



**HAL**  
open science

# Green functions for killed random walks in the Weyl chamber of $Sp(4)$

Kilian Raschel

► **To cite this version:**

Kilian Raschel. Green functions for killed random walks in the Weyl chamber of  $Sp(4)$ . 2010. hal-00425923v1

**HAL Id: hal-00425923**

**<https://hal.science/hal-00425923v1>**

Preprint submitted on 22 Oct 2009 (v1), last revised 18 Nov 2010 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Martin boundary for killed random walks in the Weyl chamber of the dual of $\mathrm{Sp}(4)$

Kilian Raschel\*

October 22, 2009

## Abstract

We consider a family of random walks absorbed at the boundary of the Weyl chamber of the dual of  $\mathrm{Sp}(4)$  and verifying the following property : for any  $n \geq 3$  there is in this family a random walk associated with a reflection group of order  $2n$ . Moreover, the case  $n = 4$  corresponds to a well-known random walk that appears naturally by studying quantum random walks on the dual of  $\mathrm{Sp}(4)$ . For the random walks in this family, we find the exact asymptotic of the Green functions along all infinite paths of states. In particular, we deduce that the Martin boundary is reduced to one point. We also calculate the exact asymptotic of the absorption probabilities as the absorption site goes to infinity.

*Keywords* : killed random walk, Green functions, Martin boundary, absorption probabilities.

*AMS 2000 Subject Classification* : primary 60G50, 31C35 ; secondary 30E20, 30F10.

## 0 Introduction

Random walks conditioned on staying in cones of  $\mathbb{Z}^d$  attract more and more attention in the mathematical community, as they appear in several distinct domains. An historically important example is constituted by the so-called non-colliding random walks, in other words by the processes  $(Z_1, \dots, Z_d)$  composed of  $d$  independent and identically distributed random walks conditioned on never leaving the Weyl chamber  $\{z \in \mathbb{R}^d : z_1 < \dots < z_d\}$ . These processes appeared in the eigenvalues description of important matrix-valued stochastic processes, see [Dys62], and are recently again very much studied, see *e.g.* [EK08] and the references therein. Another important area where random processes conditioned on never leaving cones of  $\mathbb{Z}^d$  appear is the domain of the quantum random walks, non-commutative generalization of classical random walks, see *e.g.* [Bia91] and [Bia92].

A usual way to condition a process  $Z$  on staying in a cone consists in using a Doob  $h$ -transform. Indeed, if  $h$  is harmonic for  $Z$ , positive inside of the cone and equal to zero at its boundary, then the Doob  $h$ -transform of  $Z$  will never hit the boundary of the cone. Moreover, such a function  $h$  is very naturally positive harmonic for the process  $Z$  absorbed at the boundary of the cone.

It is therefore natural to be interested in finding positive harmonic functions associated with processes absorbed at the boundary of cones, and more generally to be interested in the Martin compactification of such processes, that can be obtained *e.g.* from the exact asymptotic of the Green functions (in other words, the mean number of visits).

---

\*Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 4 Place Jussieu 75252 Paris Cedex 05, France. E-mail : kilian.raschel@upmc.fr

In this context, the case of random walks in  $\mathbb{Z}^d$  with non-zero drift has held a great and fruitful deal of attention from the mathematical community, particularly for  $d = 2$ .

This is how that in [Bia92], by studying quantum random walks in the Weyl chamber of the dual of Lie groups, P. Biane is naturally led to consider the case of  $SU(3)$  and  $Sp(4)$ , and in this way the classical random walks with non-zero drift, having transition probabilities inside of the Weyl chamber as represented on Picture 1 below and killed at the boundary of the Weyl chamber – it is actually possible to obtain classical random walks from quantum random walks by restricting the latter ones, initially defined on a non-commutative von Neumann algebra, to commutative subalgebras, for the details see [Bia92]. Thanks to Choquet-Denis theory, he finds all minimal non-negative harmonic functions associated with these random walks. Nevertheless, this approach does not allow him to find the Martin compactification of these processes.

In the more general case of the eight-nearest neighbors random walks with non-zero drift and absorbed at the boundary of the quarter plane, we have given in [KR09] the exact asymptotic of the Green functions along all infinite paths of states, as well as the asymptotic of the absorption probabilities as the absorption site goes to infinity. This result has allowed us to obtain the Martin compactification of these processes.

In the work [IR08], I. Ignatiouk-Robert obtains for all  $d \geq 1$  the Martin compactification of the random walks in the half-space  $\mathbb{Z}^d \times \mathbb{Z}_+$  killed at the boundary, with non-zero drift and with an exponential decay of the transition probabilities. I. Ignatiouk-Robert proposes there a new approach for the analysis of the Martin compactification, based on large deviations. I. Ignatiouk-Robert and C. Loree have then successively developed these methods in [IRL09] and have obtained the Martin compactification of the random walks in  $\mathbb{Z}_+^2$  killed at the boundary, having a non-zero drift and with an exponential decay of the transition probabilities. Under similar hypotheses and for all  $d \geq 3$ , I. Ignatiouk-Robert has recently found in [IR09] the Martin compactification of killed random walks in  $\mathbb{Z}_+^d$ .

However, the latter methods seem not to be powerful for a more detailed study, as *e.g.* for the computation of the exact asymptotic of the Green functions, or for the computation of the absorption probabilities at different sites of the boundary. In addition, having a non-zero drift is an essential hypothesis for the application of these methods. Also, the results of [IR09] are conditioned by the fact of been able (page 5 of [IR09]) “to identify the positive harmonic functions of a random walk on  $\mathbb{Z}^d$  which has zero mean and is killed at the first exit from  $\mathbb{Z}_+^d$ .” Unfortunately, for  $d \geq 2$  there are no general results in this domain.”

As may the existence of this open problem suggest, the results concerning Martin compactification or the asymptotic of Green functions for random walks in  $\mathbb{Z}^d$  absorbed at the boundary and with drift zero are actually scarce, even for  $d = 2$ .

The simplest case in dimension  $d = 2$  is the one of the cartesian product of two one-dimensional simple random walks with zero mean. The fact that the Martin boundary of these random walks is reduced to one point and the explicit expression of the unique harmonic function of the Martin boundary can be obtained from [PW92]. In addition, we have obtained in [Ras09b] the exact asymptotic of the Green functions associated with these processes.

From a Lie group theory point of view, the previous case corresponds to the group product  $SU(2) \times SU(2)$ , which is associated with a reducible rank-2 root system, see [Bou75]. It is then natural to be interested in the classical random walks that can be obtained from the construction made by P. Biane in [Bia92] starting from Lie groups associated with an irreducible rank-2 root system. It turns out that there are only two such Lie groups, namely  $SU(3)$  and  $Sp(4)$  : indeed, there are three Lie groups with an underlying irreducible rank-2 root system,  $SU(3)$ ,  $Sp(4)$  and the exceptional Lie group  $G_2$ , but  $G_2$  has no minuscule weight, see [Bou75], and for this reason the construction of P. Biane does not apply, see [Bia92].

In the case of  $SU(3)$ , the construction done in [Bia92] leads to consider the random walk with transition probabilities equal to  $1/3$  as represented at the left of Picture 1 below. In [Bia91], P. Biane makes explicit a suitable harmonic function  $h$  such that the associated Doob  $h$ -transform of the previous process will never hit the boundary of the Weyl chamber – he expresses  $h$  as the dimension of some representation of  $SU(3)$ . In addition, he computes the exact asymptotic of the Green functions of the killed process along all infinite paths of states *except* the ones which are tangent to the boundary of the Weyl chamber. However, the asymptotic of the Green functions along the paths of states tangent to the boundary couldn't be obtained by using these methods, and therefore neither the Martin boundary of the process. The object of the recent note [Ras09a] was precisely to obtain these two lacking results.

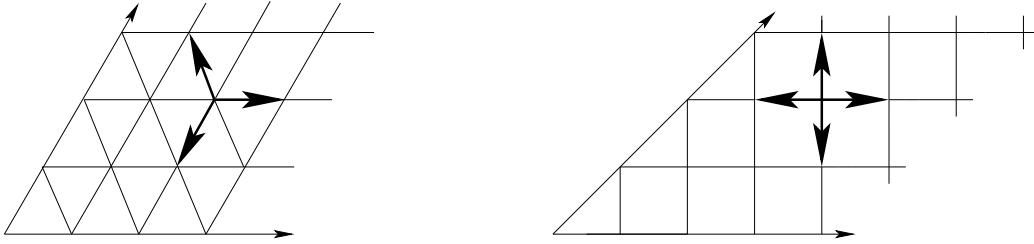


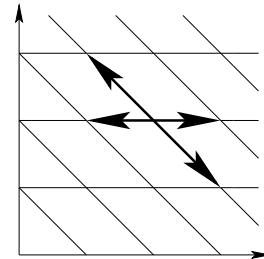
Figure 1: Random walks in the Weyl chamber of the dual of  $SU(3)$  and  $Sp(4)$

It was therefore natural to consider now the case of  $Sp(4)$ , in order to complete the case of the Lie groups associated with an irreducible rank-2 root system. Applying to  $Sp(4)$  the way described in [Bia92] in order to get classical processes starting from quantum random walks on lattices, we obtain naturally the random walk with transition probabilities equal to  $1/4$  as drawn at the right of Picture 1 above and killed at the boundary of the octant. By making a straightforward linear transformation, this process becomes the random walk  $(X(k), Y(k))_{k \geq 0}$  *spatially homogeneous inside of the quarter plane, and such that if the*  $p_{ij} = \mathbb{P}[(X(k+1), Y(k+1)) = (i_0 + i, j_0 + j) \mid (X(k), Y(k)) = (i_0, j_0)]$  *denote the transition probabilities, then we have*

$$(H1) \quad p_{10} + p_{-10} + p_{1-1} + p_{-1-1} = 1, \quad p_{10} = p_{-10}, \quad p_{1-1} = p_{-1-1},$$

$$(H2) \quad \{(i, 0) : i \geq 1\} \cup \{(0, j) : j \geq 1\} \text{ is absorbing,}$$

$$(H3) \quad p_{10} = p_{-10} = 1/4 \text{ and } p_{1-1} = p_{-1-1} = 1/4.$$



As he did in [Bia91] for the random walk in the Weyl chamber of the dual of  $SU(3)$ , P. Biane makes explicit, for this process, a suitable harmonic function  $h$  such that the underlying Doob  $h$ -transform will never hit the boundary, see [Bia92]. Note also that this random walk is the partial subject of [DO05], where the authors are interested in making explicit the exit time queues associated with some processes. Having said that, neither the Martin compactification, nor the asymptotic of the Green functions, nor the asymptotic of the absorption probabilities have been yet calculated for this process, up to our knowledge.

In order to obtain explicit formulas for the Green functions and for the absorption probabilities associated with such processes, we have wished to use methods based on [FIM99]. In this book, the authors G. Fayolle, R. Iasnogorodski and V. Malyshev elaborate a profound and ingenious analytic approach to obtain explicit expressions for the generating functions of stationary probabilities for some ergodic random walks in the quarter plane. In [KR09], we have adapted this approach to the case of the generating functions of the Green functions and of the absorption probabilities for some walks in the

quarter plane absorbed at the boundary and having a positive drift. We will see here that this approach can also be extended to the case of random walks having a drift zero : Subsections 1.1 and 4.1 of this paper, that lead to explicit formulations for the generating functions of the Green functions and of the absorption probabilities, are strongly inspired by [FIM99].

In their book, the authors of [FIM99] also highlight the importance and the influence of the *group of the random walk*. Let us define this group in the case of the random walks verifying (H1) above. For this consider the algebraic curve  $\{(x, y) \in (\mathbb{C} \cup \{\infty\})^2 : Q(x, y) = 0\}$ , where  $Q$  is the following polynomial (simple transformation of the transition probabilities generating function) :

$$Q(x, y) = xy(p_{10}x + p_{10}/x + p_{1-1}x/y + p_{1-1}y/x - 1). \quad (1)$$

If  $Q(x, y) = 0$ , then with (1) it is immediate that  $Q(\hat{\xi}(x, y)) = 0$  and  $Q(\hat{\eta}(x, y)) = 0$ , where

$$\hat{\xi}(x, y) = \left(x, \frac{x^2}{y}\right), \quad \hat{\eta}(x, y) = \left(\frac{p_{1-1}y + p_{10}}{p_{10}y + p_{1-1}x}, y\right).$$

The *group of the random walk* is then  $W = \langle \hat{\xi}, \hat{\eta} \rangle$ , the group of automorphisms of the algebraic curve  $\{(x, y) \in (\mathbb{C} \cup \{\infty\})^2 : Q(x, y) = 0\}$  generated by  $\hat{\xi}$  and  $\hat{\eta}$ .

It is already known, see *e.g.* [BMM09], that under (H1) and (H3),  $W$  is of order eight.

Moreover, we will prove later that in fact  $W$  is finite if and only if there exists a rational number  $r$  such that  $p_{10} = p_{-10} = \sin(r\pi)^2/2$  and  $p_{1-1} = p_{-11} = \cos(r\pi)^2/2$ , see Remark 3.

In this paper, we have wished to study the walks verifying (H1), (H2) and (H3'), where

$$(H3') \quad p_{10} = p_{-10} = \sin(\pi/n)^2/2 \text{ and } p_{1-1} = p_{-11} = \cos(\pi/n)^2/2.$$

We will show that under these hypotheses,  $W$  is of order  $2n$ , see Subsection 1.2 and particularly (11). Note that for  $n = 4$ , the hypotheses (H3) and (H3') are the same.

The family constituted by the union, for  $n \geq 3$ , of the random walks verifying (H1) and (H3') is therefore in some sense representative of all the random walks having a finite group  $W$  : indeed, among the set of all the walks with an underlying group  $W$  of finite order, there is in this family a representative for the subclass of the walks having a group  $W$  of order  $2n$ , for any  $n \geq 3$  – let us recall that the case of the walks with group  $W$  of order four corresponds to the product case, which has already been considered in other works. This property of being representative seemed us important and this is why we have thought that it was interesting not only to consider the particular case  $n = 4$ , *i.e.* the walk in the Weyl chamber of the dual of  $\text{Sp}(4)$ , but rather the case of all these walks, for  $n \geq 3$ .

This article is then hinged on two results.

The first of these two results concerns the asymptotic of the Green functions

$$G_{i,j}^{i_0,j_0} = \mathbb{E}_{(i_0,j_0)} \left[ \sum_{k=0}^{\infty} \mathbf{1}_{\{(X(k), Y(k)) = (i,j)\}} \right], \quad (2)$$

$\mathbb{E}_{(i_0,j_0)}$  denoting the conditional expectation given  $(X(0), Y(0)) = (i_0, j_0)$ .

We will prove in Theorem 9 of Section 3 that as  $i+j \rightarrow \infty$  and  $j/i \rightarrow \tan(\gamma)$ ,  $\gamma \in [0, \pi/2]$ ,

$$G_{i,j}^{i_0,j_0} \sim A_n f_n(i_0, j_0) \frac{\sin(n \arctan(\frac{j/i}{1+j/i} \tan(\pi/n)))}{[\cos(\pi/n)^2 (i^2 + 2ij) + j^2]^{n/2}},$$

where  $A_n$  is some positive constant depending only on the transition probabilities, and  $f_n$  a harmonic function for  $(X, Y)$  defined and studied with details in Section 2.

In particular, we will obtain from this result that the Martin boundary of the process satisfying to (H1), (H2) and (H3) is, for any  $n \geq 3$ , reduced to one point, see Corollary 11.

Therefore this paper gives a partial answer in the case  $d = 2$  to the open problem highlighted by I. Ignatiouk-Robert in [IR09], since Corollary 11 implies that there is, up to the positive multiplicative constants, only one positive harmonic function for each process in the infinite family of two-dimensional killed random walks that we consider in this paper, family which is, as already said, in a way representative of the set of all the two-dimensional killed random walks with finite group.

The second of the two results is the subject of Section 4 and concerns the absorption probabilities

$$\begin{aligned} h_i^{i_0, j_0} &= \mathbb{P}_{(i_0, j_0)} [\exists k \geq 1 : (X(k), Y(k)) = (i, 0)], \\ \tilde{h}_j^{i_0, j_0} &= \mathbb{P}_{(i_0, j_0)} [\exists k \geq 1 : (X(k), Y(k)) = (0, j)], \end{aligned} \quad (3)$$

$\mathbb{P}_{(i_0, j_0)}$  denoting the conditional probability given  $(X(0), Y(0)) = (i_0, j_0)$ .

Firstly, in Subsection 4.1, we obtain integral representations of the generating functions of the absorption probabilities

$$h^{i_0, j_0}(x) = \sum_{i=1}^{\infty} h_i^{i_0, j_0} x^i, \quad \tilde{h}^{i_0, j_0}(y) = \sum_{j=1}^{\infty} \tilde{h}_j^{i_0, j_0} y^j, \quad (4)$$

using for this some arguments based on [FIM99] and [KR09].

Secondly, in Subsection 4.2, by studying closely the function  $h^{i_0, j_0}$ , notably its behavior in the neighborhood of 1, we prove in Theorem 13 that as the site  $i$  goes to infinity,

$$h_i^{i_0, j_0} \sim B_n f_n(i_0, j_0) \frac{1}{i^{n+1}},$$

where  $B_n$  is some positive constant depending only on the transition probabilities, and  $f_n$  the same harmonic function as before. A similar result holds, of course, for the asymptotic of the  $\tilde{h}_j^{i_0, j_0}$  as the site  $j$  goes to infinity.

It is worth noting that from a technical point of view, the only moments where we use the finiteness and the order of the group  $W$  are first when we simplify the quantity  $h^{i_0, j_0}(x) + \tilde{h}^{i_0, j_0}(y) - x^{i_0} y^{j_0}$ , see Subsection 1.2 and particularly Proposition 1, and second when we find the explicit form of the conformal gluing function, see Subsection 4.1.

## Acknowledgments.

I would like to thank Professor Ignatiouk-Robert for the stimulating discussions we had together concerning the topic of this article. I also warmly thank Professor Kurkova who introduced me to this field of research and also for her constant help and support during the elaboration of this project.

## 1 Analytic approach

### 1.1 A functional equation

Subsection 1.1 consists in preparatory results and is inspired by the book [FIM99].

Denote by  $G^{i_0, j_0}(x, y) = \sum_{i, j=1}^{\infty} G_{i, j}^{i_0, j_0} x^{i-1} y^{j-1}$  the generating function of the Green functions (2). With this notation and with (4), we can state the following functional equation :

$$Q(x, y) G^{i_0, j_0}(x, y) = h^{i_0, j_0}(x) + \tilde{h}^{i_0, j_0}(y) - x^{i_0} y^{j_0}, \quad (5)$$

$Q$  being defined in (1). *A priori*, Equation (5) has a meaning in  $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$ . The proof of (5) is obtained exactly as in Part 2.1 of [KR09].

When no ambiguity on the initial state can arise, we will drop the index  $i_0, j_0$  and we will write  $G_{i,j}, G(x, y), h_i, h(x), \tilde{h}_j, \tilde{h}(y)$  for  $G_{i,j}^{i_0, j_0}, G^{i_0, j_0}(x, y), h_i^{i_0, j_0}, h^{i_0, j_0}(x), \tilde{h}_j^{i_0, j_0}, \tilde{h}^{i_0, j_0}(y)$ .

Let us now have a look to the algebraic curve  $\{(x, y) \in (\mathbb{C} \cup \{\infty\})^2 : Q(x, y) = 0\}$ , that we will note  $\{Q = 0\}$  for the sake of briefness. Start by writing the polynomial (1) alternatively

$$Q(x, y) = a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \quad (6)$$

where  $a(x) = p_{1-1}$ ,  $b(x) = p_{10}x^2 - x + p_{10}$ ,  $c(x) = p_{1-1}x^2$  and  $\tilde{a}(y) = p_{10}y + p_{1-1}$ ,  $\tilde{b}(y) = -y$ ,  $\tilde{c}(y) = p_{1-1}y^2 + p_{10}y$ . Set also  $d(x) = b(x)^2 - 4a(x)c(x)$  and  $\tilde{d}(y) = \tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y)$ . We have

$$d(x) = p_{10}^2(x-1)^2(x^2 + 2x(1 - 1/p_{10}) + 1), \quad \tilde{d}(y) = -4p_{10}p_{1-1}y(y-1)^2. \quad (7)$$

$d$  has a double root at 1 and two simple roots at positive points that we will note  $x_1 < 1 < x_4$ . As for  $\tilde{d}$ , it has a double root at 1 and a simple root at 0. We will also note  $y_1 = 0$  and  $y_4 = \infty$ . In this way these notations are consistent with the ones of [FIM99] and [KR09].

Then with (6) we remark that  $Q(x, y) = 0$  is equivalent to  $(b(x) + 2a(x)y)^2 = d(x)$  or to  $(\tilde{b}(y) + 2\tilde{a}(y)x)^2 = \tilde{d}(y)$ . In particular, it follows from the particular form of  $d$  or of  $\tilde{d}$ , see (7), that the surface  $\{Q = 0\}$  has genus zero, and is thus homeomorphic to a sphere. This Riemann surface can therefore be rationally uniformized, in the sense that it is possible to find two rational functions,  $\pi$  and  $\tilde{\pi}$ , such that  $\{Q = 0\} = \{(\pi(s), \tilde{\pi}(s)) : s \in \mathbb{C} \cup \{\infty\}\}$ . Moreover, it is remarked in Chapter 6 of [FIM99] that we can take  $\pi(s) = (x_4 + x_1)/2 + ((x_4 - x_1)/4)(s + 1/s)$ ,  $x_1$  and  $x_4$  being defined below (7); it is then possible to deduce a correct expression for  $\tilde{\pi}$ , since by construction the following equality has to hold:  $Q(\pi, \tilde{\pi}) = 0$ .

For more details about the construction of Riemann surfaces, see for instance [SG69].

## 1.2 Uniformization and meromorphic continuation

**Notation.** Throughout the whole paper, we will denote by  $\imath$  the complex number  $\imath^2 = -1$  and by  $\bar{z}$  the complex conjugate of the complex number  $z$ .

But rather than the uniformization  $(\pi, \tilde{\pi})$  proposed in [FIM99] and recalled at the end of the previous subsection, we prefer using another, that will turn out to be quite more convenient. This new uniformization, that we will call  $(x, y)$ , is just equal to  $(\pi \circ L, \tilde{\pi} \circ L)$ , where

$$L(z) = \frac{z_0 z - 1}{z - z_0}, \quad z_0 = -\exp(-\imath\pi/n).$$

Note that  $z_0$  is such that  $\pi(z_0) = \pi(\bar{z}_0) = \tilde{\pi}(z_0) = \tilde{\pi}(\bar{z}_0) = 1$  and that its explicit expression above is due to (H3'), for more details see Remark 3. Then, starting from the explicit expressions of  $(\pi, \tilde{\pi})$  and of  $L$ , we easily show that the expression of the new uniformization can be

$$x(z) = \frac{(z + z_0)(z + \bar{z}_0)}{(z - z_0)(z - \bar{z}_0)}, \quad y(z) = \frac{(z + z_0)^2}{(z - z_0)^2}. \quad (8)$$

This uniformization has the significant advantage of transforming the important cycles (*i.e.* the branch cuts  $[x_1, x_4]$  and  $[y_1, y_4]$ , the unit circles  $\{|x| = 1\}$  and  $\{|y| = 1\}$ ) into very simple cycles – what was not the case of  $(\pi, \tilde{\pi})$  –, since the following equalities hold, see also Picture 2 :

$$\begin{aligned} x^{-1}([x_1, x_4]) &= \mathbb{R} \cup \{\infty\}, & x^{-1}(\{|x| = 1\}) &= \imath\mathbb{R} \cup \{\infty\}, \\ y^{-1}([y_1, y_4]) &= z_0\mathbb{R} \cup \{\infty\}, & y^{-1}(\{|y| = 1\}) &= z_0\imath\mathbb{R} \cup \{\infty\}. \end{aligned} \quad (9)$$

To obtain (9), it is sufficient to use the explicit expressions of the branch points  $x_1, x_4, y_1, y_4$ , see below (7), as well as the explicit formulation of the uniformization, see (8).

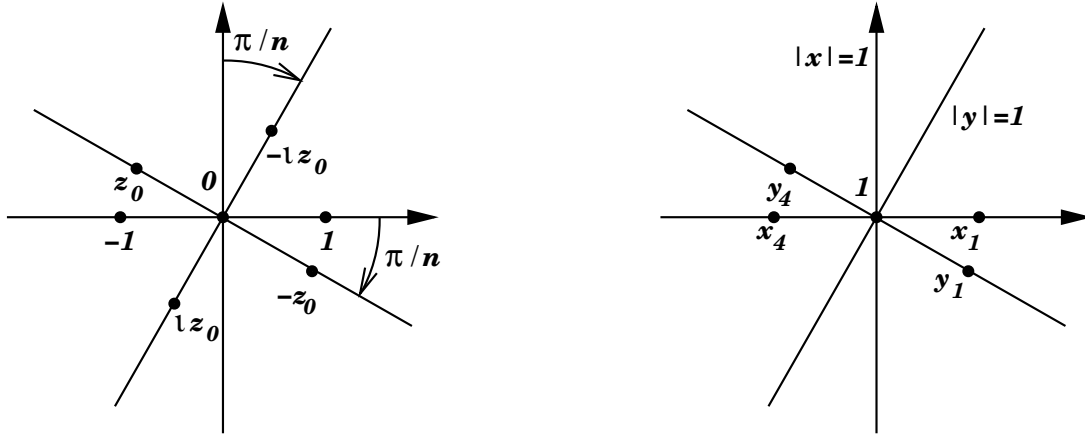


Figure 2: The uniformization space, with at the left some important elements of it, and at the right the corresponding elements through the functions  $x$  and  $y$

Let us go back to the automorphisms introduced in Section 0, namely  $\hat{\xi}$  and  $\hat{\eta}$ , the automorphisms of the algebraic curve  $\{Q = 0\}$ . Thanks to the uniformization (8), they define two automorphisms  $\xi$  and  $\eta$  on  $\mathbb{C} \cup \{\infty\}$ , which are characterized by

$$\xi^2 = \text{id}, \quad x \circ \xi = x, \quad y \circ \xi = \frac{x^2}{y}, \quad \eta^2 = \text{id}, \quad y \circ \eta = y, \quad x \circ \eta = \frac{p_{1-1}y^2 + p_{10}y}{p_{10}y + p_{1-1}} \frac{1}{x}. \quad (10)$$

Using the well-known characterization of the automorphisms of the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ , (8) and (10), we obtain that  $\xi$  and  $\eta$  have the following expressions :

$$\xi(z) = 1/z, \quad \eta(z) = \exp(-2i\pi/n)/z. \quad (11)$$

The expression of  $\eta$  in term of  $n$ , above, is due to the assumption (H3'), see Remark 3 for more details. Note also that the fact that  $\xi$  and  $\eta$  admit a particularly simple expression is an other pleasant property of the uniformization  $(x, y)$ .

As in Section 0, we will call the group generated by  $\xi$  and  $\eta$

$$W_n = \langle \xi, \eta \rangle$$

the *group of the random walk*. In our case,  $W_n$  is isomorphic to the dihedral group of order  $2n$ , *i.e.* the group of symmetries of a regular polygon with  $n$  sides,  $\xi$  and  $\eta$  playing the role of the two reflections.

We are now going to state and prove Proposition 1, which is the main result of Subsection 1.2, and that concerns the continuation of the generating functions  $h$  and  $\tilde{h}$  defined in (4). But for this we need to describe the action of the elements of  $W_n$  on some cones of the plane, and to find some fundamental domains of the plane for the action of  $W_n$  – we will say that  $D$  is a fundamental domain of the plane for the action of  $W_n$  if  $\cup_{w \in W_n} w(D) = \mathbb{C}$  and if in addition the union is disjoint.

Take the following notation : for  $\theta_1 \leq \theta_2$ ,  $\Lambda(\theta_1, \theta_2) = \{t \exp(i\theta), 0 \leq t \leq \infty, \theta_1 \leq \theta \leq \theta_2\}$  is the cone with vertex at 0 and opening angles  $\theta_1, \theta_2$ . In particular,  $\Lambda(\theta, \theta) = \exp(i\theta)\mathbb{R}_+ \cup \{\infty\}$ . Thanks to (11), we obtain that the action of  $\xi$  and  $\eta$  on these cones is given by  $\xi(\Lambda(\theta_1, \theta_2)) = \Lambda(-\theta_2, -\theta_1)$  and  $\eta(\Lambda(\theta_1, \theta_2)) = \Lambda(-\theta_2 - 2\pi/n, -\theta_1 - 2\pi/n)$ . These facts are illustrated at the left of Picture 3.

Define now, for  $k \in \{0, \dots, n\}$ ,  $D_k^+ = \Lambda(\frac{k-1}{n}\pi, \frac{k}{n}\pi)$  and  $D_k^- = \Lambda(-\frac{k+1}{n}\pi, -\frac{k}{n}\pi)$ . Sometimes we will write  $D_0$  instead of  $D_0^+ = D_0^-$  and  $D_n$  instead of  $D_n^+ = D_n^-$ . Clearly,

$$D_0 \cup D_n \cup \bigcup_{k=1}^{n-1} D_k^+ \cup \bigcup_{k=1}^{n-1} D_k^- = \mathbb{C} \cup \{\infty\}. \quad (12)$$



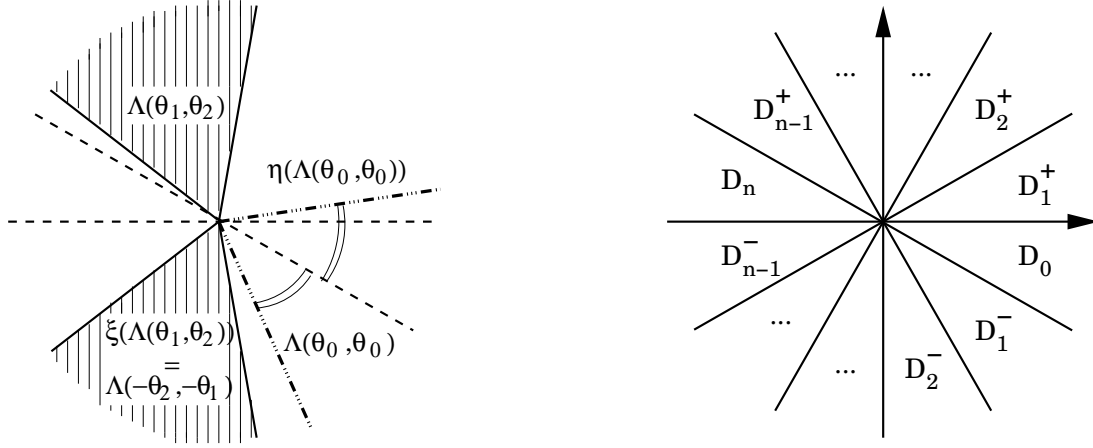


Figure 3: Important cones of the uniformization space

The definitions of  $D_k^+$  and  $D_k^-$ , as well as (12), are illustrated at the right of Picture 3.

It is immediate that for any  $k \in \{1, \dots, n\}$ ,  $D_k^+ = \xi(D_{k-1}^-)$  and  $D_k^- = \eta(D_{k-1}^+)$ . In particular, for any  $2k \in \{1, \dots, n\}$ ,  $D_{2k}^+ = (\xi \circ \eta)^k(D_0)$  and  $D_{2k}^- = (\eta \circ \xi)^k(D_0)$ . Likewise, for any  $2k+1 \in \{1, \dots, n\}$ ,  $D_{2k+1}^+ = (\xi \circ (\eta \circ \xi)^k)(D_0)$  and  $D_{2k}^- = (\eta \circ (\xi \circ \eta)^k)(D_0)$ . With (12), these equalities prove that  $\cup_{w \in W_n} w(D_0) = \mathbb{C} \cup \{\infty\}$ , and we will say that  $D_0$  is a fundamental domain for the action of  $W_n$  on  $\mathbb{C}$  – though this is not quite exact, since each half-line  $\Lambda(k\pi/n, k\pi/n)$ ,  $k \in \{0, \dots, 2n-1\}$ , appears twice in the union  $\cup_{w \in W_n} w(D_0)$ .

We are now able to state and prove Proposition 1, after the weak recall on the lifting of functions that follows : any function  $f$  of the variable  $x$  (resp.  $y$ ) defined on some domain  $D \subset \mathbb{C}$  can be lifted on  $\{z \in \mathbb{C} \cup \{\infty\} : x(z) \in D\}$  (resp.  $\{z \in \mathbb{C} \cup \{\infty\} : y(z) \in D\}$ ) by setting  $F(z) = f(x(z))$  (resp.  $F(z) = f(y(z))$ ). In particular we can lift the generating functions  $h$  and  $\tilde{h}$  and we set  $H(z) = h(x(z))$  and  $\tilde{H}(z) = \tilde{h}(y(z))$ , they are well defined on  $\{z \in \mathbb{C} \cup \{\infty\} : |x(z)| \leq 1\}$  and  $\{z \in \mathbb{C} \cup \{\infty\} : |y(z)| \leq 1\}$  respectively. In particular, on  $\{z \in \mathbb{C} \cup \{\infty\} : |x(z)| \leq 1, |y(z)| \leq 1\}$  (which is equal to  $\Lambda(-\pi/2, \pi/2 - \pi/n)$ , see Picture 2), the functional equation (5) yields  $H(z) + \tilde{H}(z) - x^{i_0} y^{j_0}(z) = 0$  – in the sequel, we will often write  $x^{i_0} y^{j_0}(z)$  instead of  $x(z)^{i_0} y(z)^{j_0}$ .

**Proposition 1.** *The functions  $H(z) = h(x(z))$  and  $\tilde{H}(z) = \tilde{h}(y(z))$  can be meromorphically continued from respectively  $\Lambda(-\pi/2, \pi/2)$  and  $\Lambda(-\pi/2 - \pi/n, \pi/2 - \pi/n)$  up to respectively  $\mathbb{C} \setminus \Lambda(\pi, \pi)$  and  $\mathbb{C} \setminus \Lambda(\pi - \pi/n, \pi - \pi/n)$ . Furthermore, these continuations verify*

$$\forall z \in \mathbb{C} : \quad H(z) = H(\xi(z)), \quad \tilde{H}(z) = \tilde{H}(\eta(z)), \quad (13)$$

and

$$H(z) + \tilde{H}(z) - x^{i_0} y^{j_0}(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus \Lambda(\pi - \pi/n, \pi) \\ - \sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) & \text{if } z \in \Lambda(\pi - \pi/n, \pi) \end{cases} \quad (14)$$

where  $l(w)$  is the length of  $w$ , that is the smallest  $r$  for which we can write  $w = w_1 \circ \dots \circ w_r$ , with  $w_1, \dots, w_r$  equal to  $\xi$  or  $\eta$ .

**Remark 2.** *As a consequence of Proposition 1, the generating functions  $h$  and  $\tilde{h}$  can be continued as meromorphic functions from the unit disc up to  $\mathbb{C} \setminus [1, x_4]$  and  $\mathbb{C} \setminus [1, y_4]$  respectively. Indeed, the formulas  $h(x) = H(z)$  if  $x(z) = x$  and  $\tilde{h}(y) = \tilde{H}(z)$  if  $y(z) = y$  define  $h$  and  $\tilde{h}$  not ambiguously, thanks to (13). Moreover, since  $x(\Lambda(\pi, \pi)) = [1, x_4]$  and  $y(\Lambda(\pi - \pi/n, \pi - \pi/n)) = [1, y_4]$ , see (9) and Picture 2, the previous formulas yield meromorphic continuations on  $\mathbb{C} \setminus [1, x_4]$  and  $\mathbb{C} \setminus [1, y_4]$  respectively.*

*Proof of Proposition 1.* In order to prove Proposition 1, we will use the decomposition (12), and more precisely we will define  $H$  and  $\tilde{H}$  piecewise, by defining them on each of the  $2n$  domains  $D$  that appear in the decomposition (12), to be equal to some functions  $H_D$  and  $\tilde{H}_D$ ; it will then suffice to show that the functions  $H$  and  $\tilde{H}$  so defined verify the conclusions of Proposition 1.

- In  $D_0 \subset \{z \in \mathbb{C} \cup \{\infty\} : |x(z)| \leq 1, |y(z)| \leq 1\}$ , we are going to use the most natural way to define  $H_{D_0}$  and  $\tilde{H}_{D_0}$ , *i.e.* their power series, and we set, for  $z \in D_0$ ,  $H_{D_0}(z) = h(x(z))$  and  $\tilde{H}_{D_0}(z) = \tilde{h}(y(z))$ .

- Then, for  $k \in \{1, \dots, n-1\}$ , we define  $H_{D_k^+}$ ,  $\tilde{H}_{D_k^+}$  on  $D_k^+$ , and  $H_{D_k^-}$ ,  $\tilde{H}_{D_k^-}$  on  $D_k^-$  by

$$\begin{aligned} \forall z \in D_k^+ = \xi(D_{k-1}^-) & : H_{D_k^+}(z) = H_{D_{k-1}^-}(\xi(z)), \quad \tilde{H}_{D_k^+}(z) = -H_{D_k^+}(z) + x^{i_0}y^{j_0}(z), \\ \forall z \in D_k^- = \eta(D_{k-1}^+) & : \tilde{H}_{D_k^-}(z) = \tilde{H}_{D_{k-1}^+}(\eta(z)), \quad H_{D_k^-}(z) = -\tilde{H}_{D_k^-}(z) + x^{i_0}y^{j_0}(z). \end{aligned}$$

- At last, for  $z \in D_n$ , we set  $H_{D_n}(z) = H_{D_{n-1}^-}(\xi(z))$  and  $\tilde{H}_{D_n}(z) = \tilde{H}_{D_{n-1}^+}(\eta(z))$ .

Therefore we have, for each of the  $2n$  domains  $D$  of the decomposition (12), defined two functions  $H_D$  and  $\tilde{H}_D$ . Then, as said at the beginning of the proof, we set  $H(z) = H_D(z)$  and  $\tilde{H}(z) = \tilde{H}_D(z)$ , for all  $z \in D$  and for all domains  $D$  that appear in (12).

With this construction, (13) and (14) are immediately obtained. In order to prove (15), we can use the fact that it is possible to express all the functions  $H_D$ ,  $\tilde{H}_D$  in terms of  $H_{D_0}$ ,  $\tilde{H}_{D_0}$  and  $x^{i_0}y^{j_0}$  only; we give, as examples, the expression of  $H_{D_{2k}^+}$ , for any  $2k \in \{1, \dots, n\}$ :

$$H_{D_{2k}^+}(z) = -\tilde{H}_{D_0}((\eta \circ \xi)^k(z)) + \sum_{p=0}^{k-1} x^{i_0}y^{j_0}(\xi \circ (\eta \circ \xi)^p(z)) - \sum_{p=1}^{k-1} x^{i_0}y^{j_0}((\eta \circ \xi)^p(z)), \quad (16)$$

and the one of  $\tilde{H}_{D_{2k}^-}$ , for any  $2k \in \{1, \dots, n\}$ :

$$\tilde{H}_{D_{2k}^-}(z) = -H_{D_0}((\xi \circ \eta)^k(z)) + \sum_{p=0}^{k-1} x^{i_0}y^{j_0}(\eta \circ (\xi \circ \eta)^p(z)) - \sum_{p=1}^{k-1} x^{i_0}y^{j_0}((\xi \circ \eta)^p(z)). \quad (17)$$

As a consequence, we obtain (15) for even values of  $n$ . Indeed, for this it is sufficient first to add  $H_{D_n}(z)$  and  $\tilde{H}_{D_n}(z)$ , in other words the equalities (16) and (17) above for  $k = n/2$ , then to use the fact that for even  $n$ ,  $(\xi \circ \eta)^{n/2} = (\eta \circ \xi)^{n/2}$ , next to use that for  $z \in D_n$ ,  $H_{D_0}((\xi \circ \eta)^{n/2}(z)) + \tilde{H}_{D_0}((\xi \circ \eta)^{n/2}(z)) = x^{i_0}y^{j_0}((\xi \circ \eta)^{n/2}(z))$ , see (14), and finally to remark that  $W_n = \{1, \eta\xi, \dots, (\eta\xi)^{n/2-1}, \xi\eta, \dots, (\xi\eta)^{n/2-1}, \xi, \dots, \xi(\eta\xi)^{n/2-1}, \eta, \dots, \eta(\xi\eta)^{n/2-1}, (\xi\eta)^{n/2}\}$ .

Likewise, we could write the expressions in terms of  $H_{D_0}$ ,  $\tilde{H}_{D_0}$  and  $x^{i_0}y^{j_0}$  of

$$H_{D_{2k}^-}, \quad \tilde{H}_{D_{2k}^+}, \quad H_{D_{2k+1}^+}, \quad \tilde{H}_{D_{2k+1}^+}, \quad H_{D_{2k+1}^-}, \quad \tilde{H}_{D_{2k+1}^-},$$

and we would verify that (15) is still true for odd values of  $n$ . Proposition 1 is proved.  $\square$

**Remark 3.** We now explain why the assumption (H3') concerning the values of the transition probabilities was both natural and necessary for our study.

If we suppose (H1) but no more (H3'), then the uniformization (8) is the same, but with  $z_0 = -(2p_{1-1})^{1/2} + i(2p_{10})^{1/2}$ . The transformations (9) of important cycles through the uniformization are also still valid, and the automorphism  $\xi$  is yet again equal to  $\xi(z) = 1/z$ . As for  $\eta$ , it takes the value  $\eta(z) = z_0^2/z$ ; in particular the group of the random walk  $W = \langle \xi, \eta \rangle$  is finite if and only if there exists an integer  $p$  such that  $z_0^{2p} = 1$ . In this case, if  $n$  denotes the smallest of these positive integers  $p$ , the group  $W$  is of order  $2n$ .

If a such  $n$  does not exist, then there is no hope to find a fundamental domain for the action of the group  $W$ , neither to obtain any equality like (15).

If a such  $n$  exists, then by using the fact that  $z_0^{2n} = 1$ , in other words the fact that  $[-(2p_{1-1})^{1/2} + i(2p_{10})^{1/2}]^{2n} = 1$ , we immediately obtain that  $p_{10} = \sin(q\pi/n)^2/2$  for some integer  $q$  having a greatest common divisor with  $n$  equal to 1.

In this last case, we have  $z_0 = -\exp(-iq\pi/n)$ , and it is easily proved that the domain bounded by the cycles  $x^{-1}([1, x_4])$  and  $y^{-1}([1, y_4])$ , namely  $\Lambda(\arg(z_0), \pi) = \Lambda(\pi - q\pi/n, \pi)$ , is a fundamental domain for the action of the group  $W$  if and only if  $q = 1$ . In particular, having an equality like (15) is possible if and only if  $q = 1$ , see the proof of Proposition 1.

But it will turn out that having an equality like (15) is essential in what follows, particularly in Section 3, where we have to know the precise behavior of  $H(z) + \bar{H}(z) - x^{i_0}y^{j_0}(z)$  near 0 and  $\infty$ . This is why we have supposed that  $p_{10} = \sin(\pi/n)^2/2$  for some integer  $n$ , in other words (H3').

## 2 Harmonic functions

The aim of this section is to introduce and to study some harmonic function associated with the process, that will be of the highest importance in the forthcoming Sections 3 and 4.

It turns out that this harmonic function will be obtained from the expansion at 0 of  $\sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z))$ , quantity which is appeared naturally in (15) ; this is why we begin here by studying closely the behavior of this sum in the neighborhood of 0.

Note first that thanks to the expression of the automorphisms  $\xi$  and  $\eta$ , see (11), we have

$$\sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) = \sum_{k=0}^{n-1} [x^{i_0} y^{j_0}(\exp(-2ik\pi/n)z) - x^{i_0} y^{j_0}(\exp(-2ik\pi/n)/z)]. \quad (18)$$

Let us now take the following notations for the expansion at 0 of the function  $x^{i_0} y^{j_0}$  :

$$x^{i_0} y^{j_0}(z) = \sum_{p=0}^{\infty} \kappa_p(i_0, j_0) z^p, \quad (19)$$

and remark that with (8), we obtain that for  $z$  close to 0,

$$x^{i_0} y^{j_0}(1/z) = \sum_{p=0}^{\infty} \bar{\kappa}_p(i_0, j_0) z^p. \quad (20)$$

In a quite general setting, if  $f$  is holomorphic in a neighborhood of 0 with expansion  $f(z) = \sum_{p=0}^{\infty} f_p z^p$ , then  $\sum_{k=0}^{n-1} f(\exp(-2ik\pi/n)z) = \sum_{k=0}^{n-1} f(\exp(2ik\pi/n)z) = \sum_{p=0}^{\infty} n f_{np} z^{np}$ . This is why, by using (19) and (20), we obtain that the sum (18) is equal to

$$\sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) = \sum_{p=1}^{\infty} n [\kappa_{np}(i_0, j_0) - \bar{\kappa}_{np}(i_0, j_0)] z^{np}. \quad (21)$$

We are now going to be interested in the term corresponding to  $p = 1$  in the sum (21) and we set

$$f_n(i_0, j_0) = n [\kappa_n(i_0, j_0) - \bar{\kappa}_n(i_0, j_0)] / [(-1)^n i]. \quad (22)$$

**Proposition 4.** *The function  $f_n$ , defined in (22), verifies the following assertions :*

- (i)  $f_n$  is a real polynomial in the variables  $i_0, j_0$  of degree exactly  $n$ .
- (ii)  $f_n$  is a harmonic function for the process  $(X, Y)$ .

(iii)  $f_n(i_0, 0) = f_n(0, j_0) = 0$  for all integers  $i_0$  and  $j_0$ .

(iv) If  $i_0 > 0$  and  $j_0 > 0$ , then  $f_n(i_0, j_0) > 0$ .

**Corollary 5.** *The Doob  $f_n$ -transform process of  $(X, Y)$  will never hit the boundary.*

**Remark 6.** *Writing an explicit formula for the function  $f_n$  is quite possible. Indeed, using the Cauchy product,  $\kappa_n$  (and therefore also  $f_n$ ) can be written in terms of the coefficients of the expansions of  $x$  and  $y$  at 0, and these coefficients are easily calculated, see Equation (23) below. As examples, we give the factorized form of  $f_3$ ,  $f_4$  and  $f_6$  :*

$$\begin{aligned} f_3(i_0, j_0) &= 24 \cdot 3^{1/2} \cdot i_0 j_0 (i_0 + 2j_0), \\ f_4(i_0, j_0) &= (256/3) \cdot i_0 j_0 (i_0 + 2j_0)(i_0 + j_0), \\ f_6(i_0, j_0) &= (288/5) 3^{1/2} \cdot i_0 j_0 (i_0 + 2j_0)(i_0 + j_0) \left( (i_0 + 2j_0/3)(i_0 + 4j_0/3) + 10/9 \right). \end{aligned}$$

We don't write the explicit expression of  $f_n$  for general values of  $n$ , since it wouldn't really be usable.

**Remark 7.** *The explicit expression and the harmonicity of the function  $f_4$  have already been obtained by P. Biane in [Bia91].*

The quantity  $f_4(i_0, j_0)$  also appears as a multiplicative factor in the asymptotic tail of the exit time of the process  $(X, Y)$  associated with  $n = 4$  starting from the initial state  $(i_0, j_0)$ . Indeed, if we denote by  $\tau = \inf\{k \geq 1 : X(k) = 0 \text{ or } Y(k) = 0\}$  the exit time of  $(X, Y)$  from the quadrant, the authors of [DO05] make explicit  $\mathbb{P}_{(i_0, j_0)}[\tau > k]$  in the case  $n = 4$ , by using an extension of the well-known reflection principle. It is then possible to deduce that the following asymptotic holds :  $\mathbb{P}_{(i_0, j_0)}[\tau > k] \sim C f_4(i_0, j_0)/k^2$ ,  $C$  being some positive constant.

In particular, we can specify Corollary 5 in the case  $n = 4$ . Indeed, by using the following equality, for  $l < k$  (obtained from the strong Markov property of the process  $(X, Y)$ ) :

$$\mathbb{P}_{(i_0, j_0)}[(X(l), Y(l)) = (i, j) | \tau > k] = \mathbb{P}_{(i_0, j_0)}[(X(l), Y(l)) = (i, j)] \frac{\mathbb{P}_{(i, j)}[\tau > k - l]}{\mathbb{P}_{(i_0, j_0)}[\tau > k]},$$

the results of [DO05] yield that the Doob  $f_4$ -transform process is equal in distribution to the limit as the time  $k$  goes to  $\infty$  of the process conditioned on the event  $\{\tau > k\}$ .

**Remark 8.** *Proposition 4 shows that for all  $n \geq 3$ , there exists at least one positive harmonic function for the process  $(X, Y)$  ; we will prove in Corollary 11 of Section 3 that for all  $n \geq 3$ ,  $f_n$  is in fact the unique positive harmonic function for the process  $(X, Y)$ , up to the positive multiplicative constants.*

*Proof of Proposition 4.* The fact that  $f_n$  takes real values is immediate from its definition. For the rest of the proof of (i), we will use the following straightforward fact : for any function  $f(z) = 1 + \sum_{p=1}^{\infty} f_{p,1} z^p$ , denote by  $1 + \sum_{p=1}^{\infty} f_{p,i} z^p$  the expansion at 0 of  $f(z)^i$  ; then  $f_{p,i}$  is a polynomial of degree equal or less than  $p$  in  $i$ , with dominant term equal to  $f_{1,1}^p i^p / p!$ . In particular,  $f_{p,i}$  is of degree exactly  $p$  if and only if  $f_{1,1} \neq 0$ .

In our case, it is immediate from (8) that  $\kappa_1(1, 0) = -4 \cos(\pi/n) \neq 0$  and  $\kappa_1(0, 1) = -4 \exp(i\pi/n) \neq 0$ . This is why, for any non-negative integer  $p$ ,  $\kappa_p(i, 0)$  is a polynomial of degree  $p$  in  $i$  and, likewise,  $\kappa_p(0, j)$  is a polynomial of degree  $p$  in  $j$ . In particular,  $\kappa_n(i, j) = \sum_{p=0}^n \kappa_p(0, j) \kappa_{n-p}(i, 0)$  is a polynomial in  $i, j$  of degree  $n$  with dominant term equal to  $\sum_{p=0}^n [\kappa_1(0, 1)^p / p!] j^p [\kappa_1(1, 0)^{n-p} / (n-p)!] i^{n-p}$ . In this way we obtain that  $f_n$  is a polynomial in  $i, j$  of degree  $n$  with dominant term equal to, after simplification,  $(2^{2n+1} / (n-1)!) \sum_{p=1}^{n-1} C_n^p \sin(p\pi/n) \cos(\pi/n)^{n-p} j^p i^{n-p}$ . Assertion (i) follows then immediately.

To prove (ii), it is enough to show that  $\kappa_n$  is harmonic. To show that, start by using the following obvious equality :  $x^{i-1}y^{j-1}(z)Q(x(z), y(z)) = 0$ , which reads  $x^i y^j(z) = p_{10}x^{i+1}y^j(z) + p_{10}x^{i-1}y^j(z) + p_{1-1}x^{i+1}y^{j-1}(z) + p_{1-1}x^{i-1}y^{j+1}(z)$ . Using (19), we get that

$$\sum_{p=0}^{\infty} [\kappa_p(i, j) - p_{10}\kappa_p(i+1, j) - p_{10}\kappa_p(i-1, j) - p_{1-1}\kappa_p(i+1, j-1) - p_{1-1}\kappa_p(i-1, j+1)] z^p$$

is identically zero ; this means that all the  $\kappa_p$ ,  $p \geq 0$ , hence in particular  $\kappa_n$ , are harmonic.

In order to prove (iii), we will need to know explicitly the expansions of  $x$  and  $y$  at 0. From (8) we immediately obtain these expansions :

$$x(z) = 1 + \frac{4}{\tan(\pi/n)} \sum_{p=1}^{\infty} (-1)^p \sin(p\pi/n) z^p, \quad y(z) = 1 + 4 \sum_{p=1}^{\infty} (-1)^p p \exp(ip\pi/n) z^p. \quad (23)$$

We prove now the first part of (iii), namely the fact that  $f_n(i, 0) = 0$  for all non-negative integer  $i$ . As it can be remarked in (23), the coefficients of  $x$  are real ; for this reason, for all integers  $i$  and  $p$ ,  $\kappa_p(i, 0)$  is also real and thus  $f_p(i, 0) = 0$  ; in particular,  $f_n(i, 0) = 0$ .

As for the second part of (iii), namely the fact that  $f_n(0, j) = 0$  for all non-negative integer  $j$ , we will show that  $\kappa_n(0, j)$  is real – however, it isn't true that for all  $j$  and  $p$ ,  $\kappa_p(0, j)$  is real.

In order to obtain  $\kappa_p(0, j)$  – that is, the  $p$ -th coefficient of the Taylor series of  $y(z)^j$  –, we will add all the terms of the form  $\kappa_{p_1}(0, 1)\kappa_{p_2}(0, 1)\cdots\kappa_{p_j}(0, 1)$  with  $p_1 + \cdots + p_j = p$ , this is nothing else but the Cauchy product of the  $j$  series  $y(z)$ . In other words, using (23), we will add terms of the form  $p_1 \cdots p_j (-1)^{p_1 + \cdots + p_j} \exp(i(p_1 + \cdots + p_j)\pi/n)$ . In particular,  $\kappa_p(0, j)$  can be written as  $\varphi_p(j)(-1)^p \exp(ip\pi/n)$ , with  $\varphi_p(j) > 0$  if  $j > 0$ .

In the particular case  $p = n$ , we obtain  $\kappa_n(0, j) = -\varphi_n(j)(-1)^n$  ;  $\kappa_n(0, j)$  is therefore real and, immediately,  $f_n(0, j) = 0$ .

We prove now (iv). With (23), it is immediate that the sequence  $\kappa_0(1, 0), \dots, \kappa_{n-1}(1, 0)$  is alternating, in the sense that for all  $p \in \{0, \dots, n-1\}$ ,  $(-1)^p \kappa_p(1, 0) > 0$ . In particular, it follows from general results on power series that the sequence  $\kappa_0(i, 0), \dots, \kappa_{n-1}(i, 0)$  is still alternating, for any  $i > 0$ .

In addition, by using the Cauchy product of  $x(z)^i$  and  $y(z)^j$  we obtain that  $\kappa_n(i, j) = \kappa_n(i, 0) + \kappa_n(0, j) + \sum_{p=1}^{n-1} (-1)^p \exp(ip\pi/n) \varphi_p(j) \kappa_{n-p}(i, 0)$ . Then, by definition of  $f_n(i, j)$ , and by using the fact that  $\kappa_n(i, 0)$  and  $\kappa_n(0, j)$  are real, we obtain

$$f_n(i, j) = 2n (-1)^n \sum_{p=1}^{n-1} (-1)^p \sin(p\pi/n) \varphi_p(j) \kappa_{n-p}(i, 0).$$

But above we have shown that  $\varphi_p(j) > 0$  if  $j > 0$ , and that  $(-1)^{n-p} \kappa_{n-p}(i, 0) > 0$  if  $i > 0$  ;  $f_n$  is thus written, above, as the sum of  $n-1$  positive terms, and is, therefore, positive.  $\square$

## 3 Asymptotic of the Green functions and Martin boundary

### 3.1 Statement of the results

**Theorem 9.** *The Green functions (2) admit the following asymptotic as  $i + j \rightarrow \infty$  and  $j/i \rightarrow \tan(\gamma)$ ,  $\gamma \in [0, \pi/2]$  :*

$$G_{i,j}^{i_0, j_0} \sim \frac{2}{\pi} \frac{(n-1)!}{4^n \sin(2\pi/n)} f_n(i_0, j_0) \frac{\sin(n \arctan(\frac{j/i}{1+j/i} \tan(\pi/n)))}{[\cos(\pi/n)^2 (i^2 + 2ij) + j^2]^{n/2}}, \quad (24)$$

**Remark 10.** Set  $N_n(j/i) = \sin(n \arctan((j/i)/(1 + j/i) \tan(\pi/n)))$  – this quantity appears in the asymptotic (24). Let  $\gamma$  be in  $[0, \pi/2]$  and suppose that  $j/i$  goes to  $\tan(\gamma)$ .

If  $\gamma \in ]0, \pi/2[$ , then  $N_n(j/i)$  goes to  $N_n(\tan(\gamma))$ , which belongs to  $]0, \infty[$ .

If  $\gamma = 0$  or  $\gamma = \pi/2$ , then  $N_n(j/i)$  goes to 0. More precisely,  $N_n(j/i) = n \tan(\pi/n)[j/i + O(j/i)^2]$  if  $\gamma = 0$  and  $N_n(j/i) = (n \tan(\pi/n)/(1 + \tan(\pi/n)^2))[i/j + O(i/j)^2]$  if  $\gamma = \pi/2$ .

**Corollary 11.** The Martin boundary of the process is reduced to one point.

## 3.2 Proofs

**Sketch of the proof of Theorem 9.** We will begin by expressing  $G_{i,j}$ , in (25), as a double integral, using for this the Cauchy formula and Equation (5). Then we will make the change of variable given by the uniformization (8) and we will apply the residue theorem ; in this way we will obtain  $G_{i,j}$  as the sum  $G_{i,j,1} + G_{i,j,2}$  of two single integrals w.r.t. the uniformization variable but on two contours *a priori* different, see (26) and (27). Then we will show, using Cauchy theorem and Proposition 1, that it is possible to move these contours of integration and in fact to have the same contours for the integrals  $G_{i,j,1}$  and  $G_{i,j,2}$ . Finally, using (15) we will obtain (28), which is the most important explicit formulation of the  $G_{i,j}$ , starting from which we will get their asymptotic. In (28),  $G_{i,j}$  will be written as an integral on  $\exp(i\theta)\mathbb{R}_+ \cup \{\infty\}$ , for some  $\theta \in [\pi - \pi/n, \pi]$ .

After having chosen an appropriate value of  $\theta$  in  $[\pi - \pi/n, \pi]$ , see (30), we will see that it is quite natural to decompose the contour into three parts, namely a neighborhood of 0, one of  $\infty$ , and an intermediate part. Indeed, the function  $x(z)^i y(z)^j$  that appears in the integrand of (28), is, on the contour, close to 1 near 0 and  $\infty$  and strictly larger than 1 elsewhere. Next we will study successively these contributions in three paragraphs, using for this essentially the Laplace method, what will conclude the proof of Theorem 9.

**Beginning of the proof of Theorem 9.** Equation (5) yields immediately that the generating function  $G$  of the Green functions is holomorphic in  $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$ . As a consequence and using again Equation (5), the Cauchy formulas allow us to write its coefficients  $G_{i,j}$  as the following double integrals :

$$G_{i,j} = \frac{1}{(2\pi i)^2} \iint_{\substack{|x|=1 \\ |y|=1}} \frac{G(x, y)}{x^i y^j} dx dy = \frac{1}{(2\pi i)^2} \iint_{\substack{|x|=1 \\ |y|=1}} \frac{h(x) + \tilde{h}(y) - x^{i_0} y^{j_0}}{x^i y^j Q(x, y)} dx dy, \quad (25)$$

where the circles  $\{|x| = 1\} = \{|y| = 1\} = \{\exp(i\theta), \theta \in [0, 2\pi[ \}$  are orientated according to the increasing values of  $\theta$ .

With (25), we can thus write  $G_{i,j}$  as the sum  $G_{i,j} = G_{i,j,1} + G_{i,j,2}$ , where

$$G_{i,j,1} = \frac{1}{(2\pi i)^2} \int_{|x|=1} \frac{h(x)}{x^i} \int_{|y|=1} \frac{dy}{y^j Q(x, y)} dx,$$

$$G_{i,j,2} = \frac{1}{(2\pi i)^2} \int_{|y|=1} \frac{\tilde{h}(y)}{y^j} \int_{|x|=1} \frac{dx}{x^i Q(x, y)} dy + \frac{1}{(2\pi i)^2} \int_{|y|=1} \frac{1}{y^{j-j_0}} \int_{|x|=1} \frac{dx}{x^{i-i_0} Q(x, y)} dy.$$

We are now going to make in  $G_{i,j,1}$  the change of variable  $x = x(z)$ . For this we remark that if  $\Lambda(\pi/2, \pi/2) = \{it, t \in [0, \infty[ \}$  is orientated according to the increasing values of  $t$ , then the following equality between orientated contours holds :  $x(\Lambda(\pi/2, \pi/2)) = -\{|x| = 1\}$ , see (9) and Picture 2. In this way and by using in addition the equality  $h(x(z)) = H(z)$  we obtain

$$G_{i,j,1} = -\frac{1}{(2\pi i)^2} \int_{\Lambda(\pi/2, \pi/2)} \frac{H(z)}{x(z)^i} \int_{|y|=1} \frac{dy}{y^j Q(x(z), y)} x'(z) dz.$$

But  $Q(x(z), y) = 0$  if and only if  $y \in \{y(z), x(z)^2/y(z)\}$ , see (10). Moreover, if  $z$  belongs to  $\Lambda(\pi/2, \pi/2) \setminus \{0, \infty\}$ , then  $|y(z)| > 1$ , see Picture 2. Therefore the residue theorem at infinity gives that for such  $z$ ,  $\int_{|y|=1} dy/(y^j Q(x(z), y)) = -2\pi i/(y(z)^j \partial_y Q(x(z), y(z)))$ . Finally we have proved that

$$G_{i,j,1} = \frac{1}{2\pi i} \int_{\Lambda(\pi/2, \pi/2)} \frac{H(z)}{x(z)^i y(z)^j} \frac{x'(z)}{\partial_y Q(x(z), y(z))} dz. \quad (26)$$

Likewise we prove that

$$G_{i,j,2} = -\frac{1}{2\pi i} \int_{\Lambda(-\pi/2 - \pi/n, -\pi/2 - \pi/n)} \frac{\tilde{H}(z) - x(z)^{i_0} y(z)^{j_0}}{x(z)^i y(z)^j} \frac{y'(z)}{\partial_x Q(x(z), y(z))} dz. \quad (27)$$

We are now going to explain why it is possible to move the contours of integration of the integrals (26) and (27) up to  $\Lambda(\theta, \theta)$ , for any  $\theta \in [\pi - \pi/n, \pi]$  – see Picture 4 below.

Start by considering  $G_{i,j,1}$  in (26). Thanks to the Cauchy theorem, it is sufficient to show that the integrand of  $G_{i,j,1}$  is holomorphic inside of  $\Lambda(\pi/2, \pi)$ , horizontally hatched on Picture 4, and this is what we are going to prove.

On one hand, in this domain, with (1) and (8), we obtain  $x'(z)/\partial_y Q(x(z), y(z)) = -i/(2(p_{10}p_{1-1})^{1/2}z)$ , which has manifestly no pole inside of  $\Lambda(\pi/2, \pi)$ . On the other hand, it is possible to deduce from the proof of Proposition 1 that the only poles of  $H$  are at  $z_0$  and  $\bar{z}_0$ . In particular, using (8), we obtain that for  $i$  or  $j$  large enough,  $H(z)/(x(z)^i y(z)^j)$  has no pole in  $\Lambda(\pi/2, \pi)$ . Therefore, for  $i$  or  $j$  large enough, the integrand of  $G_{i,j,1}$  has no pole in  $\Lambda(\pi/2, \pi)$ , and we can thus move the contour from  $\Lambda(\pi/2, \pi/2)$  to  $\Lambda(\theta, \theta)$ , for any  $\theta \in [\pi/2, \pi]$ . Note that it isn't possible to move the contour beyond  $\Lambda(\pi, \pi)$ , since  $\Lambda(\pi, \pi)$  is a singular curve for  $H$  – indeed, recall that  $\Lambda(\pi, \pi) = x^{-1}([1, x_4])$  and see Proposition 1.

By similar considerations, we obtain that it is possible to move the initial contour of integration of  $G_{i,j,2}$  up to  $\Lambda(\theta, \theta)$ , for any  $\theta \in [\pi - \pi/n, 3\pi/2 - \pi/n]$ .

In particular, if we wish to have the same contour of integration for  $G_{i,j,1}$  and  $G_{i,j,2}$ , we can chose  $\Lambda(\theta, \theta)$ , for any  $\theta \in [\pi - \pi/n, \pi] = [\pi/2, \pi] \cap [\pi - \pi/n, 3\pi/2 - \pi/n]$ .

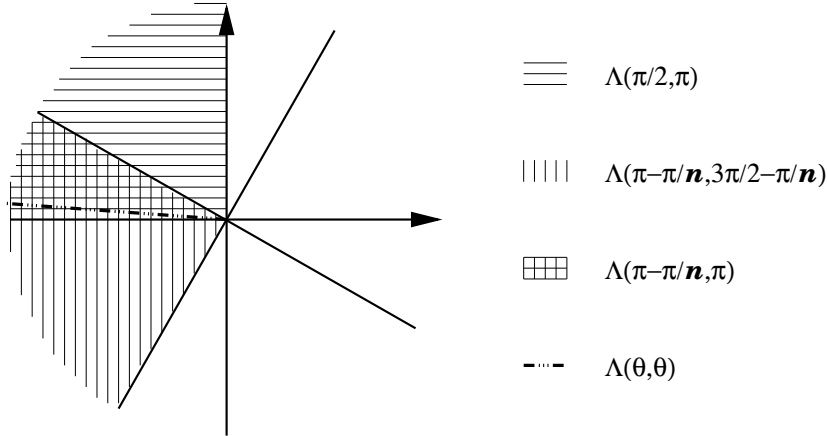


Figure 4: Change of contour of integration for the integrals (26) and (27)

Then, using the equality  $x'(z)/\partial_y Q(x(z), y(z)) = -y'(z)/\partial_x Q(x(z), y(z))$ , that comes from differentiating  $Q(x(z), y(z)) = 0$ , as well as (26), (27) and (15) – we can use (15) since  $\theta \in [\pi - \pi/n, \pi]$  and thus  $\Lambda(\theta, \theta) \subset \Lambda(\pi - \pi/n, \pi)$  –, we obtain the following final formulation for  $G_{i,j}$ ,  $\theta$  being any angle in  $[\pi - \pi/n, \pi]$ .

$$G_{i,j} = \frac{1}{4\pi(p_{10}p_{1-1})^{1/2}} \int_{\Lambda(\theta, \theta)} \left[ \frac{1}{z} \sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) \right] \frac{1}{x(z)^i y(z)^j} dz. \quad (28)$$

The function  $x(z)^i y(z)^j$  is, in  $\Lambda(\theta, \theta) \subset \Lambda(\pi - \pi/n, \pi)$ , larger than 1 in modulus, see Picture 2. Moreover, it goes to 1 when (and only when)  $z$  goes to 0 or to  $\infty$ . This is why it seems natural to decompose the contour  $\Lambda(\theta, \theta)$  into a part near 0, an other near  $\infty$  and the remaining part, and to think that the parts near 0 and  $\infty$  will lead to the asymptotic of  $G_{i,j}$ , and that the remaining part will lead to a negligible contribution. But how to find the best contour in order to achieve this idea ? In other words how to find the value of  $\theta \in [\pi - \pi/n, \pi]$  for which the calculation of the asymptotic of (28) will be the easiest ?

For this, we are going to consider with details the function  $x(z)^i y(z)^j$ , or equivalently the function  $\chi_{j/i}(z) = \log(x(z)) + (j/i) \log(y(z))$ . Incidentally, this is why from now on we will suppose that  $j/i \in [0, M]$ , for some  $M < \infty$ . Indeed, the function  $\chi_{j/i}$  is manifestly not adapted to the values  $j/i$  going to  $\infty$  ; for such  $j/i$ , we will consider, later, the function  $(i/j)\chi_{j/i}(z) = (i/j) \log(x(z)) + \log(y(z))$ . Nevertheless,  $M$  can be so large as wished, and, in what follows, we will suppose that some  $M > 0$  is fixed.

With (8), we easily obtain the explicit expansion of  $\chi_{j/i}$  at 0 :

$$\chi_{j/i}(z) = \sum_{p=0}^{\infty} \nu_{2p+1}(j/i) z^{2p+1}, \quad \nu_{2p+1}(j/i) = \frac{2}{2p+1} [z_0^{2p+1} + \bar{z}_0^{2p+1} + 2(j/i) \bar{z}_0^{2p+1}]. \quad (29)$$

Likewise, once again with (8), we get that for  $z$  near  $\infty$ ,  $\chi_{j/i}(z) = \sum_{p=0}^{\infty} \bar{\nu}_{2p+1}(j/i) 1/z^{2p+1}$ .

Consider now the steepest descent path associated with  $\chi_{j/i}$ , in other words the function  $z_{j/i}(t)$  defined by  $\chi_{j/i}(z_{j/i}(t)) = t$ . By inverting the latter equality, we easily obtain that the half-line  $(1/\nu_1(j/i))\mathbb{R}_+ \cup \{\infty\}$  is tangent at 0 and at  $\infty$  to this steepest descent path.

Let us now set

$$\rho_{j/i} = 1/\nu_1(j/i) = 1/[2(z_0 + \bar{z}_0 + 2(j/i)\bar{z}_0)]. \quad (30)$$

With this notation, we now answer the question asked above, that concerned the fact of finding the value of  $\theta$  for which the asymptotic of the Green functions (28) will be the most easily calculated : we will chose  $\theta = \arg(\rho_{j/i})$  (note that from (30) we easily obtain that  $\arg(\rho_{j/i}) \in [\pi - \pi/n, \pi]$ ) and the decomposition of the contour  $\Lambda(\theta, \theta)$  will be

$$\Lambda(\arg(\rho_{j/i}), \arg(\rho_{j/i})) = (\rho_{j/i}/|\rho_{j/i}|) [0, \epsilon] \cup (\rho_{j/i}/|\rho_{j/i}|) \epsilon, 1/\epsilon \cup (\rho_{j/i}/|\rho_{j/i}|) [1/\epsilon, \infty].$$

According to this decomposition and to (28), we consider now that  $G_{i,j}$  is the sum of three terms, and we are going to study successively the contribution of each of these three terms.

**Contribution of the neighborhood of 0.** In order to evaluate the asymptotic of the integral (28) on the contour  $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$ , we will use the expansion at 0 of the function  $(1/z) \sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z))$ . This is why we begin by studying the asymptotic of the following integral,  $k$  being some positive integer :

$$\int_{(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]} \frac{z^k}{x(z)^i y(z)^j} dz. \quad (31)$$

Using the equality  $1/(x(z)^i y(z)^j) = \exp(-i\chi_{j/i}(z))$  and the expansion (29) of  $\chi_{j/i}$  at 0, and then making the change of variable  $z = \rho_{j/i}t$ , we obtain that (31) is equal to

$$\rho_{j/i}^{k+1} \int_0^{\epsilon/|\rho_{j/i}|} t^k \exp(-it) \exp\left(-i \sum_{p=1}^{\infty} \nu_{2p+1}(j/i) (\rho_{j/i}t)^{2p+1}\right) dt. \quad (32)$$

But, with (29),  $|\nu_{2p+1}(j/i)| \leq 4(M+1)$ , and, therefore, for all  $t \in [0, \epsilon/|\rho_{j/i}|]$  we have  $|-i \sum_{p=1}^{\infty} \nu_{2p+1}(j/i) (\rho_{j/i}t)^{2p+1}| \leq i\epsilon^3 4(M+1)/(1-\epsilon^2)$ . This is why

$$\exp\left(-i \sum_{p=1}^{\infty} \nu_{2p+1}(j/i) (\rho_{j/i}t)^{2p+1}\right) = 1 + O(i\epsilon^3),$$



the  $O$  being independent of  $j/i \in [0, M]$  and of  $t \in [0, \epsilon/|\rho_{j/i}|]$ . The integral (32) can thus be calculated as

$$\rho_{j/i}^{k+1} [1 + O(i\epsilon^3)] \int_0^{\epsilon/|\rho_{j/i}|} t^k \exp(-it) dt = (\rho_{j/i}/i)^{k+1} [1 + O(i\epsilon^3)] \int_0^{i\epsilon/|\rho_{j/i}|} t^k \exp(-t) dt.$$

In the sequel we will chose  $\epsilon = 1/i^{3/4}$ , so that  $i\epsilon/|\rho_{j/i}| \rightarrow \infty$  and  $O(i\epsilon^3) = O(1/i^{5/4})$ .

We could be surprised by this choice of  $\epsilon$ ; in fact we will see in the forthcoming paragraph called ‘‘Conclusion’’ that in order to obtain the asymptotic of the Green functions along the paths of states  $(i, j)$  such that  $j/i \rightarrow \tan(\gamma) \in ]0, \infty[$ , it would have been sufficient to have  $O(i\epsilon^3) = o(1)$ , but for the paths  $(i, j)$  such that  $j/i \rightarrow 0$ , it is necessary to have  $O(i\epsilon^3) = o(1/i)$ , what affords the choice  $\epsilon = 1/i^{3/4}$ .

Finally, we obtain that for this choice of  $\epsilon$ , the integral (31) is equal to

$$\int_{(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]} \frac{z^k}{x(z)^i y(z)^j} dz = (\rho_{j/i}/i)^{k+1} k! [1 + O(1/i^{5/4})], \quad (33)$$

where  $O$  is independent of  $j/i \in [0, M]$ .

We are now ready to find the asymptotic of the integral (28) on the contour  $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$ . First, in accordance with (21), we have that this integral equals

$$\frac{1}{4\pi(p_{10}p_{1-1})^{1/2}} \sum_{p=1}^{\infty} n [\kappa_{np}(i_0, j_0) - \overline{\kappa_{np}}(i_0, j_0)] \int_{(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]} \frac{z^{np-1}}{x(z)^i y(z)^j} dz.$$

So clearly, with (33), we will obtain that the terms corresponding to  $p \geq 2$  in the sum above will be negligible w.r.t. the term associated to  $p = 1$ . More precisely, by using the definition (22) of the harmonic function  $f_n$  and (33) for  $k = pn - 1$ ,  $p \geq 1$ , we obtain that the integral (28) on the contour  $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$  is equal to

$$\frac{1}{4\pi(p_{10}p_{1-1})^{1/2}} (-1)^n (n-1)! f_n(i_0, j_0) (\rho_{j/i}/i)^n [1 + O(1/i^{5/4})]. \quad (34)$$

**Contribution of the neighborhood of  $\infty$ .** The part of the contour close to  $\infty$ ,  $(\rho_{j/i}/|\rho_{j/i}|)[1/\epsilon, \infty]$ , is related to the part  $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$  by the transformation  $z \mapsto 1/\bar{z}$ . Moreover, it is clear from (8) that for  $f = x$ ,  $f = y$ , or  $f = \sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w)$ ,

$$f(1/\bar{z}) = \overline{f(z)}.$$

Therefore, the change of variable  $z \mapsto 1/\bar{z}$  immediately gives us that the contribution of the integral (28) near  $\infty$  is the complex conjugate of its contribution near 0.

**Contribution of the intermediate part.** Let  $A_\epsilon$  be the annular domain  $\{z \in \mathbb{C} : \epsilon \leq |z| \leq 1/\epsilon\}$ . According to Picture 2, for all  $z \in \Lambda(\pi - \pi/n, \pi) \cap A_\epsilon$ ,  $|x(z)| > 1 + \eta_{x,\epsilon}$  and  $|y(z)| > 1 + \eta_{y,\epsilon}$ , with  $\eta_{x,\epsilon}$  and  $\eta_{y,\epsilon}$  positive. In fact, since  $x'(0) \neq 0$  and  $y'(0) \neq 0$ , it is possible to take  $\eta_{x,\epsilon} > \eta\epsilon$  and  $\eta_{y,\epsilon} > \eta\epsilon$  for some  $\eta > 0$  independent of  $\epsilon$  small enough.

Let us now consider

$$K = \sup_{z \in \Lambda(\pi - \pi/n, \pi)} \left| \frac{\left[ \sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) \right]}{\left[ x^{i_0} y^{j_0}(z) \right]} \right|,$$

and let us show that  $K$  is finite. For this, it is sufficient to prove that the function  $s$ , defined by  $s(z) = \left[ \sum_{w \in W_n} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) \right] / \left[ x^{i_0} y^{j_0}(z) \right]$ , has no pole in  $\Lambda(\pi - \pi/n, \pi)$ , including  $\infty$ . But by using (8) and (11), we see that the only poles of the numerator of  $s$

are the  $z_0 \exp(2ip\pi/n)$ , for  $p \in \{0, \dots, n-1\}$ . Among these  $n$  points, only  $z_0$  belongs to  $\Lambda(\pi - \pi/n, \pi)$ . But, in  $s$ , we have taken care of dividing by  $x^{i_0} y^{j_0}(z)$ , so that  $s$  is in fact holomorphic near  $z_0$ . Moreover, it is easily shown that  $s$  is holomorphic at  $\infty$ . Finally, we have proved that  $s$  has no pole in  $\Lambda(\pi - \pi/n, \pi)$ , hence  $s$  is bounded in  $\Lambda(\pi - \pi/n, \pi)$ , in other words  $K$  is finite.

The modulus of the contribution of the integral (28) on the intermediate part  $(\rho_{j/i}/|\rho_{j/i}|)\epsilon, 1/\epsilon \subset \Lambda(\pi - \pi/n, \pi) \cap A_\epsilon$  can therefore be bounded from above by

$$\frac{1}{4\pi(p_{10}p_{1-1})^{1/2}} \frac{1}{\epsilon^2} \frac{K}{(1 + \eta\epsilon)^{i-i_0} (1 + \eta\epsilon)^{j-j_0}}. \quad (35)$$

Note that the presence of the term  $1/\epsilon^2$  in (35) is due to : one  $1/\epsilon$  appears as an upper bound of the length of the contour, the other  $1/\epsilon$  comes from an upper bound of the modulus of the term  $1/z$  present in the integrand of (28).

Then we take, as before,  $\epsilon = 1/i^{3/4}$ , and we use the following straightforward upper bound, valid for  $i$  large enough :  $1/(1 + \eta/i^{3/4})^i \leq \exp(-(\eta/2)i^{1/4})$ . We finally obtain that for  $i$  large enough, (35) is equal to  $O(i^{3/2} \exp(-(\eta/2)i^{1/4}))$ .

**Conclusion.** We have seen that the contribution of the integral (28) in the neighborhood of 0 is given by (34), that the contribution of (28) in the neighborhood of  $\infty$  is equal to the complex conjugate of (34), and that the contribution of the remaining part can be written as  $O(i^{3/2} \exp(-(\eta/2)i^{1/4}))$ . Therefore with (28) and (34) we obtain

$$G_{i,j} = \frac{1}{4\pi(p_{10}p_{1-1})^{1/2}} (-1)^n (n-1)! f_n(i_0, j_0) \iota \left[ (\rho_{j/i}/i)^n - (\overline{\rho_{j/i}/i})^n \right] + O(1/i^{n+5/4}). \quad (36)$$

Moreover, starting from (30) we easily get

$$(\rho_{j/i}/i)^n - (\overline{\rho_{j/i}/i})^n = \frac{2i(-1)^{n+1} \sin(n \arctan(\frac{j/i}{1+j/i} \tan(\pi/n)))}{4^n [\cos(\pi/n)^2 (i^2 + 2ij) + j^2]^{n/2}}.$$

The latter equality, (36) and Remark 10 conclude the proof of Theorem 9 in the case of  $\gamma \in [0, \pi/2[$ . Note that having  $o(1/i^n)$  instead of  $O(1/i^{n+5/4})$  would have been sufficient for  $\gamma \in ]0, \pi/2[$ , since in this case Remark 10 implies that  $(\rho_{j/i}/i)^n - (\overline{\rho_{j/i}/i})^n \sim K_\gamma/i^n$  with  $K_\gamma \neq 0$ ; on the other hand, if  $\gamma = 0$ , then  $(\rho_{j/i}/i)^n - (\overline{\rho_{j/i}/i})^n \sim K_0 j/i^{n+1}$  with  $K_0 \neq 0$ , and it is necessary to have something like  $o(1/i^{n+1})$  in (36), as it is actually the case with  $O(1/i^{n+5/4})$ .

To prove Theorem 9 in the case  $\gamma = \pi/2$ , we would consider  $(i/j)\kappa_{j/i}$  rather than  $\kappa_{j/i}$ , and we would use then exactly the same analysis; we omit the details.

**Proof of Corollary 11.** From Theorem 9 and the classical theory of Martin boundary, see *e.g.* [Dyn69], it is immediate that the Martin boundary of the killed process is reduced to one point, since the limit of the Green kernel  $G_{i,j}^{i_0, j_0}/G_{i,j}^{1,1}$  when  $i, j > 0$ ,  $i + j \rightarrow \infty$  and  $j/i \rightarrow \tan(\gamma)$  is equal to  $f_n(i_0, j_0)/f_n(1, 1)$ , in particular it does not depend on  $\gamma \in [0, \pi/2]$  – by killed process we mean the same process as ours, but without attaching importance at the site where the process is absorbed, in other words by considering the boundary  $\{(i, 0) : i \geq 1\} \cup \{(0, j) : j \geq 1\}$  as a single state, usually called the cemetery.

As for the Martin boundary of the absorbed process, we have, in addition to the limits given by the Green kernel, to find the limits of the ratio of the absorption probabilities (see (3) for their exact definition)  $h_i^{i_0, j_0}/h_i^{1,1}$  and  $\tilde{h}_j^{i_0, j_0}/\tilde{h}_j^{1,1}$  as respectively  $i$  and  $j$  go to infinity. But, using the straightforward equalities

$$\begin{aligned} h_i^{i_0, j_0} &= \mathbb{P}_{(i_0, j_0)}[\exists k \geq 0 : (X(k), Y(k)) = (i-1, 1)] h_i^{i-1, 1}, \\ G_{i-1, 1}^{i_0, j_0} &= \mathbb{P}_{(i_0, j_0)}[\exists k \geq 0 : (X(k), Y(k)) = (i-1, 1)] G_{i-1, 1}^{i-1, 1}, \end{aligned} \quad (37)$$

we obtain  $h_i^{i_0, j_0} / h_i^{1,1} = G_{i-1,1}^{i_0, j_0} / G_{i-1,1}^{1,1}$ . Likewise, we prove that  $\tilde{h}_j^{i_0, j_0} / \tilde{h}_j^{1,1}$  and  $G_{1,j}^{i_0, j_0} / G_{1,j}^{1,1}$  have the same limit as  $j$  goes to infinity – even though there is no equality so simple as (37) for the vertical axis. For these reasons, the Martin boundary of the absorbed process is also reduced to one point, which proves Corollary 11.

## 4 Absorption probabilities

We are now going to be interested in the absorption probabilities of the random walk, properly defined in (3). More precisely, we will obtain results of two different kinds : first, in Subsection 4.1, we will make explicit the generating functions (4) of the absorption probabilities, and then, in Subsection 4.2, using the explicit formulation of these functions found previously, we will find the exact asymptotic of the absorption probabilities as the absorption site goes to infinity. Since the technical details are essentially the same for  $h$  and  $\tilde{h}$ , we are going to consider only  $h$  in Subsections 4.1 and 4.2, similar results could be obtained for  $\tilde{h}$  by the same analysis.

### 4.1 Integral representation of the generating functions

It is explained in [FIM99] how to obtain explicitly the generating functions of stationary probabilities in the case of some ergodic random walks in the quarter plane. In Section 3 of [KR09], we have adapted this approach to the case of the generating functions of absorption probabilities for some random walks in the quarter plane that are absorbed at the boundary and that have a positive drift.

In fact, the methods of [FIM99] can be similarly extended to the case of random walks in the quarter plane absorbed at the boundary and with zero drift. In other words it is possible, in our case, to obtain explicitly the generating functions of absorption probabilities  $h$  and  $\tilde{h}$ . Since the ideas are essentially the same as in [FIM99] and [KR09], we will not write the details and we refer to these two works for the technical aspects of this approach.

But before stating the result giving the explicit expression of  $h$ , we need some definitions.

Firstly, let  $X(y)$  be the two-valued function such that  $Q(X(y), y) = 0$  for all  $y$  in  $\mathbb{C} \cup \{\infty\}$ . Note that with (6) we immediately obtain  $X(y) = [-\tilde{b}(y) \pm \tilde{d}(y)^{1/2}] / [2\tilde{a}(y)]$ . Consider now the curve

$$\partial\mathcal{M} = X([y_1, 1]).$$

For  $y \in ]y_1, 1[$ ,  $\tilde{d}(y) < 0$ , see (7), and the two values of  $X(y)$  are distinct and complex conjugate one from the other. Since  $\tilde{d}(y_1) = \tilde{d}(1) = 0$ , the two values of  $X(y_1)$  and  $X(1)$  are the same (and are respectively equal to 0 and 1). In particular, the curve  $\partial\mathcal{M}$  is closed and symmetrical w.r.t. to the real axis. We will denote by  $\mathcal{M}$  the interior of the bounded domain delimited by  $\partial\mathcal{M}$ .

Secondly,  $a$ ,  $b$  and  $d$  being considered in (6) and (7) we define

$$\mu_{j_0}(t) = \frac{1}{(2a(t))^{j_0}} \sum_{k=0}^{(j_0-1)/2} C_{j_0}^{2k+1} d(t)^k (-b(t))^{j_0-(2k+1)}. \quad (38)$$

Then the result giving the explicit expression of  $h$  is the following, for the proof see (20), (25), (26) and (30) in Sections 3 and 4 of [KR09].

**Proposition 12.** *The function  $h$  can be meromorphically continued from the unit disc up to  $\mathbb{C} \setminus [1, y_4]$ , where it admits the following integral representation :*

$$\frac{1}{\pi} \int_1^{x_4} t^{i_0} \mu_{j_0}(t) \frac{(-d(t))^{1/2}}{t-x} dt + \frac{1}{\pi} \int_{x_1}^1 t^{i_0} \mu_{j_0}(t) \left[ \frac{u'(t)}{u(t)-u(x)} - \frac{1}{t-x} \right] (-d(t))^{1/2} dt + q(x).$$

Above  $q$  is a polynomial and  $u$  is a conformal gluing function for  $\partial\mathcal{M}$ , see below.

By a conformal gluing function (CGF)  $u$  for the curve  $\partial\mathcal{M}$  we mean a function (i) meromorphic in  $\mathcal{M}$  (ii) establishing a conformal mapping of  $\mathcal{M}$  onto the complex plane cut along some arc (iii) such that for all  $t$  in  $\partial\mathcal{M}$ ,  $u(t) = u(\bar{t})$ .

The existence – but not the explicit expression – of  $u$  in our situation can be obtained by using quite general results on conformal gluing, so to complete Proposition 12 it remains to find *explicitly* the CGF  $u$ .

For this we will use strongly the uniformization  $(x, y)$ . Indeed, it seems *a priori* difficult to find  $u$  because the curve  $\partial\mathcal{M}$  is not particularly ordinary ; on the other hand,  $\partial\mathcal{M}$  is the image through the uniformization of a quite ordinary curve, since with (9) we have  $\partial\mathcal{M} = x(\Lambda(-\pi/n, -\pi/n))$ , see Picture 2. In particular, we also have  $\mathcal{M} = x(\Lambda(-\pi/n, 0))$ .

Moreover, let us remark that if  $z \in \Lambda(-\pi/n, -\pi/n)$ , then  $\overline{x(z)} = x(z_0^2/z)$ .

The last paragraphs entail that by setting  $v = u \circ x$ ,  $u$  being the CGF and  $x$  the first coordinate of the uniformization (8), the problem of finding  $u$  is transformed into the following : to make explicit a function  $v$  (i) meromorphic in  $\Lambda(0, -\pi/n)$  (ii) establishing a conformal mapping of  $\Lambda(0, -\pi/n)$  onto the complex plane cut along some arc (iii) such that for all  $z$  in  $\Lambda(-\pi/n, -\pi/n)$ ,  $v(z_0^2 z) = v(z)$  (iv) such that for all  $z$  in  $\mathbb{C} \cup \{\infty\}$ ,  $v(z) = v(1/z)$  – this last condition appears since  $v = u \circ x$  and since for all  $z$  in  $\mathbb{C} \cup \{\infty\}$ ,  $x(z) = x(1/z)$ , see (10).

A solution of this new problem is easily found : we can take  $v(z) = z^n + 1/z^n$ . As a consequence, the function  $u(t) = x^{-1}(t)^n + 1/x^{-1}(t)^n$  is a CGF for the curve  $\partial\mathcal{M}$ .

The function  $x^{-1}$  has two branches,  $x_+^{-1}$  and  $x_-^{-1}$ , equal to  $x_{\pm}^{-1}(t) = [-\cos(\pi/n)(1+t) \pm (4t - \sin(\pi/n)^2(1+t)^2)^{1/2}]/[t-1]$ . In particular, since  $x_+^{-1}(t)x_-^{-1}(t) = 1$ , the CGF  $u(t)$  can be written as  $x_+^{-1}(t)^n + x_-^{-1}(t)^n$  ; therefore the choice of the determination of  $x^{-1}$  in the definition of the CGF  $u$  doesn't matter. Moreover, from the explicit expressions of  $x_+^{-1}$  and  $x_-^{-1}$  we easily deduce that  $u$  is a rational function of the form  $u(t) = P(t)/(t-1)^n$ , where  $P$  is a real polynomial of degree  $n$  such that  $P(1) \neq 0$ .

Now that we have obtained explicitly a CGF  $u$ , Proposition 12 is complete. We would like, however, to know the partial fraction expansion of  $u'(t)/(u(t) - u(x))$  – we will use it importantly in Subsection 4.2. For this, we need the following remark about the automorphisms.

Let us show that the application  $w \mapsto x \circ w \circ x^{-1}$  maps the Weyl group  $W_n$  onto  $\{x \circ (\eta \circ \xi)^p \circ x^{-1}, 0 \leq p \leq n-1\}$ , which is manifestly a group of order  $n$ . For this we remark that each automorphism  $w \in W_n$  defines naturally two automorphisms  $w_{x,+}$  and  $w_{x,-}$  by the formula  $w_{x,\pm} = x \circ w \circ x_{\pm}^{-1}$ . Since  $x \circ \xi = x$ , we have  $(\xi \circ w)_{x,\pm} = w_{x,\pm}$ , and since  $x_+^{-1}x_-^{-1} = 1$ , we have  $w_{x,+} = (w \circ \xi)_{x,-}$ . This is why  $\{w_{x,\pm}, w \in W_n\}$  is in fact not of order  $2n$  but of order  $n$ , equal to  $\{x \circ (\eta \circ \xi)^p \circ x^{-1}, 0 \leq p \leq n-1\}$ . In the sequel we will note  $\Delta_p = x \circ (\eta \circ \xi)^p \circ x^{-1}$ .

With the three last paragraphs, we can write the partial fraction expansion of  $u'(t)/(u(t) - u(x))$ . Indeed, using on the one hand that  $u$  has a pole at 1 of order  $n$  and on the other hand that  $u(t) = u(x)$  if and only if  $t = \Delta_p(x)$  for some  $p \in \{0, \dots, n-1\}$  – indeed,  $v(z) = v(y)$  if and only if  $z = w(y)$  for some  $w \in W_n$  –, we get

$$\frac{u'(t)}{u(t) - u(x)} = \sum_{p=0}^{n-1} \frac{1}{t - \Delta_p(x)} - \frac{n}{t-1}.$$

Then with Proposition 12 we immediately obtain that there exists a polynomial  $r$  such that

$$h(x) = \frac{1}{\pi} \int_1^{x_4} \frac{t^{i_0} \mu_{j_0}(t)}{t-x} (-d(t))^{1/2} dt + \sum_{p=1}^{n-1} \frac{1}{\pi} \int_{x_1}^1 \frac{t^{i_0} \mu_{j_0}(t)}{t - \Delta_p(x)} (-d(t))^{1/2} dt + r(x). \quad (39)$$

## 4.2 Asymptotic of the absorption probabilities

**Theorem 13.** *Let  $f_n$  be the function defined in (22). The probabilities of absorption (3) admit the following asymptotic as the absorption site  $i$  goes to infinity :*

$$h_i^{i_0, j_0} \sim \frac{1}{2\pi} \frac{n!}{(4 \cos(\pi/n))^n} f_n(i_0, j_0) \frac{1}{i^{n+1}}.$$

In order to prove Theorem 13, we will study closely the quantity  $h(x)$ , equal to  $\sum_{i=1}^{\infty} h_i x^i$ , see (4). More precisely we will show that  $h$  is holomorphic in the open unit disc, continuable holomorphically through every point of the unit circle except 1 and we will prove that in the neighborhood of 1,

$$h(x) = -\frac{1}{2\pi} \frac{f_n(i_0, j_0)}{(4 \cos(\pi/n))^n} (x-1)^n \log(1-x) [1 + O(x-1)] + h_1(x), \quad (40)$$

where  $h_1$  is holomorphic at 1. Then Theorem 13 will be an immediate consequence of (40) and of the well-known Pringsheim theorem, recalled in Lemma 14 below.

**Lemma 14** (Pringsheim theorem). *Let  $l(x) = \sum_{i=0}^{\infty} l_i x^i$  be a function holomorphic in the open unit disc. Suppose in addition that  $l$  is continuable holomorphically through every point of the unit circle except 1, in the neighborhood of which it can be written as  $l(x) = (x-1)^q \log(1-x) [1 + O(x-1)] + l_1(x)$ , where  $l_1$  is holomorphic at 1 and  $q$  is some integer. Then  $l_i \sim_{i \rightarrow \infty} -q! / i^{q+1}$ .*

We begin now the proof of Theorem 13. First, note that as a generating function of probabilities, it is clear that  $h$  is holomorphic in the open unit disc. Moreover, from Proposition 12 or (39) it is manifest that  $h$  is continuable holomorphically through every point of the unit circle except 1 ; so it remains to prove (40). For this we will need the following result :

**Lemma 15.** *Let  $k$  be a non-negative integer. There exist two functions  $f_k$  and  $g_k$  holomorphic at 1, such that in the neighborhood of 1 we have*

$$\int_1^{x_4} \frac{(t-1)^k}{t-s} dt = -(s-1)^k \log(1-s) + f_k(s), \quad \int_{x_1}^1 \frac{(t-1)^k}{t-s} dt = (s-1)^k \log(s-1) + g_k(s).$$

Note that the proof of Lemma 15 is straightforward, since the integrals that appear in its statement are easily explicitly calculated.

Take now the following notations (we recall that  $d$  is defined in (7) and  $\mu_{j_0}$  in (38), and we denote by  $\epsilon$  some small positive number) :

$$\forall t \in [1 - \epsilon, 1] : \quad t^{i_0} \mu_{j_0}(t) (-d(t))^{1/2} = \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) (t-1)^k, \quad (41)$$

and note that for  $t \in [1, 1 + \epsilon]$ , we have  $t^{i_0} \mu_{j_0}(t) (-d(t))^{1/2} = -\sum_{k=1}^{\infty} \alpha_k(i_0, j_0) (t-1)^k$ .

With this notation and by using (39) and Lemma 15, we obtain that for  $x$  close to 1,

$$h(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) \left[ (x-1)^k \log(1-x) + \sum_{p=1}^{n-1} (\Delta_p(x) - 1)^k \log(\Delta_p(x) - 1) \right] + h_0(x), \quad (42)$$

where  $h_0$  can be written in terms of the  $f_k$  and  $g_k$  of Lemma 15. An important fact is that  $h_0$  is symmetrical in  $\Delta_1, \dots, \Delta_{n-1}$  and is therefore holomorphic in the neighborhood of 1.

Moreover, since  $\Delta_p(1) = 1$  and  $\Delta'_p(1) \neq 0$ , we have  $\log(\Delta_p(x) - 1) = \log(1 - x) + l_p(x)$ , with  $\sum_{p=1}^{n-1} l_p(x)$  holomorphic at 1. In this way, (42) becomes

$$h(x) = \frac{\log(1-x)}{\pi} \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) \sum_{p=0}^{n-1} (\Delta_p(x) - 1)^k + h_1(x), \quad (43)$$

with  $h_1$  holomorphic in the neighborhood of 1.

Consider now  $\sum_{p=0}^{n-1} (\Delta_p(x) - 1)^k$ , that appears in (43), or rather  $\sum_{p=0}^{n-1} (\Delta_p(x(z)) - 1)^k$ , that we call  $A_k(z)$ . By definition of  $\Delta_p$ , we have  $\Delta_p \circ x = x \circ (\eta \circ \xi)^p$ , see Subsection 4.1, and in addition  $(\eta \circ \xi)^p(z) = \exp(-2ip\pi/n)z$ ; therefore  $A_k(z) = \sum_{p=0}^{n-1} (x(\exp(-2ip\pi/n)z) - 1)^k$ .

If we evaluate Equation (43) at  $x = x(z)$ , the quantity  $\sum_{k=1}^{\infty} \alpha_k(i_0, j_0) A_k(z)$  appears, and we are now going to prove that in the neighborhood of 0,

$$\sum_{k=1}^{\infty} \alpha_k(i_0, j_0) A_k(z) = ((-1)^{n+1}/2) f_n(i_0, j_0) z^n + O(z^{2n}). \quad (44)$$

We first recall a notation of Subsection 4.1 :  $x_{\pm}^{-1}(t) = [-\cos(\pi/n)(1+t) \pm (4t - \sin(\pi/n)^2(1+t)^2)^{1/2}]/[t-1]$ , and now we show that for  $t$  near 1, the following equality holds :

$$-2i \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) (t-1)^k = \sum_{q=0}^{\infty} [\kappa_q(i_0, j_0) - \overline{\kappa_q}(i_0, j_0)] x_+^{-1}(t)^q. \quad (45)$$

In order to prove the equality (45) for  $t$  in a neighborhood of 1, we are going to prove it first for  $t \in [1 - \epsilon, 1]$ . So if  $t \in [1 - \epsilon, 1]$ , then (41) yields  $-2i \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) (t-1)^k = -2it^{i_0} \mu_{j_0}(t) (-d(t))^{1/2}$ . But the latter quantity can be calculated as follows :

$$t^{i_0} ([-b(t) - \iota(-d(t))^{1/2}]/[2a(t)])^{j_0} - t^{i_0} ([-b(t) + \iota(-d(t))^{1/2}]/[2a(t)])^{j_0}.$$

In addition we easily verify that for  $t \in [x_1, 1]$ ,  $y(x_+^{-1}(t)) = [-b(t) - \iota(-d(t))^{1/2}]/[2a(t)]$  and  $y(1/x_+^{-1}(t)) = [-b(t) + \iota(-d(t))^{1/2}]/[2a(t)]$ . Therefore, for  $t \in [1 - \epsilon, 1]$  we have

$$-2i \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) (t-1)^k = x^{i_0} y^{j_0} (x_+^{-1}(t)) - x^{i_0} y^{j_0} (1/x_+^{-1}(t)).$$

Then it remains to use (19) and (20) in order to obtain (45) for all  $t \in [1 - \epsilon, 1]$ . Then, by analytic continuation, (45) holds in fact on any disc with center at 1 and with radius small enough.

In particular, if we apply (45) with  $t = \Delta_p(x(z))$  (which is close to 1 if  $z$  is close to 0), if in addition we use the fact that  $x_+^{-1}(\Delta_p(x(z))) = (\eta \circ \xi)^p(z) = \exp(-2ip\pi/n)z$ , and if finally we add these equalities w.r.t.  $p \in \{0, \dots, n-1\}$ , we obtain, by definition of the  $A_k$  :

$$-2i \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) A_k(z) = \sum_{q=0}^{\infty} [\kappa_q(i_0, j_0) - \overline{\kappa_q}(i_0, j_0)] z^q \sum_{p=0}^{n-1} \exp(-2ipq\pi/n),$$

which, by using the definition of  $f_n$ , see (22), immediately entails (44).

We are now going to conclude the proof of Equation (40). For this, note that

$$\begin{aligned} \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) \sum_{p=0}^{n-1} (\Delta_p(x) - 1)^k &= \sum_{k=1}^{\infty} \alpha_k(i_0, j_0) A_k(x_+^{-1}(x)) \\ &= ((-1)^{n+1}/2) f_n(i_0, j_0) x_+^{-1}(x)^n + O(x_+^{-1}(x)^{2n}) \\ &= \frac{-f_n(i_0, j_0)}{2(4 \cos(\pi/n))^n} (x-1)^n + O(x-1)^{n+1}. \end{aligned}$$

Indeed, the first equality above is obtained by definition of the functions  $A_k$ , the second one is immediate from (44), and the third one is a consequence of the expansion of  $x_+^{-1}$  at 1, namely  $x_+^{-1}(t) = -(t-1)/(4\cos(\pi/n)) + \dots$ . Then, with (43), we obtain that

$$h(x) = \frac{\log(1-x)}{\pi} \frac{-f_n(i_0, j_0)}{2(4\cos(\pi/n))^n} (x-1)^n [1 + O(x-1)] + h_1(x), \quad (46)$$

where  $h_1$  is holomorphic near 1. Finally, by using the theorem of Pringsheim (recalled in Lemma 14) in Equation (46), we immediately obtain Theorem 13.

## References

- [Bia91] Philippe Biane. Quantum random walk on the dual of  $SU(n)$ . *Probab. Theory Related Fields*, 89(1):117–129, 1991.
- [Bia92] Philippe Biane. Minuscule weights and random walks on lattices. In *Quantum probability & related topics*, QP-PQ, VII, pages 51–65. World Sci. Publ., River Edge, NJ, 1992.
- [BMM09] Mireille Bousquet-Mélou and Marni Mishna. Walks with small steps in the quarter plane. *Preprint available at <http://arxiv.org/abs/0810.4387>*, pages 1–30, 2009.
- [Bou75] Nicolas Bourbaki. *Éléments de mathématique*. Hermann, Paris, 1975. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles, No. 1364.
- [DO05] Yan Doumerc and Neil O’Connell. Exit problems associated with finite reflection groups. *Probab. Theory Related Fields*, 132(4):501–538, 2005.
- [Dyn69] Eugene Dynkin. The boundary theory of Markov processes (discrete case). *Uspehi Mat. Nauk*, 24(2 (146)):3–42, 1969.
- [Dys62] Freeman Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Mathematical Phys.*, 3:1191–1198, 1962.
- [EK08] Peter Eichelsbacher and Wolfgang König. Ordered random walks. *Electron. J. Probab.*, 13:no. 46, 1307–1336, 2008.
- [FIM99] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev. *Random walks in the quarter-plane*, volume 40 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1999. Algebraic methods, boundary value problems and applications.
- [IR08] Irina Ignatiouk-Robert. Martin boundary of a killed random walk on a half-space. *J. Theoret. Probab.*, 21(1):35–68, 2008.
- [IR09] Irina Ignatiouk-Robert. Martin boundary of a killed random walk on  $\mathbb{Z}_+^d$ . *Preprint available at <http://arxiv.org/abs/0909.3921>*, pages 1–49, 2009.
- [IRL09] Irina Ignatiouk-Robert and Christophe Loree. Martin boundary of a killed random walk on a quadrant. *Preprint available at <http://arxiv.org/abs/0903.0070>*, pages 1–39, 2009.
- [KR09] Irina Kurkova and Kilian Raschel. Random walks in  $(\mathbb{Z}_+)^2$  with non-zero drift absorbed at the axes. *Preprint available at <http://arxiv.org/abs/0903.5486>*, pages 1–29, 2009.
- [PW92] Massimo Picardello and Wolfgang Woess. Martin boundaries of cartesian products of Markov chains. *Nagoya Math. J.*, 128:153–169, 1992.
- [Ras09a] Kilian Raschel. Martin boundary of a killed random walk in the Weyl chamber of the dual of  $SU(3)$ . *Preprint available at <http://arxiv.org/abs/0910.1055>*, pages 1–11, 2009.
- [Ras09b] Kilian Raschel. Random walks in the quarter plane absorbed at the boundary : Exact and asymptotic. *Preprint available at <http://arxiv.org/abs/0902.2785>*, pages 1–33, 2009.
- [SG69] Giovanni Sansone and Johan Gerretsen. *Lectures on the theory of functions of a complex variable. II: Geometric theory*. Wolters-Noordhoff Publishing, Groningen, 1969.