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$(k, 1)$ -coloring of sparse graphs

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Abstract

A graph G is called $(k, 1)$ -colorable, if the vertex set of G can be partitioned into subsets V_1 and V_2 such that the graph $G[V_1]$ induced by the vertices of V_1 has maximum degree at most k and the graph $G[V_2]$ induced by the vertices of V_2 has maximum degree at most 1. We prove that every graph with a maximum average degree less than $\frac{10k+22}{3k+9}$ admits a $(k, 1)$ -coloring, where $k \geq 2$. In particular, every planar graph with girth at least 7 is $(2, 1)$ -colorable, while every planar graph with girth at least 6 is $(5, 1)$ -colorable. On the other hand, for each $k \geq 2$ we construct non- $(k, 1)$ -colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k}{4k+1}$.

1 Introduction

A graph G is called *improperly* (d_1, \dots, d_k) -colorable, or just (d_1, \dots, d_k) -colorable, if the vertex set of G can be partitioned into subsets V_1, \dots, V_k such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leq i \leq k$. This notion generalizes those of proper k -coloring (when $d_1 = \dots = d_k = 0$) and d -improper k -coloring (when $d_1 = \dots = d_k = d \geq 1$).

Proper and d -improper colorings have been widely studied. As shown by Appel and Haken [1, 2], every planar graph is 4-colorable, i.e. $(0, 0, 0, 0)$ -colorable. Eaton and Hull [9] and independently Škrekovski [12] proved that every planar graph is 2-improperly 3-colorable (in fact, 2-improper 3-choosable), i.e. $(2, 2, 2)$ -colorable. This latter result was extended by Havet and Sereni [11] to not necessarily planar sparse graphs as follows: For every $k \geq 0$, every graph G with $\text{mad}(G) < \frac{4k+4}{k+2}$ is k -improperly 2-colorable (in fact k -improperly 2-choosable), i.e. (k, k) -colorable, where

$$\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$$

is the maximum average degree of a graph G .

Let $g(G)$ denote the girth of graph G (the length of a shortest cycle in G). Glebov and Zambalava [10] proved that every planar graph G is $(1, 0)$ -colorable if $g(G) \geq 16$. This was strengthened by Borodin and Ivanova [7] by proving that every graph G is $(1, 0)$ -colorable if $\text{mad}(G) < \frac{7}{3}$, which implies that every planar graph G is $(1, 0)$ -colorable if $g(G) \geq 14$.

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This was extended by Borodin et al. [8] by proving that every graph with a maximum average degree smaller than $\frac{3k+4}{k+2}$ is $(k, 0)$ -colorable if $k \geq 2$. Note that the proof in [8] extends that in [7] but does not work for $k = 1$.

In this paper, we focus on $(k, 1)$ -colorability of graph. So, a graph G is $(k, 1)$ -colorable if its vertices can be partitioned into subsets V_1 and V_2 such that in $G[V_1]$ every vertex has degree at most k , while in $G[V_2]$ every component has at most two vertices. Our main result is:

Theorem 1 *Every graph G with $\text{mad}(G) < \frac{10k+22}{3k+9}$, where $k \geq 2$, is $(k, 1)$ -colorable.*

On the other hand, we construct non- $(k, 1)$ -colorable graphs whose maximum average degree is arbitrarily close to $\frac{14k}{4k+1}$.

Since every planar graph G satisfies $\text{mad}(G) < \frac{2g(G)}{g(G)-2}$, from Theorem 1 we have:

Corollary 1 *Every planar graph G is $(2, 1)$ -colorable if $g(G) \geq 7$, and $(5, 1)$ -colorable if $g(G) \geq 6$.*

On the other hand, there is (see [8]) a planar graph with girth 6 which is not $(k, 0)$ -colorable whatever large k , whereas every planar graph G is $(8, 0)$ -colorable if $g(G) \geq 7$ and $(4, 0)$ -colorable if $g(G) \geq 8$ (see [8]). Also note that every planar graph G with $g(G) \geq 6$ is $(2, 2)$ -colorable, while that with $g(G) \geq 8$ is $(1, 1)$ -colorable (see [11]). The results are summarized in the following table:

$g(G)$	$(k, 0)$	$(k, 1)$	$(k, 2)$
6	\times [8]	$(5, 1)$	$(2, 2)$ [11]
7	$(8, 0)$ [8]	$(2, 1)$	
8	$(4, 0)$ [8]	$(1, 1)$ [11]	

Table 1: The relationship between the girth of G and its (k, j) -colorability.

A distinctive feature of the discharging in the proof of Theorem 1 for $2 \leq k \leq 4$ is its "globality": a charge for certain vertices is collected from arbitrarily large "feeding areas", which is possible due to the existence of reducible configurations of unlimited size in the minimum counter-examples, called "soft components". Such global discharging first appears in [3] and is used, in particular, in [4, 5, 6, 7, 8, 11]. The terms "feeding area" and "soft component" are introduced in [7] and also used in our recent paper [8].

2 Non- $(k, 1)$ -colorable graphs with a small maximum average degree

Let $H_{a,b}^i$ be the graph consisted of two adjacent vertices a and b and of i vertices of degree 2 c_1, \dots, c_i linked each to a and b . We take one copy of $H_{a,b}^{k+1}$ and $k - 1$ copies of $H_{a,b}^2$ and identify all the vertices a to a single vertex a^* . Let H_{a^*} be the obtained graph. Finally, we take an odd cycle $C_{2n-1} = a_1 a_2 \dots a_{2n-1}$ and n copies of H_{a^*} , and we identify each vertex a_i with odd index with the vertex a^* of a copy of H_{a^*} . Let $G_{n,k}$ be the obtained graph. An example is given in Figure 1.

One can observe that $G_{n,k}$ is not $(k, 1)$ -colorable. Indeed, observe first that no two consecutive vertices x, y on C_{2n-1} belongs to V_2 . Otherwise we can suppose that x is of odd index on C_{2n-1} and the subgraph $H_{a,b}^{k+1}$ associated to x is not $(k, 1)$ -colorable. Due to the parity of C_{2n-1} , it follows that two consecutive vertices x, y on C_{2n-1} belongs to V_1 . Similarly, we can suppose that x is of odd index on C_{2n-1} . If $G_{n,k}$ is $(k, 1)$ -colorable, then one more vertex in each $H_{a,b}^i$ associated to x must belong to V_1 ; it follows that the degree of x in $G[V_1]$ is $k + 1$, a contradiction.

Now observe that:

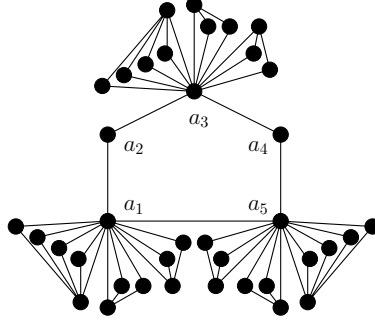


Figure 1: An example of $G_{n,k}$ with $n = 3$ and $k = 3$.

$$\begin{aligned} \text{mad}(G_{n,k}) &= \frac{2|E(G_{n,k})|}{|V(G_{n,k})|} = \frac{2(2n-1 + 5(k-1)n + n(2k+3))}{2n-1 + 3(k-1)n + n(k+2)} = \frac{2(7nk-1)}{n(4k+1)-1} \\ \lim_{n \rightarrow \infty} \text{mad}(G_{n,k}) &= \frac{14k}{4k+1} \end{aligned}$$

3 Proof of Theorem 1

Let $G = (V, E)$ be a counterexample to Theorem 1 on the fewest number of vertices. Clearly, G is connected and its minimum degree is at least 2. By definition, we have:

$$\begin{aligned} \frac{2|E|}{|V|} \leq \text{mad}(G) < \frac{10k+22}{3k+9} \\ 2|E| - |V| \frac{10k+22}{3k+9} = \sum_{v \in V} \left(d(v) - \frac{10k+22}{3k+9} \right) < 0, \end{aligned}$$

where $d(v)$ is the degree of a vertex v .

Thus, we have:

$$\sum_{v \in V} \left(\frac{3(k+3)}{2(k+1)} d(v) - \frac{5k+11}{k+1} \right) < 0. \quad (1)$$

Let the *charge* $\mu(v)$ of each vertex v of G be $\frac{3(k+3)}{2(k+1)} d(v) - \frac{5k+11}{k+1}$. We shall describe a number of structural properties of G (Section 3.1) which make it possible to vary the charges so that the new charge μ^* of every vertex becomes nonnegative for $k \geq 5$ (Section 3.2). For $2 \leq k \leq 4$ there is a difference: some vertices has a non-negative μ^* individually (Section 3.3), while the others are partitioned into disjoint subsets, called *feeding areas*, and the total charge of each feeding area is proved to be non-negative (Lemma 1 in Section 3.3). Since the sum of charges does not change, in both cases we get a contradiction with (1), which will complete the proof of Theorem 1.

A vertex of degree k (resp. at least k , at most k) is called a k -vertex (resp. k^+ -vertex, k^- -vertex). A $(k+1)^-$ -vertex is *minor*; a $(k+2)^+$ -vertex is *senior*. A *weak vertex* is a minor vertex adjacent to exactly one senior vertex. A *light vertex* is either a 2-vertex or a weak vertex. A 3_i -vertex is a 3-vertex adjacent to i 2-vertices.

Claims 2 and 3 below lead us to the following definition. A d -vertex, where $d \geq k+3$, is *soft* if it is adjacent to $d-1$ weak vertices. For $d = k+2$ the notion of soft vertex is broader: a $(k+2)$ -vertex is *soft* if it is adjacent to $k+1$ light vertices.

We will color the vertices of the subgraph of maximum degree at most k by color k and the other vertices by color 1.

3.1 Structural properties of G

Claim 1 *No 2-vertex in G is adjacent to a 2-vertex.*

PROOF. Suppose G has two adjacent 2-vertices t and u , and let s (resp. v) be the other neighbor of t (resp. u). By the minimality of G , the graph $G \setminus \{t, u\}$ has a $(k, 1)$ -coloring c . It suffices to color t and u with a color different from those of s and v respectively to extend c to the whole graph G , a contradiction. \square

Claim 2 *Every minor vertex in G is adjacent to at least one senior vertex.*

PROOF. Suppose G has a minor vertex x adjacent only to minor vertices. Take a $(k, 1)$ -coloring c of $G \setminus x$. If none of the neighbors of x has color 1, then we simply color x with 1. So suppose that at least one neighbor of x is colored with 1. We then color x with k . There is now a problem only if there exists a neighbor of x , say y , colored with k and surrounded by $k + 1$ neighbors colored with k . In this case, we recolor y with 1. We iterate this operation while a such y exists. The coloring obtained is a $(k, 1)$ -coloring of G , a contradiction. \square

Claim 3 *If a senior d -vertex is adjacent to $d - 1$ weak vertices, then it is adjacent to a non-light vertex.*

PROOF. Suppose G has a d -vertex x adjacent to vertices x_1, \dots, x_d , where x_1, \dots, x_{d-1} are weak while x_d is either weak or has $d(x_d) = 2$. We take a $(k, 1)$ -coloring of $G \setminus x$ and recolor each weak neighbor x_i with color k (followed by recoloring if necessary the neighbors of x_i 's in any order). If x_d is a 2-vertex, then we recolor it properly. Now it suffices to color x with 1; a contradiction. \square

Claim 4 *No 3-vertex is adjacent to two soft vertices and to a minor vertex.*

PROOF. Suppose G has a 3-vertex x adjacent to vertices x_1, x_2, x_3 , where x_1 and x_2 are $(k + 2)^+$ -vertices while $d(x_3) \leq k + 1$. Let $y_1^1, \dots, y_{d(x_1)-1}^1$ (resp. $y_1^2, \dots, y_{d(x_2)-1}^2$) be the other neighbors of x_1 (resp. x_2). We take a $(k, 1)$ -coloring of $G \setminus \{x, x_1, x_2\}$. We first recolor the vertices y_j^i as follows: if y_j^i has $d(y_j^i) = 2$, then we recolor y_j^i properly; otherwise if y_j^i is weak, we recolor y_j^i with k (followed by recoloring if necessary the neighbors of y_j^i 's in any order). Now if $d(x_1) \geq k + 3$, we color x_1 with 1 (observe that all colored neighbors of x_1 are colored with k). Assume $d(x_1) = k + 2$. If the color 1 appears at least twice on the y_j^i , then we color x_1 with k and with 1 otherwise. We do the same for x_2 . Finally, if a same color appear three times in the neighborhood of x , then we color x properly. Otherwise we color x with k (followed by recoloring x_3 and some y_j^i , if necessary). This gives an extension of c to the whole graph G , a contradiction. \square

An edge xy is *soft* if one of the following holds:

- $d(x) = k + 2$ while y is light, i.e. is a 2-vertex or a weak vertex, or
- x is a minor vertex while $d(y) = 2$.

The vertex x is called the *good end* of the soft edge xy .

A *soft component* SC is a subgraph of G such that $\Delta(SC) \leq k + 2$, each edge joining SC to $G \setminus SC$ is soft and each good end of the soft edges belongs to SC .

Claim 5 *G does not contain soft components.*

PROOF. Assume that G contains a soft component SC . By minimality of G , the graph $G \setminus SC$ has a $(k, 1)$ -coloring c . We will show that we can extend c to the whole graph G , a contradiction. First, for each edge xy with $x \in SC$ and $y \notin SC$, we recolor (if necessary) the vertex y such that the choice of any color for x will not create any problem on y . If y is a 2-vertex, then we just recolor y properly. If y is weak vertex, then we recolor y with 1 if it has k colored neighbors with the color

k and with k otherwise. Observe that if x is later colored with 1 or k , then that will not create a conflict for y . Now we extend the coloring c to the whole graph G as follows: we choose a coloring ϕ of SC that minimizes $\sigma = k \cdot E_{11} + E_{kk}$ where E_{ii} denotes the number of edges whose both ends are colored with i . Clearly, such a coloring exists. Moreover we will show that such a coloring is a $(k, 1)$ -coloring. Assume that the coloring ϕ of SC is not a $(k, 1)$ -coloring. So suppose that there exists a vertex u of SC colored with 1 which has two neighbors colored with 1. We just recolor u with k and obtain a coloring with a smaller σ which contradicts the choice of ϕ . Similarly, assume that there exists a vertex v of SC colored with k which has $k + 1$ neighbors colored with k . We just recolor v with 1 and obtain a coloring with a smaller σ which contradicts the choice of ϕ . \square

Corollary 2 *No $(k + 2)$ -vertex can be adjacent to $k + 2$ light vertices.*

3.2 Discharging procedure when $k \geq 5$

Set $\alpha = \frac{3k+1}{2(k+1)}$, $\gamma = \frac{k-1}{k+1}$, $\epsilon = \frac{k-5}{2(k+1)}$. Note that $2 - \alpha = \frac{k+3}{2(k+1)}$, $\frac{1}{3} \leq \gamma < 1$, $\frac{1}{2} < 2 - \alpha \leq \frac{5}{6}$, and $\gamma \geq 2 - \alpha$ when $k \geq 5$.

Our rules of discharging are as follows:

- R1.** Every d -vertex with $3 \leq d \leq k + 1$ gives $2 - \alpha$ to each adjacent 2-vertex.
- R2.** Every weak vertex gets α from its adjacent senior vertex.
- R3.** Every non-weak 2-vertex gets 1 from each neighbor.
- R4.** Every minor non-light vertex gets γ from each non-soft adjacent $(k + 2)$ -vertex, ϵ from each soft adjacent $(k + 2)$ -vertex and $2 - \alpha$ from each adjacent $(k + 3)^+$ -vertex.

We now show that $\mu^*(v) \geq 0$ for all v in $V(G)$. Let v be a d -vertex, where $d \geq 2$. Set

$$\mu_d = \frac{3(k+3)}{2(k+1)}d - \frac{5k+11}{k+1}$$

In particular, $\mu_2 = -2$ and $-\frac{1}{2} < \mu_3 \leq \frac{1}{2}$.

Case 1. $d \geq k + 3$.

Claim 6 *If $d \geq k + 3$, then $\mu_d \geq \alpha(d - 2) + 2$; in particular, $\mu_{k+3} = \alpha(k + 1) + 2$.*

PROOF.

$$\begin{aligned} \mu_d - \alpha(d - 2) - 2 &= \frac{3(k+3)}{2(k+1)}d - \frac{5k+11}{k+1} - \frac{3k+1}{2(k+1)}(d - 2) - 2 \\ &= \frac{4(d - (k + 3))}{k + 1} \geq 0 \end{aligned}$$

\square

By Claim 3, v is adjacent to at most $d - 1$ weak vertices. If v is adjacent to at most $d - 2$ weak vertices, then $\mu^*(v) \geq \mu_d - \alpha(d - 2) - 2 \times 1 \geq 0$ by R1–R4 due to Claim 6. Suppose now that v is adjacent to exactly $d - 1$ weak vertices. By Claim 3, v is adjacent to a non light vertex. So we have $\mu^*(v) \geq \mu_d - \alpha(d - 1) - (2 - \alpha) \geq 0$ by R1–R4 due to Claim 6.

Case 2. $d = k + 2$.

By Corollary 2, the vertex v is adjacent to at most $k + 1$ light vertices. By Claim 6, we have

$$\begin{aligned}
\mu_{k+2} &= \mu_{k+3} - \frac{3(k+3)}{2(k+1)} \\
&= \alpha(k+1) + 2 - \frac{3(k+3)}{2(k+1)} \\
&= \alpha k + 2\gamma
\end{aligned}$$

If v is adjacent to at most k light vertices, then this implies $\mu^*(v) \geq 0$ by R1–R4.

If v is adjacent to exactly $k+1$ light vertices, then v is soft. By Claim 3 and R1–R4, we have $\mu^*(v) \geq \alpha k + 2\gamma - \alpha(k+1) - \epsilon = 2\gamma - \alpha - \epsilon = 0$.

Case 3. $2 \leq d \leq k+1$.

By Claim 1, a 2-vertex is adjacent to 3^+ -vertices. By Claim 2, a d -vertex with $3 \leq d \leq k+1$ is adjacent to at most $d-1$ vertices of degree 2, each of which gets $2-\alpha$ from v by R1.

Subcase 3.1. v is weak.

If $d=2$, then $\mu^*(v) = -2 + (2-\alpha) + \alpha = 0$ by R1 and R3. Suppose $d(v) \geq 3$.

Claim 7 For each $d \geq 3$, it holds $\mu_d - (d-1)(2-\alpha) + \alpha = \frac{(k+3)(d-3)}{k+1}$.

PROOF.

$$\begin{aligned}
\mu_d - (d-1)(2-\alpha) + \alpha &= \frac{3(k+3)}{2(k+1)}d - \frac{5k+11}{k+1} - (d-1)\frac{k+3}{2(k+1)} + \frac{3k+1}{2(k+1)} \\
&= \frac{(k+3)(d-3)}{k+1}
\end{aligned}$$

□

The vertex v is weak. By R2, it gets α from its adjacent senior vertex and gives $2-\alpha$ to at most $d-1$ adjacent 2-vertices, it follows from Claim 7 that $\mu^*(v) \geq \frac{(k+3)(d-3)}{k+1} \geq 0$, when $d \geq 3$.

Subcase 3.2. v is not weak.

The vertex v is adjacent to two senior vertices.

If $d=2$, then $\mu^*(v) = -2 + 2 \cdot 1 = 0$ by R3.

If $d=3$, then $\mu_3 = \frac{5-k}{2(k+1)}$. Assume that v is adjacent to 2-vertex. Then v gives $2-\alpha$ by R1.

By Claim 4, v is adjacent to a non soft $(k+2)^+$ -vertex. Note that $\gamma \geq 2-\alpha > \epsilon$. By R1 and R4, we have $\mu^*(v) \geq \mu_3 - (2-\alpha) + 2-\alpha + \epsilon = 0$. Assume that v is not adjacent to a 2-vertex. Then $\mu^*(v) \geq \mu_3 + 2\epsilon = \epsilon \geq 0$.

If $d \geq 4$, then by R1, $\mu^*(v) \geq \mu_d - (d-2)(2-\alpha) = \frac{k(d-4)+3d-8}{k+1} \geq 0$.

3.3 Discharging procedure when $2 \leq k \leq 4$

3.3.1 Preliminaries

A *weak edge* between vertices x and y is either an ordinary edge xy , or a path xzy with $3 \leq d(z) \leq k+1$, where z is called the *intermediate vertex of the weak edge* xy . A *feeding area*, abbreviated to FA , is a maximal subgraph of G consisting of $(k+2)$ -vertices mutually accessible from each other along weak edges and of their intermediate vertices. An edge xy with $x \in FA$ and $y \notin FA$ is a *link*. By Claim 5, at least one of links for FA is not soft (such links will be called *rigid*). A *FA is a weak feeding area*, denoted by WFA , if it has just one rigid link xy ; in this case, the vertex y is called the *sponsor* of WFA . See Figure 2.

Sometimes a WFA with $d(y) = i$ will be denoted by $WFA(i)$, where $3 \leq i \leq k+2$. A FA with at least two rigid links is *strong* and denoted by SFA . By definition, no $WFA(k+2)$ can be joined by its rigid link to a FA , and no $WFA((k+1)^-)$ can be joined by its rigid link to a $(k+2)$ -vertex in a FA . An immediate consequence of Claim 5 is that no two $WFA((k+1)^-)$'s can be joined by their rigid link.

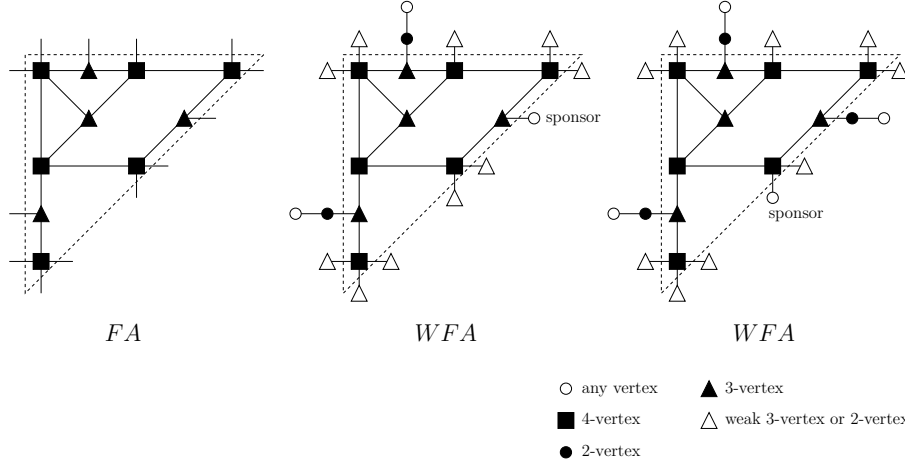


Figure 2: Examples of feeding areas for $k = 2$.

3.3.2 Discharging for $2 \leq k \leq 4$ and its consequences

Set $\alpha = \frac{3k+1}{2(k+1)}$, $\gamma = \frac{k-1}{k+1}$, $\beta = \frac{5-k}{2(k+1)}$. Observe that $2 - \alpha = \frac{k+3}{2(k+1)}$, $\alpha > 1 > 2 - \alpha > \beta$, and $2 - \alpha \geq \gamma$. Moreover, $\mu_2 = -2$ and $\mu_3 = \beta$.

A 3^* -vertex is a 3-vertex adjacent to exactly one minor vertex.

The discharging rules for $2 \leq k \leq 4$ are almost the same as Subsection 3.2. Our rules of discharging are as follows:

- R1.** Every d -vertex with $3 \leq d \leq k + 1$ gives $2 - \alpha$ to each adjacent 2-vertex.
- R2.** Every weak vertex gets α from its adjacent senior vertex.
- R3.** Every non-weak 2-vertex gets 1 from each neighbor.
- R4.** Every 3^* -vertex gets $2 - \alpha$ from each adjacent $(k + 3)^+$ -vertex.
- R5.** Every WFA gets β along the rigid link from its sponsor.

By the definition of FA , a minor vertex can belong to at most one of them. We cannot prove that each $v \in FA$ has $\mu^*(v) \geq 0$; however, it turns out that the total new charge $\mu^*(FA) := \sum_{v \in V(FA)} \mu^*(v)$ of each FA is nonnegative (see Lemma 1 below). This is also a way to arrive at a contradiction with (1).

We now prove $\mu^*(v) \geq 0$ assuming $v \notin FA$.

Case 1. $d = d(v) \geq k + 3$.

By Claim 3, the vertex v is adjacent to at most $d - 1$ weak vertices. If v is adjacent to exactly $d - 1$ weak vertices z_1, \dots, z_{d-1} , then its d^{th} neighbor z_d (which is not a 2-vertex by Claim 3) may be a 3^* -vertex or a vertex belonging to a WFA . Hence v gives α to each adjacent weak vertex by R2 and may give $2 - \alpha$ by R4 and R5 ($2 - \alpha > \beta$); it follows that $\mu^*(v) \geq \mu_d - (d - 1)\alpha - (2 - \alpha) = \mu_d - (d - 2)\alpha - 2 \geq 0$ by Claim 6. Now if v is adjacent to at most $d - 2$ weak vertices, then its two last neighbors may be 2-vertices and so $\mu^*(v) \geq \mu_d - (d - 2)\alpha - 2 \geq 0$ by Claim 6 and R2–R5 ($\alpha > 1 > 2 - \alpha > \beta$).

Case 2. $d = k + 2$.

Since every $(k + 2)$ -vertex belongs to a FA by definition, this case does not occur.

Case 3. $2 \leq d \leq k + 1$.

We consider two cases: v is weak or not.

Subcase 3.1. v is weak.

If $d = 2$, then by R1 and R2, it receives $2 - \alpha$ from its minor neighbor and α from its senior neighbor and $\mu^*(v) = -2 + 2 - \alpha + \alpha = 0$.

Suppose that $d \geq 3$. The vertex v is adjacent to $d - 1$ minor vertices, say z_1, \dots, z_{d-1} , and to a senior vertex, say z_d . By Claim 5, the edge vz_d cannot be the rigid link of a WFA. By R2, v receives α from z_d . Now, each edge vz_i may lead to a 2-vertex, and in this case, v gives $2 - \alpha$ to z_i , or, may lead to a l -vertex with $3 \leq l \leq k + 1$ belonging to a WFA (vz_i is a rigid link), and in this case, v gives β to the corresponding WFA. Since $2 - \alpha > \beta$, it follows that $\mu^*(v) \geq \mu_d - (d - 1)(2 - \alpha) + \alpha \geq 0$ due to Claim 7.

Subcase 3.2. v is not weak.

If $d = 2$, then $\mu^*(v) = -2 + 2 \cdot 1 = 0$ by R2.

Assume that $d \geq 3$. Observe that v is adjacent to at least two senior vertices (v is not weak) and at most one of them belongs to a FA (otherwise, v would belong to a FA, contradicting our assumption).

Suppose $d = 3$. If v is not a 3^* -vertex, then v is adjacent to three senior vertices and $\mu^*(v) \geq \mu_3 - \beta = 0$ by R5. If v is a 3^* -vertex, then v is adjacent to a $(k + 3)^+$ -vertex which gives $2 - \alpha$ to v by R4. Hence, $\mu^*(v) \geq \mu_3 - (2 - \alpha) - \beta + (2 - \alpha) = 0$ by R1, R4, and R5.

Suppose $d \geq 4$. By R1 and R5, v gives nothing to at least one $(k + 3)^+$ -vertex ; hence $\mu^*(v) \geq \mu_d - (d - 2)(2 - \alpha) - \beta = \frac{(2d-7)(k+3)}{2(k+1)} \geq 0$ when $d \geq 4$.

Hence we proved that $\forall v \notin FA, \mu^*(v) \geq 0$. Since the FA's in G are disjoint, to complete the proof of Theorem 1 it suffices to prove the following:

Lemma 1 *Each FA in G has*

$$\mu^*(FA) = \sum_{v \in V(FA)} \mu^*(v) \geq 0.$$

PROOF. We define now the *special rigid edge* of a FA. For a weak feeding area WFA, the *special rigid edge* is its unique rigid edge and observe that by R5 a charge β is transferred inside WFA along this edge. Now, for a strong feeding area SFA, we are sure that at least one rigid link does not lead to a WFA by Claim 5 ; we choose one of them as the *special rigid edge* of SFA. Observe that no charge is transferred along this link by R5.

In order to compute the new charge of FA, we perform now a series of operations, split in four steps, which transform FA into a feeding area FA_4 consisting of just one vertex. Each operation φ transforms a feeding area FA into a feeding area $\varphi(FA)$ of the same nature (strong or weak) preserving the special rigid edge such that $\mu^*(\varphi(FA)) \leq \mu^*(FA)$. During some of these operations we will simplify and modify the structure of FA by eliminating some parts RP of FA and adding some "loose vertices". For each removed part RP of FA, we may change locally the discharging process to be sure that $\mu^*(RP) \geq 0$. The role of the loose vertices is to retain these changes.

The resulting FA of Step i is denoted by FA_i . We will easily see at Step 5 that $\mu^*(FA_4) \geq 0$, which will complete the proof of Theorem 1.

Step 1. (*Eliminating edges between two minor vertices and subdividing $(k+2, k+2)$ -edges.*) Suppose xy is an edge between two vertices of FA . If both x and y are minor, then we delete xy and add a loose adjacent 2-vertex to each of x and y . This implies $\mu^*(\varphi(FA)) = \mu^*(FA) - 2(2 - \alpha) < \mu^*(FA)$ by R1. If $d(x) = d(y) = k + 2$, then we replace xy by a path xzy , where z is a 3_1 -vertex; clearly, $\mu^*(\varphi(FA)) = \mu^*(FA) + \mu_3 - (2 - \alpha) < \mu^*(FA)$ by R1 ($\mu_3 = \beta < 2 - \alpha$). So, our FA_1 is bipartite, every edge of FA_1 joins a $(k+2)$ -vertex with a minor 3^+ -vertex, and $\mu^*(FA_1) \leq \mu^*(FA)$.

Step 2. (*Breaking cycles.*) Suppose a cycle C in FA contains a path yxz with $d(x) \leq k + 1$ and $d(y) = d(z) = k + 2$.

If x has a neighbor t such that $t \notin \{y, z\}$ and $d(t) = k+2$, then we delete edge yx and add a loose 3_2 -vertex adjacent to y and also add a loose 2-vertex adjacent to x . This yields $V(\varphi(FA)) = V(FA)$ and this does not change the type (strong or weak) of FA since this operation does not create or destroy rigid links. By R1 and R2, we have $\mu^*(\varphi(FA)) = \mu^*(FA) - \alpha - (2 - \alpha) < \mu^*(FA)$.

Now suppose all neighbors of x other than y and z have a degree different from $k + 2$, and let r (resp. s) be the number of rigid (respectively, special) links associated to FA going out of x . We consider several cases according to the value of r :

Case $r = 0$. We make the choice to transfer a charge α from y and a charge α from z to x . Hence, $\mu^*(x) \geq \mu(x) - (d(x) - 2)(2 - \alpha) + 2\alpha \geq 0$. Now we remove x from FA . Moreover to retain the fact that we have given α twice from y and z , we add a loose adjacent 3_2 -vertex to y and a loose adjacent 3_2 -vertex to z (by R2, y and z give each α to these “virtual” 3_2 -vertices). We obtain a new feeding area $\varphi(FA)$ of the same type (strong or weak) and $\mu^*(\varphi(FA)) \leq \mu^*(FA) - \mu(x) + (d(x) - 2)(2 - \alpha) - 2\alpha \leq \mu^*(FA)$.

Case $r = 1$. We make the choice to transfer a charge α from y to x . Hence $\mu^*(x) \geq \mu(x) - (d(x) - 3)(2 - \alpha) - \beta + \alpha = \frac{k(2d(x)-3)+6d(x)-17}{2(k+1)} \geq 0$ (the rigid link incident to x may lead to a WFA). Now we remove x from FA , we add a loose 3_2 -vertex adjacent to y (by R2, y gives α to this added vertex) and a rigid link incident to z , namely a special link if $s = 1$ and non-special otherwise. We obtain a new feeding area $\varphi(FA)$ of the same type (strong or weak) and $\mu^*(\varphi(FA)) \leq \mu^*(FA) - \mu(x) + (d(x) - 3)(2 - \alpha) - \alpha \leq \mu^*(FA)$.

Case $r \geq 2$. We remove x from FA ($d(x) \geq 4$). We have $\mu^*(x) \geq \mu(x) - (d(x) - 4)(2 - \alpha) - 2\beta = \frac{k(d(x)-2)+3d(x)-10}{k+1} \geq 0$. We add a rigid link to each of y and z and make one of these links special if and only if $s = 1$. We obtain a new feeding area $\varphi(FA)$ of the same type (strong or weak) and $\mu^*(\varphi(FA)) \leq \mu^*(FA) - \mu(x) + (d(x) - 4)(2 - \alpha) \leq \mu^*(FA)$.

Thus, $\mu^*(FA_2) \leq \mu^*(FA_1)$ and FA_2 is a $((k+1)^-, (k+2))$ -alternating tree with all pendant vertices having degree $(k+2)$.

Step 3. (*Deleting and moving rigid links.*) We want to leave a SFA with just two rigid links and then we want to move all links of a FA (strong or weak) to a pendant $(k+2)$ -vertex.

Suppose xy is a non-special rigid link, where $x \in SFA$ (and $y \notin SFA$). We delete xy and join x to a loose 2-vertex if $d(x) \leq k + 1$ or to a loose 3_2 -vertex if $d(x) = k + 2$. We obtain $\mu^*(\varphi(SFA)) = \mu^*(SFA) - (2 - \alpha)$ or $\mu^*(SFA) - \alpha$ by R1 and R2; so $\mu^*(\varphi(SFA)) \leq \mu^*(SFA)$. Thus, starting with a SFA we can get an SFA with just two rigid links, one of which is special.

Note that replacing a rigid link rooted at a minor vertex of FA by a rigid link of the same kind (special or not) at a $(k+2)$ -vertex yields $\mu^*(\varphi(FA)) = \mu^*(FA) + (2 - \alpha) - \alpha < \mu^*(FA)$.

Thus, we arrive at a μ^* -minimal FA whose all rigid links are adjacent to a pendant vertex of FA , called a *root*. In particular, a SFA has one special and one non-special rigid links.

Step 4. (*Reducing pendant vertices other than the root.*) Suppose our alternating tree FA has $|V(FA)| > 1$; then there is a path zyx , where z is a pendant $(k+2)$ -vertex different from the root of FA . We make the choice to transfer a charge α from x to y and z . It follows that :

$$\begin{aligned}
\mu^*(z) + \mu^*(y) &\geq \mu(z) - (d(z) - 1)\alpha + \mu(y) - (d(y) - 2)(2 - \alpha) + \alpha \\
&\geq \alpha k + 2\gamma - (k + 1)\alpha + \mu(y) - (d(y) - 2)(2 - \alpha) + \alpha \\
&\geq \frac{k(d(y) - 4) + 3d(y) - 8}{k + 1} + 2\frac{k - 1}{k + 1} \\
&\geq \frac{k(d(y) - 2) + 3d(y) - 10}{k + 1} \geq 0
\end{aligned}$$

Then we delete z and y and add a loose 3_2 -vertex adjacent to x (by R2, x gives α to this new vertex). We obtain a new feeding area $\varphi(FA)$ of the same type (strong or weak) and $\mu^*(\varphi(FA)) \leq \mu^*(FA)$.

Step 5. (*Counting $\mu^*(FA)$ for terminal FA 's.*) Recall that each WFA gets β from its sponsor by R5, and in each SFA , no charge β is transferred along the special rigid link. It follows:

$$\begin{aligned}
\mu^*(WFA) &= \mu_{k+2} - (k + 1)\alpha + \beta \\
&= \alpha k + 2\gamma - (k + 1)\alpha + \beta \\
&= 2\gamma - \alpha + \beta \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mu^*(SFA) &\geq \mu_{k+2} - k\alpha - \beta \\
&\geq \alpha k + 2\gamma - k\alpha - \beta \\
&\geq 2\gamma - \beta \\
&\geq \frac{5k - 9}{2(k + 1)} \geq 0
\end{aligned}$$

This completes the proofs of Lemma 1 and Theorem 1. □

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