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Numerical convergence analysis of Cauchy problem with noisy data solved by minimizing an energy-like functional

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Abstract

This paper is concerned with solving Cauchy problem for elliptic equation by minimizing an energy-like error functional and by taking into account noisy Cauchy data. After giving some fundamental results, Cauchy problem is presented as an optimal control problem. Numerical convergence analysis is carried out and leads to an adapted stopping criteria for the minimization process depending on noise rate. Numerical examples involving smooth and singular data are presented.

1 Introduction

The Cauchy problem considered here consists of solving a partial differential equation on a domain for which over-specified boundary conditions are given on a part of its boundary, which means to solve a data completion problem and recover the missing boundary conditions on the remaining part of the boundary. This kind of problem arises in many industrial, engineering or biomedical applications.

Since J.Hadamard's works [1], the Cauchy problem is known to be ill-posed and an important numerical instability may occur during the numerical resolution of this kind of problem. It provides researchers with an interesting challenge to carry out numerical procedure approximating the solution of Cauchy problem in the particular case of noisy data. Much theoretical and applied works were proposed about this subject, using Steklov-Poincaré theory (see [2, 3, 4]), regularization methods (see [5, 6]), quasi-reversibility method (see [7]) or minimal error methods (see [8, 9, 10]).

In this paper, we propose to perform mathematical and numerical convergence analysis of a method based on minimization of an energy-like functional introduced in [11, 12, 13]. The objective is to establish a stopping criteria for the minimization process in order to avoid numerical implosion due to noisy Cauchy data.

The outline of the paper is as follows. In section 2, we give the Cauchy problem and report classical theoretical results. In section 3, we formulate the Cauchy problem as a data completion problem and introduce the related minimization problem, which can be written as an optimal control problem. In section 4, the finite element discretisation and convergence analysis are presented. Section 5 is devoted to the study of noisy data effects. We give a priori error estimates taking into account data noise and propose a stopping criteria to control instability of minimization process. Finally, the numerical procedure and results are presented.

2 Statement of problem

We consider a Lipschitz bounded domain Ω in \mathbb{R}^d , $d = 2, 3$ with n the outward unit normal to the boundary $\Gamma = \partial\Omega$. Assume that Γ is partitioned into two parts Γ_u and Γ_m , of non-vanishing measure and such that $\Gamma_u \cap \Gamma_m = \emptyset$.

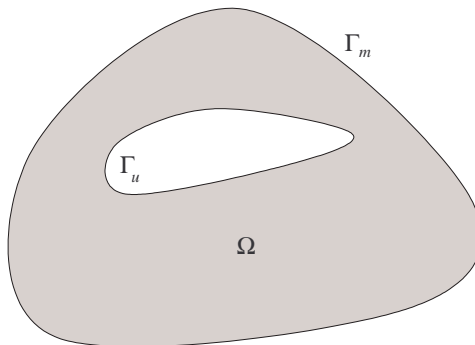


Figure 1: An example of geometry

The more common problem consists of temperature recovering in a given domain Ω assuming temperature distribution and heat flux are given over the accessible region of the boundary. Given a source term f and a conductivity field k in Ω , a flux ϕ and the corresponding temperature T on Γ_m , we would like to recover the corresponding flux and temperature on Γ_u . The Cauchy problem is then written as :

$$\begin{cases} -\nabla \cdot (k(x)\nabla u) = f & \text{in } \Omega \\ k(x)\nabla u \cdot n = \phi & \text{on } \Gamma_m \\ u = T & \text{on } \Gamma_m \end{cases} \quad (1)$$

A problem is well-posed in the sense of Hadamard (see [1, 14, 5]) if it fulfills the three following properties : uniqueness and existence of the solution and stability. The extended Holmgren's theorem to the Sobolev spaces (see [14]) guarantees uniqueness under regularity assumptions on a solution of the Cauchy problem. The well known Cauchy-Kowalevsky theorem (see [15]) being applicable only in the case of analytic data, the existence of this solution is then caution to a verification of a compatibility condition which can hardly be explicitly formulated. This compatibility condition added to the fact that, for one fixed data, the set of compatible data is dense in the set of all data (see [16]), imply that the stability assumption is not satisfied in the sense that the dependence of the solution u of (1) on the data (ϕ, T) is not continuous. Hereafter, we assume that data (ϕ, T) are compatible.

Some notations : Let x be a generic point of Ω . The space of squared integrable functions $L^2(\Omega)$ is endowed by natural inner product written $(\cdot, \cdot)_{L^2(\Omega)}$. The associated norm is written $\|\cdot\|_{0,\Omega}$. We note $H^1(\Omega)$ the Sobolev space of functions of $L^2(\Omega)$ for which their first order derivatives are also in $L^2(\Omega)$. Its norm and semi norm are written $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ respectively. Let $\gamma \subset \Gamma$, we define the space $H_{0,\gamma}^1(\Omega) = \{v \in H^1(\Omega); v|_\gamma = 0\}$ and $H_{00}^{1/2}(\gamma)$ is the space of restrictions to γ of the functions of $H^{1/2}(\Omega) = tr(H^1(\Omega))$. Its topological dual is written $H_{00}^{-1/2}(\gamma) = (H_{00}^{1/2}(\gamma))'$. The associated norms are written $\|\cdot\|_{1/2,00,\gamma}$ and $\|\cdot\|_{-1/2,00,\gamma}$ respectively and $\langle \cdot, \cdot \rangle_{1/2,00,\gamma}$ states for the duality inner product.

3 Weak formulation

3.1 Weak formulation of the Cauchy problem

Let $f \in L^2(\Omega)$, $k(x) \in L^\infty(\Omega)$ positive, $\phi \in H_{00}^{-1/2}(\Gamma_m)$ and $T \in H_{00}^{1/2}(\Gamma_m)$. The Cauchy problem can be written as a data completion problem :

Find $(\varphi, t) \in H_{00}^{-1/2}(\Gamma_u) \times H_{00}^{1/2}(\Gamma_u)$ such that there exists $u \in H^1(\Omega)$ solution of

$$\begin{cases} -\nabla \cdot (k(x)\nabla u) = f \text{ in } \Omega \\ u = T, k(x)\nabla u \cdot n = \phi \text{ on } \Gamma_m \\ u = t, k(x)\nabla u \cdot n = \varphi \text{ on } \Gamma_u \end{cases} \quad (2)$$

The functional spaces being given, we introduce more precisely the concept of density mentioned in the previous section. We recall the following theorem :

Theorem 3.1 (i) For a fixed $T \in H_{00}^{1/2}(\Gamma_m)$, the set of data ϕ for which there exists a solution $u \in H^1(\Omega)$ to the Cauchy problem (1) is everywhere dense in $H_{00}^{-1/2}(\Gamma_m)$.

(ii) For a fixed $\phi \in H_{00}^{-1/2}(\Gamma_m)$, the set of data T for which there exists a solution $u \in H^1(\Omega)$ to the Cauchy problem (1) is everywhere dense in $H_{00}^{1/2}(\Gamma_m)$.

Two proofs of this theorem, based on Hahn-Banach theorem and penalty method, are given in [16].

We introduce now two distinct fields u_1 and u_2 solution of well posed problems which differ by their boundary conditions. We attribute to each of them one data on Γ_m and one unknown on Γ_u . Then, we have :

$$\begin{cases} -\nabla \cdot (k(x)\nabla u_1) = f \text{ in } \Omega \\ u_1 = T \text{ on } \Gamma_m \\ k(x)\nabla u_1 \cdot n = \eta \text{ on } \Gamma_u \end{cases} \quad (3) \quad \begin{cases} -\nabla \cdot (k(x)\nabla u_2) = f \text{ in } \Omega \\ u_2 = \tau \text{ on } \Gamma_u \\ k(x)\nabla u_2 \cdot n = \phi \text{ on } \Gamma_m \end{cases} \quad (4)$$

We denote $a_i(\cdot, \cdot)$ and $l_i(\cdot)$, $i = 1, 2$ the bilinear and linear forms associated to problems (3) and (4) respectively. Then, we have by summation the following weak problem :

$$\begin{aligned} \text{Find } u &= (u_1, u_2) \in V \text{ such that} \\ a(u, v) &= l(v), \quad \forall v = (v_1, v_2) \in V \\ \text{with } a(u, v) &= a_1(u_1, v_1) + a_2(u_2, v_2) \\ \text{and } l(v) &= l_1(v_1) + l_2(v_2) \end{aligned} \quad (5)$$

where $V = H_{0, \Gamma_m}^1(\Omega) \times H_{0, \Gamma_u}^1(\Omega)$ and $\|v\|_V = (\|v_1\|_{1, \Omega}^2 + \|v_2\|_{1, \Omega}^2)^{1/2}$ is the norm associated to the space V . It is easy to show that the linear form $l(\cdot)$ is continuous and that the bilinear form $a(\cdot, \cdot)$ is continuous and V -elliptic. Then, by the Lax-Milgram theorem, the weak problem (5) admits a unique solution.

3.2 Energy-like minimization problem

We consider now the following energy-like functional in order to compare the fields u_1 and u_2 :

$$E(\eta, \tau) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_1(\eta) - \nabla u_2(\tau))^2 dx \quad (6)$$

and the following minimization problem :

$$\begin{cases} (\varphi, t) = \operatorname{argmin}_{(\eta, \tau) \in \mathcal{U}} E(\eta, \tau), \quad \mathcal{U} = H_{00}^{-1/2}(\Gamma_u) \times H_{00}^{1/2}(\Gamma_u) \\ \text{with } u_1 \text{ and } u_2 \text{ solutions of (3) and (4) respectively.} \end{cases} \quad (7)$$

Using convexity of the space \mathcal{U} , existence and uniqueness of the Cauchy problem solution in the case of compatible data, we are able to prove that the solution (η^*, τ^*) of the minimization problem (7), if it exists and is unique, is solution of the data completion problem up to an arbitrary additive constant for the Dirichlet unknown τ . In other words, if $(\eta_d, \tau_d) \in \mathcal{U}$ is solution of the data completion problem, $\eta^* = \eta_d$, $\tau^* = \tau_d + \kappa$, where κ is a constant.

Remarks :

- (i) When $E(\eta, \tau)$ reaches its minimum, $\nabla u_1(\eta^*) = \nabla u_2(\tau^*)$.
- (ii) Cauchy problem can be formulated as a minimization of the energy-like functional with fields u_1 and u_2 solutions of problems (3) and (4) respectively (see [11]).

3.3 Optimal control

In this part, we re-formulate the problem (7) as an optimal control problem as defined in [17]. We assume more regularity on the field u_2 , say $u_2 \in H^2(\Omega) \cap H_{0,\Gamma_m}^1(\Omega)$. Thereby, we define the operators $A \in \mathcal{L}(V', V)$ and $B \in \mathcal{L}(\mathcal{U}, V')$ given by :

$$(Au, v)_{V', V} = a(u, v) \quad \text{and} \quad (B(\eta, \tau), v)_{V', V} = \langle \eta, v \rangle_{1/2, 0, \Gamma_u} - \langle k(x) \nabla v \cdot n, \tau \rangle_{1/2, 0, \Gamma_u} \quad (8)$$

and $F \in V'$ such that $(F, v)_{V', V} = (f, v)_{L^2(\Omega)}$. Then, the weak problem (5) can be written as

$$Au(\eta, \tau) = F + B(\eta, \tau). \quad (9)$$

Furthermore, we define an operator $C \in \mathcal{L}(V, L^2(\Omega))$ such that the energy-like functional could be written as follows :

$$E(\eta, \tau) = \|Cu(\eta, \tau)\|_{L^2(\Omega)} \quad (10)$$

The functional being convex, it can be proven that, if it exists, the optimal control is unique. However, it is well known that conditions that guarantee existence can be hardly described. Nevertheless, it appears not as restrictive. Indeed, as seen later, even without this condition one can produce a stable algorithm for finding numerical solution (see [14]).

This formulation enables us to characterize the optimal control. We introduce the adjoint state $v = (v_1, v_2) \in V$. If (η, τ) is the optimal control, v_1 and v_2 are solution of the two following adjoint problems :

$$\begin{cases} \nabla \cdot (k(x) \nabla v_1) = 0 \text{ in } \Omega \\ v_1 = 0 \text{ on } \Gamma_m \\ k(x) \nabla v_1 \cdot n = \eta - k(x) \nabla u_2 \cdot n \text{ on } \Gamma_u \end{cases} \quad (11) \quad \begin{cases} \nabla \cdot (k(x) \nabla v_2) = 0 \text{ in } \Omega \\ v_2 = 0 \text{ on } \Gamma_u \\ k(x) \nabla v_2 \cdot n = \phi - k(x) \nabla u_1 \cdot n \text{ on } \Gamma_m \end{cases} \quad (12)$$

The gradient of the related functional is then given by :

$$\nabla E(\eta, \tau) = (v_1|_{\Gamma_u}, -k(x) \nabla v_2 \cdot n|_{\Gamma_u}) \quad (13)$$

Remark : This optimal control problem is equivalent to a constrained optimization problem (see [18]) by introducing the following Lagrangian :

$$\mathcal{L}(\eta, \tau, u, v) = E(\eta, \tau) - \langle v, Au - F - B(\eta, \tau) \rangle_{V', V} \quad (14)$$

4 Finite element discretisation and error estimation

4.1 Finite element discretisation

Let X_h be the finite element space for which the following classical assumptions are verified :

- (i) Ω is polyhedral domain in \mathbb{R}^d , $d = 2, 3$.
- (ii) \mathcal{T}_h is a regular triangulation of $\bar{\Omega}$ i.e. $h = \max_{K \in \mathcal{T}_h} h_K \rightarrow 0$ and $\max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \leq c$ which c independent constant on h , h_K the element K diameter and ρ_K the K inscribed circle diameter.
- (iii) Γ_u and Γ_m can be written exactly as union of faces of some finite elements $K \in \mathcal{T}_h$.
- (iv) The family (K, P_K, Σ_K) , $K \in \mathcal{T}_h$ for all h is affine-equivalent to a unique reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ of class \mathcal{C}^0 .
- (v) The following inclusion is satisfied : $P_l(\hat{K}) \subset \hat{P} \subset H^1(\hat{K})$ for $l \geq 1$.

These assumptions imply that $X_h \subset H^1(\Omega)$. We define the following spaces :

$$\begin{aligned} X_{uh} &= \{v_h \in X_h; v_h|_{\Gamma_u} = 0\} \\ X_{mh} &= \{v_h \in X_h; v_h|_{\Gamma_m} = 0\} \end{aligned}$$

and $V_h = X_{uh} \times X_{mh} \subset V$ the finite dimensional approximation space. So, we have the discrete problem associated to the weak problem (5) :

$$\begin{aligned} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h \end{aligned} \quad (15)$$

The Lax-Milgram theorem guarantees that (15) admits a unique solution.

4.2 Convergence analysis

Using standard procedure (see [19], Theorem 3.2.2.), we report the following error estimate :

Proposition 4.1 *In addition to the assumptions stated above, assume that there exists an integer $l \geq 1$ such that the following inclusion is satisfied :*

$$H^{l+1}(\hat{K}) \subset \mathcal{C}^s(\hat{K}) \text{ with continuous injection} \quad (16)$$

where s is the maximal order of partial derivatives occurring in the definition of the set $\hat{\Sigma}$.

Then if the solution $u \in V$ of the variational problem (5) is also in the space $(H^{l+1}(\hat{K}))^2$, there exists a constant C independent on h such that

$$\|u - u_h\|_V \leq Ch^l (|u_1|_{l+1, \Omega}^2 + |u_2|_{l+1, \Omega}^2)^{1/2} \quad (17)$$

where $u_h \in V_h$ is the discrete solution.

5 Noisy data, apriori error estimates and stopping criteria

5.1 A priori error estimates and data noise effects

In the case of given perturbed data, say (ϕ^δ, T^δ) , problem (15) writes as:

$$\begin{aligned} \text{Find } u_h^\delta = (u_{1h}^\delta, u_{2h}^\delta) \in V_h \text{ such that} \\ a(u_h^\delta, v_h) = l^\delta(v_h), \quad \forall v_h \in V_h \\ \text{where } l^\delta(\cdot) \text{ is the linear form with noisy data } (\phi^\delta, T^\delta) \end{aligned} \quad (18)$$

Proposition 5.1 *Under assumptions of proposition 4.1, if the solution $u \in V$ of the variational problem (5) is also in the space $(H^{l+1}(\hat{K}))^2$, then there exist two constants C_1 and C_2 independent on h and data such that*

$$\|u - u_h^\delta\|_V \leq C_1 h^l (|u_1|_{l+1, \Omega}^2 + |u_2|_{l+1, \Omega}^2)^{1/2} + C_2 (\|T - T^\delta\|_{1/2, 0, \Gamma_m}^2 + \|\phi - \phi^\delta\|_{-1/2, 0, \Gamma_m}^2)^{1/2} \quad (19)$$

where u_h^δ is the solution of the discrete problem (18) associated to the noisy Cauchy problem.

Proof Using the V -ellipticity property of the bilinear form $a(\cdot, \cdot)$, we have

$$\alpha \|u_h - u_h^\delta\|_V^2 \leq l(u_h - u_h^\delta) - l^\delta(u_h - u_h^\delta) \quad (20)$$

According to trace theorem (see [20]), the trace operator is continuous and there exists two lifting operators $\mathcal{R}_1 : H_{00}^{1/2}(\Gamma_m) \rightarrow H^1(\Omega)$ and $\mathcal{R}_2 : H_{00}^{-1/2}(\Gamma_m) \rightarrow H^1(\Omega)$ which are continuous and linear. Then, there exists a constant $C > 0$ such that

$$\alpha \|u_h - u_h^\delta\|_V^2 \leq C \|u_h - u_h^\delta\|_V (\|\mathcal{R}_1(T - T^\delta)\|_{1, \Omega}^2 + \|\mathcal{R}_2(\phi - \phi^\delta)\|_{1, \Omega}^2)^{1/2} \quad (21)$$

and by continuity, there exists two constants M_1, M_2 such that

$$\|u_h - u_h^\delta\|_V \leq \frac{C}{\alpha} (M_1 \|T - T^\delta\|_{1/2,00,\Gamma_m}^2 + M_2 \|\phi - \phi^\delta\|_{-1/2,00,\Gamma_m}^2)^{1/2} \quad (22)$$

Using the triangular inequality, we have

$$\|u - u_h^\delta\|_V = \|u - u_h + u_h - u_h^\delta\|_V \leq \|u - u_h\|_V + \|u_h - u_h^\delta\|_V \quad (23)$$

We obtain (19) by applying (17) and (22) on (23).

5.2 Stopping criteria for the minimization process

When noise is introduced on the Cauchy data, we observe during the optimization process that the error reaches a minimum before increasing very fastly and leading to a numerical implosion. At the same time, the energy-like functional attains asymptotically a minimal threshold, which is a strictly positive constant depending on the noise. Notice that this constant vanishes for compatible Cauchy data. Now, the aim is to theoretically determine this threshold in order to propose a stopping criteria depending on the noise rate. This criteria will allow to stop the minimization process just before numerical implosion. Let

$$E_h^\delta(\eta, \tau) = \frac{1}{2} \int_{\Omega} k(x) (\nabla u_{1h}^\delta(\eta) - \nabla u_{2h}^\delta(\tau))^2 dx \quad (24)$$

be the perturbed discrete functional.

Proposition 5.2 *Under assumptions of proposition 4.1, if the solution $u \in V$ of the variational problem (15) is also in the space $(H^{l+1}(\hat{K}))^2$ and if (η^*, τ^*) is the solution of the minimization problem (7), then there exists two constants C_1 and C_2 independent on h and data such that*

$$E_h^\delta(\eta^*, \tau^*) \leq C_1 h^{2l} (|u_1|_{l+1,\Omega}^2 + |u_2|_{l+1,\Omega}^2) + C_2 (\|T - T^\delta\|_{1/2,00,\Gamma_m}^2 + \|\phi - \phi^\delta\|_{-1/2,00,\Gamma_m}^2) \quad (25)$$

Proof Let (η^*, τ^*) be the solution of the minimization problem (7) with compatible Cauchy data. After some algebraic operations and taking into account the fact that $\nabla u_1(\eta^*) = \nabla u_2(\tau^*)$, we can write :

$$E_h^\delta(\eta^*, \tau^*) - E(\eta^*, \tau^*) = \frac{1}{2} \int_{\Omega} k(x) [(\nabla u_{1h}^\delta(\eta^*) - \nabla u_1(\eta^*)) - (\nabla u_{2h}^\delta(\tau^*) - \nabla u_2(\tau^*))]^2 dx \quad (26)$$

As seen previously $E(\eta^*, \tau^*) = 0$. Consequently :

$$E_h^\delta(\eta^*, \tau^*) \leq \|k\|_{L^\infty(\Omega)} (|\nabla u_{1h}^\delta(\eta^*) - \nabla u_1(\eta^*)|_{1,\Omega}^2 + |\nabla u_{2h}^\delta(\tau^*) - \nabla u_2(\tau^*)|_{1,\Omega}^2) \quad (27)$$

and then

$$E_h^\delta(\eta^*, \tau^*) \leq \|k\|_{L^\infty(\Omega)} \|u - u_h^\delta\|_V^2 \quad (28)$$

Therefore, using proposition 5.1, we derive (25).

Thus we can define a stopping criteria based on this last proposition. Indeed, when the discrete functional with noisy data (24) reaches its minimum, for h sufficiently small, we have by (25) :

$$E_h^\delta(\eta^*, \tau^*) \sim O(\|T - T^\delta\|_{1/2,00,\Gamma_m}^2 + \|\phi - \phi^\delta\|_{-1/2,00,\Gamma_m}^2) \quad (29)$$

Then, we stop the optimization algorithm when the relative variation of E_h^δ becomes smaller than noise norm. We write E_j the value of the discrete functional at the j -th iteration of the optimization algorithm. We propose then the following stopping criteria :

$$|E_j - E_{j-1}| \leq C_j (\|T - T^\delta\|_{1/2,00,\Gamma_m}^2 + \|\phi - \phi^\delta\|_{-1/2,00,\Gamma_m}^2) \quad (30)$$

where

$$C_j < \frac{E_{j-1}}{E_j} \quad (31)$$

which is consistent with (25).

6 Numerical issues

6.1 Numerical procedure

Let us describe the calculation method of the required elements for the optimization procedure, specifically the adjoint states and the gradient of the functional. Assume that the triangulation \mathcal{T}_h of Ω is characterized by n nodes. Let p and q denote the number of nodes on the boundaries Γ_u and Γ_m respectively and $(\omega_i)_{1 \leq i \leq n} = (\omega_{1i}, \omega_{2i})_{1 \leq i \leq n}$ the canonical basis of V_h . We write X_η and X_τ the unknowns. The vectors U_1 and U_2 correspond to the fields u_1 and u_2 . We introduce the following notations, $(K_1)_{kl} = a_1(\omega_{1k}, \omega_{1l})$, $(K_2)_{kl} = a_2(\omega_{2k}, \omega_{2l})$, $(F_1)_k = l_1(\omega_{1k})$, $(F_2)_k = l_2(\omega_{2k})$. The bilinear forms being similar, we note $K = K_1 = K_2$.

Laying down Dirichlet conditions, we obtain the following linear systems :

$$\begin{cases} KU_1 + L_m^T p_1 = F_1(X_\eta) \\ L_m U_1 = T^\delta \end{cases} \quad (32) \quad \begin{cases} KU_2 + L_u^T p_2 = F_2(\Phi^\delta) \\ L_u U_2 = X_\tau \end{cases} \quad (33)$$

The matrices $L_u \in \mathcal{M}_{p \times n}(\mathbb{R})$ and $L_m \in \mathcal{M}_{q \times n}(\mathbb{R})$ contain only 0 and 1.

Based on (6) and (14), we can write the discrete functional :

$$E(X_\eta, X_\tau) = \frac{1}{2}(U_1 - U_2)^T K(U_1 - U_2) \quad (34)$$

and the discrete lagrangian :

$$\mathcal{L}(U_1, U_2, \lambda_1, \lambda_2; X_\eta, X_\tau) = E(X_\eta, X_\tau) - \begin{bmatrix} \lambda_1 \\ q_1 \end{bmatrix}^T \begin{bmatrix} KU_1 + L_m^T p_1 - F_1 \\ L_m U_1 - T^\delta \end{bmatrix} \quad (35)$$

$$- \begin{bmatrix} \lambda_2 \\ q_2 \end{bmatrix}^T \begin{bmatrix} KU_2 + L_u^T p_2 - F_2 \\ L_u U_2 - X_\tau \end{bmatrix} \quad (36)$$

Let (X_η^*, X_τ^*) be the optimum. Derivating this lagrangian and given that $\frac{\partial \mathcal{L}}{\partial X_\eta}(X_\eta^*) = 0$ and $\frac{\partial \mathcal{L}}{\partial X_\tau}(X_\tau^*) = 0$, we have by identification the discrete adjoint problems :

$$\begin{cases} K\lambda_1 + L_m^T q_1 = K(U_1 - U_2) \\ L_m \lambda_1 = 0 \end{cases} \quad (37) \quad \begin{cases} K\lambda_2 + L_u^T q_2 = K(U_2 - U_1) \\ L_u \lambda_2 = 0 \end{cases} \quad (38)$$

and the gradient of the discrete functional is then given by :

$$\nabla E(X_\eta, X_\tau) = \begin{bmatrix} L_u \lambda_1 \\ L_u [K(U_2 - U_1) - K\lambda_2] \end{bmatrix} \quad (39)$$

We assume that data (T^δ, Φ^δ) are given with a noise rate $0 < a < 1$. The stopping criteria (30) can be written as follows :

$$|E_j - E_{j-1}| \leq \frac{E_{j-1}}{E_j} \left(\frac{a^2}{(1-a)^2} (\|T^\delta\|_{1/2,00,\Gamma_m}^2 + \|\Phi^\delta\|_{-1/2,00,\Gamma_m}^2) \right) \quad (40)$$

6.2 Numerical results

We consider the following Cauchy problem on the domain Ω given by figure (2) :

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f_D \text{ on } \Gamma_m \\ \frac{\partial u}{\partial n} = f_N \text{ on } \Gamma_m \end{cases} \quad (41)$$

where f_D and f_N are the Cauchy data extracted from the exact solution which we intend to approximate.

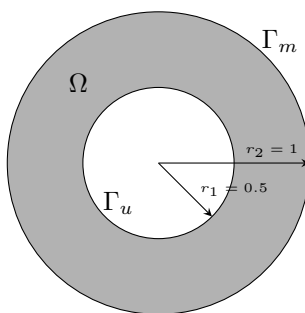


Figure 2: Ring

6.2.1 Analytic example :

The figure 3 represents the exact analytic solution $u(x, y) = e^x \cos(y)$ and the finite element solution of the data completion problem obtained by energy-like functional minimization. We can see that the recovered temperature and heat flux are close to the exact ones. The figure 6 represents the finite element discretisation error with respect to the maximum edge size of the mesh. This result is in agreement with the theoretical error estimates (17).

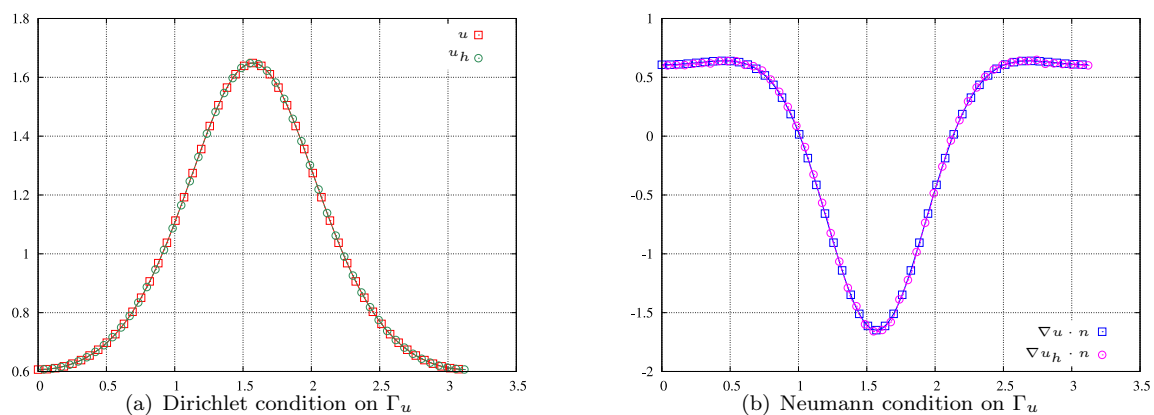


Figure 3: Exact (\square) and identified (\circ) boundary conditions, $h = 0.03$

We introduce a gaussian random noise on data with an amplitude which depends on a rate a . The figures 4 and 5 represent the error and the energy-like functional at each iteration for different noise rates. These behaviors make it necessary to introduce a criteria to stop the optimization process.

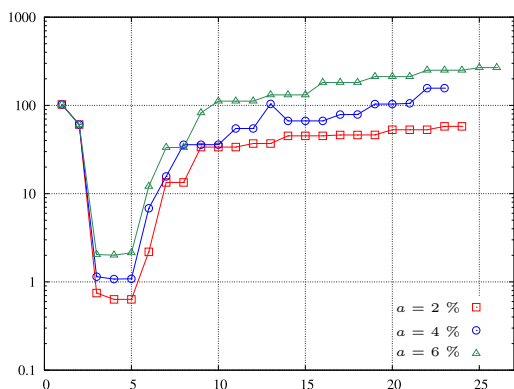


Figure 4: Evolution of $\|u - u_h\|_{1, \Omega}$ during the optimization procedure for different noise rates

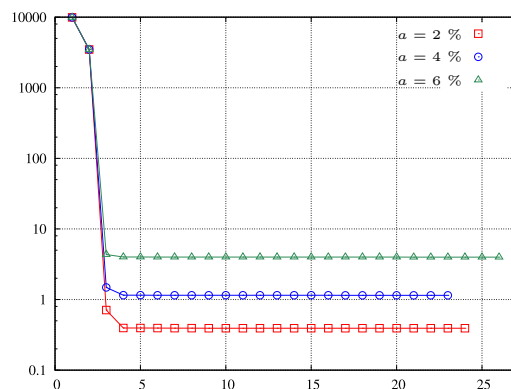


Figure 5: Evolution of $E(\eta, \tau)$ during the optimization procedure for different noise rates

Next we choose h such that finite element error could be negligible in comparison with error due to noise and we observe error and functional behaviors with respect to the noise norm. These results, presented in the figure 7, are in agreement with the error estimates (19) and (25).

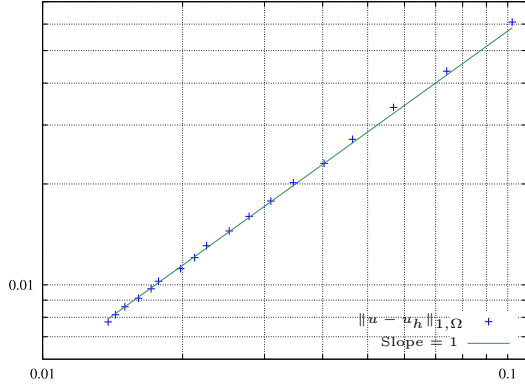


Figure 6: Evolution of $\|u - u_h\|_{1,\Omega}$ with respect to h

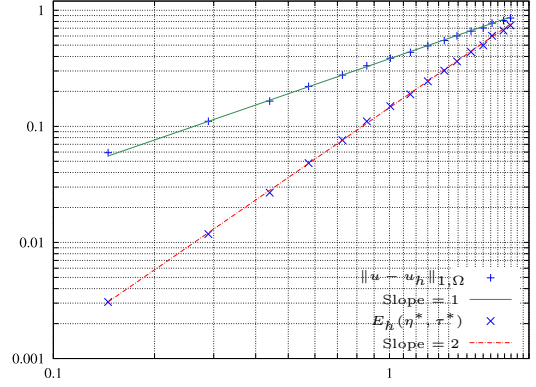


Figure 7: Evolution of $\|u - u_h\|_{1,\Omega}$ and $E(\eta, \tau)$ with respect to the noise norm.

The stopping criteria defined by (30) allows to identify a consistent solution, as shown in figure 8, otherwise the solution of the optimization algorithm numerically implodes.

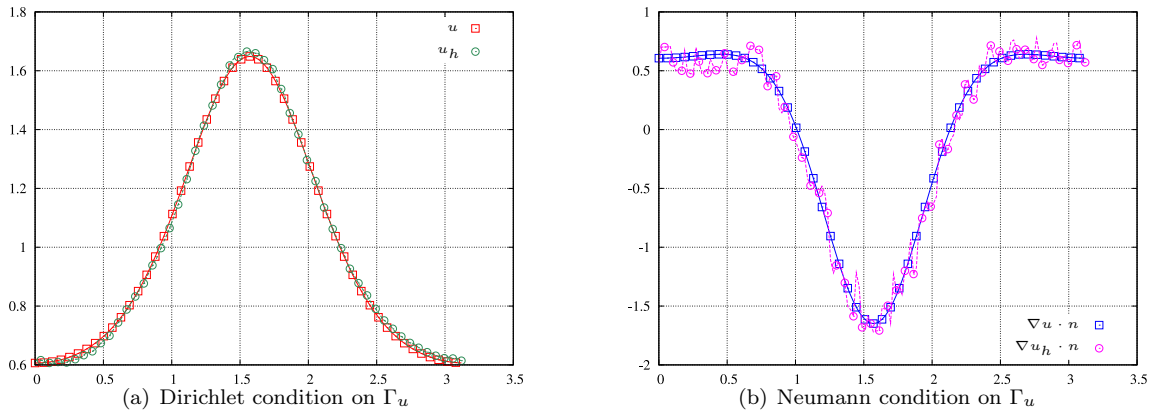


Figure 8: Exact (\square) and identified (\circ) boundary conditions with noisy data, $a = 4\%$, $h = 0.03$

6.2.2 Source point and stratified inner fluid examples :

The next source point example deals with the reconstruction of singular data, coming from

$$u(x, y) = \operatorname{Re}\left(\frac{1}{z - r}\right), \quad \text{where } z = x + iy \quad (42)$$

where r is the position of the point source on the abscissa axis. Numerical results are illustrated by figure 9 in case that the source point is in the vicinity of the inner boundary and figure 10 if the source point is in the vicinity of the outer boundary.

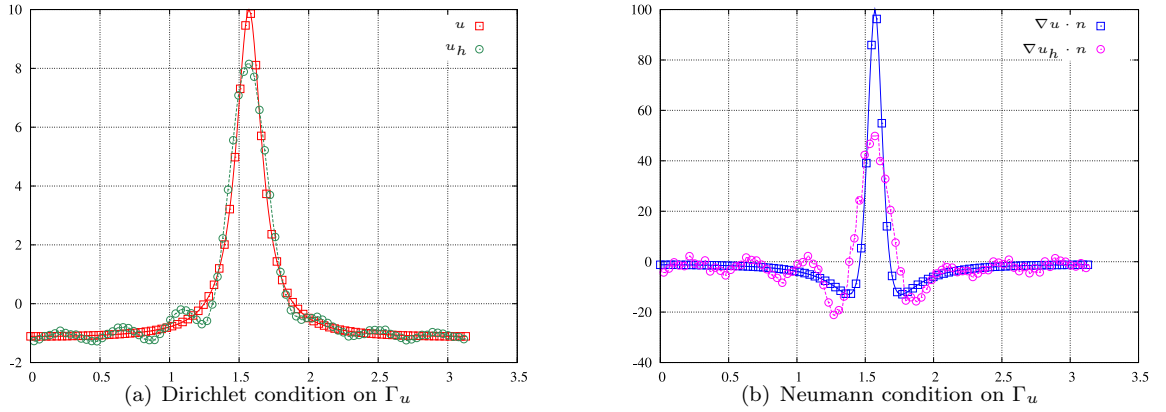


Figure 9: Exact (\square) and identified (\circ) boundary conditions with noisy data, $r = 0.4$, $a = 4\%$, $h = 0.02$

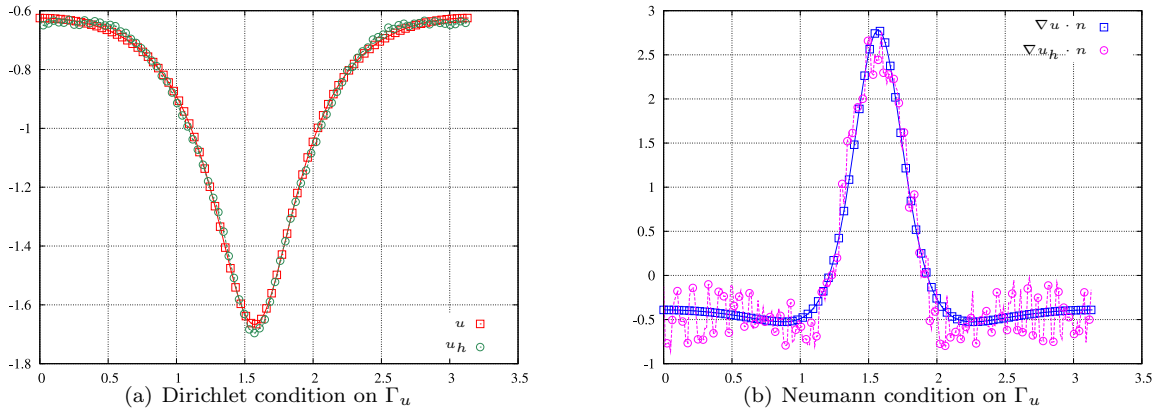


Figure 10: Exact (\square) and identified (\circ) boundary conditions with noisy data, $r = 1.1$, $a = 4\%$, $h = 0.02$

Now, we explore the efficiency of proposed stopping criteria on the so-called stratified inner fluid case. We consider therefore the reconstruction of temperature and flux in a pipeline of infinite length. This application arises in several industrial processes. Indeed, knowledge of temperature on internal wall of a pipeline is necessary for controlling the material safety : stratified inner fluid generates mechanical stresses, which may cause damages such as cracks. We assume that the temperature does not depend on the longitudinal coordinate. We consider then the following problem on the geometry defined by figure 2 :

$$\begin{cases} \nabla \cdot (k \nabla u) = 0 & \text{in } \Omega \\ k \nabla u \cdot n + \alpha u = T & \text{on } \Gamma \end{cases} \quad (43)$$

where $k = 17 \text{ W.m}^{-1}.\text{°C}^{-1}$ is the constant thermal conductivity, T is the temperature, α is the Fourier coefficient, Γ_u is partitioned into two parts, the lower half circle $\Gamma_{u,lo} = \{(x, y) \in \Gamma_u; y < 0\}$ and the upper half one $\Gamma_{u,up} = \{(x, y) \in \Gamma_u; y \geq 0\}$. The coefficients values are given in table 1.

Table 1: Coefficients values for stratified inner fluid test

	T ($^{\circ}\text{C}$)	α ($\text{W.m}^{-2}.\text{°C}^{-1}$)
Γ_m	20	12
$\Gamma_{u,up}$	250	1000
$\Gamma_{u,lo}$	50	1000

The Cauchy data are generated by solving the forward problem defined by (43). Then, a random noise is applied on Dirichlet data and we assume that the flux is exactly known on Γ_m . The figure 11 shows the recovered temperature and heat flux in comparison to the data given by numerical resolution of (43). Notice that reconstructed field is close to the solution to be recovered.

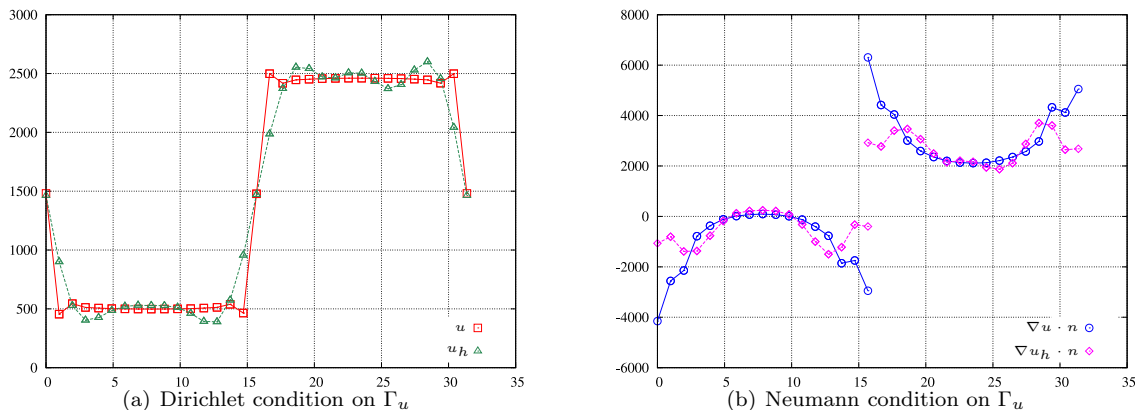


Figure 11: Identified temperature (\triangle) and flux (\diamond) on Γ_u with noisy data, $a = 4\%$, $h = 0.1$

Finally, in order to illustrate the efficiency of the given stopping criteria, we increase the noise rate up to 10% and perform numerical experiments. Figure 12 shows the solution of the generic optimization algorithm. However figure 13 shows the solution of the optimization algorithm with the stopping criteria defined by (30). The numerical implosion without the proposed stopping criteria is then clearly observed.

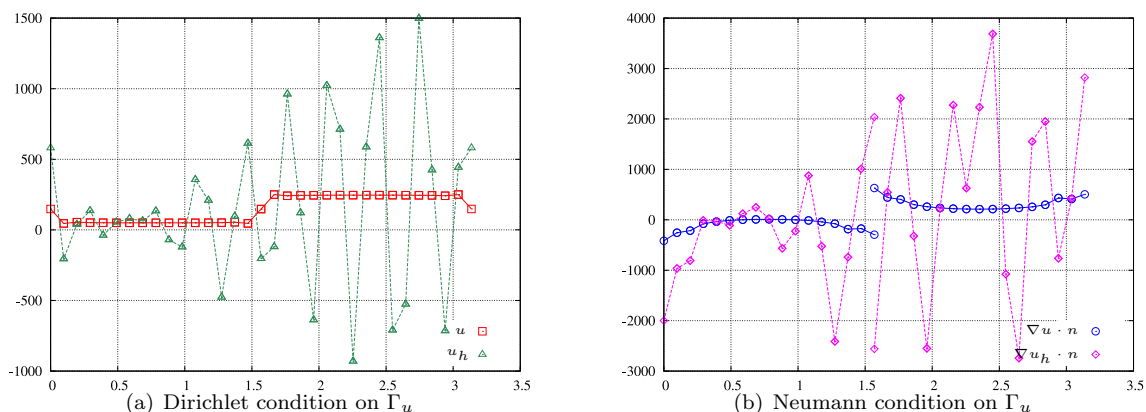


Figure 12: Identified temperature (\triangle) and flux (\diamond) on Γ_u with noisy data and classical stopping criteria, $a = 10\%$, $h = 0.1$.

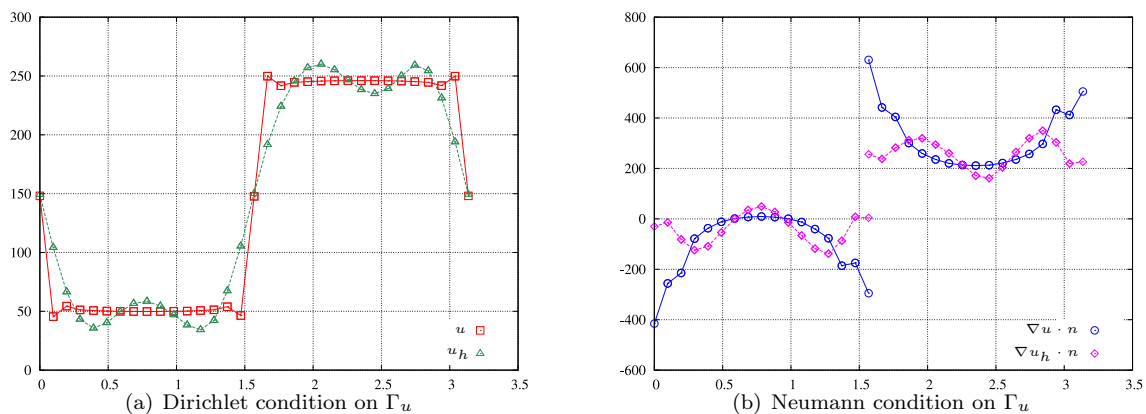


Figure 13: Identified temperature (\triangle) and flux (\diamond) on Γ_u with noisy data, $a = 10\%$, $h = 0.1$

7 Conclusion

In this work, we considered Cauchy problem as a minimization problem and presented classical theoretical results. Then, we gave the finite element discretisation and performed convergence analysis. We derived a priori error estimates taking into account noisy data. Then we proposed a stopping criteria depending on the noise rate in order to control numerical instability of the minimization process due to noisy data. We proposed a numerical procedure and performed numerical experiments in agreement with error estimates. We illustrated robustness and efficiency of the proposed stopping criteria, especially in the case of singular data. The numerical analysis of noise effects and derivation of stopping minimization criteria for parabolic (see [21, 22]) and hyperbolic problem is under consideration. It will be a subject of forthcoming paper.

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