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Martin boundary of a killed random walk in the Weyl chamber of the dual of $SU(3)$

Kilian Raschel*

October 6, 2009

Abstract

We consider a classical random walk killed at the boundary of the quarter plane \mathbb{Z}_+^2 and with drift zero, that appears quite naturally in the study of quantum random walks on the dual of $SU(3)$. In [Bia91], P. Biane has computed the asymptotic of the Green functions of this process along the paths of states $(i, j) \in \mathbb{Z}_+^2$ such that $i + j \rightarrow \infty$ and $j/i \rightarrow \tan(\gamma)$, for any $\gamma \in]0, \pi/2[$. In this note we wish to extend his results up to $\gamma \in [0, \pi/2]$. In particular, this will allow us to prove that the Martin boundary of the process is reduced to one point.

Keywords : random walk, Green functions, Martin boundary, absorption probabilities.

AMS 2000 Subject Classification : primary 60G50, 31C35 ; secondary 30F10.

1 Introduction and main results

Random walks conditioned on staying in cones of \mathbb{Z}^d attract a lot of attention in the mathematical community, particularly since the nineties. They appear actually in several distinct domains. An historically important example is constituted by the so-called non-colliding random walks, in other words by the processes (Z_1, \dots, Z_d) composed of d independent and identically distributed random walks conditioned on never leaving the Weyl chamber $\{z \in \mathbb{R}^d : z_1 < \dots < z_d\}$. These processes appeared in the eigenvalues description of important matrix-valued stochastic processes, see [Dys62], and are recently again very much studied, see e.g. [EK08] and the references therein.

Another important area where random processes conditioned on staying in cones of \mathbb{Z}^d appear is the domain of quantum random walks. We briefly recall, see e.g. [Bia08], that they are a non-commutative generalization of classical random walks, and that it is possible to obtain classical processes from quantum processes by restricting the latter ones, initially defined on a non-commutative von Neumann algebra, to commutative subalgebras.

This is how that in [Bia91], P. Biane considers a quantum random walk on the dual of $SU(n)$, and obtains two classical processes by restricting it to the two commutative subalgebras that are firstly the subalgebra generated by the dual of the diagonal matrices and secondly the center of the dual of $SU(n)$.

The first of these two classical random walks is easily described : it is the uniform n -nearest neighbors random walk on the lattice of integral forms of $SU(n)$. The noteworthy Theorem 4.1 of [Bia91] states that the second of these two classical random walks is obtained from the first by killing it at the boundary of the Weyl chamber associated with the lattice, and then by taking the Doob h -transform of this killed process, with some function h harmonic for the killed process and made explicit – it equals the dimension of some representation of the Lie group $SU(n)$.

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Moreover, P. Biane identifies this harmonic function h in the Martin boundary of the killed process. Indeed, in Proposition 5.1 of [Bia91], he computes the exact asymptotic of the Green kernel of the killed process along all paths of states lying in some cones, that he supposes to be strictly included in the Weyl chamber. Unfortunately, this latter hypothesis concerning the cones is essential, and P. Biane writes in [Bia91] that with the methods he used there, “[he] has not been able to obtain sufficiently precise estimates on the Green kernel” in order to remove this assumption. In particular, neither the asymptotic of the Green kernel along paths tangent to the boundary of the Weyl chamber, nor the Martin boundary of the process can be obtained with [Bia91].

The aim of this note is to complete the results of P. Biane in the particular case $n = 3$ by finding the asymptotic of the Green kernel along *all* paths of states, as well as the Martin compactification of the process.

In the particular case $n = 3$, it is immediate from the construction that the first classical process obtained by P. Biane in [Bia91] is the random walk spatially homogeneous on the lattice $\{m + n \exp(i\pi/3), (m, n) \in \mathbb{Z}^2\}$ with jump probabilities as represented at the left of Picture 1 below. Moreover, the Weyl chamber of this lattice is $\{m + n \exp(i\pi/3), (m, n) \in \mathbb{Z}_+^2\}$, see [Bia91].

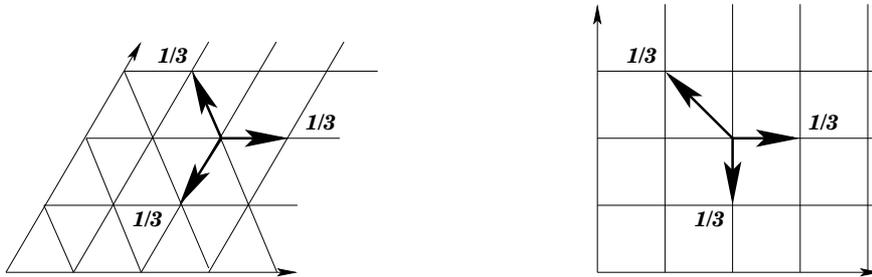


Figure 1: Random walk on the dual of the diagonal matrices of $SU(3)$

It is immediate (see Picture 1) that a suitable linear transformation maps the lattice $\{m + n \exp(i\pi/3), (m, n) \in \mathbb{Z}^2\}$ into \mathbb{Z}^2 . In this way, the Weyl chamber $\{m + n \exp(i\pi/3), (m, n) \in \mathbb{Z}_+^2\}$ becomes \mathbb{Z}_+^2 and the new jump probabilities are represented at the right of Picture 1. This is this latter point of view, namely the one of a random walk in the quarter plane, that we will embrace in the sequel.

The results that we wish to complete concern the process killed at the boundary of the Weyl chamber, this is why we are going to be interested in the random walk (X, Y) verifying :

- (H1) *the jump probabilities $p_{ij} = \mathbb{P}[(X(k+1), Y(k+1)) = (i_0 + i, j_0 + j) \mid (X(k), Y(k)) = (i_0, j_0)]$ verify $p_{0-1} = p_{-11} = p_{10} = 1/3$.*
- (H2) *the random walk is killed at the boundary $\{(i, 0) : i \geq 1\} \cup \{(0, j) : j \geq 1\}$.*

For killed processes in the quarter plane like (X, Y) , there exist methods, based on [FIM99], in order to get explicitly the generating function of the Green functions

$$G_{i,j}^{i_0,j_0} = \mathbb{E}_{(i_0,j_0)} \left[\sum_{k \geq 0} \mathbf{1}_{\{(X(k), Y(k)) = (i,j)\}} \right], \quad (1)$$

$\mathbb{E}_{(i_0,j_0)}$ denoting the conditional expectation given $(X(0), Y(0)) = (i_0, j_0)$.

Indeed, in the book [FIM99], the authors G. Fayolle, R. Iasnogorodski and V. Malyshev elaborate a profound and ingenious analytic approach in order to obtain explicit expressions of the stationary probabilities generating functions for some ergodic random walks in the quarter plane. In [KR09], we have adapted this approach to the case of the

generating functions of Green functions and of the absorption probabilities for some walks in the quarter plane absorbed at the boundary and having a positive drift. We will see here that this approach can also be extended to the case of a random walk having a drift zero : Subsection 2.1 of this paper, that leads to an explicit formulation for the generating functions of Green functions, is strongly inspired by [FIM99].

In this way we will get an explicit expression of the Green functions and we will prove in Section 3 the following :

Theorem 1. *The Green functions (1) admit the following asymptotic as $i + j \rightarrow \infty$ and $j/i \rightarrow \tan(\gamma)$, γ lying in $[0, \pi/2]$:*

$$G_{i,j}^{i_0,j_0} \sim \frac{3^{7/2}}{2\pi} i_0 j_0 (i_0 + j_0) \frac{ij(i+j)}{(i^2 + ij + j^2)^3}. \quad (2)$$

Note that for $\gamma \in]0, \pi/2[$, (2) is a consequence of [Bia91], so that we will attach a particular importance in the proof to the two cases $\gamma = 0$ and $\gamma = \pi/2$.

The function $i_0 j_0 (i_0 + j_0)$, that appears in (2), is the harmonic function used by P. Biane in [Bia91] in order to express the second classical process as the Doob transform of the first classical process killed at the boundary of the Weyl chamber. We will show that the function $i_0 j_0 (i_0 + j_0)$ is in fact the only harmonic function for (X, Y) , since as a consequence of Theorem 1, we will obtain :

Corollary 2. *The Martin boundary of the process is reduced to one point.*

Contrary to the killed random walks in the quarter plane with *non zero drift* which have often been considered and for which the Martin boundary is known, see [IRL09], results concerning the killed random walks in the quarter plane with *drift zero* are very rare ; and Corollary 2 is, up to our knowledge, only the second example of Martin boundary for such a random walk – the first example being the one of the cartesian product of two one-dimensional random walks, for which it is not hard to show, see e.g. [PW92], that the Martin boundary is also reduced to one point.

Moreover, it turns out that starting from [FIM99] and [KR09] we can very easily obtain the explicit expression of the generating functions of the absorption probabilities

$$\begin{aligned} h_i^{i_0,j_0} &= \mathbb{P}_{(i_0,j_0)} [\exists k \geq 1 : (X(k), Y(k)) = (i, 0)], \\ \tilde{h}_j^{i_0,j_0} &= \mathbb{P}_{(i_0,j_0)} [\exists k \geq 1 : (X(k), Y(k)) = (0, j)], \end{aligned} \quad (3)$$

$\mathbb{P}_{(i_0,j_0)}$ denoting the conditional probability given $(X(0), Y(0)) = (i_0, j_0)$.

Indeed, in Sections 3 and 4 of [KR09], we have, by using some methods of [FIM99], calculated these generating functions for the walks in the quarter plane verifying (H2) and $p_{0-1} + p_{-11} + p_{10} = 1$, $p_{0-1} < p_{-11} < p_{10}$ instead of (H1).

We will therefore immediately obtain the explicit expression of the generating functions of the absorption probabilities in the case $p_{0-1} = p_{-11} = p_{10} = 1/3$, see (26)-(27) ; then, by studying closely this explicit expression, notably its behavior near 1, we will prove in Section 4 the following result :

Theorem 3. *The probabilities of absorption (3) admit the following asymptotic :*

$$h_i^{i_0,j_0} \underset{i \rightarrow \infty}{\sim} \frac{3^{5/2}}{2\pi} i_0 j_0 (i_0 + j_0) \frac{1}{i^4}, \quad \tilde{h}_j^{i_0,j_0} \underset{j \rightarrow \infty}{\sim} \frac{3^{5/2}}{2\pi} i_0 j_0 (i_0 + j_0) \frac{1}{j^4}. \quad (4)$$

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2 Meromorphic continuation of the generating functions

2.1 A functional equation between the generating functions

Subsection 2.1 consists in preliminary facts and is based on [FIM99]. Start by defining

$$G^{i_0, j_0}(x, y) = \sum_{i, j \geq 1} G_{i, j}^{i_0, j_0} x^{i-1} y^{j-1}, \quad h^{i_0, j_0}(x) = \sum_{i \geq 1} h_i^{i_0, j_0} x^i, \quad \tilde{h}^{i_0, j_0}(y) = \sum_{j \geq 1} \tilde{h}_j^{i_0, j_0} y^j, \quad (5)$$

the generating functions of the Green functions (1) and of the absorption probabilities (3). With the notations (5), we can state the following functional equation :

$$Q(x, y) G^{i_0, j_0}(x, y) = h^{i_0, j_0}(x) + \tilde{h}^{i_0, j_0}(y) - x^{i_0} y^{j_0}, \quad (6)$$

where $Q(x, y) = xy(x + y/x + 1/y - 3)/3$ – note that $(x + y/x + 1/y)/3$ is nothing else but the generating function of the jump probabilities of the process.

A priori, Equation (6) has a meaning in $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$. The proof of (6) is obtained exactly as in Subsection 2.1 of [KR09].

When no ambiguity on the initial state can arise, we will drop the index i_0, j_0 and we will write $G_{i, j}, G(x, y), h_i, h(x), \tilde{h}_j, \tilde{h}(y)$ for $G_{i, j}^{i_0, j_0}, G^{i_0, j_0}(x, y), h_i^{i_0, j_0}, h^{i_0, j_0}(x), \tilde{h}_j^{i_0, j_0}, \tilde{h}^{i_0, j_0}(y)$.

The polynomial Q can be written alternatively :

$$Q(x, y) = a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \quad (7)$$

where $a(x) = 1/3$, $b(x) = x^2/3 - x$, $c(x) = x/3$, $\tilde{a}(y) = y/3$, $\tilde{b}(y) = -y + 1/3$, $\tilde{c}(y) = y^2/3$. We define also $d(x) = b(x)^2 - 4a(x)c(x)$ and $\tilde{d}(y) = \tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y)$. Note that we have :

$$d(x) = (1/9)x(x-4)(x-1)^2, \quad \tilde{d}(y) = -(4/9)(y-1/4)(y-1)^2. \quad (8)$$

In order to be consistent with the notations of [FIM99], we will note $x_1 = 0$, $x_4 = 4$, $y_1 = 1/4$ and $y_4 = \infty$.

2.2 Uniformization and meromorphic continuation

Notation. Throughout the whole paper, ι denotes the complex number : $\iota^2 = -1$.

Let us now have a look to the surface defined by $\{(x, y) \in (\mathbb{C} \cup \{\infty\})^2 : Q(x, y) = 0\}$, that we will note $\{Q = 0\}$ for the sake of briefness. Note first that with (7), $Q(x, y) = 0$ is equivalent to $(b(x) + 2a(x)y)^2 = d(x)$ or to $(\tilde{b}(y) + 2\tilde{a}(y)x)^2 = \tilde{d}(y)$. In particular, it follows from the particular form of d (two simple roots at x_1 and x_4 , a double root at 1) or of \tilde{d} (a simple root at y_1 , a double root at 1), see (8), that the surface $\{Q = 0\}$ has genus zero, and is thus homeomorphic to a sphere. Therefore, this Riemann surface can be rationally uniformized, in the sense that it is possible to find two rational functions $x(z)$ and $y(z)$, such that $\{Q = 0\} = \{(x(z), y(z)) : z \in \mathbb{C} \cup \{\infty\}\}$. Moreover, we easily verify that we can chose :

$$x(z) = \frac{(z-1)^2}{(z - \exp(2i\pi/3))(z - \exp(-2i\pi/3))}, \quad y(z) = \frac{(z-1)(z - \exp(-2i\pi/3))}{(z - \exp(2i\pi/3))^2}. \quad (9)$$

For more details about Riemann surfaces and uniformization, see for instance [SG69].

For a better understanding of the surface $\{Q = 0\}$ as well as for a coming use, we are now going to be interested in the transformations through the uniformization (x, y) of some important cycles, namely the branch cuts $[x_1, x_4]$, $[y_1, y_4]$ and the unit circles $\{|x| = 1\}$, $\{|y| = 1\}$. Using (9) we easily obtain :

$$x^{-1}([x_1, x_4]) = \mathbb{R} \cup \{\infty\}, \quad y^{-1}([y_1, y_4]) = \exp(2i\pi/3) \mathbb{R} \cup \{\infty\}. \quad (10)$$

As for the cycles $x^{-1}(\{|x| = 1\})$ and $y^{-1}(\{|y| = 1\})$, their explicit expression (calculated starting from (9)) shows that they are real elliptic curves, which are located as in the middle of Picture 2 below.

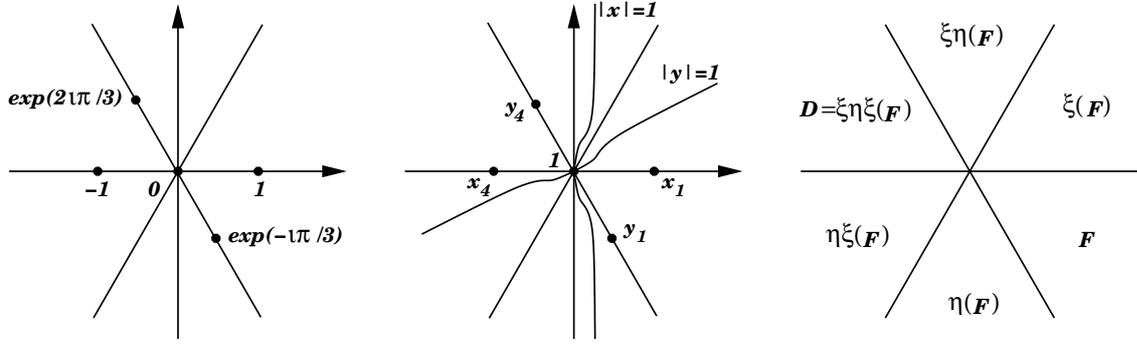


Figure 2: The uniformization space $\mathbb{C} \cup \{\infty\}$, with at the left some important elements of it, in the middle the corresponding elements through the uniformization (x, y) , and at the right the images of the cone $F = \{x \exp(i\theta), x \geq 0, -\pi/3 \leq \theta \leq 0\}$ through the elements (12) of the group of the random walk

Let us now be interested in some automorphisms naturally attached to the surface $\{Q = 0\}$. To begin with, we remark that it follows from (7) that $Q(x, y) = 0$ entails $Q(x, x/y) = 0$ and $Q(y/x, y) = 0$; it is therefore natural to consider the group generated by the two bilinear transformations $\hat{\xi}(x, y) = (x, x/y)$ and $\hat{\eta}(x, y) = (y/x, y)$, noted $\langle \hat{\xi}, \hat{\eta} \rangle$. The authors of [FIM99] call this group $\langle \hat{\xi}, \hat{\eta} \rangle$ *the group of the random walk*, and they show that in the particular case of the process we are considering in this note, it is of order six.

These automorphisms $\hat{\xi}$ and $\hat{\eta}$ define two automorphisms ξ and η of the uniformization space $\mathbb{C} \cup \{\infty\}$, characterized by :

$$\xi \circ \xi = \text{id}, \quad x \circ \xi = x, \quad y \circ \xi = x/y, \quad \eta \circ \eta = \text{id}, \quad y \circ \eta = y, \quad x \circ \eta = y/x. \quad (11)$$

Using the well-known characterization of the automorphisms of the Riemann sphere, (9) and (11), we obtain that ξ and η have the following expressions :

$$\xi(z) = 1/z, \quad \eta(z) = \exp(-2i\pi/3)/z. \quad (12)$$

In particular, it is immediate that the group $W = \langle \xi, \eta \rangle$ generated by ξ and η is isomorphic to the dihedral group of order six, i.e. the group of symmetries of a triangle, ξ and η playing the role of the two reflections.

Note also that (12) immediately yields $\xi(\exp(i\theta)\mathbb{R}_+) = \exp(-i\theta)\mathbb{R}_+$ and $\eta(\exp(i\theta)\mathbb{R}_+) = \exp(-i(\theta + 2\pi/3))\mathbb{R}_+$. In particular, if F denotes $\{x \exp(i\theta), x \geq 0, -\pi/3 \leq \theta \leq 0\}$, then

$$\bigcup_{w \in W} w(F) = \mathbb{C}, \quad (13)$$

see at the right of Picture 2.

Thanks to the group $W = \langle \xi, \eta \rangle$ and to (13), we are now going to continue the functions $H(z) = h(x(z))$ and $\tilde{H}(z) = \tilde{h}(y(z))$, what will be of the highest importance in Section 3, where we will prove Theorem 1.

Note that in the sequel, we will often write $x^{i_0}y^{j_0}(z)$ instead of $x(z)^{i_0}y(z)^{j_0}$.

Proposition 4. *The functions $H(z) = h(x(z))$ and $\tilde{H}(z) = \tilde{h}(y(z))$ can be continued meromorphically from respectively $\{z \in \mathbb{C} : |x(z)| \leq 1\}$ and $\{z \in \mathbb{C} : |y(z)| \leq 1\}$ to respectively $\mathbb{C} \setminus \exp(i\pi)[0, \infty]$ and $\mathbb{C} \setminus \exp(2i\pi/3)[0, \infty]$. These continuations verify*

$$\forall z \in \mathbb{C} : \quad H(z) = H(\xi(z)), \quad \tilde{H}(z) = \tilde{H}(\eta(z)), \quad (14)$$

and

$$H(z) + \tilde{H}(z) - x^{i_0}y^{j_0}(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus D \\ - \sum_{w \in W} (-1)^{l(w)} x^{i_0}y^{j_0}(w(z)) & \text{if } z \in D \end{cases} \quad (15a)$$

where we have set $D = \{x \exp(i\theta), x \geq 0, 2\pi/3 \leq \theta \leq \pi\}$ and $l(w)$ for the length of w , i.e. the smallest r for which we can write $w = w_1 \cdots w_r$, with w_1, \dots, w_r equal to ξ or η .

Remark 5. In $\{z : |x(z)| \leq 1, |y(z)| \leq 1\} \subset \mathbb{C} \setminus D$, (15a) follows immediately from (6).

Remark 6. As a consequence of Proposition 4, the generating functions h and \tilde{h} , see (5), can be continued into meromorphic functions on $\mathbb{C} \setminus [1, x_4]$ and $\mathbb{C} \setminus [1, y_4]$ respectively.

Proof of Proposition 4. In order to prove Proposition 4, we will strongly use the decomposition (13), and more precisely we will define H and \tilde{H} piecewise, by defining them on each of the six domains $w(F)$ that appear in the decomposition (13), to be equal to some functions H_w and \tilde{H}_w ; it will then suffice to show that the functions H and \tilde{H} so defined verify the conclusions of Proposition 4.

- In $F = \{x \exp(i\theta), x \geq 0, -\pi/3 \leq \theta \leq 0\} \subset \{z \in \mathbb{C} : |x(z)| \leq 1, |y(z)| \leq 1\}$, see Picture 1, we are going to use the most natural way to define H_1 and \tilde{H}_1 , i.e. their power series. So we set, for $z \in F$: $H_1(z) = h(x(z))$ and $\tilde{H}_1(z) = \tilde{h}(y(z))$ – the subscript 1 in H_1 and \tilde{H}_1 stands for the identity element of the group W .

- Next we define H_ξ, \tilde{H}_ξ on $\xi(F)$ and H_η, \tilde{H}_η on $\eta(F)$ by

$$\begin{aligned} \forall z \in \xi(F) & : H_\xi(z) = H_1(\xi(z)), \quad \tilde{H}_\xi(z) = -H_\xi(z) + x^{i_0}y^{j_0}(z), \\ \forall z \in \eta(F) & : \tilde{H}_\eta(z) = \tilde{H}_1(\eta(z)), \quad H_\eta(z) = -\tilde{H}_\eta(z) + x^{i_0}y^{j_0}(z). \end{aligned}$$

- Then we define $H_{\xi\eta}, \tilde{H}_{\xi\eta}$ on $\xi\eta(F)$ and $H_{\eta\xi}, \tilde{H}_{\eta\xi}$ on $\eta\xi(F)$ by

$$\begin{aligned} \forall z \in \xi\eta(F) & : H_{\xi\eta}(z) = H_\eta(\xi(z)), \quad \tilde{H}_{\xi\eta}(z) = -H_{\xi\eta}(z) + x^{i_0}y^{j_0}(z), \\ \forall z \in \eta\xi(F) & : \tilde{H}_{\eta\xi}(z) = \tilde{H}_\xi(\eta(z)), \quad H_{\eta\xi}(z) = -\tilde{H}_{\eta\xi}(z) + x^{i_0}y^{j_0}(z). \end{aligned}$$

- At last we define $H_{\xi\eta\xi}$ and $\tilde{H}_{\xi\eta\xi}$ on $\xi\eta\xi(F) = \eta\xi\eta(F)$ by

$$\forall z \in \xi\eta\xi(F) : H_{\xi\eta\xi}(z) = H_{\eta\xi}(\xi(z)), \quad \tilde{H}_{\xi\eta\xi}(z) = \tilde{H}_{\xi\eta}(\eta(z)).$$

Therefore we have, for each of the six domains $w(F)$ of the decomposition (13), defined two functions H_w and \tilde{H}_w . Then, as said at the beginning of the proof, we set $H(z) = H_w(z)$ and $\tilde{H}(z) = \tilde{H}_w(z)$ for all $z \in w(F)$ and for all automorphisms $w \in W$.

With this construction, (14) and (15a) are immediately obtained. To prove (15b), we can use the fact that it is possible to express all the functions H_w, \tilde{H}_w , in terms only of H_1, \tilde{H}_1 and $x^{i_0}y^{j_0}$: we give, as examples, the expressions of $H_{\xi\eta\xi}$ and $\tilde{H}_{\xi\eta\xi}$ on $\xi\eta\xi(F)$:

$$\begin{aligned} H_{\xi\eta\xi}(z) &= H_1(\xi\eta\xi(z)) - x^{i_0}y^{i_0}(\eta\xi(z)) + x^{i_0}y^{i_0}(\xi(z)), \\ \tilde{H}_{\xi\eta\xi}(z) &= \tilde{H}_1(\xi\eta\xi(z)) - x^{i_0}y^{i_0}(\xi\eta(z)) + x^{i_0}y^{i_0}(\eta(z)). \end{aligned}$$

We therefore obtain (15b), since with (15a) we get $H_1(\xi\eta\xi(z)) + \tilde{H}_1(\xi\eta\xi(z)) = x^{i_0}y^{j_0}(\xi\eta\xi(z))$ for $z \in \xi\eta\xi(F)$, and since $W = \{1, \xi, \eta, \eta\xi, \xi\eta, \xi\eta\xi\}$. \square

3 Asymptotic of the Green functions and Martin boundary

Proof of Theorem 1. Equation (6) yields immediately that the generating function of the Green functions G , defined in (5), is holomorphic in $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$. As a consequence and using again (6), the Cauchy formulas allow us to write its coefficients $G_{i,j}$ as the following double integrals :

$$G_{i,j} = \frac{1}{(2\pi i)^2} \iint_{|x|=|y|=1} \frac{G(x, y)}{x^i y^j} dx dy = \frac{1}{(2\pi i)^2} \iint_{|x|=|y|=1} \frac{h(x) + \tilde{h}(y) - x^{i_0}y^{j_0}}{Q(x, y) x^i y^j} dx dy.$$

Then, using the uniformization (9), the location of the cycles $\{|x| = 1\}$ and $\{|y| = 1\}$, see Picture 2, the residue theorem at infinity and the Cauchy theorem, we obtain that :

$$G_{i,j} = \frac{1}{2\pi i} \int_{\exp(i\theta)[0, \infty]} \frac{H(z) + \tilde{H}(z) - x^{i_0}y^{j_0}(z)}{\partial_y Q(x(z), y(z)) x(z)^i y(z)^j} x'(z) dz,$$

θ being any angle in $[2\pi/3, \pi] - [2\pi/3, \pi]$ because on the one hand, it is not possible to take $\theta > \pi$, since $\exp(i\pi)[0, \infty]$ is a singular curve for H , and on the other hand it is not allowed to have $\theta < 2\pi/3$, since $\exp(2i\pi/3)[0, \infty]$ is a singular curve for \tilde{H} , see Proposition 4.

Then, by using the equality $x'(z)/\partial_y Q(x(z), y(z)) = -i3^{1/2}/z$ as well as (15b) we get :

$$G_{i,j} = \frac{3^{1/2}}{2\pi} \int_{\exp(i\theta)[0, \infty]} \left[\frac{1}{z} \sum_{w \in W} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z)) \right] \frac{1}{x(z)^i y(z)^j} dz. \quad (16)$$

For any $\theta \in [2\pi/3, \pi]$, the function $x(z)^i y(z)^j$ is, on $\exp(i\theta)[0, \infty]$, larger than 1 in modulus, see Picture 2. Moreover, it goes to 1 when (and only when) z goes to 0 or to ∞ . This is why it seems natural to decompose the contour $\exp(i\theta)[0, \infty]$ into a part near 0, an other near ∞ and the remaining part, and to think that the parts near 0 and ∞ will lead to the asymptotic of $G_{i,j}$, and that the remaining part will lead to a negligible contribution. But how to find the best contour in order to achieve this idea ? In other words, how to find the value of θ for which the calculation of the asymptotic of (16) on $\exp(i\theta)[0, \infty]$ will be the easiest, among all the possibilities $\theta \in [2\pi/3, \pi]$?

For this, we are going to consider with details the function $x(z)^i y(z)^j$, or, equivalently, the function $\chi_{j/i}(z) = \ln(x(z)) + (j/i) \ln(y(z))$. Incidentally, this is why from now on we will suppose that $j/i \in [0, M]$, for some $M < \infty$. Indeed, the function $\chi_{j/i}$ is manifestly not adapted to the values j/i going to ∞ ; for such j/i we will consider later the function $(i/j)\chi_{j/i}(z) = (i/j) \ln(x(z)) + \ln(y(z))$. Nevertheless, M can be so large as wished, and, in what follows, we will suppose that some $M > 0$ is fixed.

Now we set $\chi_{j/i}(z) = \sum_{p=0}^{\infty} \nu_p(j/i) z^p$ for z in the neighborhood of 0. With (9) we obtain that $\nu_{3p}(j/i) = 0$ and that :

$$\nu_{3p+1}(j/i) = \frac{-3}{3p+1} [1 + (j/i) \exp(i\pi/3)], \quad \nu_{3p+2}(j/i) = \frac{-3}{3p+2} [1 + (j/i) \exp(-i\pi/3)]. \quad (17)$$

Likewise, we easily prove by using (9) that for z near ∞ , $\chi_{j/i}(z) = \sum_{p=0}^{\infty} \bar{\nu}_p(j/i) 1/z^p$.

Consider now the steepest descent path associated to $\chi_{j/i}$, that is the function $z_{j/i}(t)$ defined by $\chi_{j/i}(z_{j/i}(t)) = t$. By inverting this latter equality, we immediately obtain that the half-line $(1/\nu_1(j/i))[0, \infty]$ is tangent at 0 and at ∞ to the steepest descent path.

Now we set, for the sake of brevity,

$$\rho_{j/i} = 1/\nu_1(j/i) = \frac{-1}{3[1 + (j/i) \exp(i\pi/3)]}. \quad (18)$$

With this notation, we now answer the question asked above, that concerned the fact of finding the value of θ for which the asymptotic of the Green functions (16) will be the most easily calculated : we will chose $\theta = \arg(\rho_{j/i})$ and the decomposition of the contour $\exp(i\theta)[0, \infty]$ will be :

$$\exp(i\theta)[0, \infty] = (\rho_{j/i}/|\rho_{j/i}|) ([0, \epsilon] \cup \epsilon, 1/\epsilon] \cup [1/\epsilon, \infty]).$$

According to this decomposition and to (16), we consider now that $G_{i,j}$ is the sum of three terms, and we are going to study successively the contribution of each of these terms.

Contribution of the neighborhood of 0. In order to evaluate the asymptotic of the integral (16) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$, we will use the expansion of the function $(1/z) \sum_{w \in W} (-1)^{l(w)} x^{i_0} y^{j_0}(w(z))$ in the neighborhood of 0, see (22) below. This is why we begin by studying the asymptotic of the following integral, for any non-negative integer k :

$$\int_{(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]} \frac{z^k}{x(z)^i y(z)^j} dz. \quad (19)$$

Using the equality $1/(x(z)^i y(z)^j) = \exp(-i\chi_{j/i}(z))$ and the expansion (17) of $\chi_{j/i}$ near 0, and then making the change of variable $z = \rho_{j/i} t$, we obtain that (19) is equal to :

$$\rho_{j/i}^{k+1} \int_0^{\epsilon/|\rho_{j/i}|} t^k \exp(-it) \exp\left(-i\nu_2(j/i) (\rho_{j/i} t)^2\right) \exp\left(-i \sum_{p=3}^{\infty} \nu_p(j/i) (\rho_{j/i} t)^p\right) dt. \quad (20)$$

But with (17) we get $|\nu_p(j/i)| \leq 3(1+M)$, and, therefore, for all $t \in [0, \epsilon/|\rho_{j/i}|]$ we have $| -i \sum_{p=3}^{\infty} \nu_p(j/i)(\rho_{j/i}t)^p | \leq i\epsilon^3 3(1+M)/(1-\epsilon)$. This is why $\exp(-i \sum_{p=3}^{\infty} \nu_p(j/i)(\rho_{j/i}t)^p) = 1 + \mathcal{O}(i\epsilon^3)$, the \mathcal{O} being independent of $j/i \in [0, M]$ and of $t \in [0, \epsilon/|\rho_{j/i}|]$. The integral (20) can thus be calculated as

$$(\rho_{j/i}/i)^{k+1} [1 + \mathcal{O}(i\epsilon^3)] \int_0^{i\epsilon/|\rho_{j/i}|} t^k \exp(-t) \left[1 - \nu_2(j/i) \rho_{j/i}^2 t^2/i + \mathcal{O}(t^4/i^2) \right] dt.$$

In the sequel we will chose $\epsilon = 1/i^{3/4}$, so that $i\epsilon/|\rho_{j/i}| \rightarrow \infty$ and $\mathcal{O}(i\epsilon^3) = \mathcal{O}(1/i^{5/4})$ – we could have thought that having $\mathcal{O}(i\epsilon^3) = o(1)$ would suffice in order to obtain the asymptotic of the Green functions (16), but we will see in the forthcoming paragraph called “Conclusion” that for the calculation along the paths of states (i, j) such that $j/i \rightarrow 0$, it is in fact necessary to have $\mathcal{O}(i\epsilon^3) = o(1/i)$, what affords the choice $\epsilon = 1/i^{3/4}$.

Finally, we obtain that for this choice of ϵ , the integral (19) can be written as

$$(\rho_{j/i}/i)^{k+1} \left[1 + \mathcal{O}(1/i^{5/4}) \right] \left[k! - \nu_2(j/i) \rho_{j/i}^2 (k+2)!/i + \mathcal{O}(1/i^2) \right], \quad (21)$$

where the two \mathcal{O} above are independent of $j/i \in [0, M]$.

We are now ready to find the asymptotic of the integral (16) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$. To begin with, we have the following expansion in the neighborhood of 0 (directly obtained from (9) and (12)) :

$$\sum_{w \in W} (-1)^{l(w)} x^{i_0} y^{j_0} (w(z)) = -i \frac{3^{9/2}}{2} i_0 j_0 (i_0 + j_0) z^3 + \mathcal{O}(z^6). \quad (22)$$

Equation (22) implies then that the integral (16) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$ equals

$$\frac{3^{1/2}}{2\pi} \int_{(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]} \frac{-i (3^{9/2}/2) i_0 j_0 (i_0 + j_0) z^2 + \mathcal{O}(z^5)}{x(z)^i y(z)^j} dz.$$

So, with (19) and (21) applied for $k = 2$ and $k = 5$, we obtain that the integral (16) on the contour $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$ is equal to

$$\frac{-i3^5}{4\pi} i_0 j_0 (i_0 + j_0) (\rho_{j/i}/i)^3 \left[2 - 24\nu_2(j/i) \rho_{j/i}^2/i + \mathcal{O}(1/i^{5/4}) \right]. \quad (23)$$

We will see in the paragraph called “Conclusion” that the expansion $-(i3^5/(4\pi))i_0 j_0 (i_0 + j_0)(\rho_{j/i}/i)^3 [2 + o(1)]$ is accurate enough in order to find the asymptotic of the Green functions along the paths of states (i, j) such that $j/i \rightarrow \tan(\gamma) \in]0, \infty[$, but *not* for the paths such that $j/i \rightarrow 0$.

Contribution of the neighborhood of ∞ . The part of the contour close to ∞ , $(\rho_{j/i}/|\rho_{j/i}|)[1/\epsilon, \infty]$, is related to the part $(\rho_{j/i}/|\rho_{j/i}|)[0, \epsilon]$ by the transformation $z \mapsto 1/\bar{z}$. Moreover, it is immediate from (9) that for $f = x$, $f = y$, or $f = \sum_{w \in W} (-1)^{l(w)} x^{i_0} y^{j_0} (w)$, $f(1/\bar{z}) = \overline{f(z)}$. Therefore, the change of variable $z \mapsto 1/\bar{z}$ immediately entails that the contribution of the integral (16) near ∞ is the complex conjugate of its contribution near 0.

Contribution of the intermediate part. We first recall a notation of Proposition 4 : $D = \{x \exp(i\theta), x \geq 0, 2\pi/3 \leq \theta \leq \pi\}$. Let us also define A_ϵ to be the annular domain $\{z \in \mathbb{C} : \epsilon \leq |z| \leq 1/\epsilon\}$. According to Picture 2, there exist $\eta_{x,\epsilon} > 0$ and $\eta_{y,\epsilon} > 0$, such that for all $z \in D \cap A_\epsilon$, $|x(z)| \geq 1 + \eta_{x,\epsilon}$ and $|y(z)| \geq 1 + \eta_{y,\epsilon}$. In fact, since $x'(0) \neq 0$ and $y'(0) \neq 0$, it is possible to take $\eta_{x,\epsilon} \geq \eta\epsilon$ and $\eta_{y,\epsilon} \geq \eta\epsilon$ for some $\eta > 0$ independent of ϵ small enough.

Let us now consider the function $s(z) = [\sum_{w \in W} (-1)^{l(w)} x^{i_0} y^{j_0} (w(z))] / [x^{i_0} y^{j_0} (z)]$, and let us show that $\sup_{z \in D} |s(z)|$ is finite. For this, it is sufficient to prove that s has no pole in the closed domain $D \cup \{\infty\}$. But using (9) and (12), we see that the only

poles of the numerator of s , as well as the only zeros of the denominator of s , are at 1 and $\exp(\pm 2i\pi/3)$. Among these points, only $\exp(2i\pi/3)$ belongs to D . But, in s , we have taken care of dividing by $x^{i_0}y^{j_0}(z)$, so that s is in fact holomorphic near $\exp(2i\pi/3)$. Moreover, it is easily shown that s is holomorphic at ∞ . Finally, we have proved that the meromorphic function s has no pole in the closed domain $D \cup \{\infty\}$, hence s is bounded in $D \cup \{\infty\}$, in other words $\sup_{z \in D} |s(z)|$ is finite.

The modulus of the contribution of the integral (16) on the intermediate part $(\rho_{j/i}/|\rho_{j/i}|)\epsilon, 1/\epsilon \subset D \cap A_\epsilon$ can be bounded from above by

$$\frac{3^{1/2}}{2\pi} \frac{1}{\epsilon^2} \frac{\sup_{z \in D} |s(z)|}{(1 + \eta\epsilon)^{i-i_0}(1 + \eta\epsilon)^{j-j_0}}. \quad (24)$$

Note that the presence of the term $1/\epsilon^2$ in (24) is due to : one $1/\epsilon$ appears as an upper bound of the length of the contour, the other $1/\epsilon$ comes from an upper bound of the modulus of the term $1/z$ present in the integrand of (16).

Then we take, as before, $\epsilon = 1/i^{3/4}$, and we use the following straightforward upper bound, valid for i large enough : $1/(1 + \eta/i^{3/4})^i \leq \exp(-(\eta/2)i^{1/4})$. We finally obtain that, for i large enough, (24) is equal to $\mathcal{O}(i^{3/2} \exp(-(\eta/2)i^{1/4}))$. We will see soon that this contribution is negligible w.r.t. the sum of the contribution of the integral (16) in the neighborhoods of 0 and ∞ .

Conclusion. We have seen that the contribution of the integral (16) in the neighborhood of 0 is given by (23), that the contribution of (16) in the neighborhood of ∞ is equal to the complex conjugate of (23), and that the contribution of the remaining part is equal to $\mathcal{O}(i^{3/2} \exp(-(\eta/2)i^{1/4}))$. Moreover, starting from (18) we easily get that $(\rho_{j/i}/i)^3 - (\overline{\rho_{j/i}/i})^3 = i3^{-3/2}ij(i+j)/(i^2 + ij + j^2)^3$; we thus obtain that $G_{i,j}$ equals

$$i_0j_0(i_0 + j_0) \left[\frac{3^{7/2}}{2\pi} \frac{ij(i+j)}{(i^2 + ij + j^2)^3} + 2i \frac{3^6}{\pi} \frac{1}{i^4} (\nu_2(j/i)\rho_{j/i}^5 - \overline{\nu_2(j/i)\rho_{j/i}^5}) + \mathcal{O}(1/i^{3+5/4}) \right]. \quad (25)$$

If $\gamma \in]0, \pi/2[$ and $j/i \rightarrow \tan(\gamma)$ then $ij(i+j)/(i^2 + ij + j^2)^3 \sim C_\gamma/i^3$ with $C_\gamma > 0$: Theorem 1 for $\gamma \in]0, \pi/2[$ is an immediate consequence of (25). In that case, there was in fact no need to make an expansion with two terms in (23) and in (25) above, one single term would have been accurate enough.

If $\gamma = 0$ and $j/i \rightarrow 0$, then $ij(i+j)/(i^2 + ij + j^2)^3 \sim (j/i)/i^3$. By using the explicit expressions of $\nu_2(j/i)$ and $\rho_{j/i}$, see respectively (17) and (18), we easily obtain that $\nu_2(j/i)\rho_{j/i}^5 - \overline{\nu_2(j/i)\rho_{j/i}^5} = \mathcal{O}(j/i)$. So the sum of the two last terms in (25) equals $\mathcal{O}((j/i)/i^4) + \mathcal{O}(1/i^{3+5/4})$, which is obviously negligible w.r.t. $(j/i)/i^3$. Theorem 1 is therefore also proved in the case $\gamma = 0$.

In order to prove Theorem 1 in the case $\gamma = \pi/2$, we would consider $(i/j)\kappa_{j/i}$ rather than $\kappa_{j/i}$, and we would use then exactly the same analysis ; we omit the details.

Proof of Corollary 2. It follows from Theorem 1 that the limit of the Green kernel $G_{i,j}^{i_0,j_0}/G_{i,j}^{1,1}$ as $i, j > 0, i+j \rightarrow \infty$ and $j/i \rightarrow \tan(\gamma)$ is equal to $i_0j_0(i_0 + j_0)/2$, in particular it does not depend on $\gamma \in [0, \pi/2]$.

Moreover, the limits of the ratio of the absorption probabilities $h_i^{i_0,j_0}/h_i^{1,1}$ and $\tilde{h}_j^{i_0,j_0}/\tilde{h}_j^{1,1}$ (see (3) for their definition) as respectively i and j go to infinity are also equal to $i_0j_0(i_0 + j_0)/2$: indeed, starting from

$$h_i^{i_0,j_0} = \mathbb{P}_{(i_0,j_0)}[\text{to hit } (i, 1)]/3, \quad G_{i,1}^{i_0,j_0} = \mathbb{P}_{(i_0,j_0)}[\text{to hit } (i, 1)]G_{i,1}^{1,1},$$

we obtain $h_i^{i_0,j_0}/h_i^{1,1} = G_{i,1}^{i_0,j_0}/G_{i,1}^{1,1}$. And, likewise, we have $\tilde{h}_j^{i_0,j_0}/\tilde{h}_j^{1,1} = G_{1,j-1}^{i_0,j_0}/G_{1,j-1}^{1,1}$.

For these reasons and by using the classical theory of Martin boundary, see e.g. [Dyn69], we obtain that the Martin boundary of the process is reduced to one point, which proves Corollary 2.

4 Absorption probabilities

All the facts and results written in Section 4 will concern the function h , we would obtain similar ones for \tilde{h} by making exactly the same analysis.

In Sections 3 and 4 of [KR09] we have, by using ideas of [FIM99], made explicit the generating function of absorption probabilities h for the absorbed random walks having the same non-zero jump probabilities as here, but with a positive drift, i.e. verifying $p_{0-1} + p_{-11} + p_{10} = 1$ and $p_{0-1} < p_{-11} < p_{10}$ instead of (H1). There we have found a function w (page 15 of [KR09]) such that if we set $a(t) = p_{-11}$, $b(t) = p_{10}t^2 - t$, $c(t) = p_{0-1}t$, $d(t) = b(t)^2 - 4a(t)c(t)$ and

$$\mu_{j_0}(t) = \frac{1}{(2a(t))^{j_0}} \sum_{k=0}^{\lfloor (j_0-1)/2 \rfloor} \binom{j_0}{2k+1} d(t)^k (-b(t))^{j_0-(2k+1)},$$

then h can be written as the sum $h_1 + h_2 + h_3$, where (below, $0 = x_1 < x_2 < 1 < x_3 < x_4$ denote the four real roots of d)

$$h_1(x) = \frac{1}{\pi} \int_{x_3}^{x_4} t^{i_0} \mu_{j_0}(t) \frac{1}{t-x} (-d(t))^{1/2} dt, \quad (26)$$

$$h_2(x) = \frac{1}{\pi} \int_{x_1}^{x_2} t^{i_0} \mu_{j_0}(t) \left[\frac{w'(t)}{w(t) - w(x)} - \frac{1}{t-x} \right] (-d(t))^{1/2} dt, \quad (27)$$

and h_3 is a polynomial, that could also be made explicit, see [KR09].

By making precisely the same analysis in the case $p_{0-1} = p_{-11} = p_{10} = 1/3$, we obtain that the explicit integral representation of h written in (26)-(27) still holds, with the function $w(t) = t/(t-1)^3$.

In order to prove Theorem 3, we will study closely the explicit expression of h obtained just above, and more precisely we will show that h is holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1 and we will prove that in the neighborhood of 1,

$$h(x) = -\frac{3^{3/2}}{4\pi} i_0 j_0 (i_0 + j_0) (x-1)^3 \ln(1-x) [1 + \mathcal{O}(x-1)] + h_0(x), \quad (28)$$

where h_0 is holomorphic at 1. Then (5), (28) and the well-known Pringsheim's theorem (recalled in Lemma 7 below) will immediately entail Theorem 3.

Lemma 7 (Pringsheim's Theorem). *Let $l(x) = \sum_{i=0}^{\infty} l_i x^i$ be a function holomorphic in the unit disc. Suppose in addition that l is continuable holomorphically through every point of the unit circle except 1, in the neighborhood of which it can be written as $l(x) = (x-1)^q \ln(1-x) [1 + \mathcal{O}(x-1)] + l_0(x)$, where l_0 is holomorphic at 1 and q is some integer. Then $l_i \sim_{i \rightarrow \infty} -q! / i^{q+1}$.*

Now we begin the proof of Theorem 3. First, note that as a generating function of probabilities, it is clear that h is holomorphic in the unit disc. Moreover, from (26)-(27) and the expression of w given above it is manifest that h is continuable holomorphically through every point of the unit circle except 1 ; so it remains to prove (28). For this we will need the following result :

Lemma 8. *Let k be a non-negative integer. There exist two functions f_k and g_k holomorphic at 1, such that in the neighborhood of 1 we have :*

$$\int_1^{x_4} \frac{(t-1)^k}{t-s} dt = -(s-1)^k \ln(1-s) + f_k(s), \quad \int_{x_1}^1 \frac{(t-1)^k}{t-s} dt = (s-1)^k \ln(s-1) + g_k(s).$$

Note that the proof of Lemma 8 is straightforward, since the integrals that appear in its statement are easily explicitly calculated.

Now we set $t^{i_0} \mu_{j_0}(t)(t(4-t)/9)^{1/2} = \sum_{k=0}^{\infty} \alpha_k(i_0, j_0)(t-1)^k$. With this notation, (8), (26) and Lemma 8, we obtain that in the neighborhood of 1, h_1 has the following behavior :

$$h_1(x) = -\frac{\ln(1-x)}{\pi} \sum_{k=0}^{\infty} \alpha_k(i_0, j_0) (x-1)^{k+1} + h_3(x), \quad (29)$$

where h_3 can be written in terms of the f_k of Lemma 8 and is thus holomorphic at 1.

As for h_2 , we first simplify its explicit expression (27). For this we use the partial fraction expansion of $w'(t)/(w(t)-w(x))$, namely $w'(t)/(w(t)-w(x)) = 1/(t-x) + 1/(t-u_+(x)) + 1/(t-u_-(x)) - 3/(t-1)$, with $u_{\pm}(t) = [-t^2 + 3t \pm d(t)^{1/2}]/[2t]$, and we get :

$$h_2(x) = C + \frac{1}{\pi} \int_{x_1}^1 t^{i_0} \mu_{j_0}(t) \left[\frac{1}{t-u_+(x)} + \frac{1}{t-u_-(x)} \right] (-d(t))^{1/2} dt,$$

where C is some constant. In particular, with (8) and Lemma 8, we obtain that

$$h_2(x) = -\frac{\ln(1-x)}{\pi} \sum_{k=0}^{\infty} \alpha_k(i_0, j_0) \left[(u_+(x)-1)^{k+1} + (u_-(x)-1)^{k+1} \right] + h_4(x), \quad (30)$$

where h_4 can be written in terms of the g_k of Lemma 8 and of C , and is thus holomorphic in the neighborhood of 1. Note that $(u_+ - 1)^{k+1} + (u_- - 1)^{k+1}$, as any symmetric polynomial in u_+ and u_- , is a rational function. Note also that to obtain (30) we have used the fact that $\ln(u_{\pm}(x) - 1)$ has the same logarithmic singularity as $\ln(1-x)$.

Finally, with (29) and (30), we obtain that h equals

$$-\frac{\ln(1-x)}{\pi} \sum_{k=0}^{\infty} \alpha_k(i_0, j_0) \left[(x-1)^{k+1} + (u_+(x)-1)^{k+1} + (u_-(x)-1)^{k+1} \right] + h_3(x) + h_4(x).$$

But as an immediate consequence of the explicit expression of the $\alpha_k(i_0, j_0)$, it turns out that $\sum_{k=0}^2 \alpha_k(i_0, j_0) [(x-1)^{k+1} + (u_+(x)-1)^{k+1} + (u_-(x)-1)^{k+1}] = ((x-1)^3/x)(3^{3/2}/4)i_0j_0(i_0+j_0)$. Moreover, it is straightforward that $\sum_{k=3}^{\infty} \alpha_k(i_0, j_0) [(x-1)^{k+1} + (u_+(x)-1)^{k+1} + (u_-(x)-1)^{k+1}] = \mathcal{O}(x-1)^4$, so that (28) is proved, and therefore also Theorem 3.

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