-empty compact convex set. In this section, we define two properties that will help us identify classes of convex bodies.

2.2.1 Roundness

We associate to every convex body a non-negative real number called the θ-roundness which can be interpreted as a certain kind of curvature. Given a convex set C in $\mathbb{R}^N$, the *normal cone* $\mathcal{N}(x)$ to C at x is the set of unit vectors n such that $(x-y) \cdot n \geq 0$, for all points $y \in C$.

Definition 1. *Let $\theta \in [0,\pi]$ and $\varkappa \geq 0$. We say that the convex body C is* $(\theta,\varkappa)$-round *if for all points $c_1, c_2 \in C$ and all vectors $n_1 \in \mathcal{N}(c_1)$ and $n_2 \in \mathcal{N}(c_2)$, the following implication holds:*

$$\angle(n_1,n_2) \geq \theta \implies (c_1 - c_2) \cdot (n_1 - n_2) \geq \varkappa \|c_1 - c_2\|^2.$$

The θ-roundness *of C is the supremum of $\varkappa \geq 0$ such that C is $(\theta,\varkappa)$-round.*

Note that if the convex body C has roundness $\varkappa$, then C is $(\theta,\varkappa)$-round. Indeed, for all points $c_1, c_2 \in C$, for all vectors $n_1 \in \mathcal{N}(c_1)$, $n_2 \in \mathcal{N}(c_2)$ such that $\angle(n_1,n_2) \geq \theta$ and for all real number $\nu > 0$, we have

$$(c_1 - c_2) \cdot (n_1 - n_2) \geq (\varkappa - \nu) \|c_1 - c_2\|^2,$$

which also holds in the limit as ν goes to 0. If C is $(\theta, \varkappa)$-round, then C is $(\theta', \varkappa')$-round whenever $\theta \leq \theta'$ and $\varkappa' \leq \varkappa$. Since a similitude with ratio ϱ multiplies distances by ϱ and preserves angles, C is transformed under such a similitude into a $(\theta, \varkappa/\varrho)$-round convex body. It is not difficult to see that C is $(\pi, \varkappa)$-round if and only if the diameter of C is upper bounded by $\frac{2}{\varkappa}$. To get a feeling of what this means for a convex body to be $(\theta, \varkappa)$-round, let us consider two distinct points $c_1, c_2 \in C$ and two unit vectors $n_1 \in \mathcal{N}(c_1)$ and $n_2 \in \mathcal{N}(c_2)$ and study the ratio:

$$\frac{(c_1 - c_2) \cdot (n_1 - n_2)}{\|c_1 - c_2\|^2}. \tag{1}$$

First, observe that this ratio is non-negative by choice of n_i in the normal cone of c_i for $i \in \{1, 2\}$. If the boundary of C is a sphere with center z and curvature κ, then this ratio is equal to κ, since in that case $n_i = \kappa(c_i - z)$. We give a geometric interpretation of the ratio in Lemma 2 and a tight lower bound for convex bodies with a smooth boundary in Lemma 3.

Lemma 2. *For all points $c_1, c_2 \in C$, $c_1 \neq c_2$ and all vectors $n_1 \in \mathcal{N}(c_1)$ and $n_2 \in \mathcal{N}(c_2)$, we have*

$$\frac{(c_1 - c_2) \cdot (n_1 - n_2)}{\|c_1 - c_2\|^2} = \frac{1}{2}(\kappa_{c_1,n_1}(c_2) + \kappa_{c_2,n_2}(c_1)),$$

where $\kappa_{x,n}(y)$ is the curvature of the sphere passing through points x and y and with outer normal n at point x.

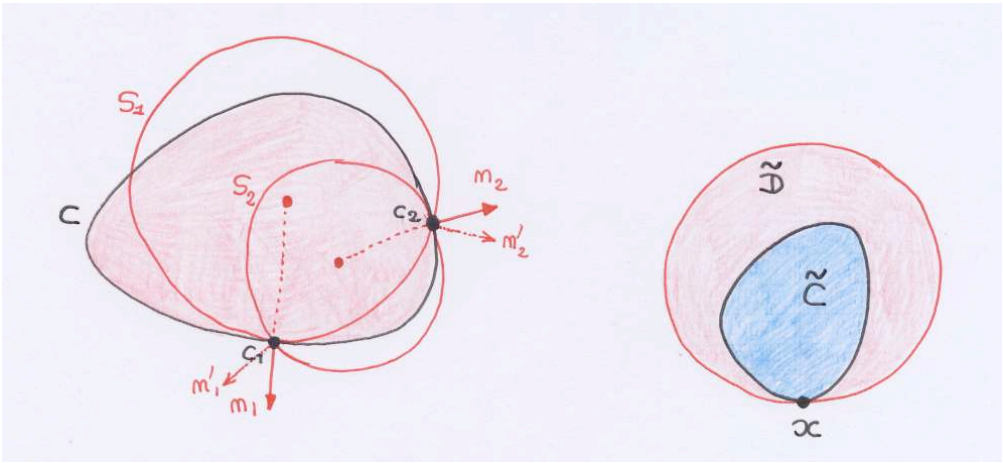


Figure 3: Notations for the proof of Lemma 2 and Lemma 3.

Proof. See Figure 3, left. Let S_i be the sphere passing through points c_1 and c_2 with outer normal n_i at c_i, for $i \in \{1, 2\}$. Let n_2' be the outer unit normal to S_1 at c_2 and n_1' be the outer unit normal to S_2 at c_1. We have

$$\begin{aligned}
(c_1 - c_2) \cdot (n_1 - n_2') &= \kappa_{c_1,n_1}(c_2)\|c_1 - c_2\|^2 \\
(c_1 - c_2) \cdot (n_1' - n_2) &= \kappa_{c_2,n_2}(c_1)\|c_1 - c_2\|^2 \\
(c_1 - c_2) \cdot (n_2' + n_1) &= 0 \\
(c_2 - c_1) \cdot (n_1' + n_2) &= 0
\end{aligned}$$

Summing up these four equations gives the result. □

Suppose the boundary of C is a C^2-smooth hypersurface in $\mathbb{R}^N$ and orient ∂C such that normals point outside the convex set. Then, for all points $x \in \partial C$, the normal cone at x is reduced to a single vector which is the normal to ∂C at x. The absolute values of the principal curvatures at point $x \in \partial C$ are non-negative real numbers $|\kappa_1(x)| \geq |\kappa_2(x)| \ldots \geq |\kappa_{N-1}(x)|$ and we let $\kappa_{\min}(C)$ be the minimum of $|\kappa_{N-1}(x)|$ over all points $x \in \partial C$. We refer to [15] for an introduction to differential geometry that define curvature. Note that if c_2 tends to c_1 along a curve γ in ∂C, then the ratio in (1) tends to the absolute value of the normal curvature of γ at c_1. In particular, if $|\kappa_{N-1}(x)|$ reaches its minimum at $x = c_1$ and the tangent line to γ at c_1 is the associated principal direction, then the ratio in (1) tends to $\kappa_{\min}(C)$. The next lemma states that the ratio in (1) is actually lower bounded by $\kappa_{\min}(C)$.

Lemma 3. *Suppose C is a convex body whose boundary is a C^2-smooth hypersurface in $\mathbb{R}^N$. Then,*

$$\inf_{c_1,c_2,n_1,n_2} \frac{(c_1 - c_2) \cdot (n_1 - n_2)}{\|c_1 - c_2\|^2} = \kappa_{\min}(C),$$

where the infimum is taken over all points $c_1, c_2 \in \partial C$, $c_1 \neq c_2$ and n_i designates the normal at c_i for $i \in \{1, 2\}$.

Proof. We only need to prove that for all points $c_1, c_2 \in \partial C$ with normals n_1 and n_2 respectively:

$$(c_1 - c_2) \cdot (n_1 - n_2) \geq \kappa_{\min}(C)\|c_1 - c_2\|^2.$$

Consider a sphere S tangent to ∂C at point x and meeting ∂C in another point $y \neq x$. Let D be the ball that S bounds. We begin by proving that $\kappa_{\min}(D) \geq \kappa_{\min}(C)$. Consider a 2-dimensional plane P passing through x and y and containing the common normal to ∂D and ∂C at point x. In particular, P passes through the center of D. We think of $\tilde{D} = D \cap P$ and $\tilde{C} = C \cap P$ as two convex bodies in $\mathbb{R}^2$. By construction, $\partial\tilde{D}$ and $\partial\tilde{C}$ are C^2-smooth curves tangent at point x and meeting at point $y \neq x$. Thus, we reduced the geometric situation in $\mathbb{R}^N$ to the same situation in $\mathbb{R}^2$. Let us prove that $\kappa_{\min}(\tilde{D}) \geq \kappa_{\min}(\tilde{C})$. Suppose for a contradiction that $\kappa_{\min}(\tilde{D}) < \kappa_{\min}(\tilde{C})$. In other words, the curvature of circle $\tilde{D}$ is smaller than the curvature at any point on the curve $\tilde{C}$. Theorem 1 in [20] tells us that $\tilde{C}$, except for x, lies in the interior of $\tilde{D}$, as illustrated in Figure 3 right. But, this contradicts the fact that $\tilde{C}$ intersects the boundary of $\tilde{D}$ in $y \neq x$. Thus, $\kappa_{\min}(\tilde{D}) \geq \kappa_{\min}(\tilde{C})$ and it follows that $\kappa_{\min}(D) = \kappa_{\min}(D \cap P) \geq \kappa_{\min}(C \cap P) \geq \kappa_{\min}(C)$, as claimed. In other words, given two points $x \neq y$ on the boundary of C and a unit vector $n \in \mathcal{N}(x)$, we have just proved that the curvature $\kappa_{x,n}(y)$ of the sphere passing through x and y with outer normal n at point x satisfies $\kappa_{x,n}(y) \geq \kappa_{\min}(C)$. Applying Lemma 2 gives the claimed inequality and therefore the result. □

As an immediate consequence of Lemma 3, a convex body C whose boundary is C^2-smooth has 0-roundness $\kappa_{\min}(C)$. For instance, the N-dimensional Euclidean ball B has 0-roundness 1. For our reconstruction theorems, we shall consider convex bodies which have the property to be $(\theta, \varkappa)$-round for a positive $\varkappa$ and a small enough θ. Specifically, we will require $\theta \leq \theta_N = \arccos(-\frac{1}{N})$. Not all convex bodies satisfy this property. To construct a counterexample, consider a convex body C which is contained in an affine space of dimension i with $0 < i < N$. Let n be a unit vector orthogonal to the smallest affine space containing C. For every point $c \in C$, both n and its opposite vector $-n$ belong to $\mathcal{N}(c)$. Consider two distinct points $c_1, c_2 \in C$ and let $n_1 = n$ and $n_2 = -n$. We have $\angle(n_1, n_2) = \pi$, $(c_1 - c_2) \cdot (n_1 - n_2) = 0$ and $\|c_1 - c_2\| \neq 0$, showing that there are no $\varkappa > 0$ such that C is $(\pi, \varkappa)$-round. Equivalently, the π-roundness of C is zero. As a counterexample with

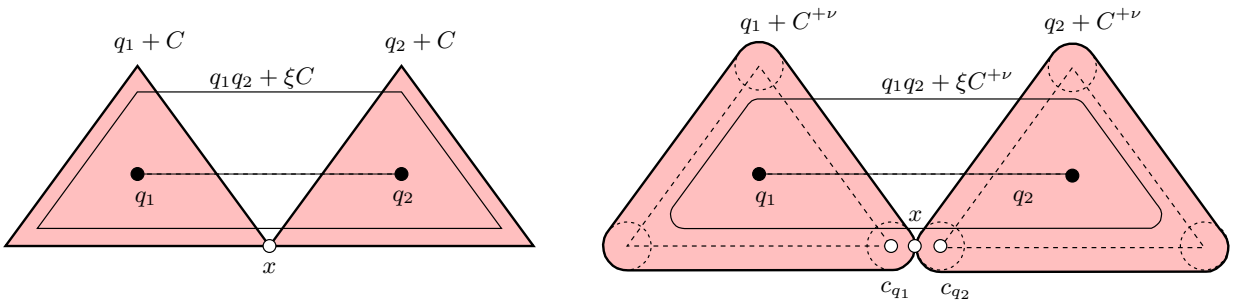


Figure 4: Left: two translated copies of a triangle C. The intersection point x does not belong to any set $\mathrm{Hull}(\{q_1, q_2\}) + \xi C$ with $\xi < 1$, thus showing that the eccentricity of C is 1. Right: bulging the triangle makes its eccentricity drops to a value smaller than one. Notations for the proof of Lemma 22.

a non-empty interior, take a triangular prism in $\mathbb{R}^3$. Its θ_3-roundness is zero because we can always find an edge of the prism whose dihedral angle is less than $\frac{\pi}{3}$ and therefore two distinct points c_1, c_2 on this edge, two vectors $n_1 \in \mathcal{N}(c_1)$, $n_2 \in \mathcal{N}(c_2)$ orthogonal to the edge and making an angle $\angle(n_1, n_2) = \frac{2\pi}{3} \geq \theta_3$ such that $(c_1 - c_2) \cdot (n_1 - n_2) = 0$ and $\|c_1 - c_2\| \neq 0$. In Appendix B.1.1, we compute the θ_2-roundness of regular polygons in the plane and in Appendix B.1.2, we give the θ_N-roundness of N-dimensional cubes in $\mathbb{R}^N$.

2.2.2 Eccentricity

In this section, we associate to every subset $C \subseteq \mathbb{R}^N$ a parameter called *eccentricity*. Intuitively, eccentricity can be thought of as a measure of how much intersections of translated copies of C centered at points in Q deviate from the convex hull of Q. We recall that the *Minkowski sum* of two subsets $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^N$ is the subset defined by $X + Y = \{x + y \mid x \in X, y \in Y\}$. To simplify notations, we shall write $x + Y$ instead of $\{x\} + Y$. Let $B = \{x \in \mathbb{R}^N \mid \|x\| \leq 1\}$ be the unit Euclidean ball centered at the origin. Given a non-negative real number r, we call the Minkowski sum $X + rB$ the Euclidean *r-offset* of X and denote it by X^{+r}.

Definition 2. *Given $\xi \geq 0$, we say that C is* ξ-eccentric *if for all compact $Q \subset \mathbb{R}^N$, the following implication holds:*

$$\bigcap_{q \in Q} (q + C) \neq \emptyset \implies \left(\bigcap_{q \in Q} (q + C) \right) \cap (\mathrm{Hull}(Q) + \xi C) \neq \emptyset.$$

The eccentricity *of C is the infimum of $\xi \geq 0$ such that C is ξ-eccentric.*

Note that the eccentricity is a real number between 0 and 1. To see this, consider a subset $Q \subset \mathbb{R}^N$ and suppose $z \in \bigcap_{q \in Q}(q + C)$ or equivalently $Q \subset z - C$. This implies that $\mathrm{Hull}(Q) \cap (z - C) \neq \emptyset$ or equivalently $z \in \mathrm{Hull}(Q) + C$. It follows that any subset C is 1-eccentric and the eccentricity is at most 1. The property to be ξ-eccentric is invariant under bijective linear transformations. To see this, consider a bijective linear map ϕ. Since ϕ is linear, we have $\phi(q + C) = \phi(q) + \phi(C)$ and $\phi(\mathrm{Hull}(Q) + \xi C) = \mathrm{Hull}(\phi(Q)) + \xi\phi(C)$. Since ϕ is bijective, two subsets X and Y intersect if and only if $\phi(X)$ and $\phi(Y)$ intersect. It follows immediately that C is ξ-eccentric if and only if

$\phi(C)$ is ξ-eccentric and $\phi(C)$ has the same eccentricity as C. A useful observation is that if the eccentricity of a compact set C is ξ, then C is ξ-eccentric. We now give eccentricities for simple objects and adopt the convention that all objects we consider in this paragraph have their centroids at the origin. The eccentricity of the unit N-dimensional Euclidean ball B is zero. Indeed, suppose $\bigcap_{q\in Q}(q+B) \neq \emptyset$ and let z be the center of the smallest Euclidean ball enclosing all points in Q. Clearly, z belongs to the non-empty common intersection $\bigcap_{q\in Q}(q+B)$ and by Lemma 14, z also belongs to Hull(Q), showing that B is 0-eccentric. More generally, i-dimensional Euclidean balls for $0 \leq i \leq N$ have eccentricity 0. Ellipsoids which can be obtained from Euclidean balls by applying a linear transformation also have eccentricity 0. We establish in Appendix B.2.1 that symmetric convex bodies in the plane have eccentricity 0 as well. At the opposite end of the spectrum, triangles have eccentricity 1 (see Figure 4). While a subset $C \subset \mathbb{R}^N$ may have an eccentricity equal to 1, we prove in Appendix B.2.2 that bulging it is enough to make its eccentricity drops to a value smaller than 1. In Appendix B.2.3, we establish that N-dimensional cubes in $\mathbb{R}^N$ have eccentricity $1-\frac{2}{N}$, for $N \geq 2$.

2.3 Reconstruction Theorems

First, we formulate a sampling condition inspired by the work in [2, 1, 7]:

Definition 3. *Given a non-negative real number ε and a subset $C \subset \mathbb{R}^N$, we say that $P \subset \mathbb{R}^N$ is an* (ε, C)-sample *of $A \subset \mathbb{R}^N$ if $A \subset P + \varepsilon C$ and $P \subset A + \varepsilon B$.*

Notice that P is an (ε, B)-sample of A if and only if the Hausdorff distance between P and A does not exceed ε. If $B \subset C$, then an (ε, B)-sample is also an (ε, C)-sample. The reason why our definition is not symmetric with respect to A and P is to enhance conditions that are used in the proofs of our reconstruction theorems. Given a compact subset A of $\mathbb{R}^N$, we recall that the *medial axis* M of A is the set of points in $\mathbb{R}^N$ which have at least two closest points in A. The *reach* of A is the infimum of distances between points in A and points in M:

$$\text{reach}(A) \;=\; \inf_{a\in A, x\in M} \|a-x\|.$$

Suppose the shape A has a positive reach. Given a convex body C and an (ε, C)-sample P of A, we would like to know whether the Minkowski sum $P + rC$ captures the topology of A. Theorem 1 answers the question when $P = A^{+\varepsilon}$ is an Euclidean ε-offset of A and Theorem 2 provides an answer when P is a finite sample of A. Before stating our results, we start with an example which illustrates that not all convex bodies C can be used to reconstruct the topology of shapes with a positive reach. Specifically, we take A to be the moment curve $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = x_1^2,\, x_3 = x_1^3\}$ and prove that for r arbitrarily small, we can always find a convex body C such that the Minkowski sum $A + rC$ is not homotopy equivalent to A. For this, let C be a segment of length 2 centered at the origin and contained in the straight-line $L_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0,\, x_3 = t^2 x_1\}$ for some positive real number t. Observe that translated copies of L_t intersect the moment curve A in at most one point, except when the translated copy meets A in the two points $a_t = (t, t^2, t^3)$ and $b_t = (-t, t^2, -t^3)$. Hence, as r increases, the Minkowski sum $A + rC$ undergoes a topology change when the two segments $a_t + rC$ and $b_t + rC$ meet (see Figure 5, left). This happens for $r = \|a_t - b_t\|/2$ which can be made arbitrarily small by choosing t small enough. Hence, besides requiring A to have a positive reach, we need conditions on the convex body C that we now list, the last condition being only for the second theorem:

(i) $B \subset C \subset \delta B$ for some $\delta \geq 1$;

(ii) C is $(\theta, \varkappa)$-round for $\theta \leq \theta_N = \arccos(-\frac{1}{N})$ and $\varkappa > 0$;

(iii) C is ξ-eccentric for $\xi < 1$.

The two conditions (ii) and (iii) do not imply each other. Indeed, an equilateral triangle in $\mathbb{R}^2$ with centroid the origin is $(\theta_2, \frac{1}{4\sqrt{3}})$-round (see Appendix B.1.1) and has eccentricity 1. Hence, it satisfies (ii) for $\theta = \theta_2$ and $\varkappa = \frac{1}{4\sqrt{3}}$ but not (iii) for any $\xi < 1$. Conversely, a segment in $\mathbb{R}^2$ has θ_2-roundness 0 and eccentricity 0. Hence, it satisfies (iii) for $\xi = 0$ but not (ii) for any $\theta \leq \theta_2$ and $\varkappa > 0$. Figure 6 gathers for different convex bodies C values of δ, θ, $\varkappa$ and ξ for which conditions (i), (ii) and (iii) hold. To state our theorems, let us introduce:

$$R_r \;=\; R - \frac{r}{4} - \sqrt{\frac{r}{4}\left(2R + \frac{r}{4}\right)},$$

and note that R_r tends to R as $\frac{r}{R} \to 0$.

Theorem 1. *Let A be a compact subset of $\mathbb{R}^N$ with positive reach R. Let C be a convex body of $\mathbb{R}^N$ satisfying conditions (i) and (ii). Then, $A^{+\varepsilon} + rC$ deformation retracts to A for all positive real numbers r and ε such that $\varepsilon + (\delta - 1)r < \min\{R - r, R_{r/\varkappa}\}$.*

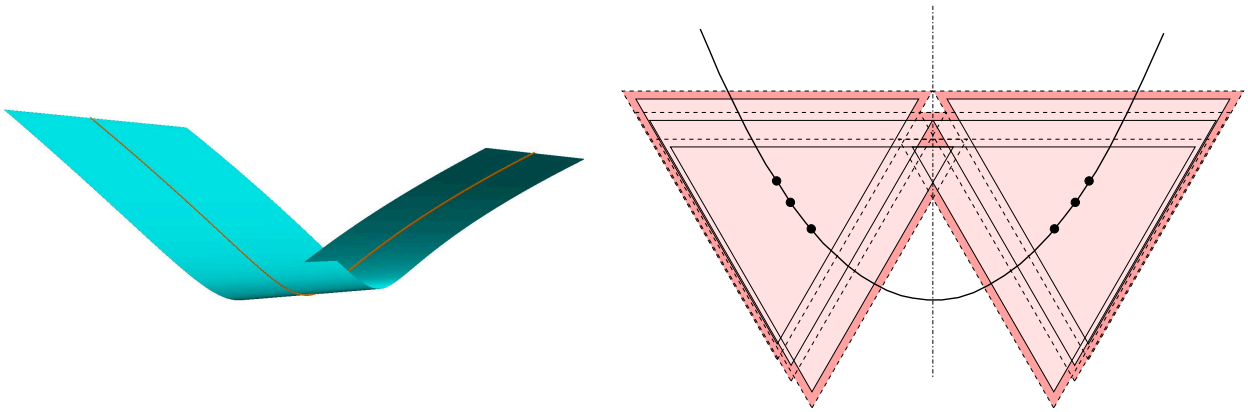


Figure 5: Left. Cycle in the Minkowski sum of the moment curve with a segment. Right. As the size of triangles increases, the hole created by the four upper points appears exactly when the hole created by the four lower points is filled up.

The proof of Theorem 1 is given in Section 3.

Theorem 2. *Let A be a compact subset of $\mathbb{R}^N$ with positive reach R. Let C be a convex body of $\mathbb{R}^N$ satisfying conditions (i), (ii) and (iii). Let P be a finite (ε, C)-sample of A. Then, the inclusion $A \hookrightarrow P + rC$ is a homotopy equivalence for all positive real numbers r and ε such that: (1) $(\delta - 1)r < \min\{R - r, R_{r/\varkappa}\}$, (2) $\delta r < R - \varepsilon$, (3) $\delta(r + \alpha_0) < R$ and*

$$(4) \quad 2R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2} - \sqrt{R^2 - \delta^2(r+\alpha_0)^2} < (1-\xi)r - \varepsilon,$$

where $\alpha_0 = \xi r + R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2}$.

The proof of Theorem 2 is given in Section 4. Notice that Theorem 2 requires that $\theta \leq \theta_N$, $\varkappa > 0$ and $\xi < 1$. If these three conditions are fulfilled, then by choosing $r = \frac{4\varepsilon}{1-\xi}$ and $\frac{\varepsilon}{R}$ small enough, all the assumptions of Theorem 2 are satisfied, implying that the inclusion $A \hookrightarrow P + rC$ is a homotopy equivalence. Given a fixed convex body C, Figure 6 gives numerical approximations of the largest value of $\frac{\varepsilon}{R}$ for which assumptions of Theorem 2 hold.

convex body C	δ	θ	$\varkappa$	ξ	R/ε	r/ε
Euclidean ball $B \subset \mathbb{R}^N$	1	0	1	0	12.9781	3.95723
cube B_∞ in $\mathbb{R}^N$	$\sqrt{N}$	$\arccos(-\frac{1}{N})$	$\varkappa(B_\infty)$	$1-\frac{2}{N}$		
cube B_∞ in $\mathbb{R}^2$	$\sqrt{2}$	$\frac{2\pi}{3}$	0.65974	0	24.9973	4.04227
cube B_∞ in $\mathbb{R}^3$	$\sqrt{3}$	$0.608\,\pi$	$\frac{1}{\sqrt{6}}$	1/3	96.4687	6.14485
cube B_∞ in $\mathbb{R}^4$	$\sqrt{4}$	$0.5804\,\pi$	1/4	1/2	247.528	8.1826
cube B_∞ in $\mathbb{R}^5$	$\sqrt{5}$	$0.5641\,\pi$	0.149071	3/5	508.183	10.2006
cube B_∞ in $\mathbb{R}^{10}$	$\sqrt{10}$	$0.5319\,\pi$	0.03953	4/5	4505.44	20.2264
cube B_∞ in $\mathbb{R}^{100}$	10	$0.503183\,\pi$	0.0010204	49/50	4948245	200.232
p-gon $\mathcal{P}_p$ in $\mathbb{R}^2$ (p even)	$\frac{1}{\cos\frac{\pi}{p}}$	$\frac{2\pi}{3}$	$\varkappa(\mathcal{P}_p)$	0		
square in $\mathbb{R}^2$	$\sqrt{2}$	$\frac{2\pi}{3}$	0.65974	0	24.9973	4.04227
hexagon in $\mathbb{R}^2$	1.1547	$\frac{2\pi}{3}$	0.69936	0	16.9858	3.99837
octagon in $\mathbb{R}^2$	1.08239	$\frac{2\pi}{3}$	0.793353	0	15.04119	3.98101
dodecagon in $\mathbb{R}^2$	1.03528	$\frac{2\pi}{3}$	0.8660254	0	13.84148	3.968
36-gon in $\mathbb{R}^2$	1.00382	$\frac{2\pi}{3}$	0.951917	0	13.07011	3.95844
360-gon in $\mathbb{R}^2$	1.00004	$\frac{2\pi}{3}$	0.9949868	0	12.97897	3.95724

Figure 6: Columns 2 to 5: values of δ, θ, $\varkappa$ and ξ for which $B \subset C \subset \delta B$ and the convex body C is $(\theta, \varkappa)$-round and ξ-eccentric. Formulas are given in Appendix B. Columns 6 and 7: values of R/ε and r/ε for which Theorem 2 holds. Numerical values are obtained by brute force, enumerating all pairs (ε, r) in a grid, checking if they satisfy conditions of Theorem 2 and keeping the one with largest ε.

To conclude this section, we prove that condition (iii) in Theorem 2 is necessary. In other words, if we take a convex body C which satisfies conditions (i) and (ii) but whose eccentricity is 1, it may happen that for some sample P, the Minkowski sum $P + rC$ does not recover the topology of A, no matter what value r takes in the interval $[\varepsilon, R - \varepsilon]$. To construct such an example, consider a parabola A in the plane with equation $y = x^2$ and a finite sample $P \subset A$ symmetric with respect to the y-coordinate axis. Furthermore, we let C be the equilateral triangle with centroid the origin and vertices $(0, -2)$, $(\sqrt{3}, 1)$ and $(-\sqrt{3}, 1)$. We note that with increasing value of r, holes appear in the Minkowski sum $P + rC$ each time two triangles scaled by r meet at a common vertex on the y-axis (see Figure 5, right). We then adjust the height of the sample points in such a way that $P + \varepsilon C$ does not have the correct topology and as r increases, a hole appears in $P + rC$ each time a hole gets destroyed, as illustrated in Figure 5, right.

3 Proof of Theorem 1

In this section, we prove Theorem 1. In other words, we prove that for r and ε small enough with respect to the reach of A, the Minkowski sum $A^{+\varepsilon} + rC$ deformation retracts to A. Our strategy is as follows. We consider two positive real numbers ε' and ε'' such that there is chain of inclusions:

$$A \quad \subset \quad A^{+\varepsilon} + rC \quad \subset \quad A^{+\varepsilon'} \quad \subset \quad A^{+\varepsilon''} + rC \tag{2}$$

and find conditions under which the third set deformation retracts to the first set and the fourth set deformation retracts to the second set. Applying the Sandwich Lemma allows us to conclude. All the difficulty comes from the second part of the proof which involves comparing the topology of two Euclidean offsets of $A + rC$, namely $A^{+\varepsilon} + rC$ and $A^{+\varepsilon''} + rC$. This leads us to study in details Euclidean offsets of Minkowski sums in Section 3.2. A powerful tool to detect changes in the topology of Euclidean offsets consists in studying the critical points of distance functions. Key results concerning distance functions are recalled in Section 3.1.

3.1 Background on distance functions

The distance function $d(\cdot, Y)$ to the compact subset Y of $\mathbb{R}^N$ associates to each point $x \in \mathbb{R}^N$ its Euclidean distance to Y:

$$d(x, Y) = \min_{y \in Y} \|x - y\|.$$

The distance function $d(\cdot, Y)$ is 1-Lipschitz, but is not differentiable in general. Nonetheless, it is possible to define a notion of critical points analogue to the classical one for differentiable functions. Specifically, Grove defines in [18, page 360] critical points for the distance function to a closed subset of a Riemannian manifold. Using Equation (1.1)' in [18, page 360], we recast this definition in our context as follows. Let $\Gamma_Y(x)$ be the set of points in Y closest to x:

$$\Gamma_Y(x) = \{y \in Y \mid d(x, Y) = \|x - y\|\}$$

Definition 4. *A point $x \in \mathbb{R}^N$ is a* critical point *of the distance function $d(\cdot, Y)$ if $x \in \mathrm{Hull}(\Gamma_Y(x))$. The* critical values *of $d(\cdot, Y)$ are the images by $d(\cdot, Y)$ of its critical points.*

Slightly recasting Proposition 1.8 in [18, page 362], we have:

Lemma 4 (Isotopy Lemma [18])**.** *Let $0 < \varepsilon \leq \varepsilon'$. If the distance function $d(\cdot, Y)$ has no critical value in the interval $[\varepsilon, \varepsilon']$, then $Y^{+\varepsilon}$ is a deformation retract of $Y^{+\varepsilon'}$.*

If furthermore Y has a positive reach R, then the projection map π_Y which associates to each point $x \in Y^{+\varepsilon}$ its closest point $\pi_Y(x)$ on Y is well defined and continuous [17, page 435]. Thus, the map $H : [0, 1] \times Y^{+\varepsilon} \to Y^{+\varepsilon}$ defined by $H(t, x) = (1 - t)x + t\pi_Y(x)$ is a deformation retraction of $Y^{+\varepsilon}$ onto Y. It follows that:

Lemma 5. *If Y has a positive reach R, then $Y^{+\varepsilon}$ deformation retracts to Y, for all $0 \leq \varepsilon < R$.*

3.2 Distance functions to Minkowski sums

In what follows, A designates a compact subset of $\mathbb{R}^N$ with positive reach R and C designates a convex body of $\mathbb{R}^N$. We begin with a technical lemma which will help us to situate critical points of the distance function to $A+C$, assuming C is round enough.

Lemma 6. *Consider a point $x \in \mathbb{R}^N$ such that $d(x, A+C) < R$. Let $y_1, y_2 \in \Gamma_{A+C}(x)$ be two points on $A+C$ with minimum distance to x. Suppose C is $(\theta, \varkappa)$-round for $\varkappa > 0$ and $\angle y_1 x y_2 \geq \theta$. Then, $d(x, A+C) \geq R_{1/\varkappa}$.*

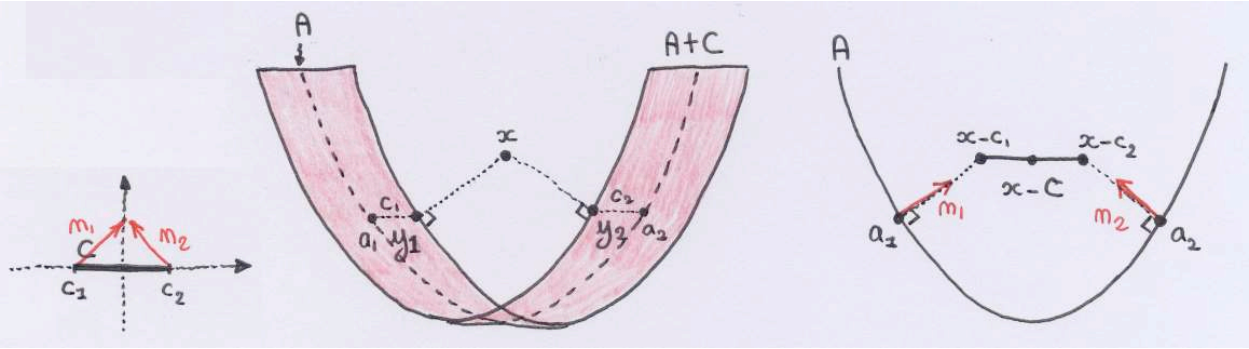


Figure 7: Notations for the proof of Lemma 6.

Proof. Let $\rho = d(x, A+C)$. For $i \in \{1,2\}$, let $y_i = a_i + c_i$ with $a_i \in A$ and $c_i \in C$. Since $\rho = \|(x-c_i) - a_i\| < R$, it follows that $x - c_i$ has a unique projection $a_i = \pi_A(x-c_i)$ onto A (see Figure 7). On the other hand, we know from [17, page 435] that the projection map π_A onto A is $\left(\frac{R}{R-\rho}\right)$-Lipschitz for points at distance less than ρ from A. Thus,

$$\|a_1 - a_2\| \;\leq\; \frac{R}{R-\rho}\,\|c_1 - c_2\|.$$

Let $n_i = \frac{x-a_i-c_i}{\|x-a_i-c_i\|}$. Squaring both sides of the above inequality and plugging $a_2 - a_1 = c_1 - c_2 + \rho(n_1 - n_2)$ into the left side, we obtain

$$2(c_1-c_2)\cdot(n_1-n_2) + \rho\,\|n_1-n_2\|^2 \;\leq\; \frac{2R-\rho}{(R-\rho)^2}\,\|c_1-c_2\|^2.$$

For $i \in \{1,2\}$, the unit vector n_i belongs to $\mathcal{N}(c_i)$. Since C is $(\theta, \varkappa)$-round and $\angle(n_1, n_2) \geq \theta$, it follows that $(c_1-c_2)\cdot(n_1-n_2) \geq \varkappa\|c_1-c_2\|^2$ and

$$\rho\|n_1-n_2\|^2 \leq \left(\frac{2R-\rho}{(R-\rho)^2} - 2\varkappa\right)\,\|c_1-c_2\|^2.$$

In particular, this implies that $2\varkappa \leq \frac{2R-\rho}{(R-\rho)^2}$ or equivalently

$$\rho^2 - 2\left(R - \frac{1}{4\varkappa}\right)\rho + R^2 - \frac{R}{\varkappa} \leq 0.$$

Solving this quadratic inequality yields to the result. □

As a consequence of the lemma above, if x is sufficiently close to $A+C$, then the angle between any two vectors connecting x to points in $\Gamma_{A+C}(x)$ is small which implies, in turn, that x is not a critical point of $d(\cdot, A+C)$. The following lemma makes this idea precise.

Lemma 7. *If C is $(\theta, \varkappa)$-round with $\theta \leq \arccos(-\frac{1}{N})$ and $\varkappa > 0$, then the distance function $d(\cdot, A+C)$ has no critical value in the interval $(0, R_{1/\varkappa})$.*

In order to prove Lemma 7, we need the following result also known as *Jung's Theorem*. Given a compact subset $K \subset \mathbb{R}^N$, we denote by $\operatorname{diam}(K) = \max_{p,q\in K} d(p,q)$ the *diameter* of K.

Lemma 8 (Jung's Theorem)**.** *The smallest ball enclosing a compact subset K of $\mathbb{R}^N$ has radius*

$$r \leq \operatorname{diam}(K)\sqrt{\frac{N}{2(N+1)}}.$$

Equality is attained for the regular N-simplex.

For a short proof of Jung's theorem, see [13].

Proof of Lemma 7. Let $x \in \mathbb{R}^N$ and $\rho = d(x, A+C)$. Suppose $0 < \rho < R_{1/\varkappa}$ and let us prove that x is non-critical. By Lemma 6, for all points $y_1, y_2 \in \Gamma_{A+C}(x)$, we have $\angle y_1 x y_2 < \theta$. It follows that $\operatorname{diam}(\Gamma_{A+C}(x)) < 2\rho \sin\frac{\theta}{2}$. Using

$$\sin\frac{\theta}{2} \;=\; \sqrt{\frac{1-\cos\theta}{2}} \;\leq\; \sqrt{\frac{N+1}{2N}},$$

and applying Jung's Theorem, we get that the smallest ball B enclosing $\Gamma_{A+C}(x)$ has radius $r < \rho$. Let S denote the sphere centered at x with radius ρ. Observe that $\Gamma_{A+C}(x) \subset S \cap B$. Since the radius of B is smaller than ρ, the radical hyperplane Π of the two spheres S and ∂B separates x from $\Gamma_{A+C}(x)$. Thus $x \notin \operatorname{Hull}(\Gamma_{A+C}(x))$ and x is non-critical. □

Remark. Using the terminology introduced in [7], Lemma 7 can be reformulated by saying that the critical function of $d(\cdot, A+C)$ does not vanish in the interval $(0, R_{1/\varkappa})$. Slightly adapting the proof, we can strengthen this result and establish that the μ-reach of $A+C$ is greater than $R_{1/\varkappa}$ for $\mu = \cos\frac{\theta}{2}$, under the same hypothesis. Shapes with positive μ-reach have nice properties [8, 9, 10].

Combining Lemma 4 and Lemma 7 and using the fact that if C is $(\theta, \varkappa)$-round, then rC is $(\theta, \frac{\varkappa}{r})$-round, we get immediately conditions under which an Euclidean offset of $A+rC$ deformation retracts to another Euclidean offset:

Lemma 9. *If C is $(\theta, \varkappa)$-round with $\theta \leq \arccos(-\frac{1}{N})$ and $\varkappa > 0$, then $A^{+\varepsilon} + rC$ is a deformation retract of $A^{+\varepsilon''} + rC$ for all positive real numbers r, ε and ε'' such that $\varepsilon \leq \varepsilon'' < R_{r/\varkappa}$.*

We are now ready to establish the proof of our first reconstruction theorem.

Proof of Theorem 1. Equation (2) holds whenever $\varepsilon' = \varepsilon + \delta r$ and $\varepsilon'' = \varepsilon + (\delta - 1)r$. Since by hypothesis $\varepsilon \leq \varepsilon'' < R_{r/\varkappa}$, Lemma 9 implies that $A^{+\varepsilon''} + rC$ deformation retracts to $A^{+\varepsilon} + rC$. By hypothesis, we have $\varepsilon' < R$ and therefore, $A^{+\varepsilon'}$ deformation retracts to A from Lemma 5. Applying the Sandwich Lemma allows us to conclude. □

4 Proof of Theorem 2

In this section, we present our proof of Theorem 2. A designates a compact subset of $\mathbb{R}^N$ whose reach R is positive, C is a convex body of $\mathbb{R}^N$ satisfying conditions (i), (ii) and (iii) and P is a finite (ε, C)-sample of A. First, we introduce a set which will play a key role. Given three positive real numbers α, β and r, we set $\mathcal{A}_p(\alpha) = (p + rC) \cap (A^{+\beta} + \alpha C)$ and define

$$\mathcal{H}(\alpha) \;=\; \bigcup_{p \in P} \operatorname{Hull} \mathcal{A}_p(\alpha).$$

Our proof uses two carefully chosen positive constants α_0 and α_1 such that for all sufficiently small β, we have the sequence of inclusions (see Figure 8):

$$A \;\subset\; \mathcal{H}(\alpha_0) \;\subset\; A^{+\beta} + \alpha_1 C \;\subset\; P + rC \tag{3}$$

Having established this sequence of inclusions in Section 4.1, we find in Section 4.2 conditions under which $\mathcal{H}(\alpha_0) \hookrightarrow P + rC$ is a homotopy equivalence. Combined with the conditions we found in Section 3 which ensure that $A \hookrightarrow A^{+\beta} + \alpha_1 C$ is a homotopy equivalence, we deduce immediately using Lemma 1 (Sandwich Lemma) conditions under which $A \hookrightarrow P + rC$ is a homotopy equivalence.

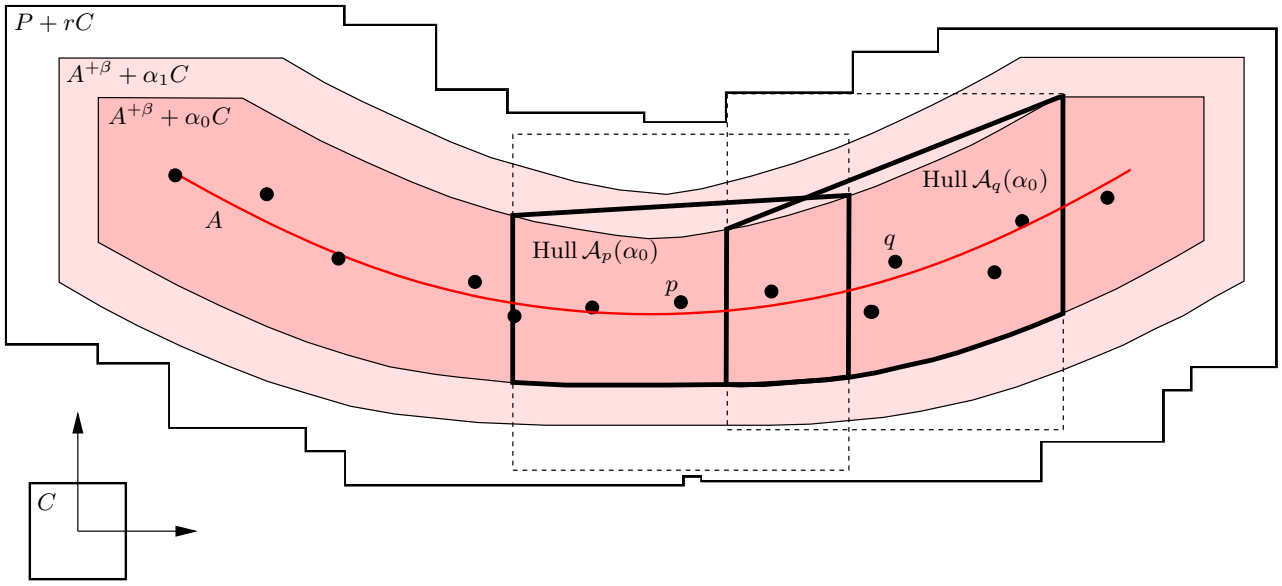


Figure 8: Nested sequence of objects considered for the proof of Theorem 2. Constants α_0 and α_1 are chosen such that Hull $\mathcal{A}_p(\alpha_0)$ is contained in $A + \alpha_1 C$ for all $p \in P$.

4.1 Establishing a key sequence of inclusions

In this section, we find conditions under which inclusions in (3) hold. To establish the middle inclusion, we need the following key inclusion, illustrated in Figure 8:

Lemma 10. *Suppose* $\delta(r + \alpha_0) < R - \beta$ *and* $\alpha_1 - \alpha_0 \geq R - \sqrt{(R-\beta)^2 - \delta^2 (r + \alpha_0)^2}$. *Then,*

$$\operatorname{Hull} \mathcal{A}_p(\alpha_0) \;\subset\; A + \alpha_1 C.$$

Proof. Let $A' = A^{+\beta} \cap (p + rC + \alpha_0(-C))$. Note that $\mathcal{A}_p(\alpha_0) \subset A' + \alpha_0 C$ for if x belongs to $\mathcal{A}_p(\alpha_0) = (p + rC) \cap (A^{+\beta} + \alpha_0 C)$, we can find $c_0, c_1 \in C$ and $a' \in A^{+\beta}$ such that $x = a' + \alpha_0 c_0 = p + rc_1$, showing that $a' \in A'$ and $x \in A' + \alpha_0 C$. Thus and using lemma 17,

$$\operatorname{Hull} \mathcal{A}_p(\alpha_0) \;\subset\; \operatorname{Hull}(A' + \alpha_0 C) \;=\; \operatorname{Hull}(A') + \alpha_0 C.$$

By construction, A' is contained in a ball of radius $\delta(r + \alpha_0)$. Applying Lemma 16 with $Q = A'$, $\varepsilon = \beta$ and $\rho = \delta(2r - \varepsilon)$, we get

$$\operatorname{Hull}(A') \;\subset\; A + (R - \sqrt{(R-\beta)^2 - \rho^2})B$$

if $\rho < R - \beta$. Thus, for all $p \in P$ we have $\operatorname{Hull} \mathcal{A}_p(\alpha_0) \subset A + \alpha_1 C$ whenever $\delta(r + \alpha_0) < R - \beta$ and $\alpha_1 - \alpha_0 \geq R - \sqrt{(R-\beta)^2 - \delta^2(r+\alpha_0)^2}$. □

Taking the union over all points $p \in P$ on both sides of the inclusion in Lemma 10 we get immediately the middle inclusion in (3), *i.e.* $\mathcal{H}(\alpha_0) \subset A^{+\beta} + \alpha_1 C$. The left-most and right-most inclusions in (3) are easy to establish. Indeed, since P is an (ε, C)-sample of A, we have $A \subset P + \varepsilon C$ and for $\varepsilon \leq r$ and $\alpha_0 > 0$, the left-most inclusion follows from

$$A \;\subset\; (P + rC) \cap (A^{+\beta} + \alpha_0 C) \;=\; \bigcup_{p \in P} \mathcal{A}_p(\alpha_0) \;\subset\; \bigcup_{p \in P} \operatorname{Hull} \mathcal{A}_p(\alpha_0) \;=\; \mathcal{H}(\alpha_0).$$

Using again $A \subset P + \varepsilon C$ and $B \subset C$, the right-most inclusion comes from

$$A^{+\beta} + \alpha_1 C \;\subset\; P + (\varepsilon + \beta + \alpha_1)C \;\subset\; P + rC$$

which holds whenever $\alpha_1 \leq r - \varepsilon - \beta$. Next lemma summarizes our findings.

Lemma 11. *The sequence of inclusions in (3) holds whenever* $\alpha_1 \leq r - \varepsilon - \beta$, $\delta(r + \alpha_0) < R - \beta$ *and* $\alpha_1 - \alpha_0 \geq R - \sqrt{(R-\beta)^2 - \delta^2(r+\alpha_0)^2}$.

4.2 A homotopy equivalence for nested collections of convex bodies

It is not difficult to see that the inclusion $\mathcal{H}(\alpha) \subset P + rC$ holds for all positive real numbers α and β. The goal of this section is to find conditions under which the inclusion map $\mathcal{H}(\alpha) \hookrightarrow P + rC$ is a homotopy equivalence. For this, we use covers of $\mathcal{H}(\alpha)$ and $P + rC$ by finite collections of convex bodies. Specifically, we have $\mathcal{H}(\alpha) = \bigcup_{p \in P} \operatorname{Hull} \mathcal{A}_p(\alpha)$ and $P + rC = \bigcup_{p \in P}(p + rC)$. Since sets in the two collections $\{\operatorname{Hull} \mathcal{A}_p(\alpha)\}_{p \in P}$ and $\{p + rC\}_{p \in P}$ are convex, we can apply Leray's theorem [21] to each, and obtain that the union of sets in each collection has the same homotopy type as its associated nerve:

$$\begin{aligned} \mathcal{H}(\alpha) &\simeq \operatorname{Nerve}\{\operatorname{Hull} \mathcal{A}_p(\alpha)\}_{p \in P} \\ P + rC &\simeq \operatorname{Nerve}\{p + rC\}_{p \in P} \end{aligned}$$

A key step consists in proving that, for suitable values of α, the nerves of the two collections are actually the same. As a consequence, $\mathcal{H}(\alpha)$ and $P + rC$ have the same homotopy type. We strengthen this result, thanks to Lemma 12, and state conditions under which the inclusion $\mathcal{H}(\alpha) \hookrightarrow P + rC$ is a homotopy equivalence in Lemma 13.

Lemma 12. *Consider two finite collections of convex bodies of $\mathbb{R}^N$, $\mathcal{C} = \{C_i\}_{i\in I}$ and $\mathcal{D} = \{D_i\}_{i\in I}$ such that $C_i \subset D_i$ for all $i \in I$ and suppose the two collections have the same nerve. Then, the inclusion $\bigcup_i C_i \hookrightarrow \bigcup_i D_i$ is a homotopy equivalence.*

From Corollary 4G.3 in [19] also known as Leray's theorem [21] or the Nerve Lemma, it is clear that $\bigcup_i C_i$ and $\bigcup_i D_i$ which share the same nerve have the same homotopy type. But, we need here a stronger result, namely that the inclusion $\bigcup_i C_i \hookrightarrow \bigcup_i D_i$ is a homotopy equivalence. Even though this fact can be deduced from a result in [4], we provide below a short proof to make the paper self-contained.

Proof. Let $K(\mathcal{C})$ be the abstract simplicial complex whose simplices are the (non-empty) subsets of indices $\sigma \subset I$ such that $\bigcap_{i\in\sigma} C_i \neq \emptyset$. Since the two collections $\mathcal{C}$ and $\mathcal{D}$ have the same nerves, $K(\mathcal{C}) = K(\mathcal{D})$ and we let $K = K(\mathcal{C})$.

For every subset of indices $\sigma \notin K$, a standard compactness argument yields a real number $\rho_\sigma > 0$ such that $\bigcap_{i\in\sigma} D_i^{+\rho_\sigma} = \emptyset$. Let $\rho = \min_{\sigma\notin K} \rho_\sigma$ and define the open set $O_i = \{x \in \mathbb{R}^N, d(x, D_i) < \rho\}$ for every $i \in I$. By construction, the nerve of the collection $\mathcal{O} = \{O_i\}_{i\in I}$ is the same as the nerve of $\mathcal{D}$ and $K(\mathcal{O}) = K$. For each $\sigma \in K$, we introduce the possibly empty open set:

$$U_\sigma = \bigcap_{i\in\sigma} O_i \setminus \bigcup_{i\notin\sigma} D_i.$$

It is obvious from the definition that $\bigcup_{\sigma\in K} U_\sigma \subset \bigcup_{i\in I} O_i$. Let us associate to each point $x \in \bigcup_{i\in I} O_i$ the subset of indices $\tau(x) = \{i \in I, x \in O_i\}$. Since $x \in U_{\tau(x)}$, it follows that:

$$\bigcup_{\sigma\in K} U_\sigma = \bigcup_{i\in I} O_i.$$

Let us consider a partition of unity $\{\phi_\sigma\}_{\sigma\in K}$ subordinate to the open cover $\{U_\sigma\}_{\sigma\in K}$ [24, page 22]. Note that the map ϕ_σ is identically zero for the simplices σ for which $U_\sigma = \emptyset$. For each simplex $\sigma \in K$, we choose an arbitrary point $c_\sigma \in \bigcap_{i\in\sigma} C_i$ and introduce the map $h : \bigcup_{i\in I} D_i \to \mathbb{R}^N$ defined by:

$$h(x) = \sum_{\sigma\in K} \phi_\sigma(x) c_\sigma.$$

By construction, h is continuous. We claim that $x \in D_i \implies h(x) \in C_i$. Indeed, if $x \in D_i$ and $\phi_\sigma(x) \neq 0$, one has $i \in \sigma$ and therefore $c_\sigma \in C_i$. Hence, the non-zero terms in the above sum is a convex combination of points in C_i and the claim follows from the convexity of C_i. Let us prove that h is a homotopy inverse of the inclusion map $g : \bigcup_i C_i \to \bigcup_i D_i$. In other words, we have to check that $g \circ h$ is homotopic to the identity of $\bigcup_i D_i$ and $h \circ g$ is homotopic to the identity of $\bigcup_i C_i$. This can be done using twice the homotopy $H(x, t) = (1 - t) \cdot x + t \cdot h(x)$, first considered as a map from $\bigcup_i D_i \times [0, 1]$ into $\bigcup_i D_i$, second considered as a map from $\bigcup_i C_i \times [0, 1]$ into $\bigcup_i C_i$. □

Lemma 13. *Consider positive real numbers r, ε, α and β such that $\delta r < R-\varepsilon$, $\delta(r+\alpha) < R-\beta$ and $\alpha \geq \xi r + R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2}$. Then, the inclusion $\mathcal{H}(\alpha) \hookrightarrow P + rC$ is a homotopy equivalence.*

Proof. We prove the lemma is three stages:
(a) First, we prove that for $\delta r < R - \varepsilon$ and $\alpha \geq \xi r + R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2}$, we have

$$\text{Nerve}\{p + rC\}_{p\in P} = \text{Nerve}\{\mathcal{A}_p(\alpha)\}_{p\in P}. \tag{4}$$

Note that this is equivalent to proving that for all subsets $Q \subset P$,

$$\bigcap_{q\in Q}(q+rC) \neq \emptyset \Longleftrightarrow \bigcap_{q\in Q}[(q+rC)\cap(A^{+\beta}+\alpha C)] \neq \emptyset.$$

One direction is trivial: if a point belongs to the intersection on the right, then it belongs to the intersection on the left. Suppose now that $\bigcap_{q\in Q}(q+rC) \neq \emptyset$. In particular, using $C \subset \delta B$ this means that Q can be enclosed in a ball of radius $\rho = \delta r$. Since C is ξ-eccentric, there exists $z \in \bigcap_{q\in Q}(q+rC)$ such that $z \in \mathrm{Hull}(Q) + \xi r C$. Since P is an (ε, C)-sample of A, we have $Q \subset P \subset A^{+\varepsilon}$. Applying Lemma 16, we get that $\mathrm{Hull}(Q) \subset A^{+(\alpha-\xi r)}$. Hence and using $B \subset C$, we get $z \in A + (\alpha - \xi r)B + \xi r C \subset A^{+\beta} + \alpha C$.
(b) Second, we prove that

$$\mathrm{Nerve}\{p+rC\}_{p\in P} \;=\; \mathrm{Nerve}\{\mathrm{Hull}\,\mathcal{A}_p(\alpha)\}_{p\in P}. \tag{5}$$

From Lemma 10, we obtain the sequence of inclusions

$$\mathcal{A}_p(\alpha) \;\subset\; \mathrm{Hull}\,\mathcal{A}_p(\alpha) \;\subset\; \mathcal{A}_p(\alpha'),$$

for $\delta(r+\alpha) < R - \beta$ and $\alpha' = \alpha + R - \sqrt{(R-\beta)^2 - \delta^2(r+\alpha)^2}$. Taking the intersection over all points $q \in Q$, we get

$$\bigcap_{q\in Q}\mathcal{A}_q(\alpha) \;\subset\; \bigcap_{q\in Q}\mathrm{Hull}\,\mathcal{A}_q(\alpha) \;\subset\; \bigcap_{q\in Q}\mathcal{A}_q(\alpha'),$$

and consequently

$$\mathrm{Nerve}\{\mathcal{A}_p(\alpha)\}_{p\in P} \;\subset\; \mathrm{Nerve}\{\mathrm{Hull}\,\mathcal{A}_p(\alpha)\}_{p\in P} \;\subset\; \mathrm{Nerve}\{\mathcal{A}_p(\alpha')\}_{p\in P}.$$

By Equation (4), the two nerves on the left and on the right are equal to $\mathrm{Nerve}\{p+rC\}_{p\in P}$, showing that $\mathrm{Nerve}\{\mathrm{Hull}\,\mathcal{A}_p(\alpha)\}_{p\in P} = \mathrm{Nerve}\{p+rC\}_{p\in P}$.
(c) Third, noticing that $\mathrm{Hull}\,\mathcal{A}_p(\alpha) \subset p + rC$ for all p, we apply Lemma 12 to the two collections of convex bodies $\mathcal{C} = \{\mathrm{Hull}\,\mathcal{A}_p(\alpha)\}_{p\in P}$ and $\mathcal{D} = \{p+rC\}_{p\in P}$. □

We conclude this section by the proof of our second reconstruction theorem.

Proof of Theorem 2. For β small enough, we have $\beta + (\delta-1)r < \min\{R-r, R_{r/\varkappa}\}$, $\delta(r+\alpha_0) < R - \beta$ and

$$2R - \sqrt{(R-\varepsilon)^2 - (\delta r)^2} - \sqrt{(R-\beta)^2 - \delta^2(r+\alpha_0)^2} \leq (1-\xi)r - \varepsilon - \beta.$$

Setting $\alpha_1 = \alpha_0 + R - \sqrt{(R-\beta)^2 - \delta^2(r+\alpha_0)^2}$, the above inequality can be rewritten as $\alpha_1 \leq r - \varepsilon - \beta$. Thus, the sequence of inclusions in (3) holds by Lemma 11. Furthermore, the inclusion $\mathcal{H}(\alpha) \hookrightarrow P + rC$ is a homotopy equivalence by Lemma 13. Since $\alpha_1 \leq r$ and $\beta + (\delta-1)r < \min\{R-r, R_{r/\varkappa}\}$ imply $\beta + (\delta-1)\alpha_1 < \min\{R-\alpha_1, R_{\alpha_1/\varkappa}\}$, the inclusion $A \hookrightarrow A^{+\beta} + \alpha_1 C$ is a homotopy equivalence by Theorem 1. Applying the Sandwich Lemma allows us to conclude. □

5 Discussion

In this paper, we have exhibited a class of convex bodies whose Minkowski sum with a sufficiently dense sample captures the topology of the sampled shape. Convex bodies in this class possess three properties: a non-empty interior, a positive θ_N-roundness and an eccentricity smaller than 1. In particular, this class contains Euclidean balls but, more interestingly, also includes N-dimensional cubes, with potential algorithmic applications in high dimensions.

The results in this paper raise a number of questions. First, it would be interesting to know what is the lowest density of sample points Theorem 2 authorized and if this number is tight, especially for N-dimensional balls. In the case of Euclidean balls, we found numerically that $\frac{\varepsilon}{R} = 0.077$ fulfills the requirements of our theorem (see Figure 6) and is the best ratio we can get with our sampling conditions. This is approximatively half less than the value $3-\sqrt{8}$ obtained in [23]. To understand the discrepancy between the two results, we have to go back to the proof of Theorem 2. In the case of Euclidean balls, the proof can be simplified as described in Appendix C, yielding the following sampling condition:

$$R - \sqrt{(R-\varepsilon)^2 - r^2} \quad < \quad r - \varepsilon$$

Setting $r = \lambda\varepsilon$, the above condition is equivalent to $\frac{\varepsilon}{R} < \frac{\lambda-2}{\lambda(\lambda-1)}$ and plugging $\lambda = 2+\sqrt{2}$ the inequality gives $\frac{\varepsilon}{R} < 3-\sqrt{8}$ exactly as in [23]. Similarly, we ask whether our upper bound on ε is tight for N-dimensional cubes? Another point is that Theorem 2 requires the sample P to be finite. Can we relax this condition? Since Lemma 12 is not true anymore if we remove the finiteness condition on the families of convex set, our proof has no obvious extension to the case where P is infinite. However, we conjecture Theorem 2 should still be true. Finally, if the convex body C is close to the Euclidean ball, in other words if C satisfies $B \subset C \subset \delta B$ for a small δ, can we derive bounds on the eccentricity and θ_N-roundness?

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A Basic properties

In this appendix, we present basic properties relating the smallest ball enclosing $Q \subset \mathbb{R}^N$ and the convex hull of Q.

Lemma 14. *The center of the smallest ball enclosing the compact subset $Q \subset \mathbb{R}^N$ lies on the convex hull of Q.*

Proof. Let $B(z,r)$ be the smallest ball enclosing Q. Suppose for a contradiction that $z \notin \text{Hull}(Q)$ and let z' be the point on $\text{Hull}(Q)$ closest to z. For every point $q \in Q$, we have $\|z'-q\| < \|z-q\| = r$, showing that there exists a ball enclosing Q centered at z' and smaller than $B(z,r)$, which contradicts the definition of $B(z,r)$. □

Lemma 15. *Consider a subset $Q \subset \mathbb{R}^N$ whose smallest enclosing ball has radius r. Then,* $\text{Hull}(Q) \subset \bigcup_{q\in Q} B(q,r)$.

Proof. For all $q \in \text{Hull}(Q)$, there are points $q_1, \ldots, q_n$ in Q and non-negative real numbers $\alpha_1, \ldots, \alpha_n$ summing up to 1, such that $q = \sum_{i=1}^n \alpha_i q_i$. Let $\pi_i(x) = \|x - q_i\|^2 - r^2$ be the power distance of $x \in \mathbb{R}^N$ from $B_i = B(q_i, r)$ and note that $B_i = \pi_i^{-1}(-\infty, 0]$. Let $\pi(x) = \sum_{i=1}^n \alpha_i \pi_i(x)$ and set $B = \pi^{-1}(-\infty, 0]$. We prove that $\bigcap_{i=1}^n B_i \subset B \subset \bigcup_{i=1}^n B_i$. Indeed, if a point x belongs to all balls B_i, then $\pi_i(x) \leq 0$ for all $1 \leq i \leq n$, which implies $\pi(x) \leq 0$. On the other hand, if $\pi(x) \leq 0$ then $\pi_i(x) \leq 0$ for at least one index i, which implies that x belongs to at least one ball B_i. Now, our choice of r as the radius of the smallest ball enclosing Q implies that $\bigcap_{i=1}^n B_i \neq \emptyset$, showing that B is non-empty. Thus, B is a ball and it is not difficult to see that its center is point q. It follows that $q \in B \subset \bigcup_{i=1}^n B_i$, which concludes the proof. □

The next lemma states that the convex hull of a set of points cannot be too far away from a shape with positive reach, assuming the set of points is close to the shape and are enclosed in a ball of small radius. Formally:

Lemma 16. *Consider a subset $Q \subset A^{+\varepsilon}$ in $\mathbb{R}^N$ and suppose Q can be enclosed in a ball of radius $\rho < R - \varepsilon$. Then,* $\text{Hull}(Q) \subset A^{+\alpha}$ *for* $\alpha \geq R - \sqrt{(R-\varepsilon)^2 - \rho^2}$.

Proof. Suppose $R < +\infty$ for otherwise, A is convex and $\text{Hull}(Q) \subset A^{+\varepsilon}$. Let x be a point on $\text{Hull}(Q)$ furthest away from A. By Lemma 15, there exists a point $q \in Q$ such that $\|x - q\| \leq \rho$. Thus, $d(x,A) \leq \|x-q\| + d(q,A) \leq \rho + \varepsilon < R$, showing that x has a unique projection a onto A. We claim that the plane H passing through x and orthogonal to the segment xa is a supporting plane of the convex hull of Q. To prove this, consider the two open half-spaces that H bounds and let H^- be the one half-space that does not contain a. Furthermore, consider the half-line with origin a that passes through x and let z be the point on this half-line at distance R from a (see Figure 9). By construction, $B(z,R)$ is tangent to A at a and its interior does not intersect A. We prove that $\text{Hull}(Q) \cap H^- = \emptyset$. Suppose for a contradiction that there exists a point $y \in \text{Hull}(Q) \cap H^-$. Then, the whole segment xy belongs to $\text{Hull}(Q)$ and in particular intersects $B(z, \|z-x\|)$. But points in the interior of $B(z, \|z-x\|)$ are furthest away from A than x, contradicting the definition of x as the point of $\text{Hull}(Q)$ furthest away from A. It follows that H is a supporting plane of the convex hull of Q as claimed. Thus, $Q \cap H$ is non-empty and can be enclosed in a ball of radius smaller or equal to ρ. The convex hull of $Q \cap H$ contains x and by Lemma 15, there exists a point $q' \in Q \cap H$ such that $\|x - q'\| \leq \rho$. On the other hand, $\|z - q'\| \geq R - \varepsilon$. It follows that $\|x - a\| = R - \sqrt{\|z-q'\|^2 - \|x-q'\|^2} \leq \alpha$. □

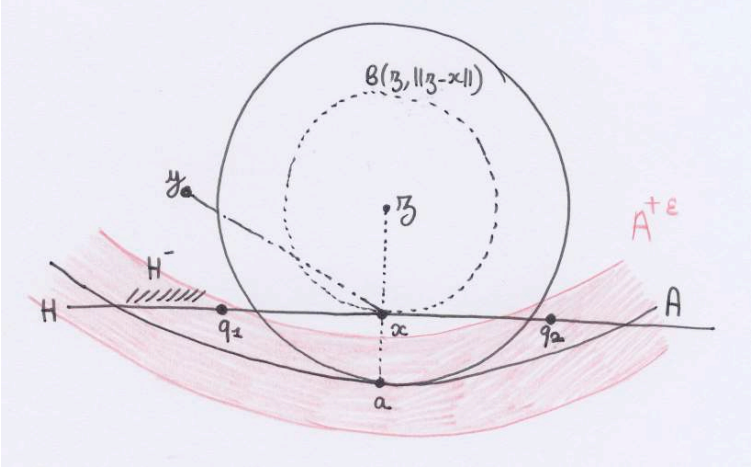

Figure 9: Notations for the proof of Lemma 16.

Lemma 17. *For any subset $Q \subset \mathbb{R}^N$ and any convex set $C \subset \mathbb{R}^N$,* $\mathrm{Hull}(Q + C) = \mathrm{Hull}(Q) + C$.

The proof is straightforward and hence omitted.

B Explicit constants in some simple cases

B.1 Roundness

Recall that $\theta_N = \arccos(-\frac{1}{N})$. In this appendix, we compute the θ_N-roundness of N-dimensional cubes and the θ_2-roundness of regular polygons. To prepare computations, we first reformulate the θ-roundness of convex bodies whose normal cones have a diameter smaller than $2\sin\frac{\theta}{2}$:

Lemma 18. *Let $\theta > 0$. Consider a convex body C and suppose that for every point $c \in C$ and every pair of vectors $n_1, n_2 \in \mathcal{N}(c)$, we have $\angle(n_1, n_2) < \theta$. Then, the θ-roundness of C can be expressed as*

$$\min_{c_1,c_2,n_1,n_2} \frac{(c_1 - c_2)\cdot(n_1 - n_2)}{\|c_1 - c_2\|^2}$$

where the minimum is taken over all points c_1, c_2 on the boundary of C and all vectors $n_1 \in \mathcal{N}(c_1)$ and $n_2 \in \mathcal{N}(c_2)$ making an angle $\angle(n_1, n_2) \geq \theta$. Furthermore, if C is polytope, the minimum is attained when c_1 and c_2 are vertices of C.

Proof. The support of the normal cycle of C is $\mathcal{N} = \{(c, n) \in C \times \mathbb{S}^{N-1} \mid n \in \mathcal{N}(c)\}$. Since $\mathcal{N}$ is homeomorphic to the boundary of an offset of C [10], $\mathcal{N}$ is compact. Define the closed set

$$E_\theta = \left\{(c_1, n_1, c_2, n_2) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^N \times \mathbb{S}^{N-1} \mid \angle(n_1, n_2) \geq \theta\right\}$$

and consider the set $D_\theta = (\mathcal{N} \times \mathcal{N}) \cap E_\theta$. By construction, D_θ which is the intersection of a compact set and a closed set is compact. For any quadruple $(c_1, n_1, c_2, n_2) \in D_\theta$, we have $c_1 \neq c_2$, for otherwise, n_1 and n_2 would belong to the same normal cone and make an angle $\angle(n_1, n_2) \geq \theta$ which contradicts our hypothesis. Hence, the function $D_\theta \to \mathbb{R}$, $(c_1, n_1, c_2, n_2) \mapsto \frac{(c_1-c_2)\cdot(n_1-n_2)}{\|c_1-c_2\|^2}$ is well-defined and attains its minimum.

For the second part of the lemma, suppose the minimum is attained at the quadruple $(c_1, n_1, c_2, n_2) \in D_\theta$. Recall that by choice of n_1 in the normal cone of c_1, we have $(c_1 - c_2) \cdot n_1 \geq 0$ and similarly $(c_2 - c_1) \cdot n_2 \geq 0$. We distinguish two cases. If the minimum vanishes, then we have that both $(c_1 - c_2) \cdot n_1 = 0$ and $(c_2 - c_1) \cdot n_2 = 0$. Hence, the minimum is still attained if we replace c_1 and c_2 by vertices of the faces to which they belong respectively. Suppose now that the minimum does not vanish and is attained at a point c_1 in the relative interior of a face F of the polytope. Let v be a unit vector parallel to F and choose $t_0 > 0$ small enough such that $c_1 + tv$ belongs to F for all $t \in [-t_0, t_0]$. Consider the function $f : [-t_0, t_0] \to \mathbb{R}$ defined by

$$f(t) \quad = \quad \frac{(c + tv) \cdot n}{\|c + tv\|^2},$$

where $c = c_1 - c_2$ and $n = n_1 - n_2$. Since f has a minimum at $t = 0$, we deduce that $f'(0) = 0$ and $f''(0) \geq 0$. Computing the first derivative of f, we obtain

$$f'(t) \quad = \quad \frac{-(v \cdot n)t^2 - 2(c \cdot n)t + \|c\|^2(v \cdot n) - 2(c \cdot n)(c \cdot v)}{\|c + tv\|^4}.$$

Using $f'(0) = 0$ and $c \cdot n = (c_1 - c_2) \cdot (n_1 - n_2) > 0$, we get that $f''(0) = \frac{-2(c \cdot n)\|v\|^2}{\|c\|^4} < 0$, which leads to a contradiction. □

Let us make a few useful observations. Suppose we are given two distinct points c_1 and c_2 on the boundary of C and we are looking for the pair of unit vectors n_1, n_2 which minimizes the ratio:

$$\frac{(c_1 - c_2) \cdot (n_1 - n_2)}{\|c_1 - c_2\|^2} \quad = \quad \frac{(c_1 - c_2) \cdot n_1}{\|c_1 - c_2\|^2} + \frac{(c_2 - c_1) \cdot n_2}{\|c_1 - c_2\|^2}$$

subject to the three constraints (a) $\angle(n_1, n_2) \geq \theta$ (b_1) $n_1 \in \mathcal{N}(c_1)$ and (b_2) $n_2 \in \mathcal{N}(c_2)$. Since the two terms on the right side are positive by choice of n_i in the normal cone of c_i, it follows that a pair of vectors which minimizes the left side subject to (a), (b_1) and (b_2) also minimizes the ith term on the right side under the two constraints (a) and (b_i) for $i \in \{1, 2\}$. If now we relax all the constraints on n_1 and n_2, then the dot product $(c_i - c_j) \cdot n_i$ has a unique minimum which is attained for $n_i = (c_j - c_i)/\|c_i - c_j\|$. Since this value can never be reached when n_i belongs to the normal cone of c_i, it shows that as we minimize the sum, one of the two constraints (a) or (b_i) must be active for all $i \in \{1, 2\}$. Specifically, if one of the two vectors n_1 or n_2 lies in the interior of its associated normal cone, then the angle $\angle(n_1, n_2)$ must exactly be equal to θ. Conversely, if $\angle(n_1, n_2) > \theta$, then both n_1 and n_2 lie on the boundary of their associated normal cone. Furthermore, if $\angle(n_1, n_2) > \theta$ for all $n_1 \in \mathcal{N}(c_1)$ and $n_2 \in \mathcal{N}(c_2)$, then the vectors n_1 and n_2 that minimize the sum both lie on the boundary of their associated normal cone.

B.1.1 Roundness of regular polygons

Recalling that $\theta_2 = \frac{2\pi}{3}$, we compute the θ_2-roundness of regular p-gons, for $p \geq 3$. Using the natural identification between $\mathbb{R}^2$ and $\mathbb{C}$, we denote by $\mathcal{P}_p$ the regular p-gon whose vertices are $v_0, v_1, \ldots, v_{p-1}$ with

$$v_k \quad = \quad \frac{1}{\cos \frac{\pi}{p}} \, e^{i \frac{2k\pi}{p}}.$$

By construction, the unit disk is tightly contained in $\mathcal{P}_p$ and we have $B \subset \mathcal{P}_p \subset \delta B$ for $\delta = (\cos\frac{\pi}{p})^{-1}$. Given $k \geq 1$, we define

$$\varkappa_k \;=\; \min_{n_0,n_k} \frac{(v_k - v_0)\cdot(n_k - n_0)}{\|v_k - v_0\|^2}, \tag{6}$$

where the minimum is taken over all vectors $n_0 \in \mathcal{N}(v_0)$ and $n_k \in \mathcal{N}(v_k)$ making an angle $\angle(n_0, n_k) \geq \frac{2\pi}{3}$. The θ_2-roundness of $\mathcal{P}_p$ is $\varkappa(\mathcal{P}_p) = \min_{1\leq k\leq \lfloor \frac{p}{2}\rfloor} \varkappa_k$. Let us compute $\varkappa_k$. Noting that the condition $\angle(n_0, n_k) \geq \frac{2\pi}{3}$ imposes $k \geq \frac{p}{3} - 1$, we consider three cases depending on the value of k.

Case 1: If $k \geq \frac{p}{3} + 1$, any pair of vectors $n_0 \in \mathcal{N}(p_0)$ and $n_k \in \mathcal{N}(p_k)$ satisfies $\angle(n_0, n_k) \geq \frac{2\pi}{3}$. Hence, the values of n_0 and n_k that minimize f are extreme values of their respective normal cones. Specifically, $n_0 = e^{i\frac{\pi}{p}}$ and $n_k = e^{i\frac{(2k-1)\pi}{p}}$. Since in that case $n_0 - n_k$ and $v_0 - v_k$ are collinear, the dot product is equal to the product of the norms and we have:

$$\varkappa_k \;=\; \cos\frac{\pi}{p} \times \frac{\|e^{i\frac{(2k-1)\pi}{p}} - e^{i\frac{\pi}{p}}\|}{\|1 - e^{i\frac{2k\pi}{p}}\|} \;=\; \cos\frac{\pi}{p} \times \frac{\sin\frac{(k-1)\pi}{p}}{\sin\frac{k\pi}{p}}.$$

Since $\varkappa_k$ increases with k, the smallest value of $\varkappa_k$ is obtained for $k = \lceil \frac{p}{3}\rceil + 1$. Note that this case may happen only if $p \geq 6$ since it requires that $\frac{p}{3} + 1 \leq k \leq \lfloor \frac{p}{2}\rfloor$.

Case 2: If $k = \lceil \frac{p}{3}\rceil$, the extreme values of the normals in their respective normal cones that minimize the ratio in (6) violate the condition $\angle(n_0, n_k) \geq \frac{2\pi}{3}$. It follows that the minimum is reached when n_0 and n_k are constrained by $\angle(n_0, n_k) = \frac{2\pi}{3}$. Therefore, the minimum is attained when the angle between $n_0 - n_k$ and $v_0 - v_k$ is maximized. This happens for instance for $n_0 = e^{i\frac{\pi}{p}}$ and $n_k = e^{i\,\pi(\frac{1}{p}+\frac{2}{3})}$. Recall that the dot product of two vectors represented by $u, v \in \mathbb{C}$, is $\mathrm{Re}\,[u\overline{v}]$ where $\overline{v}$ is the conjugate of v and $\mathrm{Re}\,[.]$ designates the real part of a complex number. Using $v_0 = 1$ and $v_k = \frac{1}{\cos\pi/p} e^{i\frac{2\lceil p/3\rceil\pi}{p}}$, we get:

$$\begin{aligned}
\varkappa_{\lceil \frac{p}{3}\rceil} \;&=\; \cos\frac{\pi}{p} \times \frac{\mathrm{Re}\left[\left(1 - e^{i\frac{2\lceil p/3\rceil\pi}{p}}\right)\left(e^{-i\frac{\pi}{p}} - e^{-i\,\pi(\frac{1}{p}+\frac{2}{3})}\right)\right]}{\mathrm{Re}\left[\left(1 - e^{i\frac{2\lceil p/3\rceil\pi}{p}}\right)\left(1 - e^{-i\frac{2\lceil p/3\rceil\pi}{p}}\right)\right]} \\
&=\; \frac{\cos\frac{\pi}{p}\left[\cos\frac{\pi}{p} - \cos\left(\frac{2\pi}{3} + \frac{\pi}{p}\right) - \cos\left(\frac{2\lceil p/3\rceil\pi}{p} - \frac{\pi}{p}\right) + \cos\left(\frac{2\lceil p/3\rceil\pi}{p} - \frac{\pi}{p} - \frac{2\pi}{3}\right)\right]}{2 - 2\cos\left(\frac{2\lceil p/3\rceil\pi}{p}\right)},
\end{aligned}$$

Case 3: If $k = \lceil \frac{p}{3}\rceil - 1$, the extreme values for the normals in their respective normal cones that minimize the ratio in (6) still violate the condition $\angle(n_0, n_k) \geq \frac{2\pi}{3}$. It follows that the minimum is reached when n_0 and n_k are constrained by $\angle(n_0, n_k) = \frac{2\pi}{3}$. Therefore, the minimum is attained when the angle between $n_0 - n_k$ and $v_0 - v_k$ is maximized. This happens for instance for $n_0 = e^{-i\frac{\pi}{p}}$

and $n_k = e^{i\,\pi(\frac{-1}{p}+\frac{2}{3})}$. Using $v_0 = 1$ and $v_k = \frac{1}{\cos \pi/p} e^{i\,2\pi\,\frac{\lceil p/3\rceil - 1}{p}}$, we get:

$$\begin{aligned}
\varkappa_{\lceil \frac{p}{3}\rceil - 1} &= \cos\frac{\pi}{p} \times \frac{\mathrm{Re}\left[\left(1 - e^{i\,\frac{(2\lceil p/3\rceil - 1)\pi}{p}}\right)\left(e^{i\,\frac{\pi}{p}} - e^{i\,\pi(\frac{1}{p}-\frac{2}{3})}\right)\right]}{\mathrm{Re}\left[\left(1 - e^{i\,\frac{(2\lceil p/3\rceil - 1)\pi}{p}}\right)\left(1 - e^{-i\,\frac{(2\lceil p/3\rceil - 1)\pi}{p}}\right)\right]} \\
&= \frac{\cos\frac{\pi}{p}\left[\cos\frac{\pi}{p} - \cos\left(\frac{2\pi}{3} - \frac{\pi}{p}\right) - \cos\left(\frac{2\lceil p/3\rceil\pi}{p} - \frac{\pi}{p}\right) + \cos\left(\frac{2\lceil p/3\rceil\pi}{p} - \frac{\pi}{p} - \frac{2\pi}{3}\right)\right]}{2 - 2\cos\left(\frac{(2\lceil p/3\rceil - 1)\pi}{p}\right)},
\end{aligned}$$

For $p \geq 6$, all three situations may happen and the roundness of $\mathcal{P}_p$ is

$$\varkappa(\mathcal{P}_p) = \min\{\varkappa_{\lceil \frac{p}{3}\rceil + 1}, \varkappa_{\lceil \frac{p}{3}\rceil}, \varkappa_{\lceil \frac{p}{3}\rceil - 1}\}.$$

All three arguments of the minimum above are converging to 1 as $p \to \infty$ but it seems that the minimum is never realized by the last expression. For all integer values in the range $p = 3, \ldots, 11$ the minimum is attained by the second expression and for values $p \geq 12$ the minimum is attained sometimes by the first and sometimes by the second argument.

For $4 \leq p \leq 5$, only the second and third cases may occur. For $p = 3$, we cannot apply Lemma 18 anymore but a closer look to this simple case shows that the roundness $\varkappa(\mathcal{P}_3)$ is equal to the second term above. We get:

$$\begin{aligned}
\varkappa(\mathcal{P}_3) &= \frac{1}{2}\cos\frac{\pi}{3} = \frac{\sqrt{3}}{4} \\
\varkappa(\mathcal{P}_4) &= \frac{1}{2}\cos\frac{\pi}{4}\left(\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\right) \\
\varkappa(\mathcal{P}_5) &= \frac{\cos\frac{\pi}{5}(\cos\frac{\pi}{5} + \cos\frac{2\pi}{15} + \cos\frac{2\pi}{5} + \cos\frac{\pi}{15})}{2 + 2\cos\frac{\pi}{5}}
\end{aligned}$$

Figure 10 shows some values of $\varkappa(\mathcal{P}_p)$.

$\mathcal{P}_p$	$\varkappa(\mathcal{P}_p)$
equilateral triangle in $\mathbb{R}^2$	0.4330127
square in $\mathbb{R}^2$	0.659739608
hexagon in $\mathbb{R}^2$	0.699358737
octagon in $\mathbb{R}^2$	0.79335334
dodecagon in $\mathbb{R}^2$	0.866025404
36-gon in $\mathbb{R}^2$	0.951917139
360-gon in $\mathbb{R}^2$	0.994986799

Figure 10: Values of roundness $\varkappa(\mathcal{P}_p)$ for regular p-gons in the plane.

B.1.2 Roundness of N-dimensional cubes

Lemma 19. *The θ_N-roundness of the N-dimensional cube $B_\infty = [-1,1]^N \subset \mathbb{R}^N$ is:*

$$\varkappa(B_\infty) \;=\; \begin{cases} \frac{1}{2\sqrt{2}}\left(\cos\frac{\pi}{4}+\cos\frac{\pi}{12}\right) & \text{if } N=2,\\ \frac{1}{\sqrt{6}} & \text{if } N=3,\\ \frac{1}{(N-2)\sqrt{N}} & \text{if } N\geq 4, \end{cases}$$

Proof. Consider two distinct vertices v, v' and let us compute the minimum of

$$f(n,n') \;=\; \frac{(v-v')\cdot(n-n')}{\|v-v'\|^2},$$

over all vectors $n \in \mathcal{N}(v)$ and $n' \in \mathcal{N}(w)$ making an angle $\angle(n,n') \geq \theta_N > \frac{\pi}{2}$. Note that when the minimum is attained, $\angle(n,n') = \theta_N$. Suppose the two vertices v and v' differ by exactly k coordinates. Without loss of generality, we may assume $v = (1,\ldots,1)$ and $v' = (-1,\ldots,-1,1,\cdots,1)$. We reformulate the minimization problem as follows. Note that a unit vector $n = (n_1,\ldots,n_N)$ belongs to $\mathcal{N}(v)$ if and only if all coordinates are non-negative and it belongs to $\mathcal{N}(v')$ if and only if $n_i \leq 0$ for $1 \leq i \leq k$ and $n_i \geq 0$ for $k+1 \leq i \leq N$. Thus, if $n' = (n'_1,\ldots,n'_N) \in \mathcal{N}(v')$, the vector $m = (-n'_1,\ldots,-n'_k,n'_{k+1},\ldots,n'_N)$ has non-negative coordinates and we set $g(n,m) = f(n,n')$. The minimum $\varkappa_k$ of f subject to $n \in \mathcal{N}(v)$, $n' \in \mathcal{N}(v')$, $\angle(n,n') \geq \theta_N$ is also the minimum of

$$g(n,m) \;=\; \frac{1}{2k}\sum_{i=1}^{k}(n_i+m_i)$$

subject to:

(i) $\sum_{i=1}^{k} n_i m_i = \frac{1}{N} + \sum_{i=k+1}^{N} n_i m_i$;

(ii) $n_i \geq 0$ and $m_i \geq 0$ for all $1 \leq i \leq k$;

(iii) $\sum_{i=1}^{N} n_i^2 = 1$, $\sum_{i=1}^{N} m_i^2 = 1$, $n_i \geq 0$ and $m_i \geq 0$ for all $k+1 \leq i \leq N$.

By Lemma 20, the minimum of g subject to (i) and (ii) is $\frac{1}{k}\sqrt{\frac{1}{N}+\sum_{i=k+1}^{N} n_i m_i}$ and this minimum is attained for $n_1 = m_1 = \sqrt{\frac{1}{N}+\sum_{i=k+1}^{N} n_i m_i}$ and $n_i = m_i = 0$ for $2 \leq i \leq k$. We consider three cases:

1. If $N \geq 3$ and $k \leq N-2$, we choose $n_1 = m_1 = \frac{1}{\sqrt{N}}$, $n_N = m_{N-1} = \sqrt{1-\frac{1}{N}}$ and set all other coordinates to zero. We note that n and m thus defined satisfy conditions (i), (ii) and (iii) and minimize g by construction. We have $\varkappa_k = \frac{1}{k\sqrt{N}} \geq \varkappa_{N-2} = \frac{1}{(N-2)\sqrt{N}}$.

2. If $N \geq 2$ and $k = N-1$, we set $n_1 = m_1 = \sqrt{\frac{1}{2}+\frac{1}{2N}}$, $n_N = m_N = \sqrt{\frac{1}{2}-\frac{1}{2N}}$ and $n_i = m_i = 0$ otherwise. The vectors n and m thus defined satisfy the three conditions (i), (ii) and (iii) and minimize g by construction. Hence, $\varkappa_{N-1} = \frac{\sqrt{N+1}}{(N-1)\sqrt{2N}}$.

3. If $N \geq 1$ and $k = N$, we use the fact that the minimum of g subject to (ii) and (iii) is $\frac{1}{N}$ and deduce that $\varkappa_N \geq \frac{1}{N}$.

$\varkappa(B_\infty)$ is the minimum of $\varkappa_k$ over all $k \in \{1, \ldots, N\}$. When $N \geq 4$, $\varkappa_{N-2} = \frac{1}{(N-2)\sqrt{N}}$ is smaller or equal than both $\varkappa_{N-1} = \frac{\sqrt{N+1}}{(N-1)\sqrt{2N}}$ and $\varkappa_N \geq \frac{1}{N}$ and this proves the lemma for the case $N \geq 4$.

When $N \leq 3$, the inequality $\varkappa_N \geq \frac{1}{N}$ is not sufficient to conclude and we have to compute $\varkappa_N$. Introducing the unit vector $e = \frac{v-v'}{\|v-v'\|}$, we note that $\varkappa_N$ is the minimum of $g(n,m) = \frac{1}{2\sqrt{N}}(e\cdot n + e\cdot m)$ subject to (i) $n \cdot m \geq \frac{1}{N}$, (ii) $n_i \geq 0$ and $m_i \geq 0$ for all $1 \leq i \leq N$ and (iii) $\|n\| = \|m\| = 1$. It is not difficult to check that the minimum of g subject to (i) and (iii) is attained when n, m and e are coplanar. Setting $\alpha = \angle(e, n)$ and constraining n and m to belong to the first orthant, we get that $\arccos\frac{1}{N} - \arccos\frac{1}{\sqrt{N}} \leq \alpha \leq \arccos\frac{1}{\sqrt{N}}$ and

$$g(n,m) \;=\; \frac{1}{2\sqrt{N}}\left[\cos\alpha + \cos\left(\arccos\frac{1}{N} - \alpha\right)\right]$$

attains its minimum for $\alpha = \arccos\frac{1}{\sqrt{N}}$, in other words when n is one of the vectors of the canonical basis of $\mathbb{R}^N$. Hence,

$$\varkappa_N = \frac{1}{2N} + \frac{1}{2\sqrt{N}}\cos\left(\arccos\frac{1}{N} - \arccos\frac{1}{\sqrt{N}}\right)$$

For $N = 2$, we get that $\varkappa_1 = \frac{\sqrt{3}}{2}$ is larger than $\varkappa_2 = \frac{1}{2\sqrt{2}}\left(\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\right)$. For $N = 3$, we have $\varkappa_1 = \frac{1}{\sqrt{3}}$, $\varkappa_2 = \frac{1}{\sqrt{6}}$ and $\varkappa_3 = \frac{4}{9}$ and the smallest value is $\frac{1}{\sqrt{6}}$. □

Lemma 20. *Consider the function $g : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ which maps the pair of vectors $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ to $g(u,v) = \sum_{i=1}^k (u_i + v_i)$. The minimum of g subject to $\sum_{i=1}^k u_i v_i = A^2$ and $u_i \geq 0$, $v_i \geq 0$ for all $i \in \{1, \ldots, k\}$ is $2A$. This minimum is attained for the k pairs of vectors (Ae_i, Ae_i) where e_i is the ith vector of the canonical basis of $\mathbb{R}^k$,* i.e. *the vector whose ith coordinate is equal 1 and whose other coordinates are equal to 0.*

Proof. First, observe that the function $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $h(x,y) = x+y$ subject to $xy = A^2$, $x \geq 0$ and $y \geq 0$ is minimized for $x = y = A$ and the minimum is $2A$. Suppose now all coordinates of u and v are fixed but u_i and v_i. Using the above observation, we deduce that the minimum of g subject to $u_i v_i = A^2 - \sum_{i\neq j} u_j v_j$, $u_i \geq 0$ and $v_i \geq 0$ is obtained for $u_i = v_i$. Repeating this argument for all i, we get that g subject to the constraints stated in the lemma attains its minimum for $u = v$. Furthermore, the minimum of $g(u,u) = 2\sum_{i=1}^k u_i$ subject to $\sum_{i=1}^k u_i^2 = A^2$ and $u_i \geq 0$ for all $i \in \{1, \ldots, n\}$ is $2A$ and this minimum is attained for the k vectors Ae_1, Ae_2, …, Ae_k. □

B.2 Eccentricity

B.2.1 Eccentricity of 2-dimensional symmetric convex bodies is 0

Recall that a subset $C \subset \mathbb{R}^N$ is *symmetric* if $C = -C$, or equivalently if for all $x \in \mathbb{R}^N$, $x \in C$ if and only if $-x \in C$. In this section we prove the following :

Lemma 21. *Any symmetric convex body C of $\mathbb{R}^2$ has eccentricity 0.*

Proof. Consider a compact subset $Q \subset \mathbb{R}^2$ such that $\bigcap_{q\in Q}(q + C) \neq \emptyset$ and let us prove that

$$\left(\bigcap_{q\in Q}(q + C)\right) \cap \operatorname{Hull} Q \neq \emptyset. \tag{7}$$

First, we claim that for any pair of points $q_1, q_2 \in Q$, the intersection $(q_1+C)\cap(q_2+C)\cap \mathrm{Hull}\, Q$ is non-empty. Consider a point x in the intersection $(q_1+C)\cap(q_2+C)$ and let $c_1, c_2 \in C$ such that $x = q_1+c_1 = q_2+c_2$. Since C is symmetric, $-c_2, -c_1 \in C$ and therefore the point $y = q_1-c_2 = q_2-c_1$ also belongs to the intersection $(q_1 + C) \cap (q_2 + C)$. By construction, the four points q_1, x, q_2, y form a parallelogram and therefore the midpoint of q_1q_2 which is also the midpoint of xy belongs to $(q_1 + C) \cap (q_2 + C)$ and since the midpoint belongs to $\mathrm{Hull}\, Q$, we get the claim.

To conclude the proof, we apply Helly's Theorem to the collection of convex sets $\{q + C\}_{q\in Q}$ completed with the set $\mathrm{Hull}\, Q$. This asserts that the collection of convex sets in $\mathbb{R}^2$ have a non-empty common intersection provided only that the same is true for each triplet of convex sets. By assumption, $(q_1 + C) \cap (q_2 + C) \cap (q_3 + C) \neq \emptyset$ for all $q_1, q_2, q_3 \in Q$ and we have just established that $(q_1 + C) \cap (q_2 + C) \cap \mathrm{Hull}\, Q \neq \emptyset$ for all $q_1, q_2 \in Q$. Hence, every triplets in the collection have a non-empty common intersection and Equation (7) follows. □

B.2.2 Eccentricity of offsets

Next lemma states that Euclidean ν-offsets have eccentricity less than 1 for any positive ν:

Lemma 22. *Let $C \subset \mathbb{R}^N$ and $\nu > 0$. The eccentricity of $C^{+\nu}$ is bounded from above by $\xi_\nu = \inf\{\xi \in \mathbb{R} \mid C \subset \xi C^{+\nu}\}$. If $C \subset \delta B$, then $\xi_\nu \leq \frac{1}{1+\frac{\nu}{\delta}}$.*

Proof. Suppose $\bigcap_{q\in Q}(q + C^{+\nu}) \neq \emptyset$. Then, for every point $q \in Q$, we can find a point $c_q \in q + C$ such that $\bigcap_{q\in Q}(c_q + \nu B) \neq \emptyset$. Because the Euclidean ball νB has eccentricity 0, it follows that

$$\left(\bigcap_{q\in Q}(c_q + \nu B)\right) \cap (\mathrm{Hull}\{c_q \mid q \in Q\}) \;\neq\; \emptyset.$$

From Lemma 17 and by definition of ξ_ν we have $\mathrm{Hull}\{c_q \mid q \in Q\} \subset \mathrm{Hull}\{q + C \mid q \in Q\} = \mathrm{Hull}(Q) + C \subset \mathrm{Hull}(Q) + \xi_\nu C^{+\nu}$. Thus, the following superset is also non-empty:

$$\left(\bigcap_{q\in Q}(q + C^{+\nu})\right) \cap (\mathrm{Hull}(Q) + \xi_\nu C^{+\nu}) \;\neq\; \emptyset,$$

showing that the eccentricity of $C^{+\nu}$ is bounded from above by ξ_ν. If $C \subset \delta B$, then $(1 + \frac{\nu}{\delta})C \subset C + \nu B$, implying the upper bound on ξ_ν. □

B.2.3 Eccentricity of N-dimensional cubes

In this section, we assume $N \geq 2$ and compute the eccentricity of the N-dimensional cube $B_\infty = \{x \in \mathbb{R}^N \mid \|x\|_\infty \leq 1\}$.

Lemma 23. *The eccentricity of B_∞ is $1 - \frac{2}{N}$*

Proof. Consider a compact subset Q in $\mathbb{R}^N$ and suppose the cubes $q + B_\infty$ centered at $q \in Q$ have a non-empty common intersection:

$$\mathcal{I}(Q) = \bigcap_{q\in Q}(q + B_\infty) \neq \emptyset.$$

The proof consists first in choosing carefully a point $z \in \mathcal{I}(Q)$ and second in finding $\xi \in \mathbb{R}$ such that $z \in \text{Hull}(Q) + \xi B_\infty$. Let Z be the smallest axis-parallel box containing Q. The z center of Z belongs to the common intersection of the cubes, $\mathcal{I}(Q)$.

We call *diagonal* of Z any segment that connects two vertices o and o' of Z which are symmetric with respect to z. To bound the distance between z and Hull(Q), we first prove that any diagonal oo' intersects Hull(Q). Suppose for a contradiction that a diagonal oo' do not intersect Hull(Q). Then, there exists a plane H parallel to oo' and separating oo' from Hull(Q) (see Figure 11 left). In particular, this implies that at least one face of Z is entirely contained in the open half-space bounded by H and containing oo'. This face does not contain any point $q \in Q$, contradicting the definition of Z as the smallest axis-parallel box containing Q.

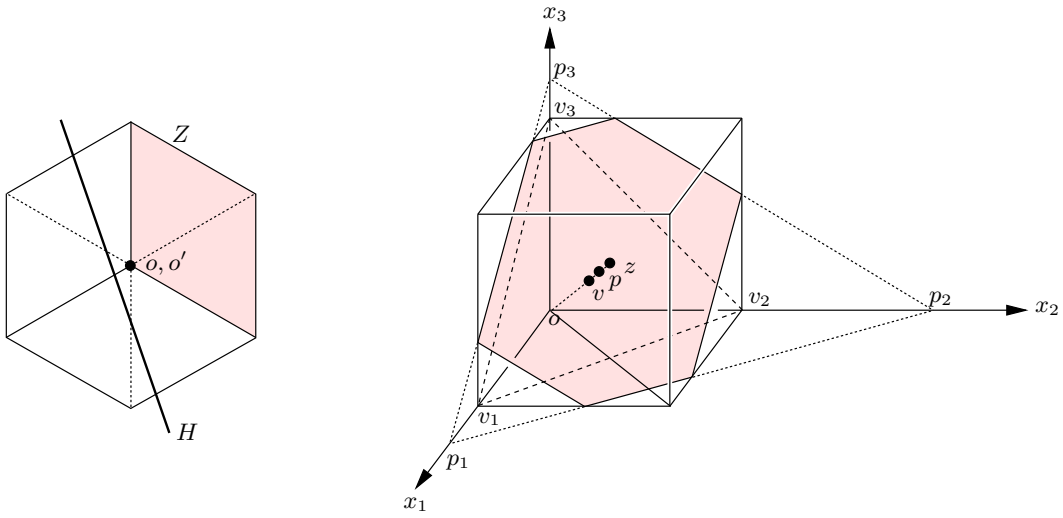

Figure 11: Left: projection of the box Z onto a plane orthogonal to the diagonal oo'. The shaded face is entirely contained in the open half-space bounded by H and passing through the diagonal oo'. Right: An axis-parallel box and its intersections with planes $p_1p_2p_3$ and $v_1v_2v_3$.

We have just proved that Z has at least one vertex o such that the segment oz intersects Hull(Q). Consider the point p on $oz \cap$Hull(Q) closest to z and let P be a plane through p supporting Hull(Q). Observe that P must intersect all faces of Z, for otherwise it would be possible to find a smaller axis-parallel box containing Q. Using this observation, we find an upper bound on $\frac{\|z-p\|_\infty}{\|z-o\|_\infty}$. Since this ratio is left unchanged by a scaling which transforms Z into a cube with edge length 1, we may assume that $o = (0, \ldots, 0)$ and $z = (\frac{1}{2}, \ldots, \frac{1}{2})$. Let v_i be the vertex of Z on the x_i-axis different from o. The vertices $v_1, \ldots, v_N$ span a plane which intersects oz in $v = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$ and

$$\frac{\|z-p\|_\infty}{\|z-o\|_\infty} \leq \frac{\|z-v\|_\infty}{\|z-o\|_\infty} = 1 - \frac{2}{N}$$

Using $\|z-o\|_\infty \leq 1$, we get $z \in \text{Hull}(Q) + (1-\frac{2}{N})B_\infty$ and B_∞ is ξ-eccentric for $\xi = 1 - \frac{2}{N}$. If we assume $Q = \{v_1, \ldots, v_N\}$, then $\mathcal{I}(Q) = \{z\}$ and v is the point in Hull(Q) closest to z for the L_∞-norm, showing that the eccentricity of B_∞ is ξ. □

C A simpler proof for Euclidean balls

In this section, we provide a simpler proof of Theorem 2 when $C = B$.

Theorem 3. *Let A be a compact subset of $\mathbb{R}^N$ with positive reach R. Let P be an (ε, B)-sample of A. Then, $P + rB$ is homotopy equivalent to A for all positive real numbers r and ε such that $r < R - \varepsilon$ and $R - \sqrt{(R-\varepsilon)^2 - r^2} < r - \varepsilon$.*

Proof. Let $\alpha = R - \sqrt{(R-\varepsilon)^2 - r^2}$. By hypothesis $\alpha < r - \varepsilon$ and because P is an (ε, B)-sample of A, we have the inclusion $A + \alpha B \subset P + rB$. Setting $\mathcal{A}_p(\alpha) = (p + rB) \cap (A + \alpha B)$, it follows that $A + \alpha B = \bigcup_{p \in P} \mathcal{A}_p(\alpha)$. Using Leray's theorem [21] and Lemma 24, we get that

$$\begin{aligned} A + \alpha B &\simeq \text{Nerve}\{\mathcal{A}_p(\alpha)\}_{p \in P} \\ P + rB &\simeq \text{Nerve}\{p + rB\}_{p \in P} \end{aligned}$$

Let us prove that $\text{Nerve}\{p + rB\}_{p \in P} = \text{Nerve}\{\mathcal{A}_p(\alpha)\}_{p \in P}$. Note that this is equivalent to proving that for all subsets $Q \subset P$,

$$\bigcap_{q \in Q} (q + rB) \neq \emptyset \Longleftrightarrow \bigcap_{q \in Q} [(q + rB) \cap (A + \alpha B)] \neq \emptyset.$$

One direction is trivial: if a point belongs to the intersection on the right, then it belongs to the intersection on the left. Suppose now that $\bigcap_{q \in Q}(q + rC) \neq \emptyset$ and let z be the center of the smallest ball enclosing Q. Clearly, z belongs to the common intersection $\bigcap_{q \in Q}(q + rC)$ and by Lemma 14, z also belongs to $\text{Hull}(Q)$. The ball centered at z with radius r encloses Q. Since P is an (ε, B)-sample of A, we have $Q \subset P \subset A^{+\varepsilon}$. Applying Lemma 16, we get that $\text{Hull}(Q) \subset A + \alpha B$ and the equality between the two nerves follows. Hence, $P + rB$ is homotopy equivalent to $A + \alpha B$ and therefore to A since $\alpha < R$, □

Lemma 24. *Consider a subset $Q \subseteq A^{+\varepsilon}$ in $\mathbb{R}^N$. Let $\mathcal{A}_p(\alpha) = (p + rB) \cap (A + \alpha B)$. Then, $\bigcap_{q \in Q} \mathcal{A}_q(\alpha)$ is either empty or contractible for $r < R - \varepsilon$ and $\alpha \geq R - \sqrt{(R-\varepsilon)^2 - r^2}$.*

Proof. Suppose $\mathcal{I} = \bigcap_{q \in Q}(q + rB)$ is non-empty. Then, $\mathcal{I}$ being convex is contractible. We prove that $\mathcal{I} \simeq \mathcal{I} \cap A^{+\alpha}$. Consider $x \in \mathcal{I} \setminus A^{+\alpha}$ (see Figure 12). For every point $q \in Q$, we have $d(x, A) \leq \|x - q\| + d(q, A) \leq r + \varepsilon < R$, showing that x has a unique orthogonal projection a onto A. We let y be the point of $A^{+\alpha}$ closest to x and prove that the segment xy is within $\mathcal{I}$. For this, we establish that for all $q \in Q$, the orthogonal projection q' of q onto the straight-line passing through x and a lies in $A^{+\alpha}$. Suppose $q' \neq a$ for otherwise clearly $q' \in A^{+\alpha}$ and consider the half-line with origin a and passing through q'. Let z be the point on this half-line whose distance to a is R. By construction, $B(z, R)$ is tangent to A at a and its interior does not intersect A. Since $Q \subseteq A^{+\varepsilon}$, points $q \in Q$ do not lie in the interior of $B(z, R - \varepsilon)$ and $\|z - q\| \geq R - \varepsilon$. By choice of q' as the orthogonal projection of q onto the straight-line through x and a, we have $\|q' - q\| \leq \|x - q\| \leq r$. Combining these two inequalities, we get an upper bound on the distance from q' to a:

$$\begin{aligned} \|a - q'\| &= R - \sqrt{\|z - q\|^2 - \|q' - q\|^2} \\ &\leq R - \sqrt{(R-\varepsilon)^2 - r^2} \\ &\leq \alpha. \end{aligned}$$

Thus, $q' \in A^{+\alpha}$ and y lies between x and q'. This shows that the distance to q decreases as we move on the segment xy, starting from x and going toward y. Since this is true for all $q \in Q$, we deduce that $xy \subseteq \mathcal{I}$. This inclusion allows us to construct a deformation retraction of $\mathcal{I}$ onto $\mathcal{I} \cap A^{+\varepsilon}$ by setting $f_t(x) = (1-t)x + ty$ if $x \in \mathcal{I} \setminus A^{+\alpha}$ and $f_t(x) = x$ otherwise. □

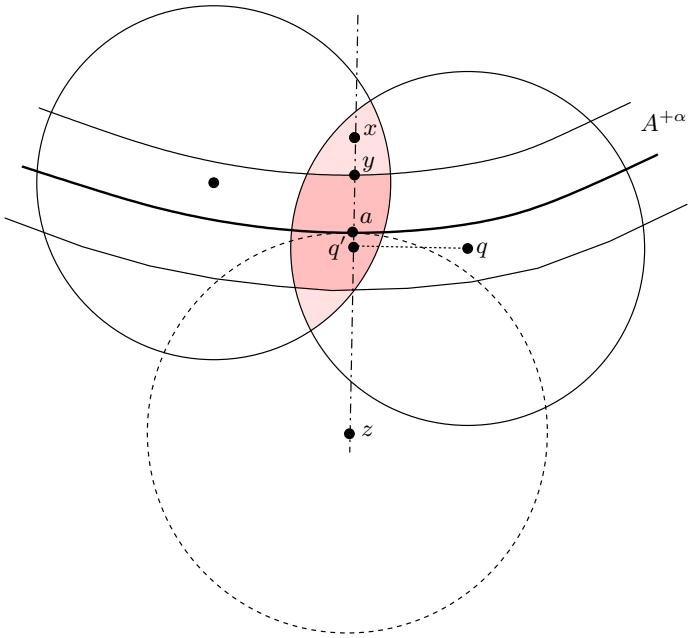

Figure 12: For the proof of Lemma 24.