A note on the acyclic 3-choosability of some planar graphs

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Abstract

An acyclic coloring of a graph G is a coloring of its vertices such that : (i) no two adjacent vertices in G receive the same color and (ii) no bicolored cycles exist in G. A list assignment of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. Let G be a graph and L be a list assignment of G. The graph G is acyclically L-list colorable if there exists an acyclic coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If G is acyclically L-list colorable for any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, then G is acyclically k-choosable. In this paper, we prove that every planar graph with neither cycles of lengths 4 to 7 (resp. to 8, to 9, to 10) nor triangles at distance less 7 (resp. 5, 3, 2) is acyclically 3-choosable.

1 Introduction

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph such that two adjacent vertices do not use the same color. A k*-coloring* of G is a proper coloring of G using k colors ; a graph admitting a k-coloring is said to be k*-colorable*. An *acyclic coloring* of a graph G is a proper coloring of G such that G contains no bicolored cycles ; in other words, the graph induced by every two color classes is a forest. A list assignment of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. Let G be a graph and L be a list assignment of G. The graph G is *acyclically* L*-list colorable* if there is an acyclic coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If G is acyclically L-list colorable for any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$, then G is *acyclically* k*-choosable*. The *acyclic choice number* of G, $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable. Borodin *et al.* [5] first investigated the acyclic choosability of planar graphs proving that:

Theorem 1 *[5] Every planar graph is acyclically 7-choosable.*

and put forward to the following challenging conjecture:

Conjecture 1 *[5] Every planar graph is acyclically 5-choosable.*

This conjecture if true strengthens Borodin's Theorem [1] on the acyclic 5-colorability of planar graphs and Thomassen's Theorem [10] on the 5-choosability of planar graphs.

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In 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [9]). This problem remains open. In 1990, Erdős suggested the following relaxation of Steinberg's Conjecture: what is the smallest integer i such that every planar graph without cycles of lengths 4 to i is 3-colorable? The best known result is $i = 7$ [6]. This question is also studied in the choosability case: what is the smallest integer i such that every planar graph without cycles of lengths 4 to i is 3-choosable? In [11], Voigt proved that Steinberg's Conjecture can not be extended to list coloring ; hence, $i \geq 6$. Nevertheless, in 1996, Borodin [2] proved that every planar graph without cycles of lengths 4 to 9 is 3-colorable ; in fact, 3-choosable. So, $i \leq 9$.

Recently the question of Erdős was studied in the acyclic choosability case: What is the smallest integer i such that every planar graph without cycles of lengths 4 to i is acyclically 3-choosable? Borodin [3] and, independently, Hocquard and Montassier [8] proved $i = 12$.

In this note we give some new sufficient conditions of the acyclic 3-choosability of planar graphs refining this last result. By $d_\Delta(G)$ denote the minimal distance (number of edges) between triangles in G. We prove:

Theorem 2 *Let G be a planar graph. Moreover, if G satisfies one of the following conditions,*

1. *G contains no cycles of length 4 to 10, and $d_\Delta(G) \geq 2$*
2. *G contains no cycles of length 4 to 9, and $d_\Delta(G) \geq 3$*
3. *G contains no cycles of length 4 to 8, and $d_\Delta(G) \geq 5$*
4. *G contains no cycles of length 4 to 7, and $d_\Delta(G) \geq 7$*

then G is acyclically 3-choosable

Notations Let G be a planar graph. We use $V(G)$, $E(G)$ and $F(G)$ to denote the set of vertices, edges and faces of G respectively. Let $d(v)$ denote the degree of a vertex v in G and $r(f)$ the length of a face f in G. A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. *$^{\geq}k$-vertex, $^{\leq}k$-vertex*). We use the same notations for faces : a *k-face* (resp. *$^{\geq}k$-face, $^{\leq}k$-face*) is a face of length k (resp. at least k, at most k).

2 Proof of Theorem 2

2.1 Preliminaries

Let G be a counterexample to Theorem 2 with the minimun order and L be a list assignment such that there does not exist an acyclic L-coloring of G.

Claim 1 *The counterexample G satisfies the following properties:*

1. *G does not contain 1-vertices.*
2. *G does not contain two adjacent 2-vertices.*
3. *G does not contain 3-vertices adjacent to two 2-vertices.*
4. *G does not contain 4-vertices adjacent to three 2-vertices.*
5. *G does not contain triangles xyz with $d(x) = 2$.*
6. *G does not contain triangles xyz such that $d(x) = d(y) = 3$, and x and y are adjacent to 2-vertices.*
7. *G does not contain paths xyz with $d(x) = d(y) = d(z) = 3$, and x, y, z are adjacent to 2-vertices.*

Proof

1. Suppose that G contains a 1-vertex u adjacent to a vertex v. By minimality of G, the graph $G' = G\backslash\{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of G'. To extend this coloring to G we just color u with $c(u) \in L(u)\backslash\{c(v)\}$. The obtained coloring is acyclic, a contradiction.

2. Suppose that G contains a 2-vertex u adjacent to a 2-vertex v. Let t and w be the other neighbors of u and v respectively. By minimality of G, the graph $G' = G\backslash\{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of G'. We show that we can extend this coloring to G. Assume first that $c(t) \neq c(v)$. Then we just color u with $c(u) \in L(u)\backslash\{c(t), c(v)\}$. Now, if $c(t) = c(v)$, we color u with $c(u) \in L(u)\backslash\{c(v), c(w)\}$. In the two cases, the obtained coloring is acyclic, a contradiction.

3. Suppose that G contains a 3-vertex u adjacent to two 2-vertices v and y. Let x, w, z be the other neighbors of u, v, y respectively. By minimality of G, the graph $G' = G \setminus \{u, v, y\}$ is acyclically 3-choosable. Hence, there exists an acyclic L-coloring c of G'. We show that we can extend this coloring to G. We first assign to u a color, different from $c(x)$, that appears at most once on w and z. If this color is different from $c(w)$ and $c(z)$, we just proper color v and y. The obtained coloring is acyclic, a contradiction. If the color assigned to u appears once on w and z, say w, then we color properly y and assign to v a color different from $c(w)$ and $c(x)$. The obtained coloring is acyclic, a contradiction.

4. Suppose that G contains a 4-vertex u adjacent to three 2-vertices v, y, and s. Let x, w, z, t be the other neighbors of u, v, y, s respectively. By minimality of G, the graph $G' = G \setminus \{u, v, y, s\}$ is acyclically 3-choosable. Hence, there exists an acyclic L-coloring c of G'. We show that we can extend this coloring to G. We first assign to u a color, different from $c(x)$, that appears at most once on w, z, and t. If this color is different from $c(w), c(z)$ and $c(t)$, we just proper color v, y, and s. The obtained coloring is acyclic, a contradiction. If the color assigned to u appears once on w, z and t, say w, then we color properly y, s and assign to v a color different from $c(w)$ and $c(x)$. The obtained coloring is acyclic, a contradiction.

5. Suppose that G contains a 2-vertex u incident to a 3-face uvw. By minimality of G, the graph $G' = G\backslash\{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of G'. We can extend this coloring to G by coloring u with $c(u) \in L(u)\backslash\{c(v), c(w)\}$, a contradiction.

6. Suppose that G contains a 3-face xyz with $d(x) = d(y) = 3$. Moreover x (resp. y) is adjacent to a 2-vertex v (resp. s). Finally let u (resp. t) be the other neighbor of v (resp. s). By minimality of G, the graph $G' = G \setminus v$ is acyclically 3-choosable. Hence, there exists an acyclic L-coloring c of G'. If $c(u) \neq c(x)$, we just color properly v and the obtained coloring is acyclic, a contradiction. Assume that $c(u) = c(x)$. If $L(v) \neq \{c(x), c(y), c(z)\}$, we color v with a color different from $c(x), c(y), c(z)$ and the obtained coloring is acyclic, a contradiction. Suppose that $L(v) = \{c(x), c(y), c(z)\}$. If $(c(x), c(y)) \neq (c(s), c(t))$, we color v with $c(y)$ and the coloring obtained is acyclic. Suppose that $(c(x), c(y)) = (c(s), c(t))$. Observe now that $L(x) = \{c(x), c(y), c(z)\}$; otherwise, we recolor x with a color different from $c(x), c(y), c(z)$ and proper color v. Similarly, $L(y) = \{c(x), c(y), c(z)\}$; otherwise, we recolor y with a color different from $c(x), c(y), c(z)$ and color v with a color different from $c(x)$ and $c(z)$. Finally we exchange the colors on x and y and proper color the vertices v and s. The obtained coloring is acyclic, a contradiction.

7. Suppose that G contains a path xyz with $d(x) = d(y) = d(z) = 3$, and x, y, z are adjacent to 2-vertices, u, v, w, respectively. Let p, q, r, s, t be the other neighbors of x, u, v, w, z, respectively. By minimality of G, the graph $G' = G\backslash\{x, y, z, u, v, w\}$ is acyclically 3-choosable. Consequently, there exists an acyclic L-coloring c of G'. We show that we can extend this coloring to G.

7.1 Suppose $L(y)\backslash\{c(p), c(r), c(t)\} \neq \emptyset$. We assign to y a color $c(y)$ different from $c(p), c(r), c(t)$.

7.1.1 If $L(x) \neq \{c(p), c(y), c(q)\}$, then we assign to x a color different from $c(p), c(y)$ and $c(q)$. Then, we color u with a color different from $c(q)$ and $c(x)$, and we assign to z a color different from $c(y)$ and $c(t)$. If $c(z) \neq c(s)$, then we just color w with a color different from $c(s)$ and $c(z)$; otherwise, we color w with a color different from $c(s)$ and $c(t)$. Finally we color v with a color different from $c(r)$ and $c(y)$, and the coloring obtained is acyclic, a contradiction.

7.1.2 Suppose now, $L(x) = \{c(p), c(y), c(q)\}$ with $c(p) \neq c(y) \neq c(q) \neq c(p)$ and, by symmetry, $L(z) = \{c(y), c(t), c(s)\}$ with $c(y) \neq c(t) \neq c(s) \neq c(y)$. We first assign to x the color $c(q)$ and we color z with the color $c(s)$. We can observe that, if $c(s) \neq c(q)$, then we assign to u a color different from $c(q)$ and $c(p)$, we color w with a color different from $c(s)$ and $c(t)$ and we color v with a color different from $c(r)$ and $c(y)$. The coloring obtained is acyclic, a contradiction. So assume that $c(s) = c(q)$, then we have two cases:

7.1.2.1 If $L(u) \neq \{c(p), c(y), c(q)\}$, then we assign to u a color different from $c(p), c(y)$ and $c(q)$. We color properly the vertex v. We color w with a color different from $c(s)$ and $c(t)$. The coloring obtained is acyclic, a contradiction.

7.1.2.2 Suppose now, $L(u) = \{c(p), c(y), c(q)\}$ and, by symmetry, $L(w) = \{c(s), c(y), c(t)\}$. Set $c(q) = 1$ and $c(y) = 2$. We have $c(q) = c(x) = c(z) = c(s) = 1$, $c(y) = 2$, $L(u) = L(x) = \{1, 2, c(p)\}$, and $L(w) = L(z) = \{1, 2, c(t)\}$. Now we recolor x and z with 2. If $c(p) \neq c(t)$, then we assign to y a color different from 2 and $c(r)$, and we color properly v. If $c(p) = c(t)$, then we color y with a color different from 2 and $c(p)$, and we color properly v. The coloring obtained is acyclic.

7.2 Assume that $L(y) = \{c(p), c(r), c(t)\}$. Set $c(r) = 1, c(p) = 2, c(t) = 3$. We first assign to the vertex y the color 1.

7.2.1 If $L(x) \neq \{1, 2, c(q)\}$, then we assign to x a color different from $1, 2$ and $c(q)$. We color properly u and z, and we color v with a color different from 1 and $c(z)$. Then, we color properly w if $c(z) \neq c(s)$; otherwise, we choose for w a color different from 3 and $c(z)$. The coloring obtained is acyclic, a contradiction.

7.2.2 Finally assume $L(x) = \{1, 2, c(q)\}$ and, by symmetry, $L(z) = \{1, 3, c(s)\}$. First, we assign the color 1 to the vertices x and z, and we recolor y properly. Finally we color properly u, v, and w. The coloring obtained is acyclic, a contradiction.

□

Lemma 1 *Let G be a connected plane graph with n vertices, m edges and r faces. Let $k \geq 2$, we have the following:*

$$\sum_{v \in V(G)} ((k-2)d(v) - 2k) + \sum_{f \in F(G)} (2r(f) - 2k) = -4k \tag{1}$$

Proof
Euler's formula $n - m + f = 2$ can be rewritten as $((2k-4)m - 2kn) + (4m - 2kf)) = -4k$. The relation $\sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} r(f) = 2m$ completes the proof. □

2.2 Proof of Theorem 2.1

Let G be a counterexample to Theorem 2.1 with the minimum order. The graph G satisfies Claim 1 and Equation (2) (given by Equation (1) for $k = 11$):

$$\sum_{v\in V(G)} (9d(v) - 22) + \sum_{f\in F(G)} (2r(f) - 22) = -44 \quad (2)$$

We apply now a discharging procedure. We define the weight function $\omega : V(G)\cup F(G) \to \mathbb{R}$ by $\omega(x) = 9d(x)-22$ if $x \in V(G)$ and $\omega(x) = 2r(x)-22$ if $x \in F(G)$. It follows from Equation (2) that the total sum of weights is equal to -44. In what follows, we will define discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is achieved. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{x\,\in\, V(G)\,\cup\, F(G)} \omega^*(x) = \sum_{x\,\in\, V(G)\,\cup\, F(G)} \omega(x) = -44 < 0 \quad (3)$$

and hence demonstrates that no such counterexample can exist.

We make the discharging procedure in two steps:

Step 1. Every $^{\geq}3$-vertex gives 2 to each adjacent 2-vertex.

We denote by $\omega'(x)$ the new charge of $x \in V(G) \cup F(G)$ after Step 1. By $n_T(v)$ denote the number of triangles at distance exactly one from v.

When Step 1 is finished, we proceed with Step 2:

Step 2. Every $^{\geq}3$-vertex v incident to a triangle T gives $\omega'(v)$ to T. Every $^{\geq}3$vertex v at distance exactly one to triangles gives $\omega'(v)/n_T(v)$ to each triangle.

Let v be a k-vertex. By Claim 1.1, $k \geq 2$.

Case $k = 2$ Observe that $\omega(v) = -4$. By Claim 1.2, v is adjacent to $^{\geq}3$-vertices. Hence, $\omega'(v) = -4 + 2 \cdot 2 = 0$ by Step 1. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 3$ Initially, $\omega(v) = 5$. By Claim 1.3, v is adjacent to at most one 2-vertex. Hence, if v is adjacent to a 2-vertex, then $\omega'(v) = 5 - 2 = 3$ and $\omega'(v) = 5$ otherwise. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 4$ Initially, $\omega(v) = 14$. By Claim 1.4, v is adjacent to at most two 2-vertices. So if v is adjacent to two (resp. one, zero) 2-vertices, then $\omega'(v) = 14 - 2 \cdot 2 = 10$ (resp. 12, 14). And by Step 2, $\omega^*(v) \geq 0$.

Case $k \geq 5$ Initially, $\omega(v) = 9k - 22$. The vertex v gives 2 to each adjacent 2-vertex in Step 1. So $\omega'(v) \geq 9k - 22 - 2k = 7k - 22 \geq 13$. And, by Step 2, $\omega^*(v) \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall v \in V(G), \omega^*(v) \geq 0$. Observe now that, after Step 1, all $^{\geq}3$-vertex can give at least $\frac{3}{2}$ to each triangle at distance exactly one during Step 2.

Let f be a k-face. Clearly, if $k \geq 11$, then $\omega^*(f) = \omega(f) = 2r(f) - 22 \geq 0$. Now, suppose that f is a 3-face xyz with $d(x) \leq d(y) \leq d(z)$. By claim 1.5, $d(x) \geq 3$. Initially, $\omega(f) = -16$. If $d(z) \geq 4$, then the vertices x, y, z gives at least $3 + 3 + 10$ to f and so $\omega^*(f) \geq 0$. Assume now that $d(x) = d(y) = d(z) = 3$. By Claim 1.6, at most one of the vertices x, y, z is adjacent to a 2-vertex. If one of these vertices is adjacent to a 2-vertex, say x, then x gives 3 to f, and the vertices y and z give each 5 to f. Now y and z are adjacent to two distinct vertices, say y_1 and z_1 (different from x, y, z), which give each at least $\frac{3}{2}$ to f. Hence $\omega^*(f) \geq -16 + 3 + 2 \cdot 5 + 2 \cdot \frac{3}{2} \geq 0$. If none of the vertices x, y, z is adjacent to a 2-vertex, we have similarly $\omega^*(f) \geq -16 + 3 \cdot 5 + 3 \cdot \frac{3}{2} \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall x \in V(G)\cup F(G), \omega^*(x) \geq 0$. The contradiction obtained by Equation (3) completes the proof.

2.3 Proof of Theorem 2.2

Let G be a counterexample to Theorem 2.2 with the minimum order. The graph G satisfies Claim 1 and Equation (4) (given by Equation (1) for $k = 10$):

$$\sum_{v \in V(G)} (4d(v) - 10) + \sum_{f \in F(G)} (r(f) - 10) = -20 \tag{4}$$

As for the proof of Theorem 2.1, we apply now a discharging procedure. We define the weight function $\omega : V(G) \cup F(G) \rightarrow \mathbb{R}$ by $\omega(x) = 4d(x) - 10$ if $x \in V(G)$ and $\omega(x) = r(x) - 10$ if $x \in F(G)$. It follows from Equation (4) that the total sum of weights is equal to -20. In what follows, we will define discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is achieved. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -20 < 0 \tag{5}$$

and hence demonstrates that no such counterexample can exist.

We make the discharging procedure in two steps:

Step 1. Every $^{\geq}$3-vertex gives 1 to each adjacent 2-vertex.

When Step 1 is finished, we proceed with Step 2:

Step 2. Every $^{\geq}$3-vertex v at distance at most one to a triangle T gives $\omega'(v)$ to T.

Notice that a vertex can be at distance one to at most one triangle. Let v be a k-vertex. By Claim 1.1, $k \geq 2$.

Case $k = 2$ Observe that $\omega(v) = -2$. By Claim 1.2, v is adjacent to $^{\geq}$3-vertices. Hence, $\omega'(v) = -2 + 2 \cdot 1 = 0$ by Step 1. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 3$ Initially, $\omega(v) = 2$. By Claim 1.3, v is adjacent to at most one 2-vertex. Hence, if v is adjacent to a 2-vertex, then $\omega'(v) = 2 - 1 = 1$ and $\omega'(v) = 2$ otherwise. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 4$ Initially, $\omega(v) = 6$. By Claim 1.4, v is adjacent to at most two 2-vertices. So if v is adjacent to two (resp. one, zero) 2-vertices, then $\omega'(v) = 6 - 2 \cdot 1 = 4$ (resp. 5, 6). And by Step 2, $\omega^*(v) \geq 0$.

Case $k \geq 5$ Initially, $\omega(v) = 4k - 10$. The vertex v gives 1 to each adjacent 2-vertex in Step 1. So $\omega'(v) \geq 4k - 10 - k = 3k - 10 \geq 5$. And, by Step 2, $\omega^*(v) \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall v \in V(G), \omega^*(v) \geq 0$. Observe now that, after Step 1, all $^{\geq}$3-vertex can give at least 1 to the triangle (if any) at distance exactly one during Step 2.

Let f be a k-face. Clearly, if $k \geq 10$, then $\omega^*(f) = \omega(f) = r(f) - 10 \geq 0$. Now, suppose that f is a 3-face xyz with $d(x) \leq d(y) \leq d(z)$. Initially, $\omega(f) = -7$. By claim 1.5, $d(x) \geq 3$. Moreover by Claim 1.6, it follows that if x and y are 3-vertices, at most once of x and y is adjacent to a 2-vertex. If $d(z) \geq 4$, then $\omega^*(f) \geq -7 + 1 + 2 + 4 = 0$. Assume now that $d(x) = d(y) = d(z) = 3$. W.l.o.g., we consider two cases: (1) x is adjacent to a 2-vertex, (2) x is not adjacent to a 2-vertex.

(1) The vertex x gives 1 to f ; the vertices y and z gives 2 to f. Moreover, the neighbors y_1, z_1 ($\neq x, y, z$) of y, z respectively are distinct and give each at least 1 to f. Hence $\omega^*(f) \geq -7 + 1 + 2 \cdot 2 + 2 \cdot 1 = 0$.

(2) The vertices x, y, z give each 2 to f. Moreover, the neighbors x_1, y_1, z_1 ($\neq x, y, z$) of x, y, z respectively are distinct and give each at least 1 to f. Hence $\omega^*(f) \geq -7 + 3 \cdot 2 + 3 \cdot 1 \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall x \in V(G) \cup F(G), \omega^*(x) \geq 0$. The contradiction obtained by Equation (5) completes the proof.

2.4 Proof of Theorem 2.3

Let G be a counterexample to Theorem 2.3 with the minimum order. The graph G satisfies Claim 1 and Equation (6) (given by Equation (1) for $k = 9$):

$$\sum_{v \in V(G)} (7d(v) - 18) + \sum_{f \in F(G)} (2r(f) - 18) = -36 \tag{6}$$

As for the proof of Theorem 2.2, we apply now a discharging procedure. We define the weight function $\omega : V(G) \cup F(G) \to \mathbb{R}$ by $\omega(x) = 7d(x) - 18$ if $x \in V(G)$ and $\omega(x) = 2r(x) - 18$ if $x \in F(G)$. It follows from Equation (6) that the total sum of weights is equal to -36. In what follows, we will define discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is achieved. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -36 < 0 \tag{7}$$

and hence demonstrates that no such counterexample can exist.

We make the discharging procedure in two steps:

Step 1. Every $\geq$3-vertex gives 2 to each adjacent 2-vertex.

When Step 1 is finished, we proceed with Step 2:

Step 2. Each $\geq$3-vertex v at distance at most two to a triangle T gives $\omega'(v)$ to T.

Let v be a k-vertex. By Claim 1.1, $k \geq 2$.

Case $k = 2$ Observe that $\omega(v) = -4$. By Claim 1.2, v is adjacent to $\geq$3-vertices. Hence, $\omega'(v) = -4 + 2 \cdot 2 = 0$ by Step 1. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 3$ Initially, $\omega(v) = 3$. By Claim 1.3, v is adjacent to at most one 2-vertex. Hence, if v is adjacent to a 2-vertex, then $\omega'(v) = 3 - 2 = 1$ and $\omega'(v) = 3$ otherwise. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 4$ Initially, $\omega(v) = 10$. By Claim 1.4, v is adjacent to at most two 2-vertices. So if v is adjacent to two (resp. one, zero) 2-vertices, then $\omega'(v) = 10 - 2 \cdot 2 = 6$ (resp. 8, 10). And by Step 2, $\omega^*(v) \geq 0$.

Case $k \geq 5$ Initially, $\omega(v) = 7k - 18$. The vertex v gives 2 to each adjacent 2-vertex in Step 1. So $\omega'(v) \geq 7k - 18 - 2k = 5k - 18 \geq 7$. And, by Step 2, $\omega^*(v) \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall v \in V(G), \omega^*(v) \geq 0$. Observe now that, after Step 1, all $\geq$3-vertex can give at least 1 to the triangle (if any) at distance at most 2 in Step 2.

Let f be a k-face. Clearly, if $k \geq 9$, then $\omega^*(f) = \omega(f) = 2r(f) - 18 \geq 0$. Now, suppose that f is a 3-face xyz with $d(x) \leq d(y) \leq d(z)$. Let xx_1x_2, yy_1y_2, and zz_1z_2 be three vertex-disjoint 2-paths starting from x, y, z respectively (these paths exist since there are no cycles of length 4 to 8). Initially, $\omega(f) = -12$. By claim 1.5, $d(x) \geq 3$. Moreover by Claim 1.6, it follows that if x and y are 3-vertices, at most once of x and y is adjacent to a 2-vertex. If $d(z) \geq 4$, then the vertices x, y, z give at least 1, 3, 10 respectively, and the vertices x_1, y_1, z_1 give at least $2 \cdot 1$; hence, $\omega^*(f) \geq -12 + 1 + 3 + 6 + 2 \cdot 1 \geq 0$. Assume now that $d(x) = d(y) = d(z) = 3$. W.l.o.g., we consider two cases: (1) x is adjacent to a 2-vertex, (2) x is not adjacent to a 2-vertex.

(1) The vertex x gives 1 to f ; the vertices y and z give 3 to f. Moreover, the vertices x_2, y_1, y_2, z_1, z_2 give each at least 1. Hence $\omega^*(f) \geq -12 + 1 + 2 \cdot 3 + 5 \cdot 1 = 0$.

(2) The vertices x, y, z give each 3 to f. The vertices $x_1, x_2, y_1, y_2, z_1, z_2$ give each at least 1. Hence $\omega^*(f) \geq -12 + 3 \cdot 3 + 6 \cdot 1 \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall x \in V(G) \cup F(G), \omega^*(x) \geq 0$. The contradiction obtained by Equation (7) completes the proof.

2.5 Proof of Theorem 2.4

Let G be a counterexample to Theorem 2.4 with the minimum order. The graph G satisfies Claim 1 and Equation (8) (given by Equation (1) for $k = 9$):

$$\sum_{v \in V(G)} (3d(v) - 8) + \sum_{f \in F(G)} (r(f) - 8) = -16 \tag{8}$$

As for the proof of Theorem 2.2, we apply now a discharging procedure. We define the weight function $\omega : V(G) \cup F(G) \to \mathbb{R}$ by $\omega(x) = 3d(x) - 8$ if $x \in V(G)$ and $\omega(x) = r(x) - 8$ if $x \in F(G)$. It follows from Equation (8) that the total sum of weights is equal to -16. In what follows, we will define discharging rules and redistribute weights accordingly. Once the discharging is finished, a new weight function ω^* is produced. However, the total sum of weights is kept fixed when the discharging is achieved. Nevertheless, we will show that $\omega^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction:

$$0 \leq \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -16 < 0 \tag{9}$$

and hence demonstrates that no such counterexample can exist.

We make the discharging procedure in two steps:

Step 1. Every $^{\geq}3$-vertex gives 1 to each adjacent 2-vertex.

When Step 1 is finished, we proceed with Step 2:

Step 2. Each $^{\geq}3$-vertex v at distance at most three to a triangle T gives $\omega'(v)$ to T.

Let v be a k-vertex. By Claim 1.1, $k \geq 2$.

Case $k = 2$ Observe that $\omega(v) = -2$. By Claim 1.2, v is adjacent to $^{\geq}3$-vertices. Hence, $\omega'(v) = -2 + 2 \cdot 1 = 0$ by Step 1. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 3$ Initially, $\omega(v) = 1$. By Claim 1.3, v is adjacent to at most one 2-vertex. Hence, if v is adjacent to a 2-vertex, then $\omega'(v) = 1 - 1 = 0$ and $\omega'(v) = 1$ otherwise. By Step 2, $\omega^*(v) \geq 0$.

Case $k = 4$ Initially, $\omega(v) = 4$. By Claim 1.4, v is adjacent to at most two 2-vertices. So if v is adjacent to two (resp. one, zero) 2-vertices, then $\omega'(v) = 4 - 2 \cdot 1 = 2$ (resp. 3, 4). By Step 2, $\omega^*(v) \geq 0$.

Case $k \geq 5$ Initially, $\omega(v) = 3k - 8$. The vertex v gives 1 to each adjacent 2-vertex in Step 1. So $\omega'(v) \geq 3k - 8 - k = 2k - 8 \geq 2$. By Step 2, $\omega^*(v) \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall v \in V(G), \omega^*(v) \geq 0$. Observe now that, after Step 1, (1) all $^{\geq}4$-vertex can give at least 2 to the triangle (if any) at distance at most 4 in Step 2, (2) a 3-vertex not adjacent to a 2-vertex can give 1 to the triangle (if any) at distance at most 4 in Step 2, and (3) the unique kind of vertices which cannot give anything is a 3-vertex adjacent to a 2-vertex. It follows by Claim 1.7:

Observation 1 *If rst is 2-path composed of $^{\geq}3$-vertices, then at least one of these vertices has a weight at least 1 after Step 1.*

Let f be a k-face. Clearly, if $k \geq 8$, then $\omega^*(f) = \omega(f) = r(f) - 8 \geq 0$.

Now, suppose that f is a 3-face xyz with $d(x) \leq d(y) \leq d(z)$. Let $xx_1x_2x_3$, $yy_1y_2y_3$, and $zz_1z_2z_3$ be three vertex-disjoint 3-paths starting from x, y, z respectively (these paths exist since there are no cycles of length 4 to 7). Initially, $\omega(f) = -5$.

We consider several cases according to the degrees of x, y, and z:

Consider the case $d(x) = 3$, $d(y) = 3$, $d(z) \geq 4$, and $d(x_1) = 2$. During Step 2, y and z give 1 and at least 2 respectively. If at least one of the vertices y_1, y_2, y_3 has degree at least 4. Then $\omega^*(f) = -5 + 1 + 2 + 2 = 0$. Assume now that $d(y_i) \leq 3$ for $i = 1, 2, 3$. By Claims 1.2, 1.3, 1.5, and 1.6, we can choose the vertices y_i such that that $d(y_i) = 3$ for $i = 1, 2, 3$. Hence by Observation 1, we are sure that at least one vertex of y_1, y_2, y_3 has a weight at least one after Step 1. This weight is transfered to f during Step 2. Similarly, by Claims 1.2, x_2 is of degree at least 3. If $d(x_2) \geq 4$, then $\omega^*(f) = -5 + 1 + 2 + 1 + 2 \geq 0$. Assume now that $d(x_2) = 3$. Let x'_3 the third neighbor of x_2 (since there are no cycles of length 4 to 7, x'_3 is distinct to $x, y, z, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$). By Claim 1.3, we have $d(x_3) \geq 3$ and $d(x'_3) \geq 3$. So by Obervation 1, at least one vertex of x_2, x_3, x'_3 has a weight at least one after Step 1. This weight is transfered to f during Step 2. Hence $\omega^*(f) = -5 + 1 + 2 + 1 + 1 = 0$.

Consider the case $d(x) = 3$, $d(y) = 3$, $d(z) \geq 4$, and $d(x_1) \geq 3$, $d(y_1) \geq 3$, $d(z_1) \geq 3$. During Step 2, x, y and z give 1, 1, and at least 2 respectively. If at least one of the vertices $x_1, x_2, x_3, y_1, y_2, y_3$ has degree at least 4. Then $\omega^*(f) = -5 + 1 + 1 + 2 + 2 \geq 0$. Assume now that $d(x_i) \leq 3$ and $d(y_i) \leq 3$ for $i = 1, 2, 3$. By Claims 1.2, 1.3, 1.5, and 1.6, we can choose x_i an y_i such that $d(x_i) = 3$ and $d(y_i) = 3$ for $i = 1, 2, 3$. Hence by Observation 1, we are sure that at least one vertex of x_1, x_2, x_3 (resp. y_1, y_2, y_3) has a weight at least one after Step 1. This weight is transfered to f during Step 2. Hence $\omega^*(f) = -5 + 1 + 1 + 2 + 1 + 1 \geq 0$.

Consider the case $d(x) = d(y) = d(z) = 3$, and $d(x_1) = 2$. During Step 2, f receives 1 from y and 1 from z. We first show that each path of $y_1y_2y_3$ and $z_1z_2z_3$ gives at least 1 to f. Consider $y_1y_2y_3$. If one of y_1, y_2, y_3 is of degree at least 4, then this path will give at least 1 to f. Otherwise, by Claims 1.2, 1.3, 1.5, and 1.6, we can assume that $d(y_i) \geq 3$ for $i = 1, 2, 3$. Hence by Observation 1, we are sure that at least one vertex of y_1, y_2, y_3 has a weight at least one after Step 1. Similarly, the path $z_1z_2z_3$ gives at least 1 to f. Now, by Claims 1.2, x_2 is of degree at least 3. If $d(x_2) \geq 4$, then $\omega^*(f) = -5+1+1+1+1+2 \geq 0$. Assume now that $d(x_2) = 3$. Let x'_3 the third neighbor of x_2 (since there are no cycles of length 4 to 7, x'_3 is distinct to $x, y, z, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$). By Claim 1.3, we have $d(x_3) \geq 3$ and $d(x'_3) \geq 3$. So by Obervation 1, at least one vertex of x_2, x_3, x'_3 has a weight at least one after Step 1. This weight is transfered to f during Step 2. Hence $\omega^*(f) = -5 + 1 + 1 + 1 + 1 + 1 = 0$.

Consider the case $d(x) = d(y) = d(z) = 3$, and $d(x_1) \geq 3$, $d(y_1) \geq 3$, $d(z_1) \geq 3$. Using similar arguments, one can prove that $\omega^*(f) \geq -5 + 1 + 1 + 1 + 1 + 1 + 1 \geq 0$.

Hence, after Steps 1 and 2, we have: $\forall x \in V(G) \cup F(G), \omega^*(x) \geq 0$. The contradiction obtained by Equation (9) completes the proof.

3 Conclusion

We conclude with some specific problems. It was recently proved by Borodin *et al.* [4] that every planar graph with girth at least 7 is acyclically 3-choosable. (We recall that the girth of graph G is the length of a shortest cycle of G.)

Problem 1 *Prove that:*

1. *Every planar graph with girth at least 6 is acyclically 3-choosable.*
2. *Every planar graph without cycles of length 4 to i is acyclically 3-choosable with* $6 \leq i \leq 11$.
3. *There exists a constant d such that every planar graph G without cycles of length 4 to 6 and $d_\triangle(G) \geq d$ is acyclically 3-choosable.*

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