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Ultimate ruin probability in discrete time with Bühlmann credibility premium adjustments

Julien Trufin ^{*} and Stéphane Loisel [†]

Abstract

In this paper, we consider a discrete-time ruin model where experience rating is taken into account. The main objective is to determine the behavior of the ultimate ruin probabilities for large initial capital in the case of light-tailed claim amounts. The logarithmic asymptotic behavior of the ultimate ruin probability is derived. Typical paths leading to ruin are studied. An upper bound is derived on the ultimate ruin probability in some particular case. The influence of the number of data points taken into account is analyzed, and numerical illustrations support the theoretical findings. Finally, we investigate the heavy-tailed case. The impact of the number of data points used for the premium calculation appears to be rather different from the one in the light-tailed case.

Key words and phrases: Discrete-time ruin model, Bühlmann's model, light-tailed claims, large deviation, ultimate ruin probability, Lundberg coefficient, path to ruin, heavy-tailed claims.

1 Introduction

Computation of ruin probabilities is an important topic in actuarial science. Standard references include Asmussen (2000), Dickson (2005), Grandell (1990), Kaas et al. (2008) and Rolski et al. (1999). Even if the study of the ruin problem resulted in a huge amount of published articles (including in the closely related fields of queueing theory and dam and storage processes), some assumptions have rarely been questioned in the literature, until recently.

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The majority of existing results require independent increments for the stochastic process describing the aggregate claim amounts filed against the insurance company. However, a lot of practical issues violate this independence assumption, as the result of the implementation of experience rating mechanisms by the insurance company. In this case, the financial result of a given calendar year then depends on the result(s) of one or several previous years.

In this paper, experience rating is taken into account. Allowing the premium amount to depend on past claims experience through an appropriate credibility mechanism is in line with insurance practice in most lines of business.

In the literature, only some authors have studied ruin problem with credibility dynamics. In Asmussen (1999), the ruin model allows adapted premiums rules. More precisely, a continuous-time risk process is considered with compound Poisson claims, where the premium rate is exclusively calculated on the basis of past claims statistics, with no reference to the market. In practice, one would use a mixture between a quantity based on past claim experience and a market premium level. The easiest way to compute credibility adjusted premiums is to use Bühlmann's credibility model. This is what Tsai and Parker (2004) have done in a discrete-time risk model. They have studied, by means of Monte-Carlo simulations, the impact of the number of past claim experience years taken into account for the calculation of the premiums on the ultimate ruin probabilities for light-tailed claims.

In this paper, we deal with a framework that is similar to the one of Tsai and Parker (2004). One of the main observation made by these authors is that for an insurance company with a large initial capital and light-tailed claims, the use of a bigger (finite) number of past claim experience years to calculate the premiums decreases the ultimate ruin probability. Among other results, until a certain number of past claim experience years, this observation is proved in this paper. We also show that the story is different for heavy-tailed claim amounts.

2 The model

Let us consider an insurance market made up of n portfolios, where portfolio j ($j = 1, \dots, n$) is characterized by independent and identically distributed annual aggregated claim amounts $Y_k^{(j)}$, with common distribution function F_j . For a randomly chosen insurance company operating on this market, let p_j be the probability that the portfolio held by this company is the portfolio j .

To motivate this setting, let us consider for example a simple market made up of 2 categories of policyholders, with n_g "good" policyholders and n_b "bad" policyholders, say. Then a company with $s \leq \min(n_g, n_b)$ policyholders would have $s + 1$ possible portfolios, i.e. $n = s + 1$. Furthermore, the probability p_j (associated to portfolio j ($j = 1, \dots, s + 1$),

characterized by $j - 1$ good policyholders) would be given by

$$p_j = \frac{s!}{(s - j + 1)!(j - 1)!} \prod_{i=0}^{j-1} \frac{n_g - i}{n_g + n_b - i} \prod_{i=0}^{s-j+1} \frac{n_b - i}{n_g + n_b - j - i}, \quad (2.1)$$

with $\prod_0^0 = 1$.

We assume that the company uses credibility premiums in order to determine, on the basis of its own claim amounts, the pure premium corresponding to its unknown portfolio. In this setting, the model proposed in Bühlmann (1967) and (1969) is largely used in practice. In this paper, we focus on this well-known credibility model.

Let us denote by Y_k the annual claim amounts of the company. The a priori mean of the Y_k 's is of course given by

$$\mu = \mathbb{E}[Y_k] = \sum_{j=1}^n \mathbb{E}[Y^{(j)}] p_j, \quad (2.2)$$

where $Y^{(j)}$ is a random variable with distribution function F_j , while the a priori variance breaks up in two terms:

$$\sigma^2 = \mathbb{V}[Y_k] = a + \nu, \quad (2.3)$$

with

$$a = \sum_{j=1}^n (\mathbb{E}[Y^{(j)}] - \mu)^2 p_j \quad (2.4)$$

and

$$\nu = \sum_{j=1}^n \mathbb{V}[Y^{(j)}] p_j. \quad (2.5)$$

Parameter a measures the part of the variance coming from the heterogeneity of the market, whereas parameter ν measures the part of hazard in the a priori variance.

As mentioned in Tsai and Parker (2004), some casualty insurers only consider a finite number of most recent periods of claim experiences to renew the premium. Let us denote by $m \in \mathbb{N}^+ \cup \{\infty\}$ the so-called "horizon of credibility", which corresponds to the number of periods taken into account for the calculation of the premiums.

So, depending on the choice of m , with the convention that $\sum_{i=1}^0 = 0$, the premium of the company in time $k - 1$ is given by

$$C_{k,m} = \left((1 - z_{k,m})\mu + z_{k,m} \frac{\sum_{i=\max(k-m,1)}^{k-1} Y_i}{\min(m, k-1)} \right) (1 + \eta), \quad (2.6)$$

where

$$z_{k,m} = \frac{\min(m, k-1) a}{\nu + \min(m, k-1) a} \quad (2.7)$$

is Bühlmann's credibility factor and $\eta > 0$ is the premium security loading. Notice that our premium rule depends on the market (via the parameters μ , a and ν) contrary to Asmussen (1999). For simplicity, we have assumed that no historical data is available. This simplifying assumption does not influence the asymptotic results derived in the sequel.

Now, if the portfolio held by the insurer is portfolio j , then the dynamics of the insurer's surplus obeys to the equation

$$U_{k,m}^{(j)} = u + \sum_{i=1}^k C_{i,m} - \sum_{i=1}^k Y_i^{(j)}, \quad (2.8)$$

where the $C_{i,m}$'s are described by equation (2.6) with $Y_i = Y_i^{(j)}$, and where u is the initial capital of the company. The corresponding ultimate ruin probability may be written as

$$\psi_m^{(j)}(u) = \Pr \left[U_{k,m}^{(j)} < 0 \text{ for some } k \mid U_{0,m}^{(j)} = u \right]. \quad (2.9)$$

Immediately, it appears the existence of a subset of "bad" portfolios, denoting b_m , such that for all $j \in b_m$, $\psi_m^{(j)}(u) = 1$ for all u . Indeed, let us define

$$\eta_m^{(j)} = \frac{(1 + \eta)((1 - z_m)\mu + z_m \mathbb{E}[Y^{(j)}])}{\mathbb{E}[Y^{(j)}]} - 1, \quad (2.10)$$

where

$$z_m = \frac{m a}{\nu + m a}. \quad (2.11)$$

Obviously, for $m < \infty$, $\eta_m^{(j)}$ corresponds to the average security loading of the premium for $k > m$ while for $m = \infty$, $\eta_m^{(j)} = \eta$, which corresponds this time to the security loading for $k \rightarrow \infty$. The subset b_m is then defined as

$$b_m = \{j = 1, \dots, n : \eta_m^{(j)} \leq 0\}. \quad (2.12)$$

Also, in the following, the complementary subset of ("not bad") portfolios is denoted by \bar{b}_m , and we shall say that $j \in g$ if and only if $\mathbb{E}[Y^{(j)}] \leq \mu$. Clearly, we have $g \subset \bar{b}_m$.

The purpose of this paper is to investigate the behavior of the ultimate ruin probabilities $\psi_m^{(j)}(u)$ for large u in the case of light-tailed annual claim amounts. First, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is derived. Then, typical paths leading to ruin are studied. Also, an upper bound on $\psi_m^{(j)}(u)$ is derived in some particular cases. Afterwards, we are able to determine, for each portfolio j , the influence of "strategy" m on $\psi_m^{(j)}(u)$ for large u . Next, whatever the portfolio j , optimal strategies, in a sense to be defined in the sequel, are deduced. As an illustration, numerical simulations are also performed. Finally, we also study the influence of m in first order on $\psi_m^{(j)}(u)$ in the heavy-tailed case. As we shall see, the conclusion is totally different than in the light-tailed case.

3 Limit results for light-tailed claims

In this Section, for simplicity, let us assume that for all $j = 1, \dots, n$, there exists $r^{(j)} > 0$ such that $\mathbb{E}[e^{rY^{(j)}}]$ exists for all $0 < r < r^{(j)}$ and such that $\lim_{r \rightarrow r^{(j)}} \mathbb{E}[e^{rY^{(j)}}] = \infty$. Clearly, this assumption implies necessarily light-tailed claims.

3.1 The case $m < \infty$

3.1.1 Asymptotic behavior of $\psi_m^{(j)}(u)$

Theorem 3.1. *Suppose that there exists a unique positive solution to the equation*

$$e^{-r(1-z_m)\mu(1+\eta)} \mathbb{E} \left[e^{rY^{(j)}(1-(1+\eta)z_m)} \right] = 1, \quad (3.1)$$

which we denote $\rho_m^{(j)}$. Furthermore, assume that for all k , the sum of the first k annual losses $Z_k = \sum_{i=1}^k (Y_i^{(j)} - C_{i,m})$ possesses a finite moment-generating function $\mathbb{E}[e^{rZ_k}]$ for $0 < r < r_0$, with $\rho_m^{(j)} < r_0$. Hence, we have

$$\lim_{u \rightarrow \infty} \frac{1}{u} \ln \psi_m^{(j)}(u) = -\rho_m^{(j)}. \quad (3.2)$$

Proof. As a consequence of the Gärtner-Ellis Theorem from large deviations (which is due to Glynn and Whitt (1994), see also Nyrhinen (1998) and Müller and Pflug (2001)), one only has to show the two following properties:

(A1) $\kappa_m(r) := \lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}]$ exists for $0 < r < r_0$,

(A2) $\rho_m^{(j)}$ is the unique positive value such that $\kappa_m(\rho_m^{(j)}) = 0$.

We have

$$\begin{aligned} \ln \mathbb{E}[e^{rZ_k}] &= -r(1+\eta)\mu \sum_{i=1}^k (1 - z_{i,m}) + \ln \mathbb{E} \left[e^{r \sum_{i=1}^k (Y_i^{(j)} - (1+\eta)\bar{z}_{i,m} \sum_{l=\max(1,i-m)}^{i-1} Y_l^{(j)})} \right] \\ &= -r(1+\eta)\mu \sum_{i=1}^k (1 - z_{i,m}) + \sum_{i=1}^k \ln \mathbb{E} \left[e^{rY^{(j)}(1-(\bar{z}_{i+1,m} + \bar{z}_{i+2,m} + \dots + \bar{z}_{\min(i+m,k),m})(1+\eta))} \right] \end{aligned}$$

where $\bar{z}_{k,m} = \frac{z_{k,m}}{\min(m,k-1)}$. Consequently

$$\lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}] = -r(1-z_m)\mu(1+\eta) + \ln \mathbb{E} \left[e^{rY^{(j)}(1-(1+\eta)z_m)} \right].$$

Hence, (A1) holds true. Concerning (A2), notice that $\kappa_m(r) = \ln h_m(r)$, where

$$h_m(r) = e^{-r(1-z_m)\mu(1+\eta)} \mathbb{E} \left[e^{rY^{(j)}(1-(1+\eta)z_m)} \right].$$

□

Notice that since $\mathbb{E}[e^{rY^{(j)}}]$ exists for all $0 < r < r^{(j)}$, result (3.2) requires in fact only the existence and the uniqueness of $\rho_m^{(j)}$, with the condition $\rho_m^{(j)} < r^{(j)}$.

3.1.2 The Lundberg coefficient $\rho_m^{(j)}$

Let us define

$$\tilde{\eta}_m^{(j)} = \frac{(1 - z_m)\mu(1 + \eta)}{\mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m)} - 1. \quad (3.3)$$

The equation (3.1) is equivalent to

$$e^{-r(1-(1+\eta)z_m)} \mathbb{E}[Y^{(j)}(1+\tilde{\eta}_m^{(j)})] \mathbb{E} \left[e^{r(1-(1+\eta)z_m)Y^{(j)}} \right] = 1, \quad (3.4)$$

and consequently

$$\rho_m^{(j)} = \frac{\tilde{\rho}_m^{(j)}}{1 - (1 + \eta)z_m}, \quad (3.5)$$

where $\tilde{\rho}_m^{(j)}$ is the classical Lundberg coefficient defined as the unique positive solution of the equation

$$e^{-r \mathbb{E}[Y^{(j)}](1+\tilde{\eta}_m^{(j)})} \mathbb{E}[e^{rY^{(j)}}] = 1. \quad (3.6)$$

Now, from classical results, it is then obvious that the inequality $(\infty >) \tilde{\eta}_m^{(j)} > 0$ is a necessary condition for the existence of $\rho_m^{(j)}$. Let us prove that the inequality $(\infty >) \tilde{\eta}_m^{(j)} > 0$ is equivalent to $m < \frac{\nu}{a\eta}$ and $j \in \bar{b}_m$.

Proposition 3.1.

$$(\infty >) \tilde{\eta}_m^{(j)} > 0 \Leftrightarrow m < \frac{\nu}{a\eta} \text{ and } j \in \bar{b}_m. \quad (3.7)$$

Proof. (\Rightarrow): If $\infty > \tilde{\eta}_m^{(j)} > 0$, then, obviously $z_m < \frac{1}{1+\eta}$, or equivalently $m < \frac{\nu}{a\eta}$, and $(1 - z_m)\mu(1 + \eta) > \mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m)$. Thus

$$\begin{aligned} \eta_m^{(j)} &= \frac{((1 - z_m)\mu + z_m \mathbb{E}[Y^{(j)}])(1 + \eta)}{\mathbb{E}[Y^{(j)}]} - 1 \\ &> \frac{\mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m) + z_m \mathbb{E}[Y^{(j)}](1 + \eta)}{\mathbb{E}[Y^{(j)}]} - 1 \\ &= (1 - (1 + \eta)z_m) + z_m(1 + \eta) - 1 = 0. \end{aligned}$$

(\Leftarrow): Since $j \in \bar{b}_m$, we have $((1 - z_m)\mu + z_m \mathbb{E}[Y^{(j)}])(1 + \eta) > \mathbb{E}[Y^{(j)}]$. Thus, $(1 - z_m)\mu(1 + \eta) > \mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m)$. Now, since $m < \frac{\nu}{a\eta}$, $z_m < \frac{1}{1+\eta}$. So, $\tilde{\eta}_m^{(j)} < \infty$ and $\mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m) > 0$. Consequently

$$\infty > \tilde{\eta}_m^{(j)} = \frac{(1 - z_m)\mu(1 + \eta)}{\mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m)} - 1 > \frac{\mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m)}{\mathbb{E}[Y^{(j)}](1 - (1 + \eta)z_m)} - 1 = 0.$$

□

This result highlights the obvious fact that the existence of $\rho_m^{(j)}$ implies that j must belong to \bar{b}_m , since $\psi_m^{(j)}(u) = 1$ for $j \in b_m$.

Let us explain the reason why the horizon of credibility $\lceil \frac{\nu}{a\eta} \rceil$, denoted by $m^{(c)}$ from now, constitutes a critical value for the existence of the Lundberg coefficient and consequently for the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$.

From $m = m^{(c)}$, the global impact of each $Y_k^{(j)}$ on the surplus process for $k > m$, i.e. $Y_k^{(j)}(1 - (1 + \eta)z_m)$, becomes negative, which is, of course, a drastic change in the nature of the insured risk. Also, $b_m = \emptyset$ for all $m \geq m^{(c)}$.

More fundamentally, in first order, the only effects selected of the credibility dynamics compared to the classical case $m = 0$ are a modification of the security loading and a decreasing of the claim amounts. Indeed, equations (3.4) and (3.5) teach us that $\rho_m^{(j)}$ is identical to that of the classical ruin model with a premium security loading equals to $\tilde{\eta}_m^{(j)}$ and annual claim amounts given by $Y_k^{(j)}(1 - (1 + \eta)z_m)$. Therefore, in this associated classical ruin model, from the critical value $m^{(c)}$, no ruin can be observed, which explains the reason why a Lundberg coefficient does not exist in this case.

So, since $\mathbb{E}[e^{rY^{(j)}}]$ exists for all $0 < r < r^{(j)}$ with $\lim_{r \rightarrow r^{(j)}} \mathbb{E}[e^{rY^{(j)}}] = \infty$, it comes that the two following conditions are of course necessary but also sufficient conditions to guarantee the existence and the uniqueness of $\rho_m^{(j)}$:

(C1) $m < m^{(c)}$,

(C2) $j \in \bar{b}_m$.

Let us recall that result (3.2) requires not only the existence and the uniqueness of $\rho_m^{(j)}$ but also the additional condition $\rho_m^{(j)} < r^{(j)}$. Firstly, we have the following property for the Lundberg coefficient $\rho_m^{(j)}$:

Property 3.2. *Assume that $\rho_{m_1}^{(j)}$ and $\rho_{m_2}^{(j)}$ exist, with $0 \leq m_1 < m_2$. Then, $\rho_{m_1}^{(j)} < \rho_{m_2}^{(j)}$.*

Proof. Since $z_{m_1} < z_{m_2}$, we have $\tilde{\eta}_{m_1}^{(j)} < \tilde{\eta}_{m_2}^{(j)}$ and consequently, by classical results, $\tilde{\rho}_{m_1}^{(j)} < \tilde{\rho}_{m_2}^{(j)}$. By (3.5), we deduce $\rho_{m_1}^{(j)} < \rho_{m_2}^{(j)}$. \square

Secondly, let us define

$$\bar{m}^{(j)} = \min\{m < m^{(c)} : \rho_m^{(j)} \text{ exist and } \rho_m^{(j)} \geq r^{(j)}\}. \quad (3.8)$$

If $\rho_m^{(j)}$ does not exist for all $m < m^{(c)}$, then we put $\bar{m}^{(j)} = 0$. Also, if for all $m < m^{(c)}$ such that $\rho_m^{(j)}$ exists, we always have $\rho_m^{(j)} < r^{(j)}$, then we put $\bar{m}^{(j)} = m^{(c)}$. Hence, if we replace condition (C1) by the stronger condition $m < \bar{m}^{(j)}$, we have now necessary and sufficient conditions to guarantee the existence and the uniqueness of $\rho_m^{(j)}$ such that $\rho_m^{(j)} < r^{(j)}$, i.e. result (3.2).

3.1.3 Large deviations path to the ruin

Let us assume that $m < \bar{m}^{(j)}$ and $j \in \bar{b}_m$, i.e. that result (3.2) is valid. The goal here is to understand how ruin occurs for large u . More particularly, we want to determine the typical shape of a path leading to ruin when the initial capital of the company is large. To this end, let us use large deviations results of Glynn and Whitt (1994). First of all, for large u , we know that ruin occurs roughly at time $\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor$. Next, the cumulant generating function $\gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor}(r)$ of $Z_{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor}$ is asymptotically changed from $\gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor}(r)$ to

$$\begin{aligned} \gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor}(r + \rho_m^{(j)}) - \gamma_{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor}(\rho_m^{(j)}) &= -r(1 + \eta)\mu \sum_{i=1}^{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor} (1 - z_{i,m}) \\ &+ \sum_{i=1}^{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor} \ln \frac{\mathbb{E} \left[e^{(r + \rho_m^{(j)})Y^{(j)} \left(1 - (\bar{z}_{i+1,m} + \bar{z}_{i+2,m} + \dots + \bar{z}_{\min(i+m, \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor), m)} \right)^{(1+\eta)} \right]}{\mathbb{E} \left[e^{\rho_m^{(j)}Y^{(j)} \left(1 - (\bar{z}_{i+1,m} + \bar{z}_{i+2,m} + \dots + \bar{z}_{\min(i+m, \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor), m)} \right)^{(1+\eta)} \right]}. \end{aligned} \quad (3.9)$$

Consequently, given that ruin occurs, the claim size distribution $Y_k^{(j)}$ is exponentially tilted by the time-dependent factor $\rho_m^{(j)}(k)$, with

$$\rho_m^{(j)}(k) = \begin{cases} \rho_m^{(j)}, & \text{for } k = 1, \dots, \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor - m \\ \left(1 - \left(\frac{\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor - k}{m} \right) z_m(1 + \eta) \right) \rho_m^{(j)}, & \text{for } k = \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor - m + 1, \dots, \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor \end{cases}.$$

Hence, for large u , the drift of the path to ruin at time k ($k = 1, \dots, \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor$), denoted as $\delta_m(k)$, is given by

$$\delta_m(k) = (1 - z_m)\mu(1 + \eta) - (1 - (1 + \eta)z_m) \frac{\mathbb{E} \left[Y^{(j)} e^{\rho_m^{(j)}(k) Y^{(j)}} \right]}{\mathbb{E} \left[e^{\rho_m^{(j)}(k) Y^{(j)}} \right]} < 0. \quad (3.10)$$

However, notice that since $u \rightarrow \infty$, the time period $(\lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor - m + 1, \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor)$ where $\delta_m(k)$ decreases with k is negligible compared with the time period $(1, \lfloor u/\kappa'_m(\rho_m^{(j)}) \rfloor - m)$ where $\delta_m(k)$ remains constant.

This analysis highlights that ruin is caused by a succession of small claims, followed by a succession of moderately large claims.

3.1.4 Upper bound on $\psi_m^{(j)}(u)$

In this subsection only, we take into account a possible claim experience of h years say, i.e. $(Y_{-h+1}^{(j)}, Y_{-h+2}^{(j)}, \dots, Y_0^{(j)})$, because, this time, as we see in the following, it will impact the

result. Hence, equation (2.6) becomes

$$C_{k,m} = \left((1 - z_{k,m})\mu + z_{k,m} \frac{\sum_{i=\max(k-m,1-h)}^{k-1} Y_i}{\min(m, h+k-1)} \right) (1 + \eta), \quad (3.11)$$

with

$$z_{k,m} = \frac{\min(m, h+k-1) a}{\nu + \min(m, h+k-1) a}. \quad (3.12)$$

Let $b = \min(m, h)$, and let us denote

$$\psi_m^{(j)}(u, y_0, \dots, y_{1-b}) = \Pr \left[U_{k,m}^{(j)} < 0 \text{ for some } k \mid U_{0,m}^{(j)} = u, Y_0^{(j)} = y_0, \dots, Y_{1-b}^{(j)} = y_{1-b} \right]. \quad (3.13)$$

Theorem 3.3. *Assume that $m < \bar{m}^{(j)}$ and $j \in \bar{b}_m$. Then*

$$\psi_m^{(j)}(u, y_0, \dots, y_{1-b}) \leq \frac{e^{-\rho_m^{(j)} \hat{u}}}{\mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_T} \mid T < \infty \right]}, \quad (3.14)$$

where $\hat{U}_k = U_{k,m}^{(j)} + (1 + \eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m-1,m}) + \dots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{m+\max(k-m+1,1-h),m})]$, $\hat{u} = \hat{U}_0$, and $T = \inf\{k \mid U_{k,m}^{(j)} < 0\}$, with $\bar{z}_{k,m} = \frac{z_{k,m}}{\min(m, h+k-1)}$.

Proof. First, we note that

$$\begin{aligned} \hat{U}_k &= U_{k,m}^{(j)} + (1 + \eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m-1,m}) \\ &\quad + \dots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{m+\max(k-m+1,1-h),m})] \\ &= U_{k-1,m}^{(j)} + C_{k,m} - Y_k^{(j)} \\ &\quad + (1 + \eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m-1,m}) \\ &\quad + \dots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{m+\max(k-m+1,1-h),m})] \\ &= U_{k-1,m}^{(j)} + (1 - z_{k,m})\mu(1 + \eta) + z_{k,m}(1 + \eta) \frac{\sum_{i=\max(k-m,1-h)}^{k-1} Y_i^{(j)}}{\min(m, k+h-1)} - Y_k^{(j)} \\ &\quad + (1 + \eta)[Y_k^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m,m}) + Y_{k-1}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m-1,m}) \\ &\quad + \dots + Y_{\max(k-m+1,1-h)}^{(j)}(\bar{z}_{k+1,m} + \dots + \bar{z}_{m+\max(k-m+1,1-h),m})] \\ &= U_{k-1,m}^{(j)} + (1 + \eta)[Y_{k-1}^{(j)}(\bar{z}_{k,m} + \dots + \bar{z}_{k+m-1,m}) + Y_{k-2}^{(j)}(\bar{z}_{k,m} + \dots + \bar{z}_{k+m-2,m}) \\ &\quad + \dots + Y_{\max(k-m,1-h)}^{(j)}(\bar{z}_{k,m} + \dots + \bar{z}_{m+\max(k-m,1-h),m})] \\ &\quad + (1 - z_{k,m})\mu(1 + \eta) - Y_k^{(j)}(1 - (1 + \eta)(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m,m})) \\ &= \hat{U}_{k-1} + (1 - z_{k,m})\mu(1 + \eta) - Y_k^{(j)}(1 - (1 + \eta)(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m,m})). \end{aligned}$$

Hence, if we denote by Φ_k the sigma-algebra generated by $\{Y_i^{(j)}, i = 1, 2, \dots, k\}$, we have

$$\mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_k} \mid \Phi_{k-1} \right] = e^{-\rho_m^{(j)} \hat{U}_{k-1}} e^{-\rho_m^{(j)} (1 - z_{k,m})\mu(1 + \eta)} \mathbb{E} \left[e^{\rho_m^{(j)} Y^{(j)}(1 - (1 + \eta)(\bar{z}_{k+1,m} + \dots + \bar{z}_{k+m,m}))} \right].$$

Now, let us define

$$h_{k,m}(r) = e^{-r(1-z_{k,m})\mu(1+\eta)} \mathbb{E} \left[e^{rY^{(j)}(1-(1+\eta)(\bar{z}_{k+1,m}+\dots+\bar{z}_{k+m,m}))} \right].$$

Obviously, for $k \geq m - h + 1$, $h_{k,m}(r) = h_m(r)$. Since $z_{k-1,m} \leq z_{k,m}$ and $\bar{z}_{k-1,m} \geq \bar{z}_{k,m}$, we clearly have $h_{k-1,m}(r) \leq h_{k,m}(r) \leq h_m(r)$. Consequently, by (3.1), $h_{k,m}(\rho_m^{(j)}) \leq h_m(\rho_m^{(j)}) = 1$. Thus, we get

$$\mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_k} | \Phi_{k-1} \right] = e^{-\rho_m^{(j)} \hat{U}_{k-1}} h_k(\rho_m^{(j)}) \leq e^{-\rho_m^{(j)} \hat{U}_{k-1}}.$$

So, the process $\{e^{-\rho_m^{(j)} \hat{U}_k}\}$ is a super-martingale.

Let w be a positive integer. By the Optional Stopping Theorem (considering the stopping time $T \wedge w = \min\{T, w\}$), we get

$$e^{-\rho_m^{(j)} \hat{U}_0} \geq \mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_{T \wedge w}} \right] = \mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_T} I_{T \leq w} \right] + \mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_w} I_{T > w} \right] \geq \mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_T} I_{T \leq w} \right].$$

Hence letting $w \rightarrow \infty$, we have

$$e^{-\rho_m^{(j)} \hat{U}_0} \geq \mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_T} | T < \infty \right] \Pr[T < \infty] = \mathbb{E} \left[e^{-\rho_m^{(j)} \hat{U}_T} | T < \infty \right] \psi_m^{(j)}(u, y_0, \dots, y_{1-b}).$$

□

Remark Inequality (3.14) becomes an equality when $h \geq m$. Indeed, in this case, for $k \geq 1$, we have $h_{k,m}(\rho_m^{(j)}) = h_m(\rho_m^{(j)}) = 1$. Thus, the process $\{e^{-\rho_m^{(j)} \hat{U}_k}\}$ is a martingale. Furthermore, the random variables $e^{-\rho_m^{(j)} \hat{U}_w} I_{T > w}$ pointwise converge to zero for $w \rightarrow \infty$ and are bounded by 1 since $\hat{U}_w > 0$ for $w < T$.

3.1.5 The case $m \geq \bar{m}^{(j)}$ and $j \in \bar{b}_m$

Now, let us consider the case $m \geq \bar{m}^{(j)}$ and $j \in \bar{b}_m$. For the next result, let us denote $\psi_m^{(j)}(u)$ by $\psi_{m,\eta}^{(j)}(u)$, $\rho_m^{(j)}$ by $\rho_{m,\eta}^{(j)}$, $\tilde{\rho}_m^{(j)}$ by $\tilde{\rho}_{m,\eta}^{(j)}$, $\tilde{\eta}_m^{(j)}$ by $\tilde{\eta}_{m,\eta}^{(j)}$ and $\eta_m^{(j)}$ by $\eta_{m,\eta}^{(j)}$ to reveal explicitly the dependence in η . Obviously, notice that $\tilde{\eta}_{m,\eta}^{(j)}$ increases with η . Hence, we know by classical results that $\tilde{\rho}_{m,\eta}^{(j)}$ also increases with η and consequently $\rho_{m,\eta}^{(j)}$ by (3.5). Furthermore, let us define

$$\eta^{(\star,1)} = \min \left\{ \tilde{\eta} \leq \eta : \eta_{m,\tilde{\eta}}^{(j)} > 0 \text{ and } m < \frac{\nu}{a\tilde{\eta}} \right\}. \quad (3.15)$$

and

$$\eta^{(\star,2)} = \max \left\{ \tilde{\eta} \leq \eta : \eta_{m,\tilde{\eta}}^{(j)} > 0 \text{ and } m < \frac{\nu}{a\tilde{\eta}} \right\}. \quad (3.16)$$

Proposition 3.2. *Assume that $\rho_{m,\eta^{(\star,1)}}^{(j)} < r^{(j)} \leq \rho_{m,\eta^{(\star,2)}}^{(j)}$. Then, we have*

$$\limsup_{u \rightarrow \infty} \frac{1}{u} \ln \psi_{m,\eta}^{(j)}(u) \leq -r^{(j)}. \quad (3.17)$$

Proof. Obviously, there exists $\eta^{(\star,1)} < \bar{\eta} \leq \eta^{(\star,2)}$ such that $\rho_{m,\bar{\eta}}^{(j)} = r^{(j)}$. For all $\eta^{(\star,1)} \leq \tilde{\eta} < \bar{\eta}$, we have

$$\limsup_{u \rightarrow \infty} \frac{1}{u} \ln \psi_{m,\tilde{\eta}}^{(j)}(u) \leq \lim_{u \rightarrow \infty} \frac{1}{u} \ln \psi_{m,\bar{\eta}}^{(j)}(u) = -\rho_{m,\bar{\eta}}^{(j)}.$$

Letting $\tilde{\eta} \uparrow \bar{\eta}$, we obtain (3.17). \square

As a particular, let us assume that $Y^{(j)} \sim \text{Exp}(\lambda_j)$. In this case, $\mathbb{E}[e^{rY^{(j)}}]$ exists all for $r < \lambda_j$ and is equal to $\frac{\lambda_j}{\lambda_j - r}$. Clearly, $r^{(j)} = \lambda_j$. Now, since

$$\lim_{u \rightarrow \infty} \frac{1}{u} \ln \Pr[U_{1,m}^{(j)} < 0] = \lambda_j \quad (3.18)$$

and that $\psi_m^{(j)}(u) \geq \Pr[U_{1,m}^{(j)} < 0]$, result (3.17) becomes

$$\liminf_{u \rightarrow \infty} \frac{1}{u} \ln \psi_m^{(j)}(u) = \limsup_{u \rightarrow \infty} \frac{1}{u} \ln \psi_m^{(j)}(u) = -\lambda_j. \quad (3.19)$$

3.1.6 Conclusion

In conclusion, on the one hand, if $j \in b_m$ then $\psi_m^{(j)}(u) = 1$ for all u . On the other hand, if $j \in \bar{b}_m$, then for $m < \bar{m}^{(j)}$, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is given by equation (3.2) while for $(\infty) > m \geq \bar{m}^{(j)}$, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is this time characterized by equation (3.17).

3.2 The case $m = \infty$

Clearly, with $z_m = 1$, i.e. $m = \infty$, the results of the previous section have no meaning. This is the reason why this important case has to be treated separately.

3.2.1 Asymptotic behavior of $\psi_\infty^{(j)}(u)$

Let us state an analogue of Theorem 3.1.

Theorem 3.4. *Suppose that there exists a unique positive solution to the equation*

$$\int_0^1 dx \ln \mathbb{E} \left[e^{rY^{(j)}(1+(1+\eta) \ln x)} \right] = 0, \quad (3.20)$$

which we denote $\rho_\infty^{(j)}$. If we assume that for all k , $\mathbb{E}[e^{rZ_k}]$ exists for $0 < r < r_0$ (with $\rho_\infty^{(j)} < r_0$), where $Z_k = \sum_{i=1}^k (Y_i^{(j)} - C_{i,\infty})$, we have

$$\lim_{u \rightarrow \infty} \frac{1}{u} \ln \psi_\infty^{(j)}(u) = -\rho_\infty^{(j)}. \quad (3.21)$$

Proof. Again, it suffices to show that $\kappa_\infty(r) := \lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}]$ exists for $0 < r < r_0$, and that $\rho_\infty^{(j)}$ is the unique positive value such that $\kappa_\infty(\rho_\infty^{(j)}) = 0$. We have

$$\ln \mathbb{E}[e^{rZ_k}] = -r(1 + \eta)\mu \sum_{i=1}^k (1 - z_{i,\infty}) + \ln \mathbb{E} \left[e^{r \sum_{i=1}^k \left(Y_k^{(j)} - (1+\eta)z_{i,\infty} \frac{\sum_{l=1}^{i-1} Y_l^{(j)}}{i-1} \right)} \right].$$

Now we have

$$\sum_{i=1}^k \left(Y_k^{(j)} - (1 + \eta)z_{i,\infty} \frac{\sum_{l=1}^{i-1} Y_l^{(j)}}{i-1} \right) = \sum_{i=1}^k Y_i^{(j)} \left(1 - (1 + \eta) \sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} \right).$$

Hence we get

$$\ln \mathbb{E}[e^{rZ_k}] = -r(1 + \eta)\mu \sum_{i=1}^k (1 - z_{i,\infty}) + \ln \mathbb{E} \left[e^{r \sum_{i=1}^k Y_i^{(j)} \left(1 - (1+\eta) \sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} \right)} \right].$$

Thus, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}] &= -r(1 + \eta)\mu \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (1 - z_{i,\infty}) \\ &\quad + \lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E} \left[e^{r \sum_{i=1}^k Y_i^{(j)} \left(1 - (1+\eta) \sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} \right)} \right]. \end{aligned}$$

It is obvious that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (1 - z_{i,\infty}) = 0.$$

Furthermore, note that

$$\sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} = \sum_{l=i}^{k-1} \frac{a}{al + \nu}.$$

Since we have

$$\int_l^{l+1} \frac{a}{al + \nu} dt \leq \frac{a}{al + \nu} \leq \int_{l-1}^l \frac{a}{al + \nu} dt,$$

we find that

$$\ln \left(\frac{ak + \nu}{ai + \nu} \right) \leq \sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} \leq \ln \left(\frac{a(k-1) + \nu}{a(i-1) + \nu} \right).$$

Consequently, letting $k \rightarrow \infty$, we get

$$-\sum_{l=i}^{k-1} \frac{z_{l+1,\infty}}{l} = \ln \left(\frac{ai}{ak + \nu} \right).$$

So, we obtain

$$\lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}] = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E} \left[e^{r \sum_{i=1}^k Y_i^{(j)} \left(1 + (1+\eta) \ln \left(\frac{ai}{ak + \nu} \right) \right)} \right].$$

Hence we have

$$\begin{aligned}
\lim_{k \rightarrow \infty} \frac{1}{k} \ln \mathbb{E}[e^{rZ_k}] &= \lim_{k \rightarrow \infty} \frac{1}{k} \left(\frac{ak + \nu}{a} \right) \sum_{i=1}^k \left(\frac{a}{ak + \nu} \right) \ln \mathbb{E} \left[e^{rY^{(j)}(1+(1+\eta)\ln(\frac{ai}{ak+\nu}))} \right] \\
&= \lim_{k \rightarrow \infty} \sum_{i=1}^k \left(\frac{a}{ak + \nu} \right) \ln \mathbb{E}[e^{rY^{(j)}(1+(1+\eta)\ln(\frac{ai}{ak+\nu}))}] \\
&= \int_0^1 dx \ln \mathbb{E} \left[e^{rY^{(j)}(1+(1+\eta)\ln x)} \right].
\end{aligned}$$

□

It appears that result (3.21) requires only the existence and the uniqueness of $\rho_\infty^{(j)}$ (since the condition $\rho_\infty^{(j)} < r^{(j)}$ is always fulfilled).

Equation (3.20) and so $\rho_\infty^{(j)}$ does not depend on market parameters μ , a and ν . It means that for $u \rightarrow \infty$, the logarithmic asymptotic behavior of $\psi_m^{(j)}(u)$ is insensitive to the initial premium and to the speed with which premiums $C_{k,\infty}$ become $\frac{Y_1^{(j)} + \dots + Y_{k-1}^{(j)}}{k-1}(1+\eta)$.

As a particular case, let us consider $Y^{(j)} \sim \sum_{i=1}^N U_i$, where N is Poisson distributed with parameter β and the U_i 's are i.i.d. random variables. We assume that N and the U_i 's are independent. We have

$$\begin{aligned}
\mathbb{E}[e^{rY^{(j)}}] &= \mathbb{E} \left[e^{r \sum_{i=1}^N U_i} \right] = \mathbb{E} \left[\mathbb{E} \left[e^{r \sum_{i=1}^N U_i} | N \right] \right] = \sum_{k=0}^{\infty} \mathbb{E} \left[e^{r \sum_{i=1}^k U_i} \right] \Pr[N = k] \\
&= \sum_{k=0}^{\infty} \mathbb{E} \left[e^{rU} \right]^k \Pr[N = k] = \sum_{k=0}^{\infty} e^{k \ln \mathbb{E}[e^{rU}]} \Pr[N = k] = \sum_{k=0}^{\infty} e^{k \ln \mathbb{E}[e^{rU}]} \Pr[N = k] \\
&= \mathbb{E} \left[e^{N \ln \mathbb{E}[e^{rU}]} \right] = e^{\beta(\mathbb{E}[e^{rU}] - 1)}
\end{aligned} \tag{3.22}$$

since $\mathbb{E}[e^{rN}] = e^{\beta(e^r - 1)}$. So, the equation (3.20) becomes

$$\int_0^1 dx \ln \mathbb{E} \left[e^{rY^{(j)}(1+(1+\eta)\ln x)} \right] = \beta \int_0^1 dx \mathbb{E} \left[e^{rU(1+(1+\eta)\ln x)} \right] - \beta = 0. \tag{3.23}$$

We recognize the equation (2.2) in Asmussen (1999). In fact, this is not surprising since $\rho_\infty^{(j)}$ is insensitive to market parameters μ , a and ν .

3.2.2 The Lundberg coefficient $\rho_\infty^{(j)}$

Since $m = \infty$, we know that $\eta_m^{(j)} = \eta > 0$ for all j , i.e. $b_m = \emptyset$.

Proposition 3.3. *Assume that $\kappa_\infty''(r) > 0$. Then, $\rho_\infty^{(j)}$ exists and is unique.*

Proof. Obviously, we have $\kappa_\infty(0) = 0$. Denoting $Y^{(j)}(1 + (1 + \eta) \ln x)$ by Z_x , we have

$$\begin{aligned}\kappa'_\infty(r) &= \int_0^1 dx \frac{\mathbb{E}[Z_x e^{rZ_x}]}{\mathbb{E}[e^{rZ_x}]} = \int_0^1 dx \frac{\mathbb{E}[Z_x e^{rZ_x} I_{Z_x > 0}] + \mathbb{E}[Z_x e^{rZ_x} I_{Z_x \leq 0}]}{\mathbb{E}[e^{rZ_x}]} \\ &\geq \int_0^1 dx \frac{\mathbb{E}[Z_x e^{rZ_x} I_{Z_x > 0}] + \mathbb{E}[Z_x I_{Z_x \leq 0}]}{\mathbb{E}[e^{rZ_x}]}.\end{aligned}$$

Hence we get

$$\kappa'_\infty(0) = \mathbb{E}[Y^{(j)}] \left(1 + (1 + \eta) \int_0^1 dx \ln x\right) = -\mathbb{E}[Y^{(j)}] \eta < 0$$

since $\eta > 0$, and $\kappa'_\infty(r)$ becomes positive for large r . Now, since $\kappa''_\infty(r) > 0$ by assumption, the proof is over. \square

3.2.3 Large deviations path to the ruin

As previously, we aim at knowing how ruin occurs for the case $m = \infty$. Of course, the ruin occurs roughly at time $\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor$. Now, the cumulant generating function $\gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor}(r)$ of $Z_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor}$ is asymptotically changed from $\gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor}(r)$ to

$$\begin{aligned}\gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor}(r + \rho_\infty^{(j)}) - \gamma_{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor}(\rho_\infty^{(j)}) &= -r(1 + \eta) \mu \sum_{i=1}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor} (1 - z_{i,\infty}) \\ &+ \sum_{i=1}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor} \ln \frac{\mathbb{E} \left[e^{(r + \rho_\infty^{(j)}) Y_i^{(j)} \left(1 - (1 + \eta) \sum_{l=i}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor - 1} \frac{z_{l+1,\infty}}{l}\right)} \right]}{\mathbb{E} \left[e^{\rho_\infty^{(j)} Y_i^{(j)} \left(1 - (1 + \eta) \sum_{l=i}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor - 1} \frac{z_{l+1,\infty}}{l}\right)} \right]}.\end{aligned}\tag{3.24}$$

Consequently, given that ruin occurs, the claim size distribution $Y_k^{(j)}$ is this time exponentially tilted by the factor $\rho_\infty^{(j)}(k)$, where

$$\rho_\infty^{(j)}(k) = \left(1 - (1 + \eta) \sum_{l=k}^{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor - 1} \frac{z_{l+1,\infty}}{l}\right) \rho_\infty^{(j)}.\tag{3.25}$$

Since $u \rightarrow \infty$, we get

$$\rho_\infty^{(j)}(k) = \left(1 + (1 + \eta) \ln \left(\frac{k}{\lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor}\right)\right) \rho_\infty^{(j)}.\tag{3.26}$$

The factor $\rho_\infty^{(j)}(k)$ is negative as long as $k < \lfloor u/\kappa'_\infty(\rho_\infty^{(j)}) \rfloor e^{-\frac{1}{1+\eta}}$.

We learn that this time, ruin is the consequence of atypically small claim amounts in a first time, followed by atypically large claim amounts in a second time. This result is similar to those obtained in Asmussen (1999) for his continuous time compound Poisson model. A typical path to ruin is shown in Figure 3.1.

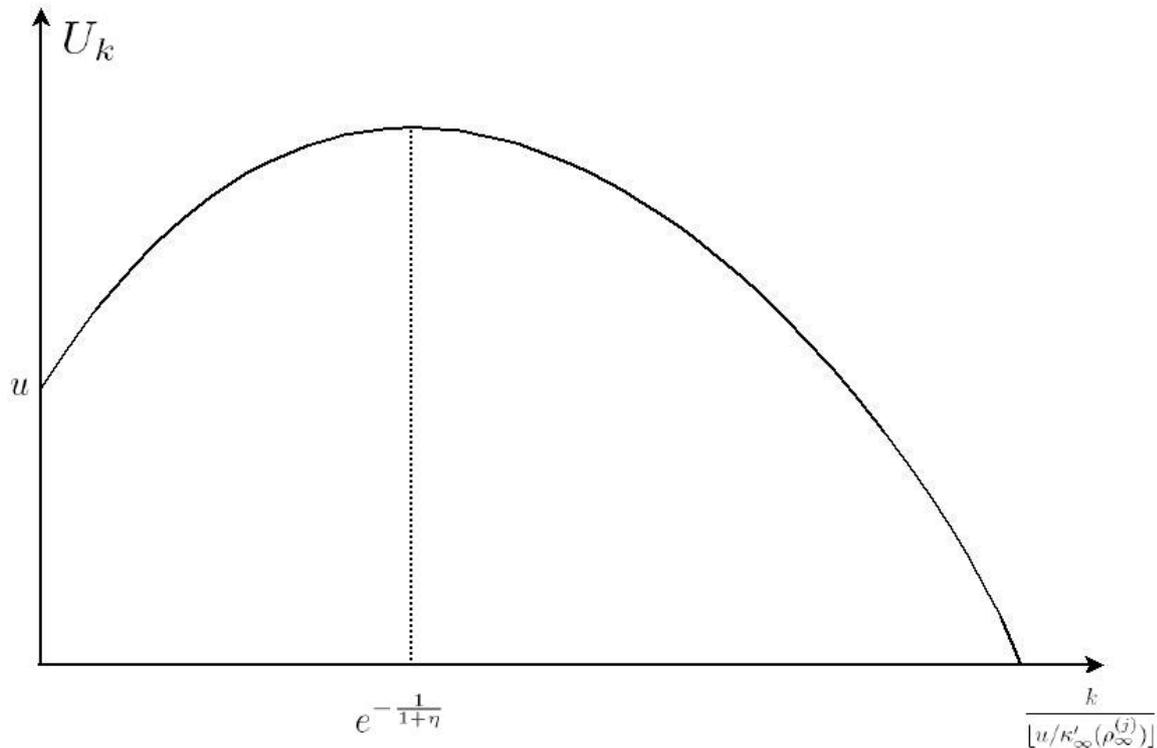


Figure 3.1:

3.3 Impact of the insurer's strategy m on $\psi_m^{(j)}(u)$ for large u

3.3.1 Comparison of $\psi_m^{(j)}(u)$ and $\psi_{m+1}^{(j)}(u)$ for large u (with $m < \infty$)

Let us assume that the insurer's strategy is to take an horizon of credibility $m < \infty$. The objective is to determine for large u if it is beneficial to take one year more to renew the premium. In other words, the objective is to compare $\psi_m^{(j)}(u)$ and $\psi_{m+1}^{(j)}(u)$ for large u .

For $j \in b_m$, we have the following Proposition:

Proposition 3.4.

$$\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u) = 1 \text{ for all } u. \quad (3.27)$$

Proof. It suffices to show that $b_{m+1} \subseteq b_m$. Let $j \in b_{m+1}$. By definition, $\eta_{m+1}^{(j)} \leq 0$. Let $h(x) = (1-x)\mu + x\mathbb{E}[Y^{(j)}]$. We have $h'(x) = \mathbb{E}[Y^{(j)}] - \mu > 0$ since $\eta_{m+1}^{(j)} \leq 0$ means $\mathbb{E}[Y^{(j)}] \geq (1+\eta)((1-z_{m+1})\mu + z_{m+1}\mathbb{E}[Y^{(j)}])$ and $\eta > 0$. Thus, since $z_{m+1} > z_m$, we have $\eta_m^{(j)} = (1+\eta)\frac{h(z_m)}{\mathbb{E}[Y^{(j)}]} - 1 < (1+\eta)\frac{h(z_{m+1})}{\mathbb{E}[Y^{(j)}]} - 1 = \eta_{m+1}^{(j)} \leq 0$, or equivalently, $j \in b_m$. \square

This means that if $j \in b_m$, it is beneficial to pass from an horizon of credibility m to an horizon $m+1$, whatever the initial capital u .

Now, for $j \in \bar{b}_m$, it could seem a priori that the comparison of $\psi_m^{(j)}(u)$ and $\psi_{m+1}^{(j)}(u)$ for large u differs according to the involved portfolio j . Indeed, two opposite a priori behaviours seem to take shape. If $j \in g$, then $h'(x) \leq 0$ and consequently $\eta_m^{(j)} \geq \eta_{m+1}^{(j)}$, which seems to be in favor of $\psi_m^{(j)}(u) \leq \psi_{m+1}^{(j)}(u)$ whatever u and in particular for large u . But, on the other hand, if $j \in \bar{b}_m \setminus g$, then $h'(x) > 0$ and consequently $\eta_m^{(j)} < \eta_{m+1}^{(j)}$, which seems this time to be in favor of the inequality $\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u)$ whatever u .

For $m < \bar{m}^{(j)}$, we have the next Proposition:

Proposition 3.5. *There exists a positive constant u_0 such that for all $u \geq u_0$, we have*

$$\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u). \quad (3.28)$$

Proof. We know that $j \in \bar{b}_m$ and $m < \bar{m}^{(j)}$. Since $b_{m+1} \subseteq b_m$, $j \in \bar{b}_{m+1}$. Hence, we have to consider two cases.

Firstly, let us assume that $m+1 < \bar{m}^{(j)}$. By Property 3.2, our result is proved.

Secondly, let us assume that $m+1 \geq \bar{m}^{(j)}$. Hence, we know that

$$\lim_{u \rightarrow \infty} \frac{1}{u} \ln \psi_m^{(j)}(u) = -\rho_m^{(j)}$$

and

$$\limsup_{u \rightarrow \infty} \frac{1}{u} \ln \psi_{m+1}^{(j)}(u) \leq -r^{(j)},$$

and since $\rho_m^{(j)} < r^{(j)}$, this completes the proof. \square

Thus, even if $j \in g$ and hence $\eta_m^{(j)} \geq \eta_{m+1}^{(j)}$, (3.28) holds for large u . In fact, this is not surprising. With light-tailed claims, the event ruin occurs after many atypically large claims. At most m is large, at most the premium reacts favorably to this large claims, i.e. $\eta_{m+1}^{(j)}$ becomes larger than $\eta_m^{(j)}$ in such circumstances. This facts are confirmed by our analysis of the path leading to ruin. Indeed, it teaches us that the ruin is caused by a succession of many atypically large claims more dangerous as m is large since $\tilde{\rho}_m^{(j)}$ increases with m .

For $m \geq \bar{m}^{(j)}$, we can nothing conclude from this kind, since the only result we have is equation (3.17).

In conclusion, assuming that the strategy of the company is m , if $j \in b_m$, then it is always beneficial for large u to increase by 1 the horizon of credibility. Furthermore, if $j \in \bar{b}_m$, for $m < \bar{m}^{(j)}$, the same conclusion comes while for $m \geq \bar{m}^{(j)}$, we cannot certify that this is also the case.

3.3.2 Sub-optimal strategies m for large u

Let us investigate sub-optimal strategies m for large u .

Definition 3.5. A strategy m_1 is said to be sub-optimal compared to a strategy m_2 if for all portfolio j , we have $\psi_{m_2}^{(j)}(u) \leq \psi_{m_1}^{(j)}(u)$ for $u \rightarrow \infty$.

Definition 3.6. A strategy m_1 is said to be sub-optimal if there exists a strategy m_2 such that m_1 is sub-optimal compared to m_2 .

We have the next Proposition:

Proposition 3.6. $m < \min_{\{j:\bar{m}^{(j)}>0\}} \bar{m}^{(j)}$ like $m = \infty$ are sub-optimal.

Proof. In view of Propositions 3.4 and 3.5, it is clear that $m < \min_{\{j:\bar{m}^{(j)}>0\}} \bar{m}^{(j)}$ is sub-optimal. For $m = \infty$, it is also obvious since $\rho_\infty^{(j)} < r^{(j)}$. \square

As we have seen, for a given portfolio j , increasing m decreases the number of "bad portfolios" and provides better premium adaptation to scenarios leading to ruin, at least as long as $m < \bar{m}^{(j)}$. An extension of this reasoning to all m would allow to think that all $m < \infty$ is sub-optimal compared to $m = \infty$. However, Proposition 3.6 shows that this is not the case. In fact, as $u \rightarrow \infty$, all $m < \infty$ becomes negligible compared to the ruin time T . Hence, the premiums will be able to react "quickly" (relatively to T) and "significantly" (at least for m "large" but finite) to a succession of large claim amounts, whatever the time where this claims occur and whatever the claim amounts observed before. But for $m = \infty$, obviously, this is not the case anymore. Indeed, as time passes, the premium reaction will become less and less effective to finish by not be able anymore to thwart several important successive claim amounts. This phenomenon is even truer if the claim amounts observed initially are small. This is well highlighted by the analysis of the path leading to ruin.

A probable extension of Proposition 3.6 would be then the following: $m = \infty$ is sub-optimal and all $m < \infty$ is sub-optimal compared to $m + 1$.

3.3.3 Numerical illustration

Let us consider an insurance market made up of 3 portfolios, i.e. $n = 3$. Let us assume that $Y^{(j)}$ ($j = 1, 2, 3$) are Exponentially distributed, with mean $1/\lambda_j$ and variance $1/\lambda_j^2$. We then

have $\mu = (1/\lambda_1)p_1 + (1/\lambda_2)p_2 + (1/\lambda_3)p_3$, $a = (1/\lambda_1 - \mu)^2 p_1 + (1/\lambda_2 - \mu)^2 p_2 + (1/\lambda_3 - \mu)^2 p_3$ and $\nu = (1/\lambda_1^2)p_1 + (1/\lambda_2^2)p_2 + (1/\lambda_3^2)p_3$. Let us assume that $1/\lambda_1 = 3/4$, $1/\lambda_2 = 1$, $1/\lambda_3 = 5/4$, $p_1 = p_2 = 1/3$ and $\eta = 0.1$. We have $j = 1, 2 \in g$ and $j = 3 \in b_m$ as long as $m < 30$. The critical horizon of credibility $m^{(c)} = 250$. For our numerical purpose, let us compare strategies $m = 0, 2, 10, 250, 1000$ and ∞ . Of course, for all u , $\psi_m^{(3)}(u) = 1$ for $m = 0, 2$ and 10 . Carrying out 100 000 simulations on an horizon of 10 000 time periods, we obtain the results below, also shown on Figures 4.1 to 4.3.

		$\psi_m^{(1)}(u)$									
u	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$m = 0$	43.758%	29.996%	20.566%	14.086%	9.844%	6.754%	4.604%	3.218%	2.268%	1.566%	1.120%
$m = 2$	44.438%	30.316%	20.572%	13.850%	9.474%	6.352%	4.232%	2.866%	1.952%	1.326%	0.890%
$m = 10$	45.892%	31.502%	21.348%	14.278%	9.568%	6.308%	4.072%	2.696%	1.772%	1.124%	0.724%
$m = 250$	46.936%	32.404%	22.034%	14.822%	9.920%	6.582%	4.252%	2.810%	1.828%	1.156%	0.758%
$m = 1000$	46.942%	32.408%	22.036%	14.827%	9.922%	6.583%	4.252%	2.810%	1.828%	1.156%	0.758%
$m = \infty$	46.948%	32.412%	22.042%	14.832%	9.924%	6.584%	4.252%	2.810%	1.828%	1.156%	0.758%
u	5.5	6	6.5	7	7.5	8	8.5	9	9.5	10	
$m = 0$	0.764%	0.540%	0.386%	0.260%	0.200%	0.148%	0.086%	0.060%	0.046%	0.036%	
$m = 2$	0.608%	0.408%	0.276%	0.214%	0.150%	0.088%	0.054%	0.042%	0.018%	0.010%	
$m = 10$	0.450%	0.308%	0.220%	0.148%	0.094%	0.046%	0.020%	0.010%	0.006%	0.000%	
$m = 250$	0.478%	0.318%	0.236%	0.160%	0.094%	0.050%	0.024%	0.008%	0.005%	0.000%	
$m = 1000$	0.478%	0.318%	0.236%	0.160%	0.094%	0.050%	0.024%	0.008%	0.004%	0.000%	
$m = \infty$	0.478%	0.318%	0.236%	0.160%	0.094%	0.050%	0.024%	0.008%	0.004%	0.000%	

		$\psi_m^{(2)}(u)$									
u	0	1	2	3	4	5	6	7	8	9	10
$m = 0$	82.416%	69.122%	57.854%	48.526%	40.738%	34.052%	28.682%	24.060%	20.122%	16.894%	14.206%
$m = 2$	82.208%	67.124%	55.578%	45.370%	37.024%	30.182%	24.640%	19.976%	16.290%	13.186%	10.594%
$m = 10$	82.932%	68.260%	52.710%	40.310%	30.182%	22.138%	16.162%	11.560%	8.202%	5.890%	4.152%
$m = 250$	91.820%	79.506%	64.450%	49.480%	36.514%	25.882%	18.026%	12.112%	7.926%	5.256%	3.402%
$m = 1000$	92.023%	79.808%	64.905%	49.754%	36.901%	26.228%	18.273%	12.300%	8.109%	5.422%	3.529%
$m = \infty$	92.232%	80.108%	65.160%	50.184%	37.088%	26.424%	18.486%	12.528%	8.278%	5.542%	3.636%
u	11	12	13	14	15	16	17	18	19	20	
$m = 0$	11.792%	9.918%	8.234%	6.904%	5.742%	4.812%	4.070%	3.374%	2.834%	2.372%	
$m = 2$	8.688%	7.016%	5.686%	4.618%	3.746%	3.018%	2.462%	1.970%	1.598%	1.328%	
$m = 10$	2.866%	2.016%	1.452%	1.004%	0.738%	0.538%	0.374%	0.280%	0.188%	0.134%	
$m = 250$	2.190%	1.360%	0.860%	0.554%	0.380%	0.224%	0.140%	0.088%	0.060%	0.038%	
$m = 1000$	2.170%	1.304%	0.823%	0.528%	0.367%	0.203%	0.128%	0.084%	0.058%	0.037%	
$m = \infty$	2.286%	1.395%	0.865%	0.530%	0.365%	0.202%	0.127%	0.084%	0.058%	0.037%	

		$\psi_m^{(3)}(u)$									
u	0	2	4	6	8	10	12	14	16	18	20
$m = 0$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%
$m = 2$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%
$m = 10$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%
$m = 250$	99.992%	99.648%	96.894%	88.370%	73.964%	58.720%	42.814%	29.850%	20.238%	13.378%	8.816%
$m = 1000$	99.994%	99.790%	97.490%	89.582%	75.532%	57.010%	41.046%	28.186%	18.682%	12.006%	7.636%
$m = \infty$	99.996%	99.810%	96.894%	88.370%	75.124%	57.010%	41.046%	28.186%	18.682%	12.006%	7.636%
u	22	24	26	28	30	32	34	36	38	40	
$m = 0$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	
$m = 2$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	
$m = 10$	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	
$m = 250$	5.714%	3.660%	2.440%	1.566%	1.034%	0.704%	0.486%	0.330%	0.240%	0.142%	
$m = 1000$	4.708%	2.862%	1.784%	1.048%	0.644%	0.412%	0.250%	0.146%	0.098%	0.038%	
$m = \infty$	4.708%	2.862%	1.784%	1.048%	0.644%	0.412%	0.250%	0.146%	0.098%	0.038%	

On the one hand, asymptotically, whatever the portfolio j , we see that the larger m is, the smaller $\psi_m^{(j)}(u)$ is, even for $m \geq \bar{m}^{(j)} (\leq m^{(c)})$. This confirms our theoretical findings and intuition, namely that (also for $m \geq \min_{\{j: \bar{m}^{(j)} > 0\}} \bar{m}^{(j)}$) m is sub-optimal compared to $m + 1$.

On the other hand, for small initial capital u , a bigger value of m may this time increase in certain cases the ultimate ruin probability. Obviously, this observation is not unexpected.

Remark Clearly, the numerical illustrations can not highlight the fact that $m = \infty$ is sub-optimal.

4 Limit results for heavy-tailed claims

We have seen that in the case where the Lundberg coefficient exists, adjusting premium with credibility may increase this one. As typical sample paths for which ruin occurs exhibit a sequence of moderately large claims, there is quite often enough time to compensate early losses with credibility-adjusted future premiums. On the opposite, in the heavy-tailed case, it is often said that ruin is likely to be caused by one large claim (see for example Theorems 1.1 and 1.2 in Asmussen and Klüppelberg (1996) in the classical risk model). It is logical to think that credibility plays a less important role for heavy-tailed claim amount distributions. In this Section, we focus on the regular variation case.

Definition 4.1. The cumulative distribution function F with support $(0, \infty)$ belongs to the regular variation class if for some $\alpha > 0$

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad \text{for } y > 0. \quad (4.1)$$

We note $F \in \mathcal{R}_{-\alpha}$. The convergence is uniform on each subset $y \in [y_0, \infty)$ ($0 < y_0 < \infty$).

In the light-tailed case, we have studied the influence of m on the Lundberg coefficient

$$\rho_m^{(j)} = - \lim_{u \rightarrow \infty} \frac{1}{u} \ln \psi_m^{(j)}(u). \quad (4.2)$$

In our heavy-tailed case, the equivalent quantity to be studied is parameter

$$\gamma_m^{(j)} = - \lim_{u \rightarrow \infty} \frac{1}{\ln u} \ln \psi_m^{(j)}(u). \quad (4.3)$$

As we consider the case $F_j \in \mathcal{R}_{-\alpha^{(j)}}$, and since credibility adjustments involve implicitly that the two first moments of claim amounts exist, we restrict to the case $\alpha^{(j)} > 2$. Now, for $m < m^{(c)}$, we shall prove that, if $j \in \bar{b}_m$, then m does not modify the value of $\gamma_m^{(j)}$, which is in accordance with our intuition.

Theorem 4.2. *Suppose that the $Y_k^{(j)}$'s are regularly varying with index $\alpha^{(j)} > 2$. If $m < m^{(c)}$ and $j \in \bar{b}_m$, i.e.*

$$\eta > \frac{\mathbb{E}[Y^{(j)}]}{(1 - z_m)\mu + z_m \mathbb{E}[Y^{(j)}]} - 1, \quad (4.4)$$

we get

$$\gamma_m^{(j)} = - \lim_{u \rightarrow \infty} \frac{1}{\ln u} \ln \psi_m^{(j)}(u) = \alpha^{(j)} - 1. \quad (4.5)$$

Proof. In the compound Poisson risk model with sub-exponential claims, Embrechts and Veraverbeke (1982) have shown that the (continuous-time) classical ruin probability $\psi(u)$ with initial surplus $u \geq 0$ (without credibility premium adjustment) behaves as $u \rightarrow \infty$ as

$$\psi(u) \sim \frac{\lambda}{c - \lambda\mu_W} \int_u^{+\infty} (1 - F_W(x)) dx,$$

where c is the (continuous-time) premium income rate, λ is the intensity of the Poisson process, F_W is the cumulative distribution function of the individual claim amounts and μ_W is the expected individual claim amount. In the $\alpha^{(j)}$ -regularly varying case with $\alpha^{(j)} > 2$ (this means that

$$1 - F_W(x) \sim x^{-\alpha^{(j)}} \text{ as } x \rightarrow \infty,$$

this corresponds to

$$\psi_{cont}(u) \sim \frac{\lambda}{c - \lambda\mu_W} \frac{1}{\alpha^{(j)} - 1} u^{-\alpha^{(j)}+1}.$$

This result may be adapted to the discrete-time model, with constant 1-period premium income and independent, identically distributed (aggregate) claim amounts on each period, with $\alpha^{(j)}$ -regularly varying distribution: if the safety loading is positive, for $\alpha^{(j)} > 1$, there exists $C > 0$ such that the (discrete-time) probability of ruin $\psi_{disc}(u)$ satisfies $\psi_{disc}(u) \sim Cu^{-\alpha^{(j)}+1}$ as $u \rightarrow \infty$. To show this, one might for example use arguments developed in Rullière and Loisel (2004) and Lefèvre and Loisel (2008).

To show that there exist $C_1 > 0$ and a function ψ_1 such that

$$\psi_m^{(j)}(u) \geq \psi_1(u) \text{ and } \psi_1(u) \sim C_1 u^{-\alpha^{(j)}+1} \text{ as } u \rightarrow \infty,$$

let us consider a modified model, in which future earned premiums due to each claim are anticipated and earned at the claim instant: if claim amount $Y_k^{(j)}$ occurs at time k , the sum of future credibility premiums associated to $Y_k^{(j)}$ corresponds to

$$Y_k^{(j)}(1 + \eta) \sum_{i=k+1}^{k+m} \bar{z}_{i,m}.$$

In our modified model, each claim amount $Y_k^{(j)}$ is replaced with

$$\tilde{Y}_k^{(j)} = Y_k^{(j)} \left[1 - (1 + \eta) \sum_{i=k+1}^{k+m} \bar{z}_{i,m} \right],$$

and premium income

$$C_{k,m} = \left((1 - z_{k,m})\mu + z_{k,m} \frac{\sum_{i=\max(k-m,1)}^{k-1} Y_i^{(j)}}{\min(m, k-1)} \right) (1 + \eta)$$

is replaced with

$$\tilde{C}_{k,m} = (1 - z_{k,m})\mu(1 + \eta).$$

For each sample path, if ruin occurs in the modified model, then it occurs as well in the initial model as credibility adjusted premiums are the same in both models but are received later in the modified model. Since from $k = m + 1$, $\sum_{i=k+1}^{k+m} \bar{z}_{i,m} = z_m$, we have from Embrechts and Veraverbeke (1982) that the ruin probability $\psi_1(u)$ in the modified model satisfies $\psi_1(u) \sim C_1 u^{-\alpha^{(j)+1}}$ for some $C_1 > 0$ as $u \rightarrow \infty$ as long as the safety loading is positive in the modified model, which is guaranteed by condition (4.4).

To show that there exist $C_2 > 0$ and a function ψ_2 such that

$$\psi_m^{(j)}(u) \leq \psi_2(u) \text{ and } \psi_2(u) \sim C_2 u^{-\alpha^{(j)+1}} \text{ as } u \rightarrow \infty,$$

let us rewrite $\psi_m^{(j)}(u)$ as

$$\psi_m^{(j)}(u) = \Pr \left[\left(\exists 1 \leq i \leq m, U_{i,m}^{(j)} < 0 \right) \text{ or } \left(\exists i > m, U_{i,m}^{(j)} < 0 \right) \right].$$

This means that

$$\psi_m^{(j)}(u) \leq \Pr \left[\exists 1 \leq i \leq m, U_{i,m}^{(j)} < 0 \right] + \Pr \left[\exists i > m, U_{i,m}^{(j)} < 0 \right].$$

The first term is $O(u^{-\alpha^{(j)}})$ and so $o(u^{-\alpha^{(j)+1}})$. The second term

$$\Pr \left[\exists i > m, U_{i,m}^{(j)} < 0 \right]$$

satisfies

$$\begin{aligned} & \Pr \left[\exists i > m, U_{i,m}^{(j)} < 0 \right] \\ & \leq \Pr \left[\exists i > m, \sum_{k=1}^i Y_k^{(j)} > u/2 + u/2 + z_m(1 + \eta) \sum_{l=1}^{i-m} Y_l^{(j)} + \mu(1 + \eta)(1 - z_m)i \right] \\ & = \Pr \left[\exists i > m, \sum_{l=i-m+1}^i Y_l^{(j)} + (1 - z_m(1 + \eta)) \sum_{k=1}^{i-m} Y_k^{(j)} > u/2 + u/2 + (1 + \eta)\mu(1 - z_m)i \right] \\ & = \Pr \left[\exists i > m, \left(\sum_{l=i-m+1}^i Y_l^{(j)} \right) + \left((1 - z_m(1 + \eta)) \sum_{k=1}^{i-m} Y_k^{(j)} \right) \right. \\ & \quad \left. > \left(u/2 + \frac{\delta}{2}i \right) + \left(u/2 + \left((1 + \eta)\mu(1 - z_m) - \frac{\delta}{2} \right) i \right) \right], \end{aligned}$$

where

$$\delta = (1 + \eta)\mu(1 - z_m) - \mathbb{E}[Y^{(j)}](1 - z_m(1 + \eta)) > 0$$

from Condition (4.4).

Hence, since for any sequences of nonnegative r.v.'s $(X_i)_{i \geq 1}$ and $(Z_i)_{i \geq 1}$, and for any sequences of nonnegative numbers $(a_i)_{i \geq 1}$ and $(b_i)_{i \geq 1}$, we have

$$\Pr[\exists i \geq 1, X_i + Z_i > a_i + b_i] \leq \Pr[\exists i \geq 1, X_i > a_i] + \Pr[\exists j \geq 1, Z_j > b_j],$$

we get that

$$\begin{aligned} & \Pr \left[\exists i > m, U_{i,m}^{(j)} < 0 \right] \\ & \leq \Pr \left[\exists i > m, \sum_{l=i-m+1}^i Y_l^{(j)} > u/2 + \frac{\delta}{2}i \right] \\ & \quad + \Pr \left[\exists i > m, (1 - z_m(1 + \eta)) \sum_{k=1}^{i-m} Y_k^{(j)} > u/2 + \left((1 + \eta)\mu(1 - z_m) - \frac{\delta}{2} \right) i \right]. \end{aligned}$$

The second term is equivalent to $C_3 u^{-\alpha^{(j)+1}}$ as $u \rightarrow \infty$ for some constant $C_3 > 0$, because it corresponds to a ruin probability with claim size distribution in $RV(\alpha^{(j)})$ and with a positive safety loading (from (4.4)). The first term

$$\Pr \left[\exists i > m, \sum_{l=i-m+1}^i Y_l^{(j)} > u/2 + \frac{\delta}{2}i \right]$$

satisfies

$$\Pr \left[\exists i > m, \sum_{l=i-m+1}^i Y_l^{(j)} > u/2 + \frac{\delta}{2}i \right] \leq \Pr \left[\exists i > m, m \max_{l=i-m+1}^i Y_l^{(j)} > u/2 + \frac{\delta}{2}i \right],$$

which leads to

$$\begin{aligned} \Pr \left[\exists i > m, \sum_{l=i-m+1}^i Y_l^{(j)} > u/2 + \frac{\delta}{2}i \right] & \leq \Pr \left[\exists i > m, mY_i^{(j)} > u/2 + \frac{\delta}{2}(i - m) \right] \\ & = \Pr \left[\exists i > 0, mY_i^{(j)} > u/2 + \frac{\delta}{2}i \right]. \end{aligned}$$

This may be rewritten as

$$\Pr \left[\exists i > m, \sum_{l=i-m+1}^i Y_l^{(j)} > u/2 + \frac{\delta}{2}i \right] \leq \Pr \left[\exists i > 0, Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m}i \right].$$

Obviously, we have

$$\Pr \left[\exists i > 0, Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m}i \right] \leq \sum_{i=1}^{\infty} \Pr \left[Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m}i \right].$$

Now, by Lemma 5.2 in Foss et al. (2009), we obtain

$$\sum_{i=1}^{\infty} \Pr \left[Y_i^{(j)} > \frac{u}{2m} + \frac{\delta}{2m} i \right] \sim \frac{2m}{\delta} \Pr \left[Y^{(j)} > \frac{u}{2m} \right] \text{ as } u \rightarrow \infty,$$

which completes the proof. \square

Let us conclude in saying that, on the one hand, if $j \in b_m$ then $\psi_{m+1}^{(j)}(u) \leq \psi_m^{(j)}(u) = 1$ by Proposition 3.4 (clearly also valid for the heavy-tailed case), but on the other hand, this time, if $j \in \bar{b}_m$, then an increasing of the horizon of credibility has asymptotically no impact on $\psi_m^{(j)}(u)$ in first order, at least for $m < m^{(c)}$.

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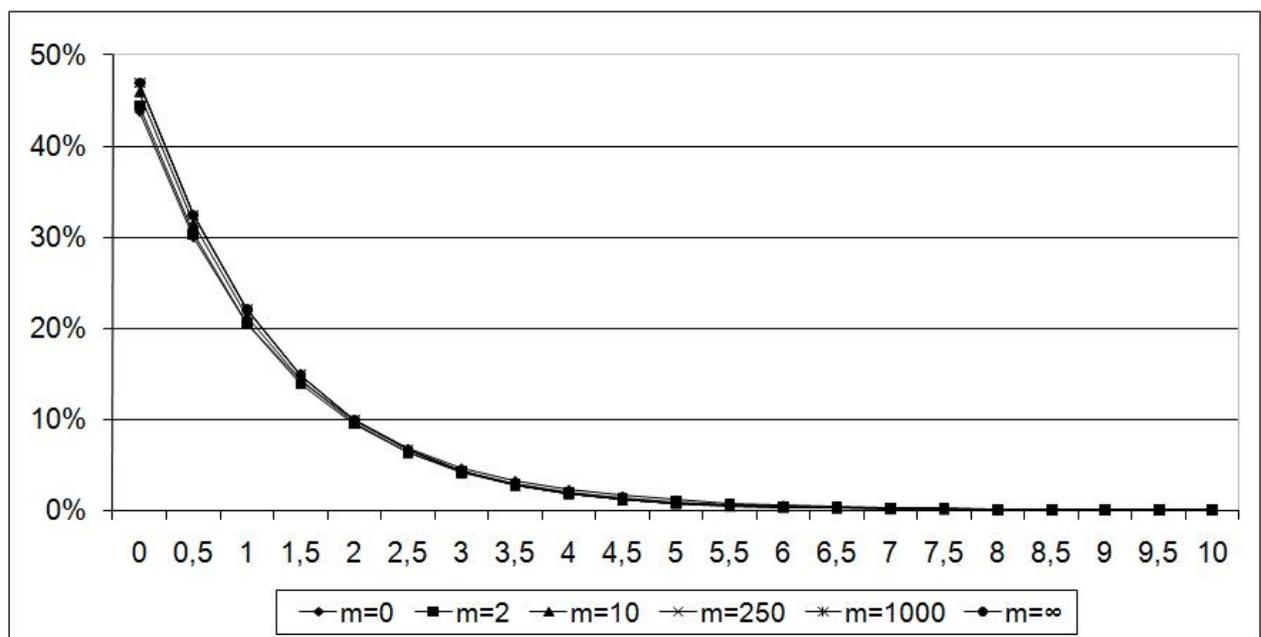


Figure 4.1: Numerical results for $\psi_m^{(1)}(u)$

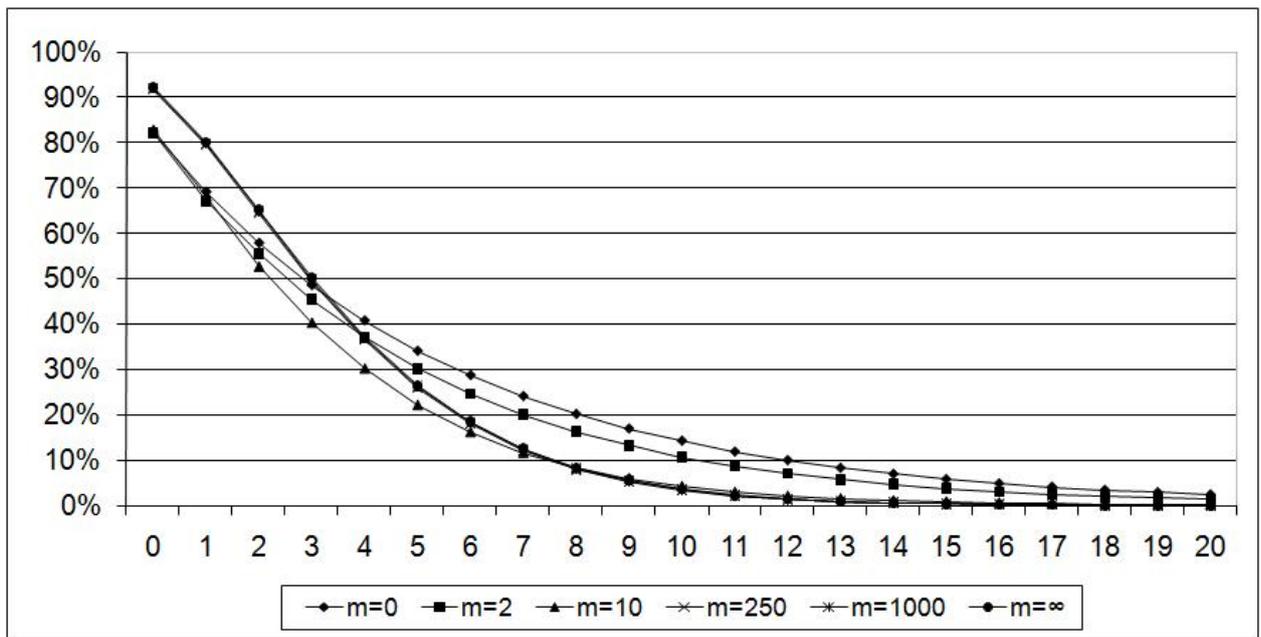


Figure 4.2: Numerical results for $\psi_m^{(2)}(u)$

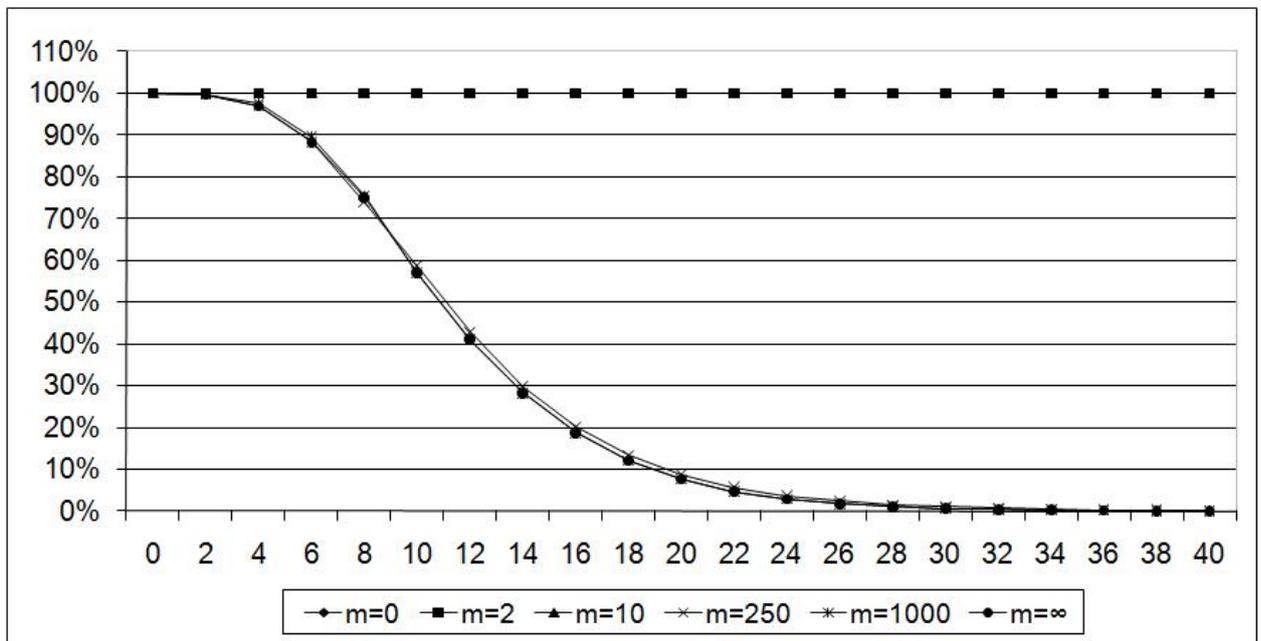


Figure 4.3: Numerical results for $\psi_m^{(3)}(u)$