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# Theoretical assessment of stress-strain curve estimates in Split Hopkinson Bar experiments

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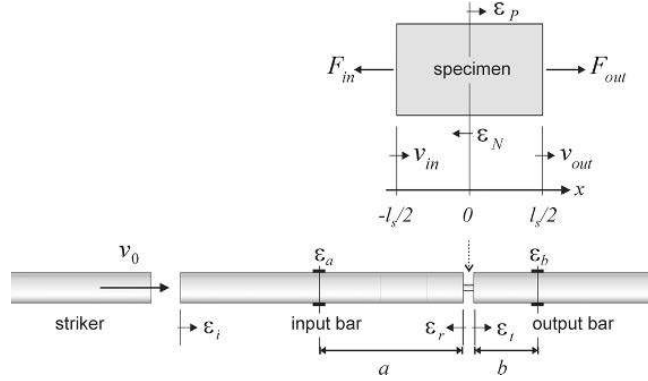
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**Abstract.** The determination of the stresses and strains based on split Hopkinson pressure bar (SHPB) measurements is an important step in the identification of reliable experimental data on the mechanical behavior of materials at high strain rates. Modern SHPB systems provide accurate measurements of the forces and velocities at the boundaries of a dynamically loaded specimen, while approximations need to be made to obtain the stress-strain curve based on these measurements. Several formulas have been proposed in the past to estimate the stress-strain curve from dynamic experiments. Here, we make use of the theoretical solution for the waves in an elastic specimen to evaluate the accuracy of these estimates. It is found that it is important to avoid an artificial time shift in the processing of the experimental data. Moreover, it is concluded that the combination of the output force based stress estimate and the average strain provides the best of the commonly used stress-strain curve estimates in standard SHPB experiments.

## 1. INTRODUCTION

A comprehensive review of developments in classical SHPB testing has been provided by Gray III [1]. The main aspects that determine the accuracy of measurements in SHPB compression tests can be classified in two types. Firstly, there are aspects related to the accuracy of the forces and velocities at the specimen boundaries provided by the SHPB system. These global quantities can be obtained from the recorded wave signals without consideration of the specimen. Aspects of the second type are related to assumptions concerning the bar-specimen interaction and the specimen behavior: interface friction, lateral inertia of the specimen, uniaxial stress distribution, and stress equilibrium.

The present conference paper closely follows [2]. It focuses on the estimation of the stress-strain curve, which involves aspects of the second type. Due to the presence of waves in dynamic experiments, both the stress and strain fields within a specimen are seldom uniform. A dynamic material test should be designed such as to minimize this inherent non-uniformity, a condition which is typically associated with “quasi-static equilibrium”. However, when testing purely elastic materials such as fiber reinforced composites or low impedance materials, the validity of this assumption needs to be checked with care. Before computers became generally available, the assumption of quasi-static equilibrium of the specimen had a special importance from an analog data processing point of view. With the general availability of numerical data acquisition and computer systems, most limitations associated with analog data processing could be overcome: The input and output bars no longer need to be identical; the waves do not need to be dispersion-free and different strain gage positions may be chosen on the input and output bars; furthermore, two independent force measurements may be obtained (so-called input and output force) which allow the evaluation of the validity of the assumption of quasi-static equilibrium. Knowing that specimen equilibrium is never achieved exactly, we seek the best of the commonly used stress-strain curve estimates in a SHPB experiment.



**Figure 1.** Schematic of conventional SHPB test set-up with detail of specimen. The input and output bar strain gages are positioned at a distance of  $a$  and  $b$  from the respective specimen/bar interfaces.

## 2. WAVE PROPAGATION IN THE SPECIMEN

We make use of the exact theoretical solution for the waves in an elastic specimen sandwiched between two Hopkinson bars to calculate the force and velocity histories at the specimen/bar interfaces. These histories are then used to estimate the elastic properties of the specimen based on common stress-strain curve approximation formulas. Subsequently, these estimates are compared with the exact elastic properties in order to evaluate their accuracy.

Consider a cylindrical specimen of length  $l_s$ , cross-sectional area  $A_s$ , Young's modulus  $E_s$ , mass density  $\rho_s$ , wave speed  $c_s = \sqrt{E_s/\rho_s}$ , and characteristic impedance  $Z_s = A_s E_s / c_s$ . Following [3], we write the strain in the specimen as

$$\hat{\varepsilon}(x, \omega) = \hat{\varepsilon}_P(\omega)e^{-i\omega x/c_s} + \hat{\varepsilon}_N(\omega)e^{i\omega x/c_s}, \quad (1)$$

where  $\hat{\varepsilon}_P(\omega)$  and  $\hat{\varepsilon}_N(\omega)$  are the strains associated with the rightward and leftward travelling waves at the mid-section of the specimen ( $\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt$ ). Thus, the force and velocity at the input bar/specimen interface ( $x = -l_s/2$ ) read

$$\hat{F}_{in}(\omega) = c_s Z_s [\alpha \hat{\varepsilon}_P(\omega) + \beta \hat{\varepsilon}_N(\omega)], \quad (2)$$

$$\hat{v}_{in}(\omega) = c_s [-\alpha \hat{\varepsilon}_P(\omega) + \beta \hat{\varepsilon}_N(\omega)], \quad (3)$$

with

$$\alpha(\omega) = e^{i\omega t_s/2}, \quad \beta(\omega) = e^{-i\omega t_s/2}, \quad (4)$$

where  $t_s = l_s/c_s$  denotes the transit time for an elastic wave propagating through the specimen. Analogously, we have the force and velocity at the output bar/specimen interface ( $x = l_s/2$ ),

$$\hat{F}_{out}(\omega) = c_s Z_s [\beta \hat{\varepsilon}_P(\omega) + \alpha \hat{\varepsilon}_N(\omega)], \quad (5)$$

$$\hat{v}_{out}(\omega) = c_s [-\beta \hat{\varepsilon}_P(\omega) + \alpha \hat{\varepsilon}_N(\omega)]. \quad (6)$$

In a SHPB compression experiment, the output bar may be considered semi-infinite (between the strain gage location and the output bar/specimen interface, there are only waves traveling away from the specimen during the interval of measurement). Thus, the output force

$$\hat{F}_{out}(\omega) = -Z_o \hat{v}_{out}(\omega) \quad (7)$$

is directly proportional to the output velocity  $\hat{v}_{out}(\omega)$  with  $Z_o$  denoting the output bar impedance.

### 3. STRESS-STRAIN CURVE ESTIMATES

Even though the forces and velocities at the boundaries of a dynamically loaded specimen can be determined to a high degree of accuracy, it can be difficult to determine the stress-strain curve from such data. The challenge is to come up with accurate estimates of the stress history  $\sigma(t)$  and the corresponding strain history  $\varepsilon(t)$  such that their combination

$$\sigma(\varepsilon) = \sigma(t) \circ \varepsilon^{-1}(t) \quad (8)$$

provides an accurate estimate of the stress-strain curve  $\sigma(\varepsilon)$  of the dynamically tested material. In the following, we investigate estimates that are widely used.

#### 3.1 Direct estimates

The spatial average of the axial strain field within the specimen is chosen to estimate the strain history. It can be expressed in terms of the interface velocities  $v_{in}(t)$  and  $v_{out}(t)$  as

$$\varepsilon_{av}^{de}(t) = \frac{1}{l_s} \int_{-l_s/2}^{l_s/2} \varepsilon(x, t) dx = \frac{1}{l_s} \int_0^t [v_{out}(t) - v_{in}(t)] dt \quad (9)$$

It is not possible to express the spatial average of the stress field in a similar manner. Instead, two distinct stress-time history estimates are considered. Firstly, the stress is estimated as the average of the forces at the input and output bar/specimen interfaces

$$\sigma_{av}^{de}(t) = \frac{F_{in}(t) + F_{out}(t)}{2A_s}. \quad (10)$$

As an alternative to Eq. (10), the stress is frequently estimated based on the output force history only, i.e.

$$\sigma_{out}^{de}(t) = \frac{F_{out}(t)}{A_s}. \quad (11)$$

Combining these two stress estimates with the average strain estimate yields two direct estimates of the stress-strain curve. These two estimates are called ‘‘direct estimates’’ as the original force and velocity measurements have not been artificially shifted on the time axis before calculating the stress-strain curve. In other words the force and velocity histories at the specimen interfaces are directly used to obtain the stress-strain curve.

#### 3.2 Foot shifting

To simplify the processing of SHPB measurements, the original measurement data are sometimes modified using a procedure which we refer to as ‘‘foot-shifting’’. The idea is to shift the strain history associated with the transmitted wave on the time axis such that it rises to non-zero values at the same time as the incident and reflected waves at the input bar/specimen interface. The ‘‘foot’’ of a strain history indicates the point on the time axis where the strain changes for the first time from zero to a non-zero value. With respect to the reflected and incident wave histories at the input bar/specimen interface, the transmitted wave at the output bar/specimen interface is delayed by the transit time  $t_s = l_s/c_s$  of an elastic wave travelling through the specimen. When using the foot-shifting procedure, the strain history associated with the transmitted wave is shifted on the time axis such that its ‘‘foot’’ coincides with that of the strain histories at the input bar/specimen interface.

Formally, the foot shifting estimates may be written as follows. The average strain in the specimen reads  $\varepsilon_{av}^{fs}(t) = (1/l_s) \int_0^t [v_{out}(t + t_s) - v_{in}(t)] dt$  which corresponds to

$$\hat{\varepsilon}_{av}^{fs}(\omega) = \frac{1}{i\omega l_s} [\hat{v}_{out}(\omega) e^{i\omega t_s} - \hat{v}_{in}(\omega)]. \quad (12)$$

The corresponding stress estimate reads

$$\hat{\sigma}_{out}^{fs}(\omega) = \frac{1}{A_s} \hat{F}_{out}(\omega) e^{i\omega t_s}. \quad (13)$$

### 3.3 Kolsky estimate

In the present context, the term ‘‘Kolsky estimate’’ is used to refer to one particular type of estimate that is based on assumptions presented in Kolsky [4]. Neglecting the dispersion in two identical bars and assuming quasi-static equilibrium, Kolsky assumed

$$\hat{\varepsilon}_i(\omega) + \hat{\varepsilon}_r(\omega) \cong \hat{\varepsilon}_t(\omega) \quad (14)$$

to estimate the strain as

$$\hat{\varepsilon}_{Ko}(\omega) = -\frac{2c_o}{i\omega l_s} \hat{\varepsilon}_r(\omega). \quad (15)$$

In terms of the force and velocity at the input specimen/bar interface, this strain estimate becomes

$$\hat{\varepsilon}_{Ko}(\omega) = \frac{1}{i\omega l_s} \left[ -\frac{\hat{F}_{in}(\omega)}{Z_o} - \hat{v}_{in}(\omega) \right]. \quad (16)$$

At the same time, Kolsky used the output force to estimate the stress-time history (see Eq. (11)).

## 4. EVALUATION

The evaluation is limited to the elastic case where the choice of estimate appears to have the greatest importance. In this case, the quality of the stress-strain curve estimates may be evaluated by comparing the apparent modulus  $E(\omega)$  with the real modulus  $E_s$  of the elastic specimen material. Given the stress history  $\hat{\sigma}(\omega)$ , and the strain-time history  $\hat{\varepsilon}(\omega)$ , we have the apparent complex modulus

$$E(\omega) = E'(\omega) + iE''(\omega) = \frac{\hat{\sigma}(\omega)}{\hat{\varepsilon}(\omega)}, \quad (17)$$

where  $E'(\omega)$  and  $E''(\omega)$  denote the real and imaginary parts, respectively. For a perfect estimate,  $E'(\omega)$  should be constant and equal the Young’s modulus,  $E'(\omega) = E_s$ , while the imaginary part should be zero,  $E''(\omega) = 0$ .

The exact derivations of the modulus according to different stress-strain curve estimates (see [4]) reveal that all modulus estimates depend on the normalized angular frequency  $\omega t_s$ . This dimensionless number is small within the range of significant frequencies of a typical SHBP compression test. For evaluation purposes, we make use of the second-order Taylor expansion of the estimated moduli:

(i) Direct estimate, average force based stress and average strain:

$$E_I(\omega) \cong E_s \left[ 1 - \frac{1}{12} (\omega t_s)^2 \right] \quad (18)$$

(ii) Direct estimate, output force based stress and average strain:

$$E_{II}(\omega) \cong E_s [1 - i(\omega t_s/2)Z_s/Z_o] \quad (19)$$

(iii) Foot-shifted estimate, output force based stress and average strain:

$$E_{III}(\omega) \cong \frac{E_s}{1 - Z_s/Z_o} (1 + i\omega t_s) \quad (20)$$

(iv) Kolsky estimate, output force based stress and reflected wave based strain:

$$E_{IV}(\omega) \cong \frac{E_s}{1 - (Z_s/Z_o)^2} \left[ 1 + \frac{1}{6} (\omega t_s)^2 \right] \quad (21)$$

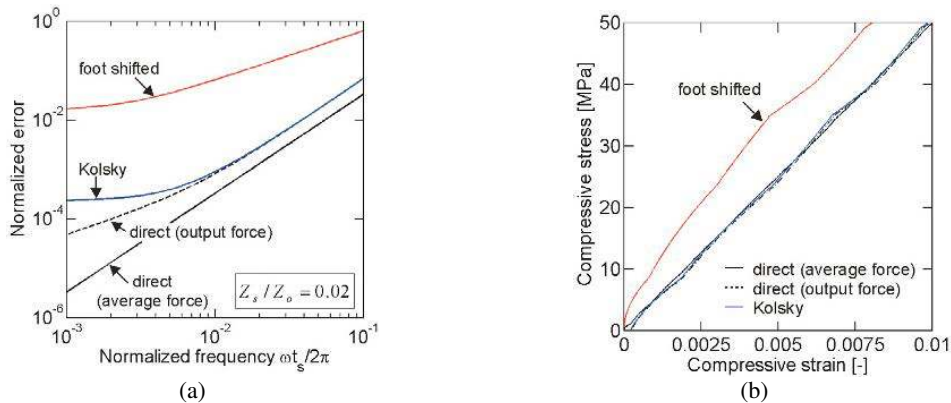
In order to quantify the error in the stress-strain curve estimates, we define the normalized distance between the estimated complex modulus  $E_i(\omega)$  and the true material modulus  $E_s$

$$e_i = \frac{|E_i(\omega) - E_s|}{E_s}. \quad (22)$$

In Figure 2a, these error functions are depicted for a large impedance mismatch ( $Z_s/Z_o = 0.02$ ). This example corresponds to the testing of 10 mm diameter PMMA specimen in a 20 mm diameter steel bar system. The smallest error is observed for the average force based direct estimate  $E_I(\omega)$  while the error for the output force based estimate  $E_{II}(\omega)$  appears to be sandwiched between the curve for  $E_I(\omega)$  and the Kolsky estimate  $E_{IV}(\omega)$ . The error of the foot-shifting based estimate  $E_{III}(\omega)$  is the largest among the present estimates. For the direct estimates, the error vanishes at low frequencies. As shown in Figure 4, the error of the Kolsky estimates does not vanish at low frequencies. The same holds true for the foot-shifted estimate where the error at low frequencies is still larger by a factor of  $Z_o/Z_s$  as compared to the Kolsky estimate.

To illustrate the error in the different stress-strain curve estimates in the time domain, we performed a one-dimensional numerical simulation of a SHPB experiment on a PMMA specimen ( $E_s = 5000$  MPa,  $\rho_s = 1.2$  g/cm<sup>3</sup>,  $D_s = 20$  mm,  $l_s = 20$  mm). The SHPB systems comprises 20 mm diameter steel input and output bars ( $E_b = 210$  GPa,  $\rho_s = 7.8$  g/cm<sup>3</sup>); we generated an incident wave with rise time  $50 \mu\text{s}$  that imitates a striker impact at 5 m/s.

The black solid line in Figure 2b depicts the average force based direct stress estimate  $\sigma_{av}^{de}$  as a function of the average strain estimate  $\varepsilon_{av}^{de}$ . As predicted by the frequency space analysis, this curve provides the best representation of the response of the linear elastic material. The plot of the foot-shifting based stress-strain curve confirms the conclusion of the theoretical analysis: the foot-shifted estimate provides the least accurate representation of the stress-strain curve and deviates substantially from the linear stress-strain relationship predicted by the other estimates.



**Figure 2.** (a) Modulus errors as a function of the normalized angular frequency for different stress-strain curve estimates; (b) Plot of the estimated stress strain curves for a dynamic compression experiment on PMMA.

## 5. CONCLUSIONS

It is concluded from the evaluation of the modulus estimates, that all of them provide reasonable results except the foot-shifted one. Irrespective of the specimen/bar impedance mismatch and frequency, the foot-shifted estimate yields poor results for the stress-strain relationship. The results demonstrate that the so-called direct estimates, which are based on the force and displacement time histories at the specimen boundaries without artificial time shifts, provide the most accurate estimates of the stress-strain curve. Unless accurate input force measurements are available, the combination of the average strain with the output force based stress estimate is recommended for standard SHPB experiments.

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