

# An Application of Martin-Löf Randomness to Effective Probability Theory

Mathieu Hoyrup<sup>1</sup> and Cristóbal Rojas<sup>2</sup>

<sup>1</sup> INRIA Nancy

<sup>2</sup> Institut de Mathématiques de Luminy  
(FRANCE)

# Initial problem

Having effective versions of measure-theoretical notions adapted to Martin-Löf randomness.

# Martin-Löf randomness

## Short review

- In every (computable) probability space  $(X, \mu)$  is defined the set of **Martin-Löf random** points:

$$\begin{aligned} \text{ML}_\mu &\subseteq X \\ \mu(\text{ML}_\mu) &= 1. \end{aligned}$$

- Most classical probability theorems like

$P(x)$  holds for almost every  $x$

can be converted into

$P(x)$  holds for every ML-random  $x$ .

# Martin-Löf randomness

Example: Strong Law of Large Numbers (SLLN)

## Theorem (Classical)

Let  $f_i$  be i.i.d. bounded **measurable** functions with mean  $m$ . For **almost every**  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{f_0(x) + \dots + f_{n-1}(x)}{n} = m.$$

## Theorem (SLLN for random points)

Let  $f_i$  be i.i.d. bounded **computable** functions with mean  $m$ . For **every random**  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{f_0(x) + \dots + f_{n-1}(x)}{n} = m.$$

Every computable function is continuous. What about discontinuous functions?

# Content of the paper

- We review, compare and generalize some effective versions of measure-theoretical notions:
  - Measurable set,
  - Measurable function,
  - Integrable function.
- We introduce the new framework of **layerwise computability**, which unifies effective measure theory and Martin-Löf randomness.

# The context

- $X$  is a computable metric space,
- $\mu$  is a computable Borel probability measure over  $X$ .

## Examples

- $(\{0, 1\}^{\mathbb{N}}, \mathbb{B}_p)$ : Bernoulli process,
- $([0, 1], \lambda)$ : uniform measure,
- $(\mathcal{K}(\mathbb{R}^n), \mu)$ : random Cantor sets,
- $(\mathcal{C}([0, 1]), \mathcal{W})$ : Brownian motion.

Martin-Löf randomness is defined on such spaces.

## 1 Effective measure theory

- First approach: review
- Second approach: review
- Third approach: layerwise computability

# A first approach (Šanin, Ko, Wu, Ding, Edalat, Bosserhoff)

Let  $\mathcal{A} = \{A_0, A_1, \dots\}$  be an effective enumeration of the finite unions of basic balls. Members of  $\mathcal{A}$  are called **simple sets**.

## Definition

A set  $A$  is  **$\mu$ -recursive** if there is a computable function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n$ ,

$$\mu(A \Delta A_{\varphi(n)}) < 2^{-n}.$$



# A first approach (Šanin, Ko, Wu, Ding, Edalat, Bosserhoff)

Let  $\mathcal{A} = \{A_0, A_1, \dots\}$  be an effective enumeration of the finite unions of basic balls. Members of  $\mathcal{A}$  are called **simple sets**.

## Definition

A set  $A$  is  $\mu$ -**recursive** if there is a computable function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n$ ,

$$\mu(A \triangle A_{\varphi(n)}) < 2^{-n}.$$

Another formulation:  $A$  is  $\mu$ -recursive if there is a type-two machine  $M$  that, given  $n$ , decides membership of  $A$  with probability of error  $< 2^{-n}$ .

# A first approach (Šanin, Ko, Wu, Ding, Edalat, Bosserhoff)

Let  $\mathcal{A} = \{A_0, A_1, \dots\}$  be an effective enumeration of the finite unions of basic balls. Members of  $\mathcal{A}$  are called **simple sets**.

## Definition

A set  $A$  is  **$\mu$ -recursive** if there is a computable function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n$ ,

$$\mu(A \Delta A_{\varphi(n)}) < 2^{-n}.$$

Another formulation:  $A$  is  $\mu$ -recursive if there is a type-two machine  $M$  that, given  $n$ , decides membership of  $A$  with probability of error  $< 2^{-n}$ .

## Definition

A function  $f : X \rightarrow X'$  is  **$\mu$ -recursive** if all the  $f^{-1}(A'_i)$  are  $\mu$ -recursive, uniformly in  $i$ .

# A first approach (Šanin, Ko, Wu, Ding, Edalat, Bosserhoff)

Can we expect the following?

## Theorem (Virtual)

Let  $f_i$  be i.i.d. bounded  $\mu$ -recursive functions with mean  $m$ . For every random  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{f_0(x) + \dots + f_{n-1}(x)}{n} = m.$$

# A first approach (Šanin, Ko, Wu, Ding, Edalat, Bosserhoff)

Can we expect the following?

## Theorem (Virtual)

Let  $f_i$  be i.i.d. bounded  $\mu$ -recursive functions with mean  $m$ . For every random  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{f_0(x) + \dots + f_{n-1}(x)}{n} = m.$$

No!

Assuming that  $f$  is  $\mu$ -recursive does not imply anything about its value at a given random point.

## 1 Effective measure theory

- First approach: review
- **Second approach: review**
- Third approach: layerwise computability

## A second approach

### Theorem (Classical result)

For every Borel set  $A$  and every  $\epsilon > 0$  there is an open set  $U$  and a closed set  $F$  such that

- $F \subseteq A \subseteq U$ ,
- $\mu(U \setminus F) < \epsilon$ .

## A second approach

### Theorem (Classical result)

For every Borel set  $A$  and every  $\epsilon > 0$  there is an open set  $U$  and a closed set  $F$  such that

- $F \subseteq A \subseteq U$ ,
- $\mu(U \setminus F) < \epsilon$ .

### Definition (Edalat)

A set  $A$  is **effectively  $\mu$ -measurable** if there are open sets  $U_n$  and closed sets  $F_n$  such that:

- $U_n, F_n$  are constructive, uniformly in  $n$ ,
- $F_n \subseteq A \subseteq U_n$ ,
- $\mu(U_n \setminus F_n) < 2^{-n}$ .

# A second approach

Comparison with the first approach

## Theorem (H. and Rojas)

Let  $A$  be a Borel set. The following statements are equivalent:

- 1  $A$  is  $\mu$ -recursive,
- 2  $A$  is equivalent to an effectively  $\mu$ -measurable set.



# A second approach

Comparison with the first approach

## Theorem (H. and Rojas)

Let  $A$  be a Borel set. The following statements are equivalent:

- 1  $A$  is  $\mu$ -recursive,
- 2  $A$  is equivalent to an effectively  $\mu$ -measurable set.

## Definition

A function  $f : X \rightarrow X'$  is **effectively  $\mu$ -measurable** if all the  $f^{-1}(A'_i)$  are effectively  $\mu$ -measurable, uniformly in  $i$ .

# A second approach

Comparison with the first approach

## Theorem (H. and Rojas)

Let  $A$  be a Borel set. The following statements are equivalent:

- 1  $A$  is  $\mu$ -recursive,
- 2  $A$  is equivalent to an effectively  $\mu$ -measurable set.

## Definition

A function  $f : X \rightarrow X'$  is **effectively  $\mu$ -measurable** if all the  $f^{-1}(A'_i)$  are effectively  $\mu$ -measurable, uniformly in  $i$ .

## Theorem (H. and Rojas)

Let  $f : X \rightarrow X'$  be a function. The following statements are equivalent:

- 1  $f$  is  $\mu$ -recursive,
- 2  $f$  is equivalent to an effectively  $\mu$ -measurable function.

# A second approach

Relation with Martin-Löf randomness

## Theorem (H. and Rojas)

- If  $A, B$  are effectively  $\mu$ -measurable and  $\mu(A \triangle B) = 0$ , then  $A \cap \text{ML}_\mu = B \cap \text{ML}_\mu$ .
- If  $f, g$  are effectively  $\mu$ -measurable and  $f = g$  almost everywhere, then  $f = g$  on  $\text{ML}_\mu$ .

# A second approach

Relation with Martin-Löf randomness

## Theorem (H. and Rojas)

- If  $A, B$  are effectively  $\mu$ -measurable and  $\mu(A \triangle B) = 0$ , then  $A \cap \text{ML}_\mu = B \cap \text{ML}_\mu$ .
- If  $f, g$  are effectively  $\mu$ -measurable and  $f = g$  almost everywhere, then  $f = g$  on  $\text{ML}_\mu$ .

Now we can expect:

## Theorem (Virtual)

Let  $f_i$  be i.i.d. bounded **effectively  $\mu$ -measurable** functions with mean  $m$ . For **every random**  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{f_0(x) + \dots + f_{n-1}(x)}{n} = m.$$

## 1 Effective measure theory

- First approach: review
- Second approach: review
- Third approach: layerwise computability

# Layerwise computability

## Layering

The existence of a universal Martin-Löf test [Martin-Löf, 1966] induces a canonical decomposition of  $ML_\mu$ :

- $ML_\mu = \bigcup_n K_n$
- with  $K_n \subseteq K_{n+1}$
- and  $\mu(K_n) > 1 - 2^{-n}$ .

### Theorem (H. and Rojas, ICALP09)

*If the space  $X$  is complete, then the sets  $K_n$  are effectively compact, uniformly in  $n$ .*

We call the sequence  $(K_n)_{n \in \mathbb{N}}$  the **Martin-Löf layering** of the space  $(X, \mu)$ .

# Layerwise computability

## Definitions

### Definition

A set  $A$  is **layerwise decidable** if it is decidable on each  $K_n$ , uniformly in  $n$ .



### Definition

A function  $f : X \rightarrow Y$  is **layerwise computable** if it is computable on each  $K_n$ , uniformly in  $n$ .



# Layerwise computability

## Equivalences

### Theorem (H. and Rojas)

Let  $A$  be a set. The following statements are equivalent:

- 1  $A$  is effectively  $\mu$ -measurable,
- 2  $A$  is layerwise decidable.

### Theorem (H. and Rojas)

Let  $f : X \rightarrow X'$  be a function. The following statements are equivalent:

- 1  $f$  is effectively  $\mu$ -measurable,
- 2  $f$  is layerwise computable.



# Layerwise computability

## Equivalences

### Theorem (H. and Rojas)

Let  $A$  be a set. The following statements are equivalent:

- 1  $A$  is effectively  $\mu$ -measurable,
- 2  $A$  is layerwise decidable.

### Theorem (H. and Rojas)

Let  $f : X \rightarrow X'$  be a function. The following statements are equivalent:

- 1  $f$  is effectively  $\mu$ -measurable,
- 2  $f$  is layerwise computable.

### Application (ICALP09)

Theorems for random points with weaker hypotheses: functions can be assumed to be **layerwise computable** instead of **computable**.

# Conclusion

Martin-Löf randomness, through the decomposition  $ML = \bigcup_n K_n$ , provides an alternative way of expressing effective measure theory (ICALP09 and ongoing work)

- Eff.  $\mu$ -measurable set
  - Eff.  $\mu$ -measurable function
  - Eff.  $\mu$ -a.s. convergence
  - $\mu \ll_{\text{eff}} \nu$
  - $\mu \ll \nu$  and  $\frac{d\mu}{d\nu} \in L^\infty(\nu)$
- Set decidable on each  $K_n$
  - Function computable on each  $K_n$
  - Eff. uniform convergence on each  $K_n$
  - $K_n^\mu \subseteq K_{\varphi(n)}^\nu$
  - $K_n^\mu \subseteq K_{n+c}^\nu$

# Conclusion

Martin-Löf randomness, through the decomposition  $ML = \bigcup_n K_n$ , provides an alternative way of expressing effective measure theory (ICALP09 and ongoing work)

- Eff.  $\mu$ -measurable set
  - Eff.  $\mu$ -measurable function
  - Eff.  $\mu$ -a.s. convergence
  - $\mu \ll_{\text{eff}} \nu$
  - $\mu \ll \nu$  and  $\frac{d\mu}{d\nu} \in L^\infty(\nu)$
- Set decidable on each  $K_n$
  - Function computable on each  $K_n$
  - Eff. uniform convergence on each  $K_n$
  - $K_n^\mu \subseteq K_{\varphi(n)}^\nu$
  - $K_n^\mu \subseteq K_{n+c}^\nu$

Thank you.