

Ultrabroadband Elastic Cloaking in Thin Plates

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Control of waves with metamaterials is of great topical interest, and is fueled by rapid progress in broadband acoustic and electromagnetic cloaks. We propose a design for a cloak to control bending waves propagating in isotropic heterogeneous thin plates. This is achieved through homogenization of a multilayered concentric coating filled with piecewise constant isotropic elastic material. Significantly, our cloak displays no phase shift for both backward and forward scattering. To foster experimental efforts, we provide a simplified design of the cloak which is shown to work in a more than two-octave frequency range (30 Hz to 150 Hz) when it consists of 10 layers using only 6 different materials overall. This metamaterial should be easy to manufacture, with potential applications ranging from car industry to anti-earthquake passive systems for smart buildings, depending upon the plate dimensions and wavelengths.

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Transformation based solutions to the conductivity and Maxwell's equations in curvilinear coordinate systems, recently reported by Greenleaf, Lassas, and Uhlmann [1], Pendry, Schurig, and Smith [2], and Leonhardt [3], have paved the way towards a better control of acoustic and electromagnetic waves around arbitrarily sized and shaped solids. The experimental validation of these theoretical considerations was given by Schurig *et al.* [4], who used a cylindrical cloak consisting of concentric arrays of split ring resonators which made a copper cylinder invisible to an incident plane wave at 8.5 GHz. However, in order to broaden the range of frequencies over which a cloaking metamaterial works, one needs to explore other routes avoiding resonant elements [5–9] with the first experimental results covering the range of 13 to 16 GHz [10].

However, Milton, Briane, and Willis [11] have shown that the elasticity equations are not invariant under coordinate transformations and consequently that if cloaking exists for such classes of waves, it would be of a different nature than its acoustic and electromagnetic counterparts.

A systematic investigation of acoustic cloaking started with Cummer and Schurig [12], who analyzed the two-dimensional cloaking for pressure waves in a transversely anisotropic fluid by exploiting the analogy with TE electromagnetic waves. Chen and Chan [13] and Cummer *et al.* [14] further noticed that a three-dimensional acoustic cloaking for pressure waves in a fluid can be envisaged since the wave equation retains its form under geometric changes. Farhat *et al.* also looked at the limitations of square acoustic cloaks for antiplane shear waves [15].

But when one moves to the area of coupled pressure and shear elastic waves, the isomorphism between the (tensor) governing equations and the wave equation is lost and computations become more involved. Norris investigated some general types of acoustic cloaks with finite mass consisting of so-called pentamode materials, which display an

anisotropic stiffness [16]. Brun, Guenneau, and Movchan studied a cylindrical cloak for in-plane elastic waves which is described by a rank-4 (nonsymmetric) elasticity tensor with 2⁴ Cartesian entries and an isotropic density [17]. Whereas the former structured metamaterial might already represent a technological challenge for mechanical engineers, the latter proposal imposes even severer constraints on the material parameters. Moreover, the required material properties for a three-dimensional elastic cloak remain elusive thus far, as these would involve a rank-4 elasticity tensor with up to 3⁴ spatially varying nonvanishing Cartesian entries. However, in the special case of thin-elastic plates, whose spectral properties require *ad hoc* numerical techniques [18], it has been shown [19] that the elasticity tensor can be represented in a cylindrical basis by a diagonal matrix with two (spatially varying) nonvanishing entries. It seems therefore quite natural to start designing such a cloak before investigating the other cases.

A promising avenue towards the realization of acoustic metamaterials was opened by Torrent and Sanchez-Dehesa [20] and Cheng *et al.* [21], who independently investigated cloaking for concentric multilayers behaving as anisotropic fluids in the homogenization limit. Using a similar approach, Farhat *et al.* [22] demonstrated cloaking of surface liquid waves for a microstructured metallic cloak which was experimentally validated at 10 Hz.

In this Letter, we show that a heterogeneous orthotropic cloak can be performed via a radially symmetric multilayered cloak filled with piecewise constant isotropic Young modulus E and mass density ρ to make an object surrounded by such a coat neutral for flexural waves in thin-elastic plates.

The equations for bending of plates are well known and can be found in many textbooks, such as those of Timoshenko or Graff [23,24]. The wavelength λ is supposed to be large enough compared to the thickness of the

plate h and small compared to its in-plane dimension L ; i.e., $h \ll \lambda \ll L$. In this case we can adopt the hypothesis of the theory of von Kármán [23,24]. If we choose to work in cylindrical coordinates, with all the above assumptions, the out-of-plane displacement $\mathbf{u} = [0, 0, U(r, \theta)]$ in the x_3 direction (along the vertical axis) is a solution of

$$\langle \lambda \rangle \nabla \cdot \{ \underline{\xi}^{-1} \nabla [\langle \lambda \rangle \nabla \cdot (\underline{\xi}^{-1} \nabla U)] \} - \beta_0^4 U = 0, \quad (1)$$

which is the homogenized biharmonic equation where $\langle \lambda \rangle = \int_0^1 \rho^{-1/2}(r) dr$ and $\beta_0^4 = \omega^2 \rho_0 h / D_0$ with $\underline{\xi}$ a homogenized rank-2 diagonal tensor (a case encompassed in [23,24]) which has the physical dimensions of a homogenized anisotropic Young's modulus $\underline{E} = \text{Diag}(E_r, E_\theta)$ given by

$$\underline{\xi} = \text{Diag}(\langle \xi^{-1} \rangle^{-1}, \langle \xi \rangle) = \underline{E}^{-1/2}. \quad (2)$$

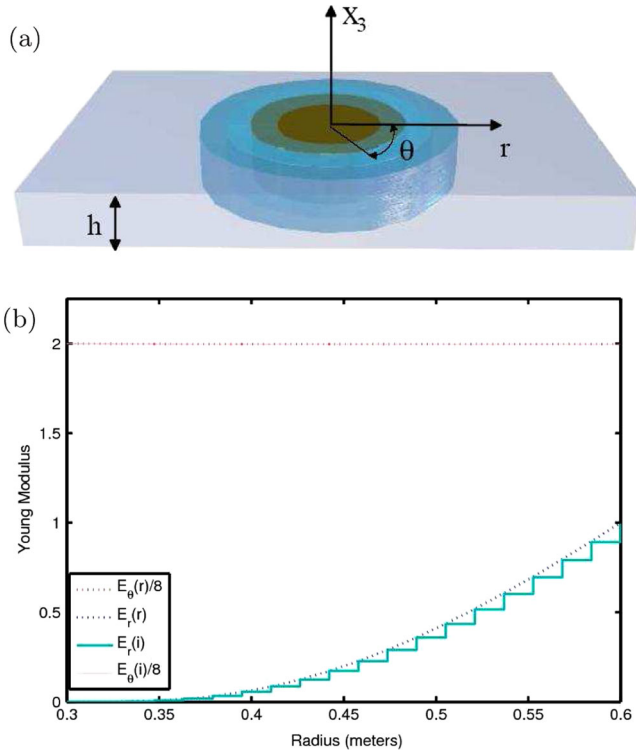


FIG. 1 (color online). (a) Sketch of the geometry of the thin metamaterial plate of thickness $h = 10^{-2}$ m, density $\rho = 10^3 \text{ kg m}^{-3}$, and flexural rigidity $D_0 = 10^2 \text{ N m}^2$. The Young modulus is isotropic and piecewise constant in the 40 isotropic homogeneous concentric layers constituting the cloak of inner radius $a = 0.3$ m and outer radius $b = 0.6$ m. The transverse dimensions of the plate are assumed to be large in comparison to the working wavelength. (b) Radial dependence of reduced elastic parameters $E_r(r)$ and $E_\theta(r)$ (in GPa units) obtained from the geometric transform (dotted lines) and corresponding elastic parameters $E_r(i)$ and $E_\theta(i)$, $i = 1, \dots, 20$, for the layered thin plate consisting of 10 cells alternating two isotropic homogeneous layers (solid lines). The density $\rho = 1 \text{ kg m}^{-3}$ and is not displayed.

We note that if the cloak consists of an alternation of two homogeneous isotropic layers of thicknesses d_A and d_B and Young's moduli E_A , E_B and densities ρ_A and ρ_B , we have

$$\frac{1}{E_r} = \frac{1}{1 + \eta} \left(\frac{1}{E_A} + \frac{\eta}{E_B} \right), \quad E_\theta = \frac{E_A + \eta E_B}{1 + \eta}, \quad (3)$$

$$\rho = \frac{\rho_A + \eta \rho_B}{1 + \eta},$$

where $\eta = d_B/d_A$ is the ratio of thicknesses for layers A and B and $d_A + d_B = 1$.

We now note that the coordinate transformation $r' = R_1 + r \frac{R_2 - R_1}{R_2}$ can compress the region $r < R_2$ into the ring $R_1 < r < R_2$, provided that the thin plate is described by the following reduced material parameters (the density is constant): [19]

$$E'_r = \left(\frac{R_2}{R_2 - R_1} \right)^4 \left(\frac{r - R_1}{r} \right)^4, \quad E'_\theta = \left(\frac{R_2}{R_2 - R_1} \right)^4, \quad \rho' = 1, \quad (4)$$

where R_1 and R_2 are the interior and the exterior radii of the elastic coat of thickness h . This ideal cloak is therefore not only anisotropic, but also spatially varying along the radius r .

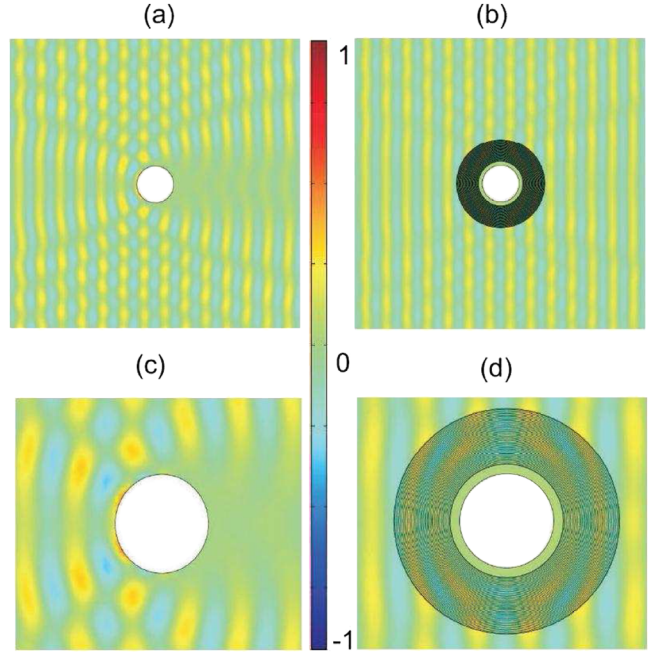


FIG. 2 (color online). Left: Real part of the displacement field U' scattered by a rigid clamped obstacle of radius 0.25 m for an incoming plane wave of frequency $\nu = 250$ Hz with (a) overview, (c) closer view. Right: Real part of the displacement field U' scattered by a rigid clamped obstacle surrounded by a multilayered isotropic cloak [inner radius $a = 0.3$ m, outer radius $b = 0.6$ m, and 40 layers, for an incoming plane wave of frequency $\nu = 250$ Hz with (b) overview, (d) closer view].

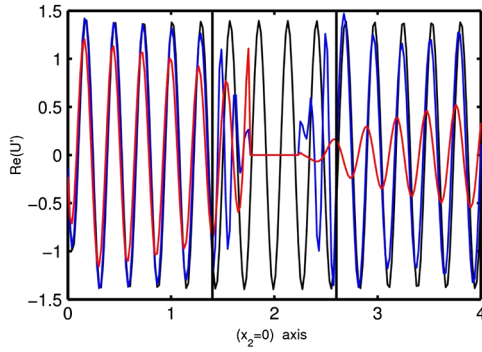


FIG. 3 (color online). Real part of the displacement U' along the line $x_2 = 0$ for a plane elastic wave of frequency $\nu = 250$ Hz incident from the left: on a clamped obstacle, as shown in Fig. 2 [light grey (red)]; on the layered cloak [dark grey (blue)]; in isotropic homogeneous plate (black).

To mimic these reduced parameters, we proceed in two steps, following [21]: we first approximate the ideal cloak by a multilayered cloak with M anisotropic homogeneous concentric layers (as shown in Fig. 1). We then approximate each layer i , $i = 1, \dots, M$ by N thin isotropic layers through the homogenization process described above. This means the overall number NM of isotropic layers can be fairly large.

We now turn to the numerical analysis of the field radiated by a plane wave vibrating harmonically in the x_3 direction and generating a harmonic vibration on the plate. Note that all the lengths are scaled in meters. A flexural plane wave at frequency $\nu = 250$ Hz is propagating from the left in Fig. 2 and is scattered by a clamped circular obstacle (left panels a, c) and by the layered cloak (right panels b, d). When it is surrounded by the heterogeneous cloak, which consists of a circular coating of radii $a = 0.3$ and $b = 0.6$ meters, both forward and backward scattering nearly vanish, with interestingly no phase shift between the elastic wave propagating in homogeneous plate and the wave bent by the cloak (whereas the elastic path is obviously much different); see Fig. 3. We note that this phase shift cannot be avoided in metamaterial electromagnetic cloaks [5,6]. The elastic parameters of the proposed layered cloak are characterized by a scalar density ρ' and a

spatially varying rank 2 tensor \underline{E}' given by (4) (see Fig. 1). We also show numerically that the cloak is ultrabroadband in nature, with the elastic cloaking extremely well preserved from $\nu = 115$ Hz to $\nu = 450$ Hz. Significantly, both forward and backward scattering worsen faster at low frequencies from comparisons between wave patterns at 50 Hz and at 1 kHz (see Fig. 1 in the supplementary material [25]).

To further demonstrate the validity of our approach to broadband cloaking, we now propose an original design for a cloak with 10 concentric layers of 6 materials (see Table I for a complete material description) which was numerically shown to greatly reduce forward and backward scattering for an incoming flexural plane wave ranging from 30 to 150 hertz. We report in Fig. 4 our computations for $\nu = 60$ hertz (panels a and b) and $\nu = 150$ hertz (panels c and d): Note that the geometry of the cloak has been changed in comparison to numerical simulations in order to minimize the variation of the Young's modulus and facilitate experiments. We hope this simplified set of material parameters will pave the way towards an experimental realization in the near future. It is important to note that ranges of working frequencies can be easily tuned by considering plates of various thicknesses, and other materials. The small-scale experiment we suggest would be directly applicable to antivibrating systems for the car industry. However, on a larger scale, one could design a shield for seismic waves which could be implemented within the infrastructure of smart buildings or around given zones to protect in cities. The main constraint in our broadband route to elastic cloaking is that the wavelength be of a larger magnitude than the plate thickness and also the layers within the cloak, but we have demonstrated that it could be built with materials at hand.

In conclusion, we have studied theoretically and numerically the extension of electromagnetic and acoustic cloaking mechanisms to the domain of flexural waves propagating in thin infinite elastic plates. We have proposed a design of a broadband multilayered cloak consisting of a large number of thin homogeneous isotropic layers. For this, we first derived the homogenized biharmonic equation using a multiscale asymptotic approach. We found that the homogenized elastic parameters are

TABLE I. This table describes the material parameters used in our simplified design of a thin-multilayered polymer cloak used in Fig. 4, and more details can be found in [26].

Material name	Density (10^3 kg m^{-3})	Young modulus (GPa)
Seals Eastern 7130 Fluorocarbon-Elastomer	1.92	0.0388
Westlake Plastics Clariflex [®] TFE/VF2/HFP	1.98	0.222
Daikin NEOFLON M-300H PCTFE Granular Molding Powder	2.10–2.14	1.03–2.06
LNP (SABIC Innovative Plastics) CYCOLOY [®] DC6305 ABS + PC Alloy	1.9	2.59
ITW Devcon Metal Repair Wear Resistant Epoxy (WR-2), Dark Grey	1.99	5.17
Plenco 1530 Polyester, Briquette, Injection Molded	1.99	20.2
Lord Adhesives Thermostat [®] ME-525 Underfill Encapsulant	1.7	4000

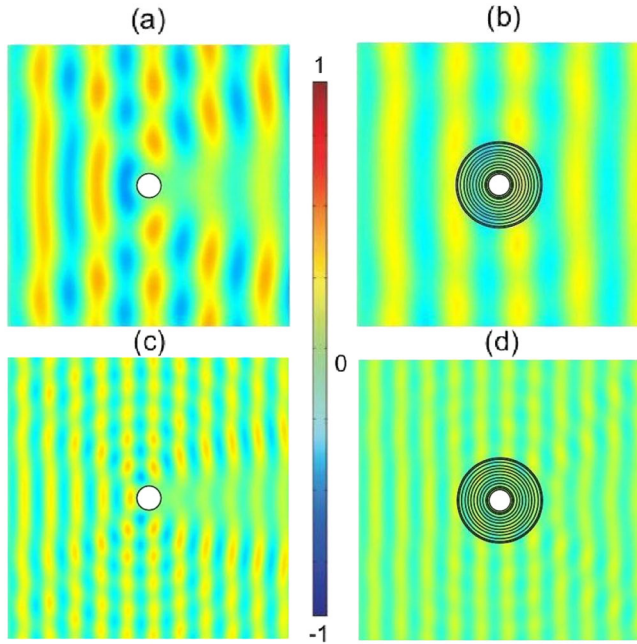


FIG. 4 (color online). Left: Real part of the displacement field U' scattered by a rigid clamped obstacle of radius 0.17 m for an incoming plane wave of frequency (a) $\nu = 60$ Hz, (c) $\nu = 150$ Hz. Right: Real part of the displacement field U' scattered by a rigid clamped obstacle surrounded by a multilayered isotropic cloak [inner radius $a = 0.175$ m, outer radius $b = 0.6$ m, and 10 layers of materials described in Table I, for an incoming plane wave of frequency (b) $\nu = 60$ Hz, (d) $\nu = 150$ Hz].

described by a rank-2 tensor (a generalized Young's modulus) and a scalar density, both of which are functions of the radius. We then performed numerical computations based on the finite elements method which proved that a rigid obstacle surrounded by a coating consisting of 40 concentric layers alternating two types of isotropic Young moduli and mass densities is neutral (vanishing backward and forward scattering) for the bending plane waves, over the ultralarge frequency range [50, 1000] hertz. We further proposed a simplified design of the cloak involving only 10 concentric layers of 6 materials. Such a metamaterial works over the frequency range [30, 150] hertz, but it should be easy to manufacture.

We note that unlike its acoustic and electromagnetic counterparts obtained through homogenization of radially symmetric metallic (or rigid) structures [6,22], the proposed flexural cloak does not induce any phase shift which is usually associated with the longer path followed by the waves smoothly bent by the cloak. Moreover, it is very easy to scale up or down the cloak in order to cover other ranges of frequencies. We thus believe such a structured plate could be manufactured with potential applications in isolating structures from vibrations of polymer (or metallic) plates in aeronautic, car, and ship industries. On a

larger scale, our design opens the way to make earthquake resistant infrastructures in smart buildings.

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 - [26] <http://www.matweb.com>.