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Rational solutions for the Riccati-Schrödinger equations associated to translationally shape invariant potentials

Y. Grandati and A. Bérard

Institut de Physique, ICPMB, IF CNRS 2843, Université Paul Verlaine, 1 Bd Arago, 57078 Metz, Cedex 3, France

We develop a new approach to build the eigenfunctions of a translationally shape-invariant potential. For this we show that their logarithmic derivatives can be expressed as terminating continued fractions in an appropriate variable. We give explicit formulas for all the eigenstates, their specific form depending on the Barclay-Maxwell class to which the considered potential belongs.

PACS numbers:

I. INTRODUCTION

In the framework of SUSY quantum mechanics, the concept of shape invariance is a key feature of explicit exact solvability^{7,8,9,10}. Combining the hierarchy of SUSY partner hamiltonians with the shape-invariance condition, it is possible, when the SUSY is unbroken, to obtain the entire spectrum of a SIP in terms of the functions characterizing the partners' correspondence. Moreover it gives an access to the eigenfunctions via a generalization of the harmonic creation-annihilation operators. Among all SIP those for which the partner's parameters are related by a translation (TSIP) play a particular role. Indeed they are so far the only ones for which we have closed-form expressions for the superpotentials and then for the eigenfunctions.

In this paper we propose an alternate way to determine the eigenfunctions of a TSIP. For this we construct analytically the logarithmic derivatives of the eigenfunctions, which we call hereafter the Riccati-Schrödinger (RS) functions of the system. They are solutions of a particular type of Riccati equations depending upon the energy as a parameter that we will call here Riccati-Schrödinger or RS equations. Using the finite-difference Bäcklund algorithm^{1,2,3,4,5,6} we obtain a terminating continued fraction expression for the RS functions in terms of the superpotential (a result outlined in a different way and in an incomplete form by Kazimierz¹¹). Then we consider two categories of potentials which can be reduced to an harmonic or isotonic form by a change of variable satisfying a constant coefficient Riccati equation. We show that their ground state RS function is a first degree polynomial or a first degree Laurent polynomial (depending on the category under consideration) in the new variable and that they are translationally shape invariant, giving general simple algebraic formula for the energy spectrum. Combining these results, we obtain exact rational expressions for the RS functions in terms of the new variable, which permits to recover in a simple way the eigenfunctions of the system. Finally we establish the equivalence between these two categories and the two Barclay-Maxwell classes of TSIP^{12,13} which shows that the above construction applies in fact to the whole set of TSIP.

II. BÄCKLUND ALGORITHM

A. Invariance group on the set of Riccati equations

As established by Cariñena et al.^{1,2}, the finite-difference Bäcklund algorithm is a consequence of the invariance of the set of Riccati equations under a subset of the group \mathcal{G} of smooth $SL(2, \mathbb{R})$ -valued curves $Map(\mathbb{R}, SL(2, \mathbb{R}))$. For any element $A \in \mathcal{G}$ characterized by the matrix:

$$A(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix}, \quad \det A(x) = \alpha(x)\delta(x) - \beta(x)\gamma(x) = 1, \quad (1)$$

the action of A on $Map(\mathbb{R}, \overline{\mathbb{R}})$ is given by:

$$w(x) \xrightarrow{A} \tilde{w}(x) = \frac{\alpha(x)w(x) + \beta(x)}{\gamma(x)w(x) + \delta(x)} = \frac{\alpha(x)}{\gamma(x)} - \frac{1}{\gamma(x)} \frac{1}{\gamma(x)w(x) + \delta(x)}. \quad (2)$$

If A acts on a solution of the Riccati equation:

$$w'(x) = a_0(x) + a_1(x)w(x) + a_2(x)w^2(x) \quad (3)$$

we obtain a solution of a new Riccati equation:

$$\tilde{w}'(x) = \tilde{a}_0(x) + \tilde{a}_1(x)\tilde{w}(x) + \tilde{a}_2(x)\tilde{w}^2(x) \quad (4)$$

the coefficients of which being given by $\vec{u}(x) = \begin{pmatrix} u_2(x) \\ u_1(x) \\ u_0(x) \end{pmatrix}$:

$$\vec{a}(x) = M(A)\vec{u}(x) + \vec{W}(x), \quad (5)$$

where:

$$M(A) = \begin{pmatrix} \delta^2(x) & -\gamma(x)\delta(x) & \gamma^2(x) \\ -2\beta(x)\delta(x) & \alpha(x)\delta(x) + \beta(x)\gamma(x) & -2\alpha(x)\gamma(x) \\ \beta^2(x) & -\alpha(x)\beta(x) & \alpha^2(x) \end{pmatrix}, \quad (6)$$

$$\vec{W}(x) = \begin{pmatrix} W(\gamma, \delta; x) \\ W(\delta, \alpha; x) + W(\beta, \gamma; x) \\ W(\alpha, \beta; x) \end{pmatrix} \quad (7)$$

and $W(f, g; x) = f(x)g'(x) - f'(x)g(x)$ is the wronskian of $f(x)$ and $g(x)$ in x . As noted in¹, Eq.(5) defines an affine action of \mathcal{G} on the set of general Riccati equations.

B. Riccati-Schrödinger equations

To a one-dimensional Schrödinger equation ($\hbar = 1, m = \frac{1}{2}$) for a potential $V(x)$:

$$\psi''(x) + (E - V(x))\psi(x) = 0 \quad (8)$$

the transformation:

$$w(x) = -\frac{\psi'(x)}{\psi(x)} \quad (9)$$

associates a particular Riccati equation of the form:

$$-w'(x) + w^2(x) = V(x) - E \quad (10)$$

which corresponds to Eq.(3) with the coefficients $a_0(x) = E - V(x)$, $a_1(x) = 0$ and $a_2(x) = 1$.

We'll call such an equation a Riccati-Schrödinger (RS) equation and $w(x)$ a RS function. Note that, up to a i factor, the RS function identifies with the quantum momentum function at energy E in the Quantum Hamilton-Jacobi formalism (QHJ) of Leacock and Padgett^{14,15} From the knowledge of the RS function, we recover immediately the corresponding wave function via:

$$\psi(x) \sim \exp\left(-\int^x w(s)ds\right) \quad (11)$$

and the Schrödinger equation Eq(8) can be rewritten:

$$H\psi(x) = (L^+L + E)\psi(x) = E\psi(x) \quad (12)$$

with

$$L = \frac{d}{dx} + w(x). \quad (13)$$

To each node of $\psi(x)$ (which is necessarily simple) corresponds a simple pole of $w(x)$. Moreover $w(x)$ decreases in the interval $[x_1, x_2]$, x_1 and x_2 being the turning points of the classical motion.

C. Finite difference Bäcklund algorithm

When applied to the RS equation Eq.(10), Eq(5) gives:

$$\begin{cases} \tilde{a}_2(x) = \delta^2(x) + (E - V(x))\gamma^2(x) + W(\gamma, \delta; x) \\ \tilde{a}_1(x) = -2\beta(x)\delta(x) - 2\alpha(x)\gamma(x)(E - V(x)) + W(\delta, \alpha; x) + W(\beta, \gamma; x) \\ \tilde{a}_0(x) = \beta^2(x) + \alpha^2(x)(E - V(x)) + W(\alpha, \beta; x). \end{cases} \quad (14)$$

The most general elements of \mathcal{G} preserving the subset of RS equations has been determined in¹. Among them we find in particular the elements of the form:

$$A(x) = \frac{1}{\sqrt{\lambda}} \begin{pmatrix} \phi(x) & \lambda - \phi^2(x) \\ -1 & \phi(x) \end{pmatrix}, \quad \lambda > 0 \quad (15)$$

where $\phi(x)$ satisfies an RS equation with the same potential as Eq.(10) but with a shifted energy:

$$-\phi'(x) + \phi^2(x) = V(x) - (E - \lambda) \quad (16)$$

With this choice Eq(14) becomes

$$\begin{cases} \tilde{a}_2(x) = \frac{1}{\lambda} (-\phi'(x) + \phi^2(x) + E - V(x)) = 1 \\ \tilde{a}_1(x) = \frac{2\phi(x)}{\lambda} (-\phi'(x) + \phi^2(x) + E - V(x)) - 2\phi(x) = 0 \\ \tilde{a}_0(x) = \frac{\phi^2(x)}{\lambda} (-\phi'(x) + \phi^2(x) + E - V(x)) + \lambda - 2\phi^2(x) - \phi'(x) = V(x) + 2\phi'(x) \end{cases} \quad (17)$$

and $\tilde{w}(x)$ satisfies the RS equation:

$$-\tilde{w}'(x) + \tilde{w}^2(x) = \tilde{V}_\phi(x) - \lambda \quad (18)$$

where $\tilde{V}_\phi(x) = V(x) + 2\phi'(x)$.

Suppose then that the spectrum of the hamiltonian H associated to $V(x)$ is $(E_l, \psi_l(x))$ with $l \geq 0$ not necessarily discrete. The corresponding RS functions $w_l(x) = -\psi_l'(x)/\psi_l(x)$ satisfy the RS equations associated with the different values of the energy:

$$-w_l'(x) + w_l^2(x) = V(x) - E_l. \quad (19)$$

Starting from a given $w_k(x)$, for every value of l such that $E_l > E_k$, we can build a element $A_{kl} \in \mathcal{G}$ of the form:

$$A_{kl}(x) = \frac{1}{\sqrt{E_l - E_k}} \begin{pmatrix} w_k(x) & E_l - E_k - w_k^2(x) \\ -1 & w_k(x) \end{pmatrix} \quad (20)$$

which transforms w_l as:

$$w_l(x) \xrightarrow{A_{kl}} \tilde{w}_l(x) = w_{kl}(x) = -w_k(x) + \frac{E_l - E_k}{w_k(x) - w_l(x)}, \quad (21)$$

where w_{kl} is a solution of the RS equation:

$$-w'_{kl}(x) + w_{kl}^2(x) = \tilde{V}_k(x) - E_l \quad (22)$$

with the same energy E_l as in Eq(19) but with a modified potential $\tilde{V}_k(x) = V(x) + 2w'_k(x) = w'_k(x) + w_k^2(x) + E_k$. This is the content of the finite-difference Bäcklund algorithm^{1,3,4,5,6}.

In the following we use a slightly different formulation of the Bäcklund algorithm:

If $w_{kl}(x)$ satisfy equation Eq(22), then for every l such that $E_l > E_k$:

$$w_l(x) = w_k(x) - \frac{E_l - E_k}{w_k(x) + w_{kl}(x)}. \quad (23)$$

In particular, choosing $k = 0$, we have for every excited level $E_l > E_0$:

$$w_l(x) = w_0(x) - \frac{E_l - E_0}{w_0(x) + w_{0l}(x)}, \quad (24)$$

where $w_{0l}(x)$ is a solution of:

$$-w'_{0l}(x) + w_{0l}^2(x) = \tilde{V}_0(x) - E_l \quad (25)$$

with $\tilde{V}_0(x) = V(x) + 2w'_0(x)$.

III. RS FUNCTIONS FOR SHAPE INVARIANT POTENTIALS AS TERMINATING CONTINUED FRACTIONS

A. Basics elements

We recall some basic elements concerning SUSY quantum mechanics and shape invariance^{7,8,9,10}.

Let $H_- = -d^2/dx^2 + V_-(x)$ be an hamiltonian the ground state energy E_0 of which is supposed to be zero: $E_0 = 0$. The RS function $w_0(x)$ associated to the corresponding ground state is called the superpotential of the system and we can write:

$$H_- = L^+L, \quad (26)$$

where $L = d/dx + w_0(x)$. The supersymmetric partner of the potential is V_- defined via:

$$H_+ = LL^+ = -\frac{d^2}{dx^2} + V_+(x). \quad (27)$$

We then have:

$$V_{\pm}(x) = \pm w'_0(x) + w_0^2(x). \quad (28)$$

The potential $V_-(x)$ is said to be shape invariant (SIP) if it depends upon a (multi)parameter $a \in \mathbb{R}^N$ and if we have the relation:

$$V_+(x, a) = V_-(x, f(a)) + R(a) \quad (29)$$

$R(a) \in \mathbb{R}$ and $f(a) \in \mathbb{R}^N$ being two given functions of a .

The complete spectrum of H_- is then given by:

$$E_n(a) = R(a) + R(a_1) + \dots + R(a_{n-1}) = \sum_{k=0}^{n-1} R(a_k), \quad (30)$$

where $a_k = f^{(k)}(a) = \overbrace{f \circ \dots \circ f}^{\text{k times}}(a)$.

When f is a simple translation $f(a) = a + \varepsilon$, $\varepsilon \in \mathbb{R}^N$, V_- is said to be translationally shape invariant and we call it a TSIP.

B. SIP and Bäcklund algorithm

We suppose below that the potential of the system is an SIP, $V_-(x, a)$, with a discrete spectrum part $(E_n(a))_{n \geq 0}$, the $E_n(a)$ forming an increasing sequence for every value of a with $E_0(a) = 0$.

Eq(25) becomes

$$-w'_{0n}(x, a) + w_{0n}^2(x, a) = \tilde{V}_0(x, a) - E_n(a) = V_-(x, a) + 2w'_0(x, a) - E_n(a), \quad (31)$$

that is

$$-w'_{0n}(x, a) + w_{0n}^2(x, a) = V_+(x, a) - E_n(a). \quad (32)$$

Using the shape-invariance condition Eq(29), we obtain:

$$-w'_{0n}(x, a) + w_{0n}^2(x, a) = V_-(x, a_1) + R(a) - E_n(a), \quad (33)$$

that is, for every m

$$-(w_{0n}(x, a) - w_m(x, a_1))' + w_{0n}^2(x, a) - w_m^2(x, a_1) = E_m(a_1) - E_n(a) + R(a). \quad (34)$$

If we take for m the specific value $m = n - 1$ and use Eq.(30), Eq.(34) becomes

$$-(w_{0n}(x, a) - w_{n-1}(x, a_1))' + w_{0n}^2(x, a) - w_{n-1}^2(x, a_1) = 0 \quad (35)$$

which is satisfied by

$$w_{0n}(x, a) = w_{n-1}(x, a_1). \quad (36)$$

Eq(24) can be now rewritten as

$$w_n(x, a) = w_0(x, a) - \frac{E_n(a)}{w_0(x, a) + w_{n-1}(x, a_1)} = w_0(x, a) - \frac{\sum_{k=0}^{n-1} R(a_k)}{w_0(x, a) + w_{n-1}(x, a_1)}. \quad (37)$$

By a direct iteration, using a standard notation for continued fractions, we arrive at:

* For $n = 1$:

$$w_1(x, a) = w_0(x, a) - \frac{R(a)}{w_0(x, a) + w_0(x, a_1)} \quad (38)$$

* For $n = 2$:

$$w_2(x, a) = w_0(x, a) - \frac{R(a) + R(a_1)}{w_0(x, a) + w_0(x, a_1)} \uparrow \frac{R(a_1)}{w_0(x, a_1) + w_0(x, a_2)} \quad (39)$$

* For $n = 3$:

$$w_3(x, a) = w_0(x, a) - \frac{R(a) + R(a_1) + R(a_2)}{w_0(x, a) + w_0(x, a_1)} \uparrow \frac{R(a_1) + R(a_2)}{w_0(x, a_1) + w_0(x, a_2)} \uparrow \frac{R(a_2)}{w_0(x, a_2) + w_0(x, a_3)} \quad (40)$$

...

and more generally:

$$w_n(x, a) = w_0(x, a) - \frac{E_n(a)}{w_0(x, a) + w_0(x, a_1)} \uparrow \dots \uparrow \frac{E_n(a) - E_{j-1}(a)}{w_0(x, a_{j-1}) + w_0(x, a_j)} \uparrow \dots \uparrow \frac{E_n(a) - E_{n-1}(a)}{w_0(x, a_{n-1}) + w_0(x, a_n)} \quad (41)$$

In¹¹ an incomplete version of such a formula has been outlined in a very different way. Note also that, working to connect the QHJ formalism and SUSY quantum mechanics, Rasinariu et al.¹⁶ had obtained the recursion relation Eq(37) for $n = 1$ in a distinct context and again in a very different way. Their exploitation of this result diverges from ours.

From the knowledge of the superpotential $w_0(x, a)$, Eq.(41) offers, on the basis of purely algebraic manipulations, a direct access to the whole set of discrete excited states. It has to be compared to the known formula^{7,8} giving the eigenstates of a SIP via the application of differential operators (generalizing the usual creator-annihilator ones) on the ground state:

$$\psi_n(x, a) \sim L^+(a)\psi_{n-1}(x, a_1) \sim L^+(a)\dots L^+(a_{n-1})\psi_0(x, a_n) \quad (42)$$

with $L(a_j) = d/dx + w_0(x, a_j)$. Note that Eq(42) can be easily retrieved from the recursion relation Eq(37). Indeed we have from this last:

$$w_{n-1}(x, a_1) = -w_0(x, a) + \frac{E_n(a)}{w_0(x, a) - w_n(x, a)} \quad (43)$$

with

$$E_n(a) = -(w_0(x, a) - w_n(x, a))' + w_0^2(x, a) - w_n^2(x, a_1) \quad (44)$$

Consequently

$$w_{n-1}(x, a_1) = w_n(x, a) - \frac{(w_0(x, a) - w_n(x, a))'}{w_0(x, a) - w_n(x, a)} \quad (45)$$

and

$$\psi_{n-1}(x, a_1) \sim (w_0(x, a) - w_n(x, a)) \psi_n(x, a) = L(a)\psi_n(x, a) \quad (46)$$

In both cases the knowledge of the ground state permits a complete reconstitution of the spectrum. Nevertheless Eq(41) avoids to use successive differentiations and is invariant with respect to changes of the position variable, a property which is useful in the following. We see how the preceding results apply to simple systems, namely the one-dimensional harmonic oscillator, the isotonic oscillator, the effective radial potential for the Kepler problem and the Morse potential.

C. One dimensional harmonic oscillator

The one dimensional harmonic oscillator potential is⁷:

$$V_-(x) = \frac{\omega^2}{4}x^2 + V_0 \quad (47)$$

and the RS equation for the ground state ($E_0 = 0$) is

$$-w'_0(x) + w_0^2(x) = \frac{\omega^2}{4}x^2 + V_0 \quad (48)$$

or

$$\left(\frac{d}{dx} - \left(w_0(x) + \frac{\omega}{2}x \right) \right) \left(w_0(x) - \frac{\omega}{2}x \right) = -V_0 - \frac{\omega}{2} \quad (49)$$

Choosing $V_0 = -\frac{\omega}{2}$, this equation admits clearly a polynomial solution:

$$w_0(x) = \frac{\omega}{2}x \quad (50)$$

which is such that $w'(x) > 0$ between the turning points of the classical motion (for a bound state we must have in this domain $E < V(x)$). As required, the ground-state eigenfunction associated to this superpotential does not present any nodes on the real line.

The SUSY partner of $V_-(x)$ is then:

$$V_+(x) = w'_0(x) + w_0^2(x) = \frac{\omega^2}{4}x^2 + \frac{\omega}{2} = V_-(x) + \omega \quad (51)$$

We recognize a SIP (cf Eq(29)) characterized by:

$$a = \frac{\omega}{2} = f(a), \quad a_k = a, \quad R(a) = 2a = \omega \quad (52)$$

that is a TSIP with a zero translation amplitude.

Consequently:

$$\begin{cases} E_n(a) - E_{j-1}(a) = \sum_{k=j-1}^{n-1} \omega = (n-j+1)\omega \\ w_0(x, a_{j-1}) + w_0(x, a_j) = \omega x \end{cases} \quad (53)$$

and Eq(41) becomes in this case:

$$w_n(x, \omega) = \frac{\omega}{2}x - \frac{n\omega}{\omega x - 1} \uparrow \dots \uparrow \frac{(n-j+1)\omega}{\omega x - 1} \uparrow \dots \uparrow \frac{1}{x}. \quad (54)$$

In particular:

$$\begin{cases} w_1(x) = \frac{\omega}{2}x - \frac{1}{x} \\ w_2(x) = \frac{\omega}{2}x - \frac{2\omega}{\omega x - 1/x} = \frac{\omega}{2}x - \frac{2\omega x}{\omega x^2 - 1} \\ w_3(x) = \frac{\omega}{2}x - \frac{3\omega}{\omega x - 2\omega/(\omega x - 1/x)} = \frac{\omega}{2}x - \frac{3(\omega x^2 - 1)}{x(\omega x^2 - 3)} \end{cases} \quad (55)$$

from which, with Eq(50), we deduce immediately the corresponding four first eigenstates:

$$\left\{ \begin{array}{l} \psi_0(x) = \exp\left(-\int^x w_0(s)ds\right) \sim \exp\left(-\frac{\omega}{4}x^2\right) \\ \psi_1(x) = \exp\left(-\int^x w_1(s)ds\right) \sim \exp\left(-\frac{\omega}{4}x^2\right) \exp\left(\int^x \frac{1}{s}ds\right) = x \exp\left(-\frac{\omega}{4}x^2\right) \\ \psi_2(x) = \exp\left(-\int^x w_2(s)ds\right) \sim \exp\left(-\frac{\omega}{4}x^2\right) \exp\left(\int^x \frac{2\omega s}{\omega s^2 - 1}ds\right) = (\omega x^2 - 1) \exp\left(-\frac{\omega}{4}x^2\right) \\ \psi_3(x) = \exp\left(-\int^x w_3(s)ds\right) \sim \exp\left(-\frac{\omega}{4}x^2\right) \exp\left(\int^x \frac{3(\omega s^2 - 1)}{\omega s^3 - 3s}ds\right) = (\omega x^3 - 3x) \exp\left(-\frac{\omega}{4}x^2\right). \end{array} \right. \quad (56)$$

D. Isotonic oscillator

The isotonic oscillator potential is^{17,18}:

$$V_-(x) = \frac{\omega^2}{4}x^2 + \frac{l(l+1)}{x^2} + V_0, \quad l > 0 \quad (57)$$

and the RS equation for the ground state ($E_0 = 0$):

$$-w'_0(x) + w_0^2(x) = \frac{\omega^2}{4}x^2 + \frac{l(l+1)}{x^2} + V_0 \quad (58)$$

Looking for a solution of the form $w_0(x) = \lambda x - \mu/x$, we obtain immediately that Eq(58) is satisfied when $\lambda = \omega/2$, $\mu = l + 1$ and $V_0 = -\omega\left(l + \frac{3}{2}\right)$, giving for the superpotential:

$$w_0(x) = \frac{\omega}{2}x - \frac{l+1}{x} \quad (59)$$

The signs have been chosen in order that $w'(x) > 0$ between the turning points of the classical motion. The SUSY partner of $V_-(x)$ is then:

$$V_+(x) = w'_0(x) + w_0^2(x) = \frac{\omega^2}{4}x^2 + \frac{(l+2)(l+1)\omega}{x^2} - \omega\left(l + \frac{3}{2}\right) + \omega. \quad (60)$$

We recognize a TSIP (cf Eq(29)) characterized by:

$$a = \left(\frac{\omega}{2}, l+1\right), \quad f(a) = \left(\frac{\omega}{2}, l+2\right), \quad a_k = \left(\frac{\omega}{2}, l+1+k\right), \quad R(a) = 2\omega. \quad (61)$$

Consequently

$$\left\{ \begin{array}{l} E_n(a) - E_{j-1}(a) = \sum_{k=j-1}^{n-1} 2\omega = 2(n-j+1)\omega \\ w_0(x, a_{j-1}) + w_0(x, a_j) = \omega x - (2(l+j)+1)/x \end{array} \right. \quad (62)$$

and Eq(41) becomes in this case:

$$w_n(x, a) = \frac{\omega}{2}x - \frac{l+1}{x} - \frac{2n\omega}{\omega x - (2l+3)/x} \uparrow \dots \uparrow \frac{2(n-j+1)\omega}{\omega x - (2(l+j)+1)/x} \uparrow \dots \uparrow \frac{2\omega}{\omega x - (2(l+n)-1)/x} \quad (63)$$

with in particular:

$$w_1(x, a) = \frac{\omega}{2}x - \frac{l+1}{x} - \frac{2\omega}{\omega x - (2l+3)/x}. \quad (64)$$

From Eq(59) and Eq(64) we deduce immediately the corresponding two first eigenstates:

$$\begin{cases} \psi_0(x) = \exp\left(-\int^x w_0(s, a)ds\right) \sim \exp\left(-\frac{\omega}{4}x^2\right) \exp\left((l+1)\int^x \frac{1}{s}ds\right) = x^{l+1} \exp\left(-\frac{\omega}{4}x^2\right) \\ \psi_1(x) = \exp\left(-\int^x w_1(s, a)ds\right) \sim x^{l+1} \exp\left(-\frac{\omega}{4}x^2\right) \exp\left(\int^x \frac{2\omega}{\omega s - (2l+3)/s}ds\right) = x^{l+1} (\omega x^2 - (2l+3)) \exp\left(-\frac{\omega}{4}x^2\right). \end{cases} \quad (65)$$

E. Effective radial potential for the Kepler-Coulomb problem

The effective radial potential for the Kepler-Coulomb is⁷:

$$V_-(x) = -\frac{\gamma}{x} + \frac{l(l+1)}{x^2} + V_0, \quad k > 0 \quad (66)$$

Defining $y = 1/x$, V_- takes the form:

$$V_-(y) = -\gamma y + l(l+1)y^2 + V_0, \quad k > 0 \quad (67)$$

and the RS equation for the ground state ($E_0 = 0$):

$$y^2 w_0'(y) + w_0^2(y) = -\gamma y + l(l+1)y^2 + V_0 \quad (68)$$

Looking for a solution of the form $w_0(x) = -\lambda y + \mu$, we obtain immediately that Eq(58) is satisfied when $\lambda = l+1$, $\mu = \gamma/2(l+1)$ and $V_0 = \gamma^2/4(l+1)^2$, giving for the superpotential:

$$w_0(y) = -(l+1)y + \frac{\gamma}{2(l+1)} \quad (69)$$

The signs have been chosen in order that $w'(x) > 0$ between the turning points of the classical motion. The SUSY partner of $V_-(x)$ is then:

$$V_+(y) = -2y^2 w_0'(y) + V_-(y) = -\gamma y + (l+1)(l+2)y^2 + \frac{\gamma^2}{4(l+1)^2} \quad (70)$$

We recognize a TSIP (cf Eq(29)) characterized by:

$$a = l+1, \quad f(a) = l+2, \quad a_k = l+1+k, \quad R(a_k) = \frac{\gamma^2}{4a_k^2} - \frac{\gamma^2}{4a_{k+1}^2}. \quad (71)$$

Consequently

$$E_n(a) = \sum_{k=0}^{n-1} R(a_k) = \frac{\gamma^2}{4a^2} - \frac{\gamma^2}{4a_n^2} = \frac{\gamma^2}{4(l+1)^2} - \frac{\gamma^2}{4(l+1+n)^2} \quad (72)$$

and

$$\left\{ \begin{array}{l} E_n(a) - E_{j-1}(a) = \frac{\gamma^2}{4a_{j-1}^2} - \frac{\gamma^2}{4a_n^2} = \frac{\gamma^2}{4} \left(\frac{1}{(l+j)^2} - \frac{1}{(l+1+n)^2} \right) \\ w_0(x, a_{j-1}) + w_0(x, a_j) = -(a_{j-1} + a_j)y + \frac{\gamma}{2} \left(\frac{1}{a_{j-1}} + \frac{1}{a_j} \right) = -2 \left(l + j + \frac{1}{2} \right) y + \frac{\gamma}{2} \left(\frac{1}{l+j} + \frac{1}{l+1+j} \right) \end{array} \right. \quad (73)$$

Eq(41) becomes in this case:

$$\begin{aligned} w_n(y, a) &= -ay + \frac{\gamma}{2a} - \frac{\gamma^2/4a^2 - \gamma^2/4a_n^2}{-(a+a_1)y + \gamma/2(1/a+1/a_1)} \uparrow \\ &\dots \uparrow \frac{\gamma^2/4a_{j-1}^2 - \gamma^2/4a_n^2}{-(a_{j-1}+a_j)y + \gamma/2(1/a_{j-1}+1/a_j)} \uparrow \\ &\dots \uparrow \frac{\gamma^2/4a_{n-1}^2 - \gamma^2/4a_n^2}{-(a_{n-1}+a_n)y + \gamma/2(1/a_{n-1}+1/a_n)} \end{aligned} \quad (74)$$

In particular:

$$w_1(y, a) = w_0(y) + \left(\frac{\gamma}{2aa_1} \right)^2 \frac{1}{y - \gamma/2aa_1} \quad (75)$$

or

$$w_1(x, a) = w_0(x) + \frac{1}{2aa_1/\gamma - x} - \frac{\gamma}{2aa_1} \quad (76)$$

From Eq(69) and Eq(76) we deduce immediately the corresponding two first eigenstates:

$$\psi_0(x) = \exp \left(- \int w_0(x, a) dx \right) = \exp \left(\int \left(\frac{(l+1)}{x} - \frac{\gamma}{2(l+1)} \right) dx \right) \quad (77)$$

$$\sim x^{l+1} \exp \left(- \frac{\gamma}{2(l+1)} x \right) \quad (78)$$

and:

$$\begin{aligned} \psi_1(x) &= \exp \left(- \int w_1(x, a) dx \right) = \exp \left(- \int w_0(x, a) dx \right) \exp \left(\int \left(\frac{1}{x - 2aa_1/\gamma} + \frac{\gamma}{2aa_1} \right) dx \right) \\ &\sim x^{l+1} \left(x - \frac{2(l+1)(l+2)}{\gamma} \right) \exp \left(- \frac{\gamma}{2(l+2)} x \right). \end{aligned} \quad (79)$$

F. Morse potential

The Morse potential is⁷:

$$V_-(x) = A^2 + B^2 e^{-2\alpha x} - 2B \left(A + \frac{\alpha}{2} \right) e^{-\alpha x}, \quad \alpha > 0 \quad (80)$$

Defining $y = \exp(-\alpha x)$, V_- takes the form:

$$V_-(y) = (By - A)^2 - \alpha By. \quad (81)$$

In terms of the y variable, the associated RS equation is:

$$-\alpha y w'_n(y) = E_n - (By - A)^2 + \alpha By + w_n^2(y)$$

and for the ground state, $E_0 = 0$, we have

$$\left(\alpha y \frac{d}{dy} + (w_0(y) - (By - A)) \right) (w_0(y) + (By - A)) = 0 \quad (82)$$

a solution of which is immediately obtained as

$$w_0(y) = -By + A \quad (83)$$

The SUSY partner of $V_-(x)$ is then:

$$V_+(y) = -\alpha y w'_0(y) + w_0^2(y) = (By - A + \alpha)^2 - \alpha By + \alpha^2 - 2\alpha A \quad (84)$$

We recognize a TSIP (cf Eq(29)) characterized by:

$$a = A, \quad f(a) = A - \alpha, \quad R(a) = \alpha^2 - 2\alpha = a^2 - a_1^2 \quad (85)$$

Consequently

$$E_n(a) = \sum_{k=0}^{n-1} R(a_k) = a^2 - a_n^2 = n\alpha(2a - n\alpha) \quad (86)$$

and

$$\begin{cases} E_n(a) - E_{j-1}(a) = \sum_{k=j-1}^{n-1} (a_k^2 - a_{k+1}^2) = a_{j-1}^2 - a_n^2 \\ w_0(x, a_{j-1}) + w_0(x, a_j) = a_{j-1} + a_j - 2By \end{cases} \quad (87)$$

Eq(41) becomes in this case:

$$w_n(y, a) = a - By - \frac{a^2 - a_n^2}{a + a_1 - 2By} \uparrow \dots \uparrow \frac{a_{j-1}^2 - a_n^2}{a_{j-1} + a_j - 2By} \uparrow \dots \uparrow \frac{a_{n-1}^2 - a_n^2}{a_{n-1} + a_n - 2By} \quad (88)$$

In particular:

$$\begin{cases} w_1(y, a) = a - By - \frac{a^2 - a_1^2}{a + a_1 - 2By} \\ w_2(y, a) = a - By - \frac{a^2 - a_2^2}{a + a_1 - 2By - (a_1^2 - a_2^2)/(a_1 + a_2 - 2By)} \end{cases} \quad (89)$$

From Eq(83) and Eq(89) we deduce immediately the corresponding two first eigenstates:

$$\psi_0(x) = \exp\left(-\int w_0(x, a) dx\right) = \exp\left(\frac{1}{\alpha} \int \frac{w_0(y, a)}{y} dy\right) \sim y^{A/\alpha} \exp\left(-\frac{B}{\alpha} y\right) = \exp\left(-Ax - \frac{B}{\alpha} \exp(-\alpha x)\right) \quad (90)$$

and:

$$\begin{aligned} \psi_1(x) &= \exp\left(-\int w_1(x, a) dx\right) = \exp\left(\frac{1}{\alpha} \int \frac{w_1(y, a)}{y} dy\right) \sim y^{(A-\alpha)/\alpha} \exp\left(-\frac{B}{\alpha} y\right) \left(By - \left(A - \frac{1}{2}\alpha \right) \right) \\ &\sim \exp\left(-\left(A - \alpha \right) x - \frac{B}{\alpha} e^{-\alpha x}\right) \left(B e^{-\alpha x} - \left(A - \frac{1}{2}\alpha \right) \right). \end{aligned} \quad (91)$$

IV. POTENTIALS OF FIRST AND SECOND CATEGORIES

A. Definitions

1. First category

We say that a one-dimensional potential is of first category if there exists a change of variable $x \rightarrow u$ transforming the potential into an harmonic one $V(x) \rightarrow V(u) = \tilde{\lambda}_2 u^2 + \tilde{\lambda}_1 u + \tilde{\lambda}_0$, such that $u(x)$ satisfies a constant coefficient Riccati equation:

$$\frac{du(x)}{dx} = A_0 + A_1 u(x) + A_2 u^2(x), \quad (92)$$

$du(x)/dx$ being of constant sign in all the range of values of x and u .

The one-dimensional harmonic oscillator corresponds to the special case $A_1 = A_2 = 0$ and the Morse potential is generically associated to the case $A_1 \neq 0$, $A_2 = 0$. These two examples have been treated above.

We note $\Delta = A_1^2 - 4A_2A_0$ the discriminant of the right hand side polynomial in Eq(92).

If $A_2 \neq 0$, $\Delta \neq 0$, it is always possible, via an affine change of variable $u = ay + b$, to reduce Eq(92) to the canonical form:

$$\frac{dy}{dx} = \alpha \pm \alpha y^2(x) > 0. \quad (93)$$

If $A_2 \neq 0$, $\Delta = 0$, a straightforward affine change of variable brings Eq(92) to the form:

$$\frac{dy}{dx} = \alpha y^2(x). \quad (94)$$

that is $y = \frac{1}{\alpha(x-x_0)}$ which corresponds to the radial effective Kepler-Coulomb potential studied above.

In the following we then consider that all the first category potentials are of the type:

$$\begin{cases} V(x) \rightarrow V(y) = \lambda_2 y^2 + \lambda_1 y + \lambda_0, & \lambda_2 > 0 \\ dy/dx = \alpha \pm \alpha y^2(x) > 0. \end{cases} \quad (95)$$

The new variable is then $y = \tan(\alpha x + \varphi_0)$ for the positive sign and $y = \tanh(\alpha x + \varphi_0)$ or $y = \coth(\alpha x + \varphi_0)$ (depending on the sign of α) for the negative sign.

In this category we find all the potentials listed in the Annexe A which contains also the three exceptional cases corresponding to the harmonic, Morse and effective radial Kepler-Coulomb potentials.

2. Second category

We say that a one dimensional potential is of second category if there exists a change of variable $x \rightarrow u$ transforming the potential into an isotonic one $V(x) \rightarrow V(u) = \tilde{\lambda}_2 u^2 + \tilde{\lambda}_0 + \tilde{\mu}_2/u^2$, such that $u(x)$ satisfies a constant coefficient Riccati equation of the form

$$\frac{du(x)}{dx} = A_0 + A_2 u^2(x) \quad (96)$$

$du(x)/dx$ being of constant sign in all the range of values of x and u .

When A_2 and A_0 are nonzero, a linear change of variable $u = ay$ reduces Eq(96) to the canonical form:

$$\begin{cases} V(x) \rightarrow V(y) = \lambda_2 y^2 + \frac{\mu_2}{y^2} + \lambda_0, & \lambda_2, \mu_2 > 0 \\ dy/dx = \alpha \pm \alpha y^2(x) > 0. \end{cases}, \quad (97)$$

Again, the new variable is then $y = \tan(\alpha x + \varphi_0)$ for the positive sign and $y = \tanh(\alpha x + \varphi_0)$ for the negative sign (due to the $y \rightarrow 1/y$ symmetry of the functional form of V , we can limit ourselves to this two solutions).

The example of the isotonic potential, treated above, is generically associated to the cases $A_0 = 0$ or $A_2 = 0$.

In this category we find all the potentials listed in the Annexe B.

V. POLYNOMIAL SOLUTIONS FOR THE RS EQUATIONS ASSOCIATED TO FIRST AND SECOND CATEGORY POTENTIALS

A. First category

The RS equation associated to a first category potential $V(x)$ is in terms of the y variable(see Eq.(95)):

$$-(\alpha_0 + \alpha_1 y + \alpha_2 y^2) w'(y) + w^2(y) = \lambda_2 y^2 + \lambda_1 y + \lambda_0 - E \quad (98)$$

($\alpha_0 = \alpha = \pm\alpha_2$, $\alpha_1 = 0$).

We look for a polynomial solution of degree N of this equation, describing a bound state of the system. Inserting

$$w(y) = \sum_{n=0}^N b_n y^n, \quad b_N \neq 0 \quad (99)$$

into Eq.(98), we obtain:

$$-\sum_{l=0}^{N+1} \left(\sum_{\substack{n=0 \\ n+m=l}}^{N-1} \sum_{m=0}^2 (n+1) \alpha_m b_{n+1} \right) y^l + \sum_{l=0}^{2N} \left(\sum_{\substack{n,m=0 \\ n+m=l}}^N b_n b_m \right) y^l = \sum_{n=0}^2 (\lambda_n - E \delta_{n,0}) y^n. \quad (100)$$

Since $b_N \neq 0$, we must have $N+1 = 2N \Leftrightarrow N = 1$ or $2 = 2N \Leftrightarrow N = 1$. The polynomial is necessarily of degree 1:

$$w(y) = b_0 + b_1 y. \quad (101)$$

Since $y(x)$ satisfies a constant coefficient Riccati equation, we recover the first Gendenshtein ansatz⁹.

For a bound state $E < V(x)$, that is $w'(x) > 0$, between the turning points of the classical motion and b_1 is of the same sign as $y'(x)$, that is positive. In this domain $w(x)$ is not singular and the wave function does not present any node, which corresponds to the ground state.

Inserting Eq.(101) into Eq.(100) we obtain for the unknown coefficients b_0, b_1 a set of 3 equations:

$$\begin{cases} -\alpha b_1 + b_0^2 = \lambda_0 - E \\ 2b_0 b_1 = \lambda_1 \\ \mp \alpha b_1 + b_1^2 = \lambda_2 > 0 \end{cases} \quad (102)$$

which imply a constraint on E fixing the value of the ground state energy.

After straightforward calculations we arrive at

$$\begin{cases} b_0 = \lambda_1 / 2\beta_{\pm}(\lambda_2) \\ b_1 = \beta_{\pm}(\lambda_2) > 0 \end{cases} \quad (103)$$

with

$$\beta_{\pm}(\lambda) = \pm \alpha / 2 + \sqrt{(\alpha/2)^2 + \lambda}. \quad (104)$$

The ground state energy is given by

$$E_0 = \lambda_0 \pm \alpha \beta_{\pm}(\lambda_2) - (\lambda_1 / 2\beta_{\pm}(\lambda_2))^2 \quad (105)$$

and the corresponding RS function:

$$w_0(y) = \frac{\lambda_1}{2\beta_{\pm}(\lambda_2)} + \beta_{\pm}(\lambda_2) y \quad (106)$$

B. Second category

The RS equation associated to a first category potential $V(x)$ is in terms of the y variable(see Eq.(97)):

$$-(\alpha_0 + \alpha_1 y + \alpha_2 y^2) w'(y) + w^2(y) = \lambda_2 y^2 + \lambda_0 + \frac{\mu_2}{y^2} - E \quad (107)$$

($\alpha_0 = \alpha = \pm \alpha_2$, $\alpha_1 = 0$).

We look for a solution of Eq.(107) which is a Laurent polynomial, that is a polynomial in y and $1/y$. Inserting

$$w(y) = \sum_{n=-M}^N b_n y^n, \quad b_N, b_{-M} \neq 0 \quad (108)$$

into Eq.(107) we obtain:

$$-\sum_{l=-M+1}^{N+1} \left(\sum_{\substack{n=-M-1 \\ n+m=l}}^{N-1} \sum_{m=0}^2 (n+1) A_m b_{n+1} \right) y^l + \sum_{l=-2M}^{2N} \left(\sum_{\substack{n,m=-M \\ n+m=l}}^N b_n b_m \right) y^l = \lambda_2 y^2 + \lambda_0 + \frac{\mu_2}{y^2} - E \quad (109)$$

Since $b_N, b_{-M} \neq 0$, we must have $N+1 = 2N \Leftrightarrow N = 1$ or $2 = 2N \Leftrightarrow N = 1$ and $2M = M-1 \Leftrightarrow M = 1$ or $2 = 2M \Leftrightarrow M = 1$. The Laurent polynomial is then of degree 1, as for the regular part as for the singular part:

$$w(y) = b_0 + b_1 y + \frac{b_{-1}}{y}. \quad (110)$$

Since $y(x)$ satisfies now a constant coefficients Riccati equation without a term of first degree, we recover the second Gendenshtein ansatz⁹.

For a bound state $E < V(x)$, that is $w'(x) > 0$, between the turning points of the classical motion and $b_1 - b_{-1}/y^2$ is of the same sign as $y'(x)$, that is positive. This necessitates that b_1 is positive and b_{-1} negative. In this domain $w(x)$ is not singular and the wave function does not present any node, which corresponds to the ground state.

Inserting Eq.(110) into Eq.(109), we obtain for the unknown coefficients b_0, b_1 and b_{-1} a set of 4 equations:

$$\left\{ \begin{array}{l} E - \lambda_0 = \alpha b_1 - b_0^2 \mp \alpha b_{-1} - 2b_1 b_{-1} \\ b_0 b_{-1} = b_0 b_1 = 0 \\ b_1^2 \mp b_1 \alpha = \lambda_2 > 0 \\ b_{-1}^2 + b_{-1} \alpha = \mu_2 > 0 \end{array} \right. \quad (111)$$

which imply a constraint on E , fixing the value of the ground state energy.

Eq.(111) gives (see Eq.(104)):

$$\left\{ \begin{array}{l} b_0 = 0 \\ b_1 = \beta_{\pm}(\lambda_2) > 0 \\ b_{-1} = -\beta_{+}(\mu_2) < 0. \end{array} \right. \quad (112)$$

The ground-state energy is given by:

$$E_0 = \lambda_0 + \alpha (\beta_{\pm}(\lambda_2) \pm \beta_{+}(\mu_2)) + 2\beta_{\pm}(\lambda_2) \beta_{+}(\mu_2) \quad (113)$$

and the corresponding RS function is

$$w_0(y) = \beta_{\pm}(\lambda_2) y - \frac{\beta_{+}(\mu_2)}{y} \quad (114)$$

VI. SHAPE INVARIANCE PROPERTIES OF FIRST AND SECOND CATEGORY POTENTIALS

A. First category

We consider a first category potential (see Eq.(95)) $V_-(y) = \lambda_2 y^2 + \lambda_1 y + \lambda_0$, $\lambda_2 > 0$, for which $dy/dx = \alpha \pm \alpha y^2(x) > 0$ and $E_0 = 0$. This last constraint implies (see Eq.(105)) that

$$\lambda_0 = \lambda_0(\lambda_2) = \left(\frac{\lambda_1}{2\beta_{\pm}(\lambda_2)} \right)^2 - \alpha\beta_{\pm}(\lambda_2).$$

The SUSY partner of $V_-(y)$ is

$$V_+(y) = V_-(y) + 2w'_0(x) = V_-(y) + 2w'_0(y) \frac{dy}{dx} \quad (115)$$

$$= y^2 (\lambda_2 \pm 2\alpha\beta_{\pm}(\lambda_2)) + \lambda_1 y + \lambda_0(\lambda_2) + 2\alpha\beta_{\pm}(\lambda_2). \quad (116)$$

We make the following change of parameter:

$$a = \beta_{\pm}(\lambda_2) \quad (117)$$

in which case

$$\lambda_2 = a(a \mp \alpha), \quad \lambda_0(a) = \left(\frac{\lambda_1}{2a} \right)^2 - \alpha a.$$

The initial potential is now

$$V_-(y, a) = a(a \mp \alpha)y^2 + \lambda_1 y + \lambda_0(a) \quad (118)$$

and its SUSY partner is

$$\begin{aligned} V_+(y, a) &= y^2 a(a \pm \alpha) + \lambda_1 y + \lambda_0(a) + 2\alpha a \\ &= y^2 a_1(a_1 \mp \alpha) + \lambda_1 y + \lambda_0(a_1) + R(a) \end{aligned} \quad (119)$$

with:

$$\begin{cases} a_1 = a \pm \alpha \\ R(a) = \lambda_0(a) - \lambda_0(a_1) + 2\alpha a = \phi_{1,\pm}(a) - \phi_{1,\pm}(a_1), \end{cases} \quad (120)$$

where

$$\phi_{1,\pm}(a) = \mp a^2 + \frac{\lambda_1^2}{4a^2} \quad (121)$$

We recognize (see Eq.(29)) the characteristic behaviour of a TSIP. More generally

$$\begin{cases} a_k = a \pm k\alpha \\ R(a_k) = \phi_{1,\pm}(a_k) - \phi_{1,\pm}(a_{k+1}), \end{cases} \quad (122)$$

The spectrum (see Eq.(30)) becomes

$$E_n(a) = \phi_{1,\pm}(a) - \phi_{1,\pm}(a_n) \quad (123)$$

or

$$E_n(a) = \phi_{1,\pm}(a) \pm \alpha^2 \left(n \pm \frac{a}{\alpha} \right)^2 - \frac{\lambda_1^2}{4\alpha^2} \frac{1}{(n \pm a/\alpha)^2} \quad (124)$$

E_n is then an isotonic function of $n \pm a/\alpha$. As for the corresponding superpotential, it takes the form (see Eq.(106)):

$$w_0(y, a) = ay + \frac{\lambda_1}{2a}, \quad a > 0 \quad (125)$$

Using Eq.(41), Eq.(123) and Eq.(125) we deduce finally the general form for the RS function associated to the n^{th} level of the spectrum of an arbitrary potential of the first category:

$$\begin{aligned} w_n(y, a) &= ay + \frac{\lambda_1}{2a} - \frac{\phi_{1,\pm}(a) - \phi_{1,\pm}(a_n)}{(a + a_1)y + \frac{\lambda_1}{2}(1/a + 1/a_1)} \uparrow \dots \\ &\uparrow \frac{\phi_{1,\pm}(a_{j-1}) - \phi_{1,\pm}(a_n)}{(a_{j-1} + a_j)y + \frac{\lambda_1}{2}(1/a_{j-1} + 1/a_j)} \uparrow \dots \\ &\uparrow \frac{\phi_{1,\pm}(a_{n-1}) - \phi_{1,\pm}(a_n)}{(a_n + a_{n-1})y + \frac{\lambda_1}{2}(1/a_{n-1} + 1/a_n)} \end{aligned} \quad (126)$$

B. Second category

We consider a second category potential (see Eq.(97)) $V_-(y) = \lambda_2 y^2 + \lambda_0 + \frac{\mu_2}{y^2}$, for which $dy/dx = \alpha \pm \alpha y^2(x) > 0$ and $E_0 = 0$. This last constraint implies (see Eq.(113)) that

$$\lambda_0 = -\alpha (\beta_{\pm}(\lambda_2) \pm \beta_+(\mu_2)) - 2\beta_{\pm}(\lambda_2) \beta_+(\mu_2).$$

The SUSY partner of $V_-(y)$ is:

$$\begin{aligned} V_+(y) &= V_-(y) + 2w'_0(x) = V_-(y) + 2w'_0(y) \frac{dy}{dx} \\ &= y^2 (\lambda_2 \pm 2\alpha\beta_{\pm}(\lambda_2)) + \frac{1}{y^2} (\mu_2 + 2\alpha\beta_+(\mu_2)) + \lambda_0 + 2\alpha (\beta_{\pm}(\lambda_2) \pm \beta_+(\mu_2)) \end{aligned} \quad (127)$$

We define the following multiparameter:

$$a = (\lambda, \mu) \begin{cases} \lambda = \beta_{\pm}(\lambda_2) > 0, \quad \lambda_2 = \lambda(\lambda \mp \alpha) \\ \mu = \beta_+(\mu_2) > 0, \quad \mu_2 = \mu(\mu - \alpha). \end{cases} \quad (128)$$

The initial potential is now

$$V_-(y, a) = \lambda(\lambda \mp \alpha) y^2 + \frac{\mu(\mu - \alpha)}{y^2} + \lambda_0(a) \quad (129)$$

with

$$\lambda_0(a) = -\alpha(\lambda \pm \mu) - 2\lambda\mu \quad (130)$$

Its SUSY partner is

$$\begin{aligned} V_+(y, a) &= \lambda(\lambda \pm \alpha) y^2 + \frac{\mu(\mu + \alpha)}{y^2} + \lambda_0(a) + 2\alpha(\lambda \pm \mu) \\ &= V_-(y, a_1) + R(a), \end{aligned} \quad (131)$$

where

$$\begin{cases} a_1 = (\lambda_1, \mu_1) = (\lambda \pm \alpha, \mu + \alpha) = a + A_{\pm} \\ R(a) = \lambda_0(a) - \lambda_0(a_1) + 2\alpha(\lambda \pm \mu) = \pm(\phi_{2,\pm}(a_1) - \phi_{2,\pm}(a)), \end{cases} \quad (132)$$

with $A_{\pm} = (\pm\alpha, \alpha)$ and:

$$\phi_{2,\pm}(a) = \phi_{2,\pm}(\lambda, \mu) = (\lambda \pm \mu)^2. \quad (133)$$

We recognize (see Eq.(29)) the characteristic behaviour of a TSIP. More generally

$$\begin{cases} a_k = a + kA = (\lambda_k, \mu_k) = (\lambda \pm k\alpha, \mu + k\alpha) \\ R(a_k) = \pm(\phi_{2,\pm}(a_{k+1}) - \phi_{2,\pm}(a_k)), \end{cases} \quad (134)$$

The spectrum (see Eq.(30)) becomes

$$E_{n,\pm}(a) = \pm(\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a)) \quad (135)$$

or

$$E_{n,\pm}(a) = \pm 4\alpha^2 \left(n - \frac{\lambda \pm \mu}{2\alpha} \right)^2 \mp \phi_{2,\pm}(a) \quad (136)$$

$E_{n,\pm}$ is then an harmonic function of $n + (\mu \pm \lambda)/2\alpha$. As for the corresponding superpotential, it takes the form (see Eq.(114)):

$$w_0(y, a) = \lambda y - \frac{\mu}{y}. \quad (137)$$

Using Eq.(41), Eq.(135) and Eq.(137), we finally deduce the general form for the RS function associated to the n^{th} level of the spectrum of an arbitrary potential of the second category of the considered type:

$$\begin{aligned} w_{n,\pm}(y, a) &= \lambda y - \frac{\mu}{y} \mp \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a)}{(\lambda + \lambda_1)y - (\mu + \mu_1)/y \mp} \mp \dots \\ &\mp \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a_{j-1})}{(\lambda_{j-1} + \lambda_j)y - (\mu_{j-1} + \mu_j)/y \mp} \mp \dots \\ &\mp \frac{\phi_{2,\pm}(a_n) - \phi_{2,\pm}(a_{n-1})}{(\lambda_{n-1} + \lambda_n)y - (\mu_{n-1} + \mu_n)/y} \end{aligned} \quad (138)$$

It is possible to authorize a complex phase φ_0 in the definition of $y(x)$ (see Eq.(97)) which becomes complex. The real part of corresponding to a shift of the variable, we can suppose φ_0 purely imaginary: $\varphi_0 = i\theta_0$. Consider the case where:

$$\frac{dy}{dx} = \alpha - \alpha y^2(x), \quad (139)$$

with

$$y(x) = \tanh(\alpha x + i\theta_0) = -i \frac{\tanh(\alpha x) + i \tan(\theta_0)}{\tanh(\alpha x) + i \cot(\theta_0)}. \quad (140)$$

If $\theta_0 = \pi/4$, $1/y(x) = \overline{y(x)}$ and the reality of $w_0(y, a)$ and consequently of every $w_n(y, a)$ is ensured when $\mu = -\bar{\lambda}$. We see immediately that $\phi_{2,-}(a)$, that is $E_{n,-}(a)$ (cf Eq.(133) and Eq.(135)), and $V_-(y, a)$ (cf Eq.(129)) are also real. We recover in this case the Scarf II potential (see Annexe B).

Since $\tanh(\theta_0)$ cannot be equal to 1, the same reasoning cannot be used when $dy/dx = \alpha + \alpha y^2(x)$, that is, when $y(x) = \tan(\alpha x + \varphi_0)$. It cannot be neither adapted for the first category potentials.

C. Comparison with the Barclay-Maxwell classification of TSIP

1. Barclay-Maxwell classification of TSIP

Concerning the case of TSIP, using exactness arguments of the SWKB condition^{7,19,20}, Barclay and Maxwell^{12,13} have established that their associated superpotentials obey one or other of the following equations:

* Class I potentials:

$$w_0'(x) = \alpha_0 + \alpha_1 w_0(x) + \alpha_2 w_0^2(x) \quad (141)$$

$\alpha_i \in \mathbb{R}$, $i = 0, 1, 2$

* Class II potentials:

$$w_0'(x) = \alpha_0 + \alpha_1 w_0(x) \sqrt{\alpha_0 + \alpha_2 w_0^2(x)} + \alpha_2 w_0^2(x) \quad (142)$$

$\alpha_i \in \mathbb{R}$, $i = 0, 1, 2$

Moreover Barclay has shown²¹:

* For Class I potentials, there exists a change of variable $x \rightarrow v$ in which the potential is transformed into that of an harmonic oscillator $V(x) \rightarrow V(v) = v^2 + V_0$, $v(x)$ being solution of a Riccati equation with constant coefficients:

$$\frac{dv(x)}{dx} = A_0 + A_1 v(x) + A_2 v^2(x) \quad (143)$$

with

$$V_0 = -\frac{\alpha_1^2}{4(1-\alpha_2)} - \alpha_0. \quad (144)$$

* For class II potentials, there exists a change of variable $x \rightarrow y$ in which the potential is transformed into that of an harmonic oscillator $V(x) \rightarrow V(y) = y^2 + V_0$, $y(x)$ being solution of a first-order ODE of the form:

$$\frac{dy(x)}{dx} = A_0 + A_1 y(x) \sqrt{A_0 + A_2 y^2(x)} + A_2 y^2(x) \quad (145)$$

with

$$V_0 = -\alpha_0 - \frac{\alpha_0(1-\alpha_2)}{2\alpha_2} \left(1 - \sqrt{1 - \frac{\alpha_1^2 \alpha_2}{(1-\alpha_2)^2}} \right). \quad (146)$$

2. First category

From Eq.(143) and Eq.(92), it appears clearly that the first category of potentials coincides with the above first class. Eq.(141) is still easily verified using the fact that the superpotential associated to a first category potential $V(u) = \tilde{\lambda}_2 u^2 + \tilde{\lambda}_1 u + \tilde{\lambda}_0$ is an affine function in the y variable (see Eq.(101)) and then also in the u variable.

Conversely, if $w_0(x)$ obeys Eq.(141), then, if we define $w_0(x) = cu + d$, $c, d \in \mathbb{R}$, $V(x) = -w_0'(x) + w_0^2(x)$ becomes quadratic in u and $u(x)$ obeys a constant coefficient Riccati equation.

3. Second category

For the second category potential $V(y) = \lambda_2 y^2 + \lambda_0 + \mu_2/y^2$, with $dy(x)/dx = A_0 + A_2 y^2(x)$, we have obtained in Eq.(137) that the superpotential is given by:

$$w_0 = b_1 y + \frac{b_{-1}}{y}. \quad (147)$$

Then:

$$w'_0(x) = y'(x) \left(b_1 - \frac{b_{-1}}{y^2} \right) = (A_0 b_1 - A_2 b_{-1}) + A_2 b_1 y^2 - A_0 b_{-1} \frac{1}{y^2}. \quad (148)$$

From Eq.(147) we have also that

$$\begin{cases} y = \frac{1}{2b_1} \left(w_0 \pm \sqrt{w_0^2 - 4b_1 b_{-1}} \right) \\ \frac{1}{y} = \frac{1}{2b_{-1}} \left(w_0 \mp \sqrt{w_0^2 - 4b_1 b_{-1}} \right) \end{cases} \quad (149)$$

which combined with Eq.(148) gives

$$w'_0(x) = A + B w_0^2(x) + C \sqrt{A + B w_0^2(x)} \quad (150)$$

with

$$\begin{cases} A = 2(A_0 b_1 - A_2 b_{-1}) \\ B = -A/4b_1 b_{-1} \\ C = \pm (A_0 b_1 + A_2 b_{-1}) / b_1 b_{-1} \sqrt{B}. \end{cases} \quad (151)$$

Consequently any second-category potential is a Class-II potential.

Conversely, suppose $V(x) = -w'_0(x) + w_0^2(x)$ is a Class II potential (see Eq.(142)). We then define the y variable via:

$$w_0 = b_1 y + \frac{b_{-1}}{y}, \quad (152)$$

where $b_1 b_{-1} = -\alpha_0/4\alpha_2$. Then using Eq.(149) we have

$$\sqrt{w_0^2 + \frac{\alpha_0}{\alpha_2}} = b_1 y - \frac{b_{-1}}{y}$$

and from Eq.(142) we deduce that

$$\begin{aligned} \frac{y'}{y} \left(b_1 y - \frac{b_{-1}}{y} \right) &= \alpha_2 \left(\frac{\alpha_0}{\alpha_2} + w_0^2 \right) + \alpha_1 \sqrt{\alpha_2} w_0 \sqrt{\frac{\alpha_0}{\alpha_2} + w_0^2} \\ &= \alpha_2 \left(b_1 y - \frac{b_{-1}}{y} \right)^2 + \alpha_1 \sqrt{\alpha_2} \left(b_1 y + \frac{b_{-1}}{y} \right) \left(b_1 y - \frac{b_{-1}}{y} \right), \end{aligned} \quad (153)$$

that is,

$$y' = A_0 + A_2 y^2 \quad (154)$$

with

$$\begin{cases} A_0 = (\alpha_1\sqrt{\alpha_2} - \alpha_2) b_{-1} \\ A_2 = (\alpha_1\sqrt{\alpha_2} + \alpha_2) b_1. \end{cases} \quad (155)$$

As for the potential, it takes the form

$$\begin{aligned} V &= -\alpha_0 - \alpha_1\sqrt{\alpha_2}w_0\sqrt{\frac{\alpha_0}{\alpha_2} + w_0^2} + (1 - \alpha_2)w_0^2 \\ &= -\alpha_0 - \alpha_1\sqrt{\alpha_2}\left(b_1y + \frac{b_{-1}}{y}\right)\left(b_1y - \frac{b_{-1}}{y}\right) + (1 - \alpha_2)\left(b_1y + \frac{b_{-1}}{y}\right)^2 \\ &= \lambda_2y^2 + \lambda_0 + \frac{\mu_2}{y^2} \end{aligned} \quad (156)$$

with

$$\begin{cases} \lambda_2 = b_1^2(1 - \alpha_1\sqrt{\alpha_2} - \alpha_2) \\ \mu_2 = b_{-1}^2(1 + \alpha_1\sqrt{\alpha_2} - \alpha_2) \\ \lambda_0 = -\alpha_0/2(1 + 1/\alpha_2) \end{cases} \quad (157)$$

Consequently any Class-II potential is a second-category potential.

Our two categories of potentials coincide then respectively with the two Barclay-Maxwell classes. Since every TSIP is necessarily a member of either one of these classes^{12,13}, we can finally conclude that the rational representations of the excited states of the RS functions given in Eq.(126) and Eq.(138) are valid for the whole set of TSIP.

VII. CONCLUSION

In this paper we have given an exact expression for the RS functions associated with the excited bound states of every SIP in terms of terminating continued fractions built on the superpotential (see Eq.(142)). Considering the set of translationally shape-invariant potentials (TSIP), we have shown that it can be divided in two categories, coinciding with the Barclay-Maxwell classes^{12,13} but using a different characterization. More precisely any TSIP can be brought into an harmonic or isotonic form using a change of variable resting on a constant coefficient Riccati equation without a term of first degree (see Eq.(95) and Eq.(97)). It can be noticed that, according to the Chalykh-Veselov theorem, the harmonic and isotonic potentials are the only rational isochronous ones and they present equispaced spectra^{22,23}. In terms of this new variable, the superpotential is a first-degree polynomial or a first-degree Laurent polynomial (see Eq.(125) and Eq.(137)) recovering the Gendenshtein ansätze⁹. With this formulation, the shape invariance characteristics of the potential appear in a very transparent manner offering a compact expression for the spectrum of any TSIP (see Eq.(123) and Eq.(135)). Interestingly, we obtain a kind of duality between the TSI potentials and the associated energy spectrums. For a second category potential which can be brought into an isotonic form, the energy is an harmonic function of the shifted quantum number $n \pm a/\alpha$ (see Eq.(124)). Reciprocally, for a first category potential which can be brought into an harmonic form, the energy is an isotonic function of $n + (\mu \pm \lambda)/2\alpha$ (see Eq.(136)). Collecting these results together we obtain compact rational representations (see Eq.(126) and Eq.(138) for all the excited states RS functions associated to a TSIP.

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IX. ANNEXE A

Rosen-Morse I The Rosen-Morse I potential with zero-energy ground state is given by^{7,8}:

$$V(x) = \frac{A(A - \alpha)}{\sin^2(\alpha x)} + \frac{2B}{\tan(\alpha x)} + \frac{B^2}{A^2} - A^2.$$

Defining:

$$\begin{cases} y(x) = \tan(\alpha x - \frac{\pi}{2}) = -\cot(\alpha x), & x \in]0, \frac{\pi}{\alpha}[\\ y'(x) = \alpha + \alpha y^2(x), \end{cases} \quad (158)$$

we obtain

$$V(y) = a(a - \alpha)y^2 + \lambda_1 y + \frac{\lambda_1^2}{4a^2} - \alpha a, \quad (159)$$

with $a = A$ and $\lambda_1 = -2B$.

The superpotential is:

$$w_0(x) = ay + \frac{\lambda_1}{2a} = -A \cot(\alpha x) - \frac{B}{A}. \quad (160)$$

Rosen-Morse II The Rosen-Morse II potential with zero-energy ground state is given by^{7,8}:

$$V(y) = -\frac{A(A + \alpha)}{\cosh^2(\alpha x)} + 2B \tanh(\alpha x) + \frac{B^2}{A^2} + A^2, \quad B < A^2. \quad (161)$$

Defining:

$$\begin{cases} y(x) = \tanh(\alpha x), & x \in \mathbb{R} \\ y'(x) = \alpha - \alpha y^2(x), \end{cases} \quad (162)$$

we obtain:

$$V(y) = a(a + \alpha)y^2 + \lambda_1 y + \frac{\lambda_1^2}{4a^2} - \alpha a, \quad (163)$$

with $a = A$ and $\lambda_1 = 2B$.

The superpotential is:

$$w_0(x) = ay + \frac{\lambda_1}{2a} = A \tanh(\alpha x) + \frac{B}{A}. \quad (164)$$

Eckardt The Eckardt potential with zero-energy ground state is given by^{7,8}:

$$V(x) = \frac{A(A - \alpha)}{\sinh^2(\alpha x)} - 2B \coth(\alpha x) + \frac{B^2}{A^2} + A^2, \quad B > A^2. \quad (165)$$

Defining:

$$\begin{cases} y(x) = \coth((-\alpha)x), & x \in \mathbb{R} \\ y'(x) = (-\alpha)(1 - y^2(x)), \end{cases} \quad (166)$$

we obtain:

$$V(x) = a(a - (-\alpha))y^2 + \lambda_1 y + \frac{\lambda_1^2}{4a^2} - (-\alpha)a, \quad (167)$$

with $a = A$ and $\lambda_1 = 2B$.

The superpotential is:

$$w_0(x) = ay + \frac{\lambda_1}{2a} = A \coth((-\alpha)x) + \frac{B}{A} \quad (168)$$

X. ANNEXE B

Pöschl-Teller The Pöschl-Teller potential with zero-energy ground state is given by^{7,8}:

$$V(x) = A^2 + \frac{(A^2 + B^2 + A\alpha)}{\sinh^2(\alpha x)} - B(2A + \alpha) \frac{\coth(\alpha x)}{\sinh(\alpha x)}, \quad B > A, \quad x > 0 \quad (169)$$

Defining:

$$\begin{cases} y(x) = \tanh\left(\frac{\alpha x}{2}\right) \\ y'(x) = \frac{\alpha}{2} - \frac{\alpha}{2}y^2(x), \end{cases} \quad (170)$$

we obtain:

$$V(y) = \lambda(\lambda + \alpha/2)y^2 + \frac{\mu(\mu - \alpha/2)}{y^2} - \frac{\alpha}{2}(\lambda - \mu) - 2\lambda\mu, \quad (171)$$

with $\lambda = (A + B)/2$ and $\mu = (B - A)/2$.

The superpotential is:

$$w_0(x) = \lambda y - \frac{\mu}{y} = \frac{A+B}{2} \tanh\left(\frac{\alpha x}{2}\right) - \frac{(B-A)/2}{\tanh\left(\frac{\alpha x}{2}\right)} = \frac{A}{\tanh(\alpha x)} - \frac{B}{\sinh(\alpha x)}. \quad (172)$$

Pöschl-Teller I The Pöschl-Teller I potential with zero-energy ground state is given by^{7,8}:

$$V(x) = -(A+B)^2 + \frac{A(A-\alpha)}{\cos^2(\alpha x)} + \frac{B(B-\alpha)}{\sin^2(\alpha x)}, \quad x \in \left]0, \frac{\pi}{2\alpha}\right[\quad (173)$$

Defining:

$$\begin{cases} y(x) = \tan(\alpha x) \\ y'(x) = \alpha + \alpha y^2(x). \end{cases} \quad (174)$$

we obtain:

$$V(y) = \lambda(\lambda - \alpha)y^2 + \frac{\mu(\mu - \alpha)}{y^2} - \alpha(\lambda + \mu) - 2\lambda\mu, \quad (175)$$

with $\lambda = A$ and $\mu = B$.

The superpotential is:

$$w_0(x) = \lambda y - \frac{\mu}{y} = A \tan(\alpha x) - \frac{B}{\tan(\alpha x)}. \quad (176)$$

Pöschl-Teller II The Pöschl-Teller II potential with zero-energy ground state is given by^{7,8}:

$$V(x) = (A - B)^2 - \frac{A(A + \alpha)}{\cosh^2(\alpha x)} + \frac{B(B - \alpha)}{\sinh^2(\alpha x)}, \quad B < A, \quad x > 0 \quad (177)$$

Defining:

$$\begin{cases} y(x) = \tanh(\alpha x) < 1 \\ y'(x) = \alpha - \alpha y^2(x). \end{cases} \quad (178)$$

we obtain:

$$V(y) = \lambda(\lambda + \alpha)y^2 + \frac{\mu(\mu - \alpha)}{y^2} - \alpha(\lambda - \mu) - 2\lambda\mu, \quad (179)$$

with

$$\begin{cases} \lambda = A \\ \mu = B. \end{cases} \quad (180)$$

The superpotential is:

$$w_0(y) = \lambda y - \frac{\mu}{y} = A \tanh(\alpha x) - \frac{B}{\tanh(\alpha x)}. \quad (181)$$

Scarf I The Scarf I potential with zero-energy ground state is given by^{7,8}:

$$V(x) = -A^2 + \frac{(A^2 + B^2 - A\alpha)}{\sin^2(\alpha x)} - B(2A - \alpha) \frac{\cot(\alpha x)}{\sin(\alpha x)}, \quad B < A, \quad x \in \left]0, \frac{\pi}{2\alpha}\right[\quad (182)$$

Defining:

$$\begin{cases} y(x) = \tan\left(\frac{\alpha x}{2}\right) \\ y'(x) = \frac{\alpha}{2} + \frac{\alpha}{2}y^2(x). \end{cases} \quad (183)$$

we obtain:

$$V(y) = \lambda(\lambda - \alpha/2)y^2 + \frac{\mu(\mu - \alpha/2)}{y^2} - \frac{\alpha}{2}(\lambda + \mu) - 2\lambda\mu, \quad (184)$$

with $\lambda = (A + B)/2$ and $\mu = (A - B)/2$.

The superpotential is:

$$w_0(x) = \lambda y - \frac{\mu}{y} = \frac{A + B}{2} \tan\left(\frac{\alpha x}{2}\right) - \frac{(A - B)/2}{\tan\left(\frac{\alpha x}{2}\right)} = -\frac{A}{\tan(\alpha x)} + \frac{B}{\sin(\alpha x)}. \quad (185)$$

Scarf II The Scarf II potential with zero-energy ground state is given by^{7,8}:

$$V(x) = A^2 + \frac{(B^2 - A^2 - A\alpha)}{\cosh^2(\alpha x)} - B(2A + \alpha) \frac{\tanh(\alpha x)}{\cosh(\alpha x)}. \quad (186)$$

Defining:

$$\begin{cases} y(x) = \tanh\left(\frac{\alpha x}{2} + i\frac{\pi}{4}\right) \\ y'(x) = \frac{\alpha}{2} - \frac{\alpha}{2}y^2(x). \end{cases} \quad (187)$$

we obtain:

$$V(y) = \lambda(\lambda + \alpha/2)y^2 + \frac{\mu(\mu - \alpha/2)}{y^2} - \frac{\alpha}{2}(\lambda - \mu) - 2\lambda\mu, \quad (188)$$

with $\lambda = (A + iB)/2$ and $\mu = (-A + iB)/2$.

The superpotential is:

$$w_0(x) = \lambda y - \frac{\mu}{y} = \frac{A + iB}{2} \tanh\left(\frac{\alpha x}{2} + i\frac{\pi}{4}\right) - \frac{(-A + iB)/2}{\tanh\left(\frac{\alpha x}{2} + i\frac{\pi}{4}\right)} = A \tanh(\alpha x) + \frac{B}{\cosh(\alpha x)}. \quad (189)$$

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