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On the confinement of a tokamak plasma.

Daniel Han-Kwan*

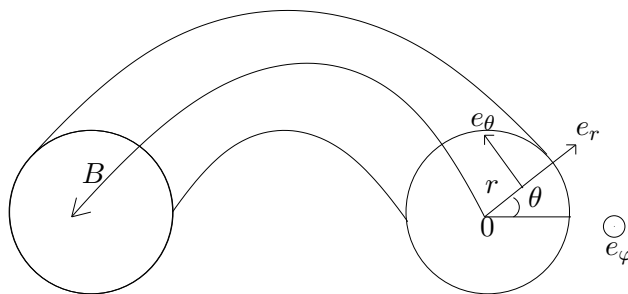
October 22, 2009

Abstract

The goal of this paper is to understand from a mathematical viewpoint the magnetic confinement of plasmas for fusion. Following Frénod and Sonnendrücker [5], we first use 2-scale convergence tools to derive a gyrokinetic system for a plasma submitted to a large magnetic field with a slowly spatially varying intensity. We finally investigate the behaviour of the plasma in such a regime and we prove non-linear stability or instability depending on which side of the tokamak we are looking at. In our analysis we will also show that there exists a temperature gradient threshold beyond which one can expect stability, even in the “bad” side : this is the so-called H-mode.

1 Introduction

Fusion is undoubtedly one of the most promising research fields in order to find new sources of energy. For the time being, magnetic confinement fusion represents one of the two main approaches (the other one being inertial confinement fusion). The principle consists basically in using a magnetic field in order to confine the very high temperature plasma. Good confinement is absolutely compulsory since the plasma could otherwise damage the surrounding materials.



A first step towards confinement is to use a tokamak¹, i.e. a torus-shaped box and consider a large purely toroidal magnetic field B , in other words $B = \frac{B}{\epsilon} e_\varphi$ with $\epsilon > 0$ small. One can formally show that at leading order in ϵ , particles gently oscillate around the magnetic field lines.² The drawback of this technique is that there is in fact many drifts appearing at higher order, some due to the geometry of B and one we are specifically

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¹Actually there are other possibilities, like stellarators. These kinds of devices are much more difficult to study from the mathematical viewpoint, since they have a very complex structure.

²However, for the moment, notice that there exists to our knowledge no rigorous mathematical justification of this statement. Anyway, this is not the point here.

concerned with, which is called the electric drift $v_E = E \wedge B$ (E denotes the electric field). Since the electric field is induced by the plasma itself, one can not precisely predict its qualitative behaviour so that the associated drift prevents us from getting a good confinement property : if we wait long enough, particles may stop to perfectly turn around the torus and start drifting toward the edge of the tokamak. In order to overcome the effects of the electric drift, the idea is basically to take advantage of the other drifts due to the geometry of B .

In the present paper, we make the assumption that the ions of the plasma are at thermodynamic equilibrium and describe the distribution of electrons by a kinetic equation. For the sake of simplicity, we restrict to the 2D problem (that is in the plane orthogonal to B) in order to understand the behaviour of the particles in the slice. We take a magnetic field given by $B = \frac{\mathcal{B}}{\epsilon} e_\varphi$ with $\epsilon > 0$ a small parameter and \mathcal{B} to be chosen later. We consider the Finite Larmor Radius scaling (see [5] for a reference in the mathematical literature) which consists in considering a characteristic spatial length with the same order as the Larmor radius (which is of order ϵ). This scaling allows a better description of the orthogonal motion and is expected to make the electric drift appear in the limit $\epsilon \rightarrow 0$. The density $f_\epsilon(t, x, v)$ (with $t > 0, x \in \mathbb{T}^2, v \in \mathbb{R}^2$) of the electrons is then given by the following dimensionless Vlasov Poisson system :

$$\begin{cases} \partial_t f_\epsilon + \frac{v}{\epsilon} \cdot \nabla_x f_\epsilon + (E_\epsilon + v^\perp \frac{\mathcal{B}}{\epsilon}) \cdot \nabla_v f_\epsilon = 0 \\ f_{\epsilon, |t=0} = f_0 \\ E_\epsilon = -\nabla_x V_\epsilon \\ -\Delta_x V_\epsilon = \int f_\epsilon dv - 1 \end{cases} \quad (1.1)$$

We denote $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $v^\perp = \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}$, in the local orthogonal basis (see figure 1).

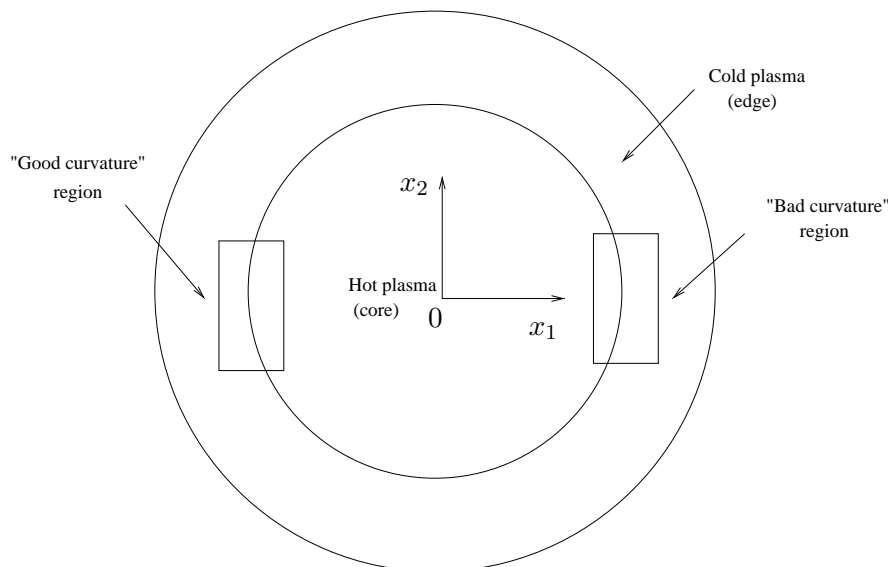


Figure 1: A slice of tokamak

Following Grandgirard et al. ([6]), we consider the explicit formula for \mathcal{B} :

$$\mathcal{B} = \frac{R_0}{R_0 + \epsilon r \cos \theta} = \frac{R_0}{R_0 + \epsilon x_1} \quad (1.2)$$

denoting by R_0 the small radius of the torus. (We recall that the characteristic spatial length is of order ϵ)

We consider that $R_0 \sim 1$; consequently at first order in ϵ we get:

$$\mathcal{B} = 1 - \epsilon x_1 \quad (1.3)$$

leading to the following system:

$$\begin{cases} \partial_t f_\epsilon + \frac{v}{\epsilon} \cdot \nabla_x f_\epsilon + (E_\epsilon + \frac{v^\perp}{\epsilon} - x_1 v^\perp) \cdot \nabla_v f_\epsilon = 0 \\ f_{\epsilon,|t=0} = f_0 \\ E_\epsilon = -\nabla_x V_\epsilon \\ -\Delta_x V_\epsilon = \int f_\epsilon dv - 1 \end{cases} \quad (1.4)$$

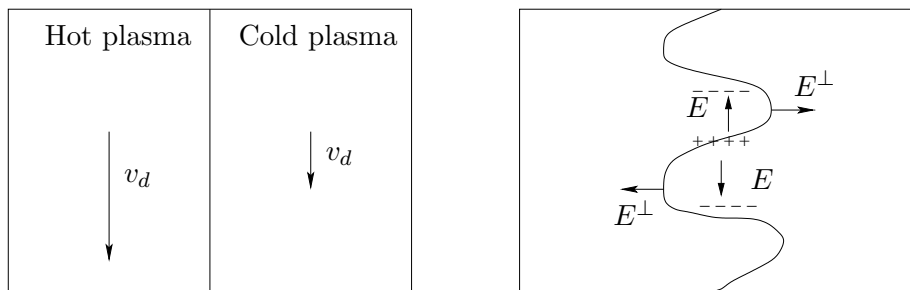
We will see that taking an inhomogeneous intensity for the magnetic field, even at order 1 in ϵ , leads to a quite different behaviour for the plasma.

Indeed, in the limit $\epsilon \rightarrow 0$, we find a system which is more or less equivalent to the following system (see sections 2 and 3):

$$\begin{cases} \partial_t f - \frac{1}{2} v^2 \partial_{x_2} f + E^\perp \cdot \nabla_x f = 0 \\ f|_{t=0} = f_0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f dv - 1 \end{cases} \quad (1.5)$$

Observe here that E^\perp corresponds to the electric drift that we mentioned earlier; the additional drift $v_d = -\frac{1}{2} v^2 e_2$ is due to the inhomogeneity of the magnetic field intensity. The remarkable point is that this drift has a fixed direction; it makes the particles “fall” toward the “bottom” of the slice. At this point of the modeling, we now have to distinguish between the plasma-core and the plasma edge (see figure 1), the only difference between the two we are concerned with, being that the core is much hotter than the edge. This means from a kinetic point of view that the velocities are much smaller in the edge.

We now separate the slice in two and denote the part $x_1 > 0$ the “**bad curvature**” side and the part $x_1 < 0$ the “**good curvature**” side : indeed, we expect the plasma in the “good curvature” side to be well confined, while the plasma in the “bad curvature” region is badly confined. This behaviour can be easily predicted with the following heuristic study in the “bad curvature” side:



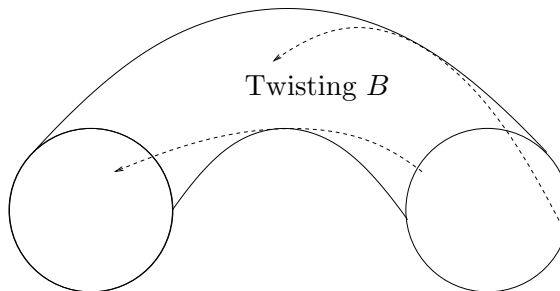
Particles in the hot plasma drift faster (left figure), so if there is any perturbation (right figure), there appears a separation of charge creating an electric field E , which entails a drift E^\perp that accentuates the perturbation: in other words, the equilibrium is unstable.

One can lead the same qualitative analysis in the “good curvature” side and show in this case stability.

Remark 1.1. This instability issue may intuitively look like the classical Kelvin-Helmholtz instability of fluid dynamics; actually things are completely different. For example Kelvin-Helmholtz instability does not depend on the relative position of the fast and the slow fluid.

One may also think of the classical Rayleigh-Taylor instability, recently rigorously studied by Hwang and Guo [9] and Desjardins and Grenier [4].

The next step towards confinement is to consider a magnetic field with a variable direction, i.e. $B = B_0 e_\varphi + B_1 e_\theta$. At leading order, particles still follow the magnetic field lines: consequently, with such a twisting field, particles from the “bad curvature” region travel every now and then to the “good curvature” region. We accordingly expect overall confinement for the plasma. This scenario will nevertheless not be considered here.



The main aim of this work is to study the system (1.5) and provide a mathematical justification of good or bad confinement. Namely, we want to investigate the stability properties of the solutions in each region.

The present paper is organized as follows: section 2 is devoted to the study of the limit $\epsilon \rightarrow 0$ for the system (1.4). In section 3 we present the simplified model (a nonlinear transport equation) we use in order to investigate stability and instability for the plasma. Section 4 is dedicated to the study of the linearized system; in particular we show the existence of unstable eigenfunctions in the “bad curvature” region, provided that the gradient of temperatures is small enough with respect to the size of the box. Finally in section 6, we are concerned with a stability property for the “good curvature” region and for the “bad curvature” region for large enough temperature gradients (referred to as the high confinement mode in plasma physics), which will be achieved by exhibiting a Lyapunov functional.

2 Gyrokinetic derivation of the equations

Following Frénod and Sonnendrücker ([5]), we can use two-scale convergence tools in order to derive the gyrokinetic equation we are interested in. We shall not dwell on the rigorous derivation of this system since the justifications in two dimensions are essentially done in [5].

First of all, let us recall precisely the two-scale convergence notions (due to Nguetseng [10] and Allaire [1]) we will use in the sequel.

Definition. Let X be a separable Banach space, X' be its topological dual space and (\cdot, \cdot) the duality bracket between X' and X . For all $\alpha > 0$, denote by $\mathcal{C}_\alpha(\mathbb{R}, X)$ (respectively $L_\alpha^q(\mathbb{R}; X')$) the space of α -periodic continuous (respectively L^q) functions on \mathbb{R} with values in X . Let $q \in [1; \infty[$.

Given a sequence (u_ϵ) of functions belonging to the space $L^q(0, t; X')$ and a function $U^0(t, \theta) \in L^q(0, T; L_\alpha^q(\mathbb{R}; X'))$ we say that

$$u_\epsilon \text{ 2-scale converges to } U^0$$

if for any function $\Psi \in L^q(0, T; \mathcal{C}_\alpha(\mathbb{R}, X))$ we have:

$$\lim_{\epsilon \rightarrow 0} \int_0^T \left(u_\epsilon(t), \Psi \left(t, \frac{t}{\epsilon} \right) dt \right) = \frac{1}{\alpha} \int_0^T \int_0^\alpha (U^0(t, \tau), \Psi(t, \tau)) d\tau dt \quad (2.1)$$

The new variable τ has to be understood as a “fast-time variable” which describes the fast oscillations. As for weak-star convergence in L^p spaces, one can show that boundedness implies 2-scale convergence in L^p spaces:

Theorem 2.1. *Given a sequence (u_ϵ) bounded in $L^{q'}(0, t; X')$, there exists for all $\alpha > 0$ a function $U_\alpha^0 \in L^{q'}(0, T; L_\alpha^{q'}(\mathbb{R}; X'))$ such that up to a subsequence,*

$$u_\epsilon \text{ 2-scale converges to } U_\alpha^0$$

The profile U_α^0 is called the α -periodic two scale limit of u_ϵ and the link between U_α^0 and the weak-* limit u of u_ϵ is given by:

$$\frac{1}{\alpha} \int_0^\alpha U^0 d\tau = u \quad (2.2)$$

We can now state the main result of this section:

Proposition 2.1. *For each ϵ , let f_ϵ be a global weak solution to (1.4) in the sense of Arsenev .*

Then, up to a subsequence as $\epsilon \rightarrow 0$, f_ϵ 2-scale converges to a function F such that there exists a function G with:

$$F(t, \tau, x, v) = G(t, x + \mathcal{R}(\tau)v, R(\tau)v) \quad (2.3)$$

and G satisfies:

$$\left\{ \begin{array}{l} \partial_t G + \left(\frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau + \begin{pmatrix} -v_1(v_2 - x_1) \\ v_2 x_1 - \frac{1}{2}(v_1^2 + 3v_2^2) \end{pmatrix} \right) \cdot \nabla_x G \\ + \left(\frac{1}{2\pi} \int_0^{2\pi} R(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau + \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} \right) \cdot \nabla_v G = 0 \\ G|_{t=0} = f_0 \\ E = -\nabla_x V \\ -\Delta V = \int G(t, x + \mathcal{R}(\tau)v, R(\tau)v) dv - 1 \end{array} \right. \quad (2.4)$$

denoting by R and \mathcal{R} the linear operators defined by :

$$R(\tau) = \begin{bmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{bmatrix}, \mathcal{R}(\tau) = (R(-\pi/2) - R(-\pi/2 + \tau))$$

Remark 2.1. The difference with an homogeneous intensity $\mathcal{B} = 1$ is represented by the terms:

$$\begin{pmatrix} -v_1(v_2 - x_1) \\ v_2 x_1 - \frac{1}{2}(v_1^2 + 3v_2^2) \end{pmatrix} \cdot \nabla_x G + \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} \cdot \nabla_v G,$$

which will be interpreted after the proof of the proposition.

Proof. We do not wish to develop the very beginning of the proof since it is strictly identical to the one given in [5].

The first step consists in deriving the so-called constraint equation. To this end, let $\Psi(t, \tau, x, v)$ be a 2π -periodic oscillating test function in τ and define:

$$\Psi^\epsilon \equiv \Psi\left(t, \frac{t}{\epsilon}, x, v\right)$$

Then we can write the weak formulation of the Vlasov equation against Ψ^ϵ and pass to the two-scale limit. We find that the two-scale limit of $f_\epsilon(t, x, v)$, denoted by $F(t, \tau, x, v)$, satisfies the following equation:

$$\partial_\tau F + v_\perp \cdot \nabla_x F + v \wedge e_z \cdot \nabla_v F_\alpha = 0, \quad (2.5)$$

As a consequence, F is constant along the characteristics so that there exists a profile G with:

$$F(t, \tau, x, v) = G(t, x + \mathcal{R}(\tau)v, R(\tau)v) \quad (2.6)$$

where R and \mathcal{R} are defined in the proposition.

The next step is to determine the profile G . We introduce the filtered function g_ϵ :

$$g_\epsilon(t, x, v) = f_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v, R(-t/\epsilon)v) \quad (2.7)$$

which represents the number density from which we have removed the essential oscillations. Notice that this function is chosen so that g_ϵ two-scale converges, as well as weakly-* converges to G .

We easily compute the equation satisfied by g_ϵ :

$$\begin{aligned} & \partial_t g_\epsilon + \mathcal{R}(t/\epsilon)E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v) \cdot \nabla_x g_\epsilon \\ & \quad + R(t/\epsilon)E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v) \cdot \nabla_v g_\epsilon \\ & \quad - \mathcal{R}(t/\epsilon) \left((x + \mathcal{R}(-t/\epsilon)v)_1 \times (R(-t/\epsilon)v)^\perp \right) \cdot \nabla_x g_\epsilon \\ & \quad - R(t/\epsilon) \left((x + \mathcal{R}(-t/\epsilon)v)_1 \times (R(-t/\epsilon)v)^\perp \right) \cdot \nabla_v g_\epsilon = 0 \end{aligned} \quad (2.8)$$

We now pass to the limit in the sense of distributions. One can prove (see [5] in the $2D$ case) that the following convergence holds for the nonlinear terms:

$$\mathcal{R}(t/\epsilon)E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v) \cdot \nabla_x g_\epsilon \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau \cdot \nabla_x G \quad (2.9)$$

$$R(t/\epsilon)E_\epsilon(t, x + \mathcal{R}(-t/\epsilon)v) \cdot \nabla_v g_\epsilon \rightharpoonup \frac{1}{2\pi} \int_0^{2\pi} R(\tau)\mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau \cdot \nabla_v G \quad (2.10)$$

Likewise, we have the following convergences for the last two terms (here there is basically nothing to justify since these are linear quantities):

$$\begin{aligned} & -\mathcal{R}(t/\epsilon) \left((x + \mathcal{R}(-t/\epsilon)v)_1 \times (R(-t/\epsilon)v)^\perp \right) \cdot \nabla_x g_\epsilon \\ \rightarrow & -\frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(\tau) \left((x + \mathcal{R}(-\tau)v)_1 \times (R(-\tau)v)^\perp \right) d\tau \cdot \nabla_x G \end{aligned} \quad (2.11)$$

$$\begin{aligned} & -R(t/\epsilon) \left((x + \mathcal{R}(-t/\epsilon)v)_1 \times (R(-t/\epsilon)v)^\perp \right) \cdot \nabla_v g_\epsilon \\ \rightarrow & -\frac{1}{2\pi} \int_0^{2\pi} R(\tau) \left((x + \mathcal{R}(-\tau)v)_1 \times (R(-\tau)v)^\perp \right) d\tau \cdot \nabla_v G \end{aligned} \quad (2.12)$$

We then easily compute the following quantities:

$$-\frac{1}{2\pi} \int_0^{2\pi} R(\tau) ((x + \mathcal{R}(-\tau)v)_1 \times (R(-\tau)v)^\perp) d\tau = \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} \quad (2.13)$$

$$-\frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(\tau) ((x + \mathcal{R}(-\tau)v)_1 \times (R(-\tau)v)^\perp) d\tau = \begin{pmatrix} -v_1(-x_1 + v_2) \\ v_2x_1 - \frac{1}{2}(v_1^2 + 3v_2^2) \end{pmatrix} \quad (2.14)$$

We consequently derive the following gyrokinetic system:

$$\begin{cases} \partial_t G + \left(\frac{1}{2\pi} \int_0^{2\pi} \mathcal{R}(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau + \begin{pmatrix} -v_1(v_2 - x_1) \\ v_2x_1 - \frac{1}{2}(v_1^2 + 3v_2^2) \end{pmatrix} \right) \cdot \nabla_x G \\ + \left(\frac{1}{2\pi} \int_0^{2\pi} R(\tau) \mathcal{E}(t, \tau, x + \mathcal{R}(-\tau)v) d\tau + \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} \right) \cdot \nabla_v G = 0 \\ G|_{t=0} = f_0 \\ E = -\nabla_x V \\ -\Delta V = \int G(t, x + \mathcal{R}(\tau)v, R(\tau)v) dv - 1 \end{cases} \quad (2.15)$$

□

Qualitative interpretation of the gyrokinetic system

Let us imagine that there is no electric field in the asymptotic equation. Then, the characteristics are given by the following curves (x, v) :

$$\begin{cases} \frac{dx}{dt} = \begin{pmatrix} -v_1(v_2 - x_1) \\ v_2x_1 - \frac{1}{2}(v_1^2 + 3v_2^2) \end{pmatrix} \\ \frac{dv}{dt} = \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} \end{cases} \quad (2.16)$$

At first sight, this dynamical system seems a bit complicated with unpleasant quadratic terms. Actually, it is an easy game to explicitly compute the solutions!

First, notice that $\frac{d}{dt}(x_1 - v_2) = 0$. This means that $x_1 = v_2 + C_1$ (with $C_1 = x_1(0) - v_2(0)$). The equation for v can now be written in the simple form:

$$\frac{dv}{dt} = \begin{pmatrix} -C_1v_2 \\ C_1v_1 \end{pmatrix} \quad (2.17)$$

The velocity is thus periodic (and we could compute it easily). Notice also that $\frac{d}{dt}(v_1^2 + v_2^2) = 0$, so that $v_1^2 + v_2^2 = C_2$ (with $C_2 = v_1^2(0) + v_2^2(0)$).

We get as well a periodic motion for x_1 (since $x_1 = v_2 + C_1$). Finally we compute for x_2 :

$$\frac{dx_2}{dt} = v_2x_1 - \frac{1}{2}(v_1^2 + 3v_2^2) = C_1v_2 - \frac{1}{2}C_2 \quad (2.18)$$

The motion along the e_2 direction is hence a sum of a periodic motion plus a fall which only depends on the initial velocity of the particles (and not on their position). Such a drift of the particles “in the bottom” of the tokamak depending on the square of their velocity is predicted by physicists and is often referred to as the $\nabla \mathcal{B}$ drift.

To support our discussion we give some graphs of the characteristic curves (figures 2 and 3).

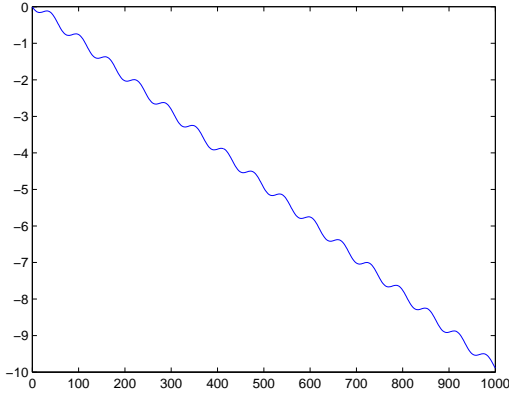


Figure 2: The graph of x_2 (in the x-axis: time and in the y-axis: x_2)

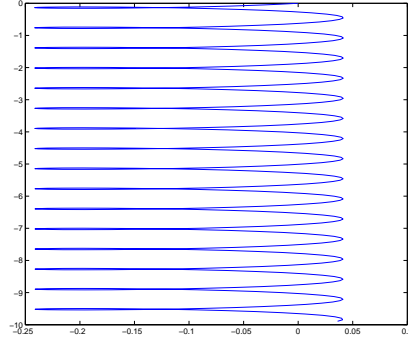


Figure 3: Motion of a particle starting at $(0,0)$ in the slice of the tokamak (in the x-axis: x_1 and in the y-axis: x_2)

3 The mathematical model

System (2.4) seems to be too complicated in order to investigate for bad or good confinement. Consequently we will make several approximations (some of them being quite rough) on (2.4) in order to get a simplified model.

A1 Reduction to a drift-fluid equation We first want to get rid of the fast-time variable and in the same time to get a system whose dynamics is very closed to the one of (2.4).

Following the previous qualitative study of characteristics, we make the same change of variables as Frénod and Sonnendrücker in [5], that is to say: we introduce $w = -v^\perp$ the Larmor radius variable, $x_c = x + w$ the so-called guiding center variable.

We observe that if a density f satisfies the transport equation:

$$\partial_t f + \begin{pmatrix} -v_1(v_2 - x_1) \\ v_2 x_1 - \frac{1}{2}(v_1^2 + 3v_2^2) \end{pmatrix} \cdot \nabla_x f + \begin{pmatrix} v_2(-x_1 + v_2) \\ -v_1(-x_1 + v_2) \end{pmatrix} \cdot \nabla_v f = 0$$

then the modified density $\tilde{f}(t, x_c, w) = f(t, x, v)$ satisfies:

$$\partial_t \tilde{f} - 1/2 w^2 \partial_{x_c} \tilde{f} + \nabla_w \left(\begin{pmatrix} w_2 x_{c1} \\ -w_1 x_{c1} \end{pmatrix} \tilde{f} \right) = 0$$

In order to get a simplified fluid model, we furthermore assume that the density is of the form:

$$\tilde{f}(t, x_c, w) = n(t, x_c) \nu(w) \tag{3.1}$$

where ν is a rotation-invariant positive measure with total mass 1, n is a positive density with total mass 1. For instance, we could assume that this function is a Maxwellian, i.e.

$$\tilde{f}(t, x_c, w) = \frac{n(t, x_c)}{2\pi\sigma} e^{-\frac{w^2}{2\sigma^2}}$$

We simply notice that if we integrate with respect to w , the last term of the previous transport equation disappears.

Likewise, Frénod and Sonnendrücker have shown in [5] that in this new setting, the terms in (2.4) involving the electric field were equivalent to the drift $E^\perp \cdot \nabla_{x_c} := \begin{pmatrix} E_2 \\ -E_1 \end{pmatrix} \cdot \nabla_{x_c}$, if we neglect the gyroaverage operator (that is the average of E around a Larmor circle):

$$\frac{1}{2\pi} \int_0^{2\pi} E^\perp(t, x_c + R(\tau)w) d\tau \rightarrow E^\perp(t, x_c)$$

Then if we integrate with respect to w , we get the macroscopic system (which is nothing but the incompressible Euler equation in its 2D vorticity formulation with an additional drift term) :

$$\begin{cases} \partial_t n - \frac{1}{2} \underbrace{\int w^2 d\nu(w)}_{:=v^2} \partial_{x_{c2}} n + E^\perp \cdot \nabla_{x_c} n = 0 \\ E = -\nabla_{x_c} V \\ -\Delta_{x_c} V = n - 1 \end{cases} \quad (3.2)$$

with $x \in [0, L] \times \mathbb{R}/L\mathbb{Z}$.

Here, the constant $L > 0$ stands for the size of the box. The periodicity with respect to x_2 is physically justified if we consider that L is small enough with respect to the size of the tokamak, so that we can decompose it in many identical cells of size L (see Figure 1).

Remark 3.1. The kinetic version of this fluid model is the following drift-kinetic equation, which is a physical model used for numerical simulations (see for instance the GYSELA code [6]):

$$\begin{cases} \partial_t f - \frac{1}{2} w^2 \partial_{x_{c2}} f + E^\perp \cdot \nabla_{x_c} f = 0 \\ E = -\nabla_{x_c} V \\ -\Delta_{x_c} V = \int f dv - 1 \end{cases} \quad (3.3)$$

A2 Boundary conditions We now have to impose relevant boundary conditions on $x_1 = 0, L$:

- For the Poisson equation, we decide to choose a Dirichlet boundary condition on $x_1 = 0, L$, which entails, thanks to the simple geometry of the boundaries, $E_2 = -\partial_{x_2} V = 0$ on $x_1 = 0, L$. One has to be aware that this choice is made due to mathematical technical reasons (which will be clear in the following) and is not really physically relevant.
- For the transport equation, we actually do not need any boundary condition. There is indeed no entering or leaving trajectories, since the “free transport” operator only entails a motion along the e_2 direction, and $E_2 = 0$ on the boundaries $x_1 = 0, L$.

A3 Modeling of the plasma equilibria and the “bi-temperature” approximation

We now consider the following equilibria, in order to model what is happening in the “good curvature” or the “bad curvature” side of the plasma, the only difference being the relative position between the hot and the cold plasma:

- in the “bad curvature” region:

$$\mu^{bad}(x_1, v) = \underbrace{\frac{x_1}{L} \mathbb{1}_{w=0}}_{\text{cold plasma}} + \underbrace{\left(1 - \frac{x_1}{L}\right) \mathbb{1}_{w^2=v_+^2}}_{\text{hot plasma}} \quad (3.4)$$

- in the “good curvature” region:

$$\mu^{good}(x_1, v) = \left(1 - \frac{x_1}{L}\right) \mathbb{1}_{w=0} + \frac{x_1}{L} \mathbb{1}_{w^2=v_+^2} \quad (3.5)$$

with $v_+^2 > 0$ a fixed temperature for the hot plasma.

Observe here that we make several approximations:

1. We have approximated the curved edge of the tokamak by a straight line.
2. We have opted for a barycentric transition between the hot and the cold plasma. Other transitions may be more physically relevant; this one is actually chosen for the sake of simplicity.
3. We have chosen a “bi-temperature” modelling, saying that the cold plasma is still ($w = 0$) while the hot plasma is fast ($w^2 = v_+^2$). For the more general case $v_- \neq 0$, see Remark 4.2.

In the distributional sense, these are stationary solutions of (3.3) (and the associated electric field is zero).

The idea is now to also look for “bi-temperature” solutions of (3.3), i.e. under the form:

$$f(t, x) = f^+(t, x) \mathbb{1}_{w^2=v_+^2} + f^-(t, x) \mathbb{1}_{w=0}$$

Gathering the pieces together, we finally consider the following nonlinear drift-fluid equation instead of (3.3):

$$\begin{cases} \partial_t f^+ - \frac{1}{2} v_+^2 \partial_{x_2} f^+ + E^\perp \cdot \nabla_x f^+ = 0 \\ \partial_t f^- + E^\perp \cdot \nabla_x f^- = 0 \\ E = -\nabla_x V \\ -\Delta_x V = f^+ + f^- - 1 \\ V = 0 \text{ on } x_1 = 0, L \\ (f^+, f^-)_{|t=0} = (f_0^+, f_0^-) \text{ with } \int f_0^+ + f_0^- = 1 \end{cases} \quad (3.6)$$

for $t \geq 0, x \in [0, L] \times \mathbb{R}/L\mathbb{Z}$.

Remark 3.2. The kinetic version of this drift-fluid model is the following drift-kinetic equation:

$$\begin{cases} \partial_t f^+ - \frac{1}{2} v^2 \partial_{x_2} f^+ + E^\perp \cdot \nabla_x f^+ = 0 \\ \partial_t f^- + E^\perp \cdot \nabla_x f^- = 0 \\ E = -\nabla_x V \\ -\Delta_x V = \int f^+ dv + \int f^- dv - 1 \end{cases} \quad (3.7)$$

It might be interesting to model the equilibria and look at the stability problem in this case.

From now on, we denote $f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix}$, so that the transport part also states:

$$\partial_t f + \begin{pmatrix} -\frac{1}{2} v^2 \\ 0 \end{pmatrix} \cdot \partial_{x_2} f + E^\perp \cdot \nabla_x f = 0 \quad (3.8)$$

with the notation: for $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $E \cdot \nabla_x g = \begin{pmatrix} E \cdot \nabla_x g_1 \\ E \cdot \nabla_x g_2 \end{pmatrix}$.

Then $\mu^{bad} = \begin{pmatrix} 1 - \frac{x_1}{L} \\ \frac{x_1}{L} \end{pmatrix}$ (and μ^{good}) are stationary solutions of (3.6) .

Following the classical works for the 2D incompressible Euler system (see [3] for a review on this topic), we get the following existence result:

Theorem 3.1. *Let $f_0 = (f_0^1, f_0^2) \in (L^1((0, L) \times \mathbb{R}/L\mathbb{Z}))^2$ with f_0^1, f_0^2 non-negative and $\int (f_0^1 + f_0^2) dx = 1$.*

If $f_0 \in (L^\infty)^2$, then there exists a non-negative weak solution $f \in (L^1 \cap L^\infty)^2$ to (3.6) with initial data f_0 .

If f_0 is smooth then the solution f is smooth and unique.

Let us now define precisely the stability and instability notions that we will work on until the end of the paper.

Definition. *Let g be a solution of (3.6).*

This solution is said to be stable with respect to the X norm if for any $\epsilon > 0$, there exists $\delta > 0$ such that: for any solution f of the system, the initial estimate $\|f(0) - g(0)\|_X \leq \delta$ implies that for any $t \geq 0$, $\|f(t) - g(t)\|_X \leq \epsilon$.

Otherwise, the solution g is said to be unstable with respect to the X norm.

We now investigate stability and instability for μ^{bad} and μ^{good} .

4 Linear instability in the “bad curvature” region

We first consider the case of the “bad curvature” region, for which we expect to obtain instability. The equilibrium writes:

$$\mu^{bad}(x_1) = \begin{pmatrix} 1 - \frac{x_1}{L} \\ \frac{x_1}{L} \end{pmatrix}$$

The first step before trying to prove any instability property for the nonlinear transport equation consists in investigating the problem of instability for the linearized operator around μ^{bad} . We accordingly consider the following linearized system (for $t > 0$, $x \in [0, L] \times \mathbb{R}/L\mathbb{Z}$):

$$\begin{cases} \partial_t f^+ - \frac{1}{2} v_+^2 \partial_{x_2} f^+ - \frac{E_2}{L} = 0 \\ \partial_t f^- + \frac{E_2}{L} = 0 \\ E = -\nabla_x V \\ -\Delta_x V = f^+ + f^-, V = 0 \text{ on } x_1 = 0, L \\ (f^+, f^-)|_{t=0} = (f_0^+, f_0^-) \text{ with } \int f_0^+ + f_0^- = 0 \end{cases} \quad (4.1)$$

For the sake of readability we will from now on write v^2 instead of v_+^2 (there is no risk of misunderstanding since it is the only velocity we deal with).

In the following, we will also give some remarks about what is happening for the linearized operator in “good curvature” region, in which case the system states:

$$\begin{cases} \partial_t f^+ - \frac{1}{2} v^2 \partial_{x_2} f^+ + \frac{E_2}{L} = 0 \\ \partial_t f^- - \frac{E_2}{L} = 0 \\ E = -\nabla_x V \\ -\Delta_x V = f^+ + f^-, V = 0 \text{ on } x_1 = 0, L \end{cases} \quad (4.2)$$

4.1 Looking for unstable eigenfunctions

We look for special solutions under the form $f_k(t, x) = \begin{pmatrix} \rho_k^+(t)g_k(x) \\ \rho_k^-(t)g_k(x) \end{pmatrix}$. The equations (4.1) can be restated as:

$$\begin{cases} \partial_t(\rho_k^+(t)g_k(x)) - \frac{1}{2}v^2\partial_{x_2}(\rho_k^+(t)g_k(x)) - \frac{E_2}{L} = 0 \\ \partial_t(\rho_k^-(t)g_k(x)) + \frac{E_2}{L} = 0 \\ E = -\nabla_x V \\ -\Delta_x V = (\rho_k^-(t) + \rho_k^+(t))g_k(x), \quad V = 0 \text{ on } x_1 = 0, L \end{cases} \quad (4.3)$$

We take g_k with the particular form $g_k(x) = \sin\left(\frac{k_1}{L}\pi x_1\right)e^{i2\pi\frac{k_2}{L}x_2}$ (with $k_1, k_2 \in \mathbb{Z}^*$), so that g_k is an eigenfunction for the laplacian with the considered boundary conditions. It satisfies indeed:

$$\Delta g_k(x) = -\pi^2 \left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2} \right) g_k(x) \quad (4.4)$$

and also $g_k = 0$ on $x_1 = 0, L$.

The solution of the Poisson equation is then given by:

$$V_k = \frac{1}{\pi^2 \left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2} \right)} (\rho_k^-(t) + \rho_k^+(t))g_k(x)$$

hence $E_2 = \frac{-i2k_2/L}{\pi \left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2} \right)} (\rho_k^-(t) + \rho_k^+(t))g_k(x)$, which lead us to study the following first order ordinary differential equation:

$$\partial_t \begin{pmatrix} \rho_k^+(t) \\ \rho_k^-(t) \end{pmatrix} + \frac{1}{L} \begin{pmatrix} -i\pi v^2 k_2 + \frac{i2k_2}{\pi \left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2} \right)} & \frac{i2k_2}{\pi \left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2} \right)} \\ -\frac{i2k_2}{\pi \left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2} \right)} & -\frac{i2k_2}{\pi \left(\frac{k_1^2}{L^2} + 4\frac{k_2^2}{L^2} \right)} \end{pmatrix} \begin{pmatrix} \rho_k^+(t) \\ \rho_k^-(t) \end{pmatrix} = 0 \quad (4.5)$$

We want to compute the eigenvalues of the matrix; its characteristic polynomial states:

$$X^2 + i\pi k_2 v^2 X - \frac{2k_2^2 v^2}{\frac{k_1^2}{L} + 4\frac{k_2^2}{L}}$$

and its discriminant:

$$\Delta = -\pi^2 k_2^2 v^4 + \frac{8k_2^2 v^2}{\frac{k_1^2}{L} + 4\frac{k_2^2}{L}} = \pi^2 k_2^2 v^4 \left(\frac{8}{\pi^2 \left(\frac{k_1^2}{L} + 4\frac{k_2^2}{L} \right) v^2} - 1 \right) \quad (4.6)$$

We can now distinguish between two cases:

- First case: $\frac{8}{\pi^2 \left(\frac{k_1^2}{L} + 4\frac{k_2^2}{L} \right)} > v^2$. In other words the box is big enough with respect to the drift velocity; then there exist k_1, k_2 such that $\Delta > 0$. We consequently obtain two complex roots, one of which has a strictly negative real part equal to $-\frac{\sqrt{\Delta}}{2}$. In other words, this shows the existence of an unstable mode.
- Second case: $\frac{8}{\pi^2 \left(\frac{k_1^2}{L} + 4\frac{k_2^2}{L} \right)} \leq v^2$. In the case where the box is sufficiently small, we always have $\Delta \leq 0$ and consequently, there is linear stability! Roughly speaking, there is a kind of homogenization process that appears: the large velocity drift together with the periodicity makes that everything is averaged.

This phenomenon may at first sight look like a mathematical artifact due to the periodicity constraint in the x_2 direction. Nevertheless the existence of such a threshold is well known in plasma physics, beyond which one can expect tremendous confinement properties (see for instance [12], [13] and references therein) and there seems to be both experimental and numerical evidence of its relevance. In very rough terms: the hotter, the more stable! The stable mode is referred to as the H-mode (where H stands for high, by opposition to the L-mode, where L stands for low).

Remark 4.1. In the “good curvature” region, with the same method, we obtain the following ordinary differential equation:

$$\partial_t \begin{pmatrix} \rho_k^+(t) \\ \rho_k^-(t) \end{pmatrix} + 1/L \begin{pmatrix} -i\pi v^2 k_2 - \frac{i2k_2}{\pi \left(\frac{k_1^2}{L} + 4 \frac{k_2^2}{L} \right)} & -\frac{i2k_2}{\pi \left(\frac{k_1^2}{L} + 4 \frac{k_2^2}{L} \right)} \\ \frac{i2k_2/L}{\pi \left(\frac{k_1^2}{L} + 4 \frac{k_2^2}{L} \right)} & \frac{i2k_2}{\pi \left(\frac{k_1^2}{L} + 4 \frac{k_2^2}{L} \right)} \end{pmatrix} \begin{pmatrix} \rho_k^+(t) \\ \rho_k^-(t) \end{pmatrix} = 0 \quad (4.7)$$

We consequently have to look for the roots of the polynomial:

$$X^2 + i\pi k_2 v^2 X + \frac{2k_2^2 v^2}{\frac{k_1^2}{L} + 4 \frac{k_2^2}{L}}$$

In this case, the discriminant is always strictly negative, so that the roots always have a vanishing real part. As a result, we do not find unstable mode by this method. We only consider this fact as a good and encouraging indication for stability around μ^{good} . Actually, we will never use it when we will prove nonlinear stability in section 6.

Remark 4.2. If we consider the case $v_-^2 \neq 0$ (with $v_- < v_+$), i.e. the system:

$$\begin{cases} \partial_t f^+ - \frac{1}{2} v_+^2 \partial_{x_2} f^+ - \frac{E_2}{L} = 0 \\ \partial_t f^- - \frac{1}{2} v_-^2 \partial_{x_2} f^- + \frac{E_2}{L} = 0 \\ E = -\nabla_x V \\ -\Delta_x V = f^+ + f^-, V = 0 \text{ on } x_1 = 0, L \end{cases} \quad (4.8)$$

we can make the same calculations so that we get the following discriminant to study:

$$\Delta = \pi^2 k_2^2 (v_-^2 - v_+^2)^2 \left(\frac{8}{\pi^2 \left(\frac{k_1^2}{L} + 4 \frac{k_2^2}{L} \right) (v_+^2 - v_-^2)} - 1 \right) \quad (4.9)$$

This shows that the size of the box has to be rather compared to the difference of the two velocities to the power 2 (i.e. more or less the difference of the temperatures). In more physical terms, if the temperature gradient is large enough then one can expect stability.

Remark 4.3. We finally mention that the quest for unstable modes seems more difficult in the kinetic case, since one has to deal with the continuous velocity space. A very famous criterion in the Vlasov-Poisson case was given by Penrose [11] and rigorously studied later on by Guo and Strauss [8].

4.2 Wellposedness of the linearized operator around μ^{bad} on $L^p([0, 1] \times \mathbb{R}/\mathbb{Z})$, $1 < p < \infty$ for big enough boxes.

We assume here that the box is big enough to have unstable modes, or equivalently that v^2 is small, so for simplicity we take $L = 1$.

The main tool we use in this paragraph is a variant of a theorem by Weyl, stated for instance in the paper of Guo and Strauss [8]. Basically, it gives informations on the wellposedness of linear compact perturbations of some linear operator for which we already know wellposedness and it entails the existence of a dominant unstable eigenvalue provided that we know the existence of one.

Theorem 4.1 (Weyl). *Let Y be a Banach space and A be a linear operator that generates a strongly continuous semigroup on Y such that $\|e^{-tA}\| \leq M$ for all $t \geq 0$. Let K be a compact operator from Y to Y . Then $(A + K)$ generates a strongly continuous semigroup $e^{-t(A+K)}$ and $\sigma(-A - K)$ consists of a finite number of eigenvalues of finite multiplicity in $\{\operatorname{Re} \lambda > \delta\}$ for every $\delta > 0$. These eigenvalues can be labeled by:*

$$\operatorname{Re} \lambda_1 \geq \operatorname{Re} \lambda_2 \geq \dots \operatorname{Re} \lambda_N \geq \delta$$

Furthermore, for any $\gamma > \operatorname{Re} \lambda_1$, there exists some constant C_γ such that

$$\|e^{-t(A+K)}\|_{Y \rightarrow Y} \leq C_\gamma e^{t\gamma} \quad (4.10)$$

Corollary 4.1. *Let $Y = \{y_1, y_2 \in L^p([0, 1] \times \mathbb{R}/\mathbb{Z})^2, \int (y_1 + y_2) dx = 0\}$. Let L be the linear operator defined by:*

$$g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \in Y \mapsto L \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} -1/2v^2 \partial_{x_2} g^+ - E_2 \\ E_2 \end{pmatrix} \quad (4.11)$$

with $E_2 = -\partial_{x_2} V$, $-\Delta V = (g^+ + g^-)$ and Dirichlet boundary conditions on $x_1 = 0, 1$. Then there exists an eigenvalue λ with non-vanishing and maximal real part associated to an eigenvector in L^p .

Furthermore for any $\gamma > \operatorname{Re} \lambda$, there is a constant C_γ such that

$$\|e^{-tL}\|_{L^p \rightarrow L^p} \leq C_\gamma e^{t\gamma} \quad (4.12)$$

Proof. The linear operator A is defined by $f \mapsto -\frac{1}{2}v^2 \partial_{x_2} f$ (for which we know how to explicitly solve the semi-group and which is wellposed on Y). The operator K is defined by $f \mapsto \begin{pmatrix} -E_2 \\ E_2 \end{pmatrix}$. This operator is compact thanks to standard elliptic estimates.

Moreover, we have shown in the last paragraph the existence of an unstable eigenfunction for the linearized operator that belongs to any L^p . We can therefore apply once more Weyl's theorem which gives the existence of an eigenfunction associated to an eigenvalue with a non-vanishing and maximal real part. At last, the estimate in the corollary follows directly from the estimate given in Weyl's theorem. □

Remark 4.4. With the same proof we can show the wellposedness in stronger regularity spaces, like Sobolev spaces $W^{k,p}$. We can also obviously prove that the linearized operator around the equilibrium μ^{good} is wellposed.

In the following, we denote for any $h \in L^2$ with $\int h dx = 0$, $\Delta^{-1}h$ the unique solution in H^1 to the problem:

$$\begin{cases} -\Delta u = h \\ u = 0 \text{ on } x_1 = 0, L \end{cases}$$

We give now a lemma which tells us that any eigenvector associated to a non vanishing eigenvalue for the linearized operator on L^p for $p \geq 2$ is actually smooth.

Lemma 4.1. *Let $f = (f^+, f^-) \in (L^2)^2$ with $\int (f^+ + f^-) dx = 0$ and $\lambda \neq 0$ such that:*

$$\begin{aligned} -\frac{1}{2}v^2\partial_{x_2}f^+ - \partial_{x_2}\Delta^{-1}(f^+ + f^-) &= \lambda f^+ \\ \partial_{x_2}\Delta^{-1}(f^+ + f^-) &= \lambda f^- \end{aligned}$$

then $(f^+, f^-) \in \mathcal{C}^\infty([0, 1] \times \mathbb{R}/\mathbb{Z})$

Proof. The principle of the proof is to show by recursion that $f \in H^k$, for any $k \in \mathbb{N}^*$.

For $k = 1$, we already observe, thanks to elliptic estimates, that $f^- \in H^1$. We also have the identity:

$$-\frac{1}{2}v^2\partial_{x_2}f^+ = \lambda(f^+ + f^-) \quad (4.13)$$

Hence, $\partial_{x_2}f^+ \in L^2$.

We can apply the differential operator ∂_{x_1} to the equation satisfied by f^+ , which entails:

$$-\frac{1}{2}v^2\partial_{x_1}\partial_{x_2}f^+ - \partial_{x_1}\partial_{x_2}\Delta^{-1}(f^+ + f^-) = \lambda\partial_{x_1}f^+$$

Then multiply by $\partial_{x_1}f^+$ and integrate with respect to x :

$$\lambda\|\partial_{x_1}f^+\|_{L^2}^2 = \int -\frac{1}{2}v^2\partial_{x_1}\partial_{x_2}f^+\partial_{x_1}f^+ dx - \int \partial_{x_1}\partial_{x_2}\Delta^{-1}(f^+ + f^-)\partial_{x_1}f^+ dx$$

Thanks to the periodicity with respect to x_2 , we get the identity:

$$\int \partial_{x_1}\partial_{x_2}f^+\partial_{x_1}f^+ dx = 1/2 \int \partial_{x_2}(\partial_{x_1}f^+)^2 dx = 0$$

Then using Cauchy-Schwarz inequality:

$$\lambda\|\partial_{x_1}f^+\|_{L^2}^2 \leq \|\partial_{x_1}\partial_{x_2}\Delta^{-1}(f^+ + f^-)\|_{L^2}\|\partial_{x_1}f^+\|_{L^2}$$

As a result we showed that:

$$\lambda^2\|\partial_{x_1}f^+\|_{L^2} \leq \|\partial_{x_1}\partial_{x_2}\Delta^{-1}(f^+ + f^-)\|_{L^2} \quad (4.14)$$

By standard elliptic estimates the right-hand side is finite since f^+ et f^- belong to L^2 . We can then go on by recursion. \square

5 Nonlinear instability in the “bad curvature” region for big enough boxes

What we intend to show now is a property of nonlinear instability in the “bad curvature” region for big enough boxes (or equivalently for small v), so for readability we take $L = 1$. This can be interpreted as a bad confinement property. We first recall that the equilibrium in this case is the following:

$$\mu^{bad}(x_1) = \begin{pmatrix} 1 - x_1 \\ x_1 \end{pmatrix}$$

Thanks to the existence of an eigenvalue with maximal positive real part for the linearized operator around μ^{bad} , we can prove nonlinear instability theorems.

5.1 A first result

The first result we have in mind is very reminiscent of the paper of Guo and Strauss [8]: the principle is to show that for a well chosen initial data, the linear part entails an explosive behaviour that can “control” in some sense the nonlinear dynamics. With this kind of proof, we can only get instability with strong norms (here the Sobolev norm $W^{1,q}$) since with a weaker estimate, we would not be able to control the nonlinearity.

Theorem 5.1. *For any $q > 4$, μ^{bad} is nonlinearly unstable for the norm $\|\cdot\|_{W^{1,q}}$:*

There exist (f^n) a sequence of solutions to the system with inial data f_0^n and times $t_n \geq 0$ such that:

$$\|f_0^n - \mu^{bad}\|_{W^{1,q}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

but $\|f^n(t_n) - \mu^{bad}\|_{W^{1,q}}$ does not tend to 0.

More precisely there exists $\epsilon_1 > 0$ such that for any $\delta < 1$, there exists f_0 satisfying

$$\|f_0 - \mu^{bad}\|_{W^{1,q}} < \delta$$

and $\|f(T_\delta) - \mu^{bad}\|_{W^{1,q}} \geq \epsilon_1$ for $T_\delta = O(|\log \delta|)$.

Proof. First of all, we fix $q > 2$. Let $p > 2$ such that $q > \frac{p^2}{p-1}$ (and notice that this implies $q > p$). The proof follows by contradiction: suppose that for all $\epsilon > 0$, there exists $\delta > 0$ such that any solution f satisfies:

$$\|f - \mu^{bad}\|_{L_t^\infty(W^{1,q})} \leq \epsilon \quad (5.1)$$

provided that $\|f(0) - \mu^{bad}\|_{W^{1,q}} \leq \delta$. From now on and until the end of the section, we write μ instead of μ^{bad} for the sake of simplicity. Since we work on a bounded domain, observe that (5.1) implies for any $r < q$,

$$\|f - \mu\|_{L_t^\infty(W^{1,r})} \leq \epsilon \quad (5.2)$$

Without loss of generality we can choose $\epsilon < 1$ and $\delta < 1$.

We have obtained in the previous section the existence of an eigenfunction associated to an eigenvalue λ whose real part is positive and maximal for the linearized operator

$$g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \in \{f^+, f^- \in (L^p)^2; \int f^+ + f^- = 0\} \mapsto L \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} -1/2v^2 \partial_{x_1} g^+ - E_2 \\ E_2 \end{pmatrix}$$

with $E_2 = \partial_{x_2} \Delta^{-1}(g^+ + g^-)$ and with Dirichlet boundary conditions for the potential on $x_1 = 0, 1$.

We have further established in lemma 4.1 that such an eigenvector is smooth: it belongs to $\mathcal{C}^\infty((0, 1) \times \mathbb{R}/\mathbb{Z})$. Let R be such an eigenvector, such that its $W^{1,q}$ norm is equal to 1. We then set:

$$f(0) - \mu = \delta R \quad (5.3)$$

For the sake of simplicity we will assume that the eigenvalue with maximal real part is real. In the general case, since the linearized operator is real, the conjugate of this eigenvalue is also an eigenvalue so that one can then consider by linearity the real part of R instead of R and the following of the proof remains the same.

Let us first recall that μ does not generate any electric field. The difference $(f - \mu)$ then satisfies the equation:

$$\partial_t(f - \mu) + \begin{pmatrix} -1/2v^2 \partial_{x_2}(f^+ - \mu^+) \\ 0 \end{pmatrix} + E^\perp \cdot \nabla_x \mu = -E^\perp \cdot \nabla_x(f - \mu) \quad (5.4)$$

with

$$\begin{aligned} -\Delta V &= f^+ + f^- - \mu^+ - \mu^- = f^+ + f^- - 1 \\ V &= 0 \text{ on } x_1 = 0, 1 \\ E &= -\nabla V \end{aligned}$$

We consequently have the following Duhamel's formula:

$$f(t) - \mu = e^{-tL}(\delta R) - \int_0^t e^{-L(t-s)}(\nabla_x^\perp V \cdot \nabla_x(f(s) - \mu))ds \quad (5.5)$$

$$= \delta R e^{\lambda t} - \int_0^t e^{-L(t-s)}(\nabla_x^\perp V \cdot \nabla_x(f(s) - \mu))ds \quad (5.6)$$

Let:

$$T = \sup \left\{ t', \|f(t) - \mu - \delta e^{-tL} R\|_{L^p} \leq \frac{1}{2} \delta e^{\operatorname{Re}(\lambda)t} \|R\|_{L^p} \text{ for } 0 \leq t \leq t' \right\} \quad (5.7)$$

We get $T > 0$ by continuity (the left-hand side is equal to 0 for $t = 0$). For any $t \leq T$ we thus have:

$$\|f(t) - \mu\|_{L^p} \leq \frac{3\delta}{2} e^{t \operatorname{Re}(\lambda)} \|R\|_{L^p} \quad (5.8)$$

We now prove a lemma which will help us to control the nonlinear part.

Lemma 5.1. *With the same notations as before, we have the estimate for $t \leq T$:*

$$\|\nabla_x(f - \mu)\|_{L^p} \leq C \delta^{1/p} e^{t \frac{\operatorname{Re} \lambda}{p}} \|R\|_{L^p}^{1/p} \quad (5.9)$$

Remark 5.1. The natural (and painless!) estimate $\|f - \mu\|_{L_t^\infty(W^{1,q})} \leq \epsilon$ is not enough since one is not able to estimate δ with respect to ϵ : if $\delta \ll \epsilon$, the estimate we prove is more precise for small times.

Proof. The quantity $(f - \mu)$ satisfies the equation:

$$\partial_t(f - \mu) + \begin{pmatrix} -1/2v^2 \partial_{x_2}(f^+ - \mu^+) \\ 0 \end{pmatrix} + E^\perp \cdot \nabla_x(f - \mu) = -E^\perp \cdot \nabla_x \mu \quad (5.10)$$

If we derive once with respect to x_1 , we obtain for the first coordinate $g^+ = f^+ - \mu^+$:

$$\partial_t \partial_{x_1} g^+ - 1/2v^2 \partial_{x_2} \partial_{x_1} g^+ + E^\perp \cdot \nabla_x \partial_{x_1} g^+ = -\partial_{x_1} E^\perp \cdot \nabla_x g^+ + \partial_{x_1} E_2 \quad (5.11)$$

We then multiply by $p|\partial_{x_1} g^+|^{p-1} \operatorname{sign}(\partial_{x_1} g^+)$ and integrate with respect to the space variable. We consequently get, using periodicity with respect to x_2 :

$$\begin{aligned} \frac{d}{dt} \|\partial_{x_1} g^+\|_{L^p}^p &= -p \int \partial_{x_1} E^\perp \cdot \nabla_x g^+ |\partial_{x_1} g^+|^{p-1} \operatorname{sign}(\partial_{x_1} g^+) dx \\ &+ p \int \partial_{x_1} E_2 |\partial_{x_1} g^+|^{p-1} \operatorname{sign}(\partial_{x_1} g^+) dx \\ &\leq Cp \left(\underbrace{\|\partial_{x_1} E^\perp\|_{L^p} \|\nabla_x g^+\|_{L^{\frac{p^2}{p-1}}}^p}_{\leq \|\nabla_x g^+\|_{L^q}^p} + \|\partial_{x_1} E_2\|_{L^p} \underbrace{\|\nabla_x g^+\|_{L^p}^{p-1}}_{\leq \|\nabla_x g^+\|_{L^q}^{p-1}} \right) \\ &\leq Cp \left(\|f - \mu\|_{L^p} \|\nabla_x g^+\|_{L^q}^p + \|f - \mu\|_{L^p} \|\nabla_x g^+\|_{L^p}^{p-1} \right) \\ &\leq C \frac{3\delta}{2} e^{t \operatorname{Re} \lambda} \|R\|_{L^p} (\epsilon^p + \epsilon^{p-1}) \end{aligned}$$

thanks to Hölder's inequality and to standard elliptic estimates.

Hence:

$$\begin{aligned}\|\partial_{x_1} g^+\|_{L^p}^p &\leq C \frac{3\delta}{2} e^{t \operatorname{Re} \lambda} \|R\|_{L^p} (\epsilon^p + \epsilon^{p-1}) + \|\partial_{x_1} g^+(0)\|_{L^p}^p \\ &\leq C \frac{3\delta}{2} e^{t \operatorname{Re} \lambda} \|R\|_{L^p} (\epsilon^p + \epsilon^{p-1}) + \delta^p\end{aligned}$$

that is to say, since $\epsilon, \delta < 1$:

$$\|\partial_{x_1} g^+\|_{L^p} \leq C \left(\delta e^{t \operatorname{Re} \lambda} \|R\|_{L^p} \right)^{1/p} \quad (5.12)$$

where C depends neither on δ , nor on ϵ . The same arguments give the expected estimates for the other derivative and for the other coordinate. \square

Let γ such that $\operatorname{Re} \lambda < \gamma < \operatorname{Re} \lambda(1 + \frac{1}{p})$. By Weyl's theorem, we get

$$\|e^{-tL}\|_{L^p \rightarrow L^p} \leq C e^{\gamma t}.$$

As a consequence, for any $t \leq T$, using Sobolev's embedding from $W^{1,p}$ into L^∞ :

$$\begin{aligned}\|f(t) - \mu - \delta e^{-tL} R\|_{L^p} &\leq C \int_0^t e^{\gamma(t-s)} \|\nabla_x^\perp V(s)\|_{L^\infty} \|\nabla_x(f(s) - \mu)\|_{L^p} ds \\ &\leq C \int_0^t e^{\gamma(t-s)} \|f(s) - \mu\|_{L^p} \|\nabla_x(f(s) - \mu)\|_{L^p} ds \\ &\leq C \int_0^t e^{\gamma(t-s)} \frac{3\delta}{2} e^{s \operatorname{Re} \lambda} \delta^{1/p} e^{s \frac{\operatorname{Re} \lambda}{p}} \|R\|_{L^p}^{1/p} ds \\ &\leq C \left(\|R\|_{L^p} \delta e^{\operatorname{Re} \lambda t} \right)^{1+1/p}\end{aligned}$$

where C depends neither on δ , nor on ϵ .

Hence, for $t \leq T$:

$$\|f(t) - \mu\|_{L^p} \geq \|R\|_{L^p} \delta e^{t \operatorname{Re} \lambda} - C \left(\|R\|_{L^p} \delta e^{\operatorname{Re} \lambda t} \right)^{1+1/p} \quad (5.13)$$

We intend to make the right-hand side as large as possible, so that we choose t_0 defined by:

$$\|R\|_{L^p} \delta e^{t_0 \operatorname{Re} \lambda} = \left(\frac{1}{2C} \right)^p \quad (5.14)$$

If δ is chosen small enough, we do have $t_0 > 0$.

There remains to check that we have $t_0 \leq T$. If $T = \infty$, there is nothing to do; if T is finite, we have by definition:

$$\|f(T) - \mu - \delta R e^{\operatorname{Re} \lambda T}\|_{L^p} = \frac{1}{2} \delta e^{\operatorname{Re} \lambda T} \|R\|_{L^p} \leq C \left(\|R\|_{L^p} \delta e^{\operatorname{Re} \lambda T} \right)^{1+1/p}$$

this latest inequality can be re-written as $e^{t_0 \operatorname{Re} \lambda} \leq e^{T \operatorname{Re} \lambda}$, whence we can conclude that $t_0 \leq T$.

Eventually, we obtain $\|f(t_0) - \mu\|_{L^p} \geq 1/2 \left(\frac{1}{2C} \right)^p$, which is contradictory with

$$\|f - \mu\|_{L_t^\infty(W^{1,q})} \leq \epsilon \quad (5.15)$$

since we can choose ϵ as small as we want to.

For the estimate on T_δ , one can remark that for arguing by contradiction, one has just to assume that

$$\|f - \mu^{bad}\|_{L_t^\infty(W^{1,q})} \leq \epsilon \quad (5.16)$$

for $t \leq t_0$ and do the same proof. \square

5.2 A refined result

Using the method introduced by Grenier [7], we can also pass from the linear spectral instability to the nonlinear instability. This can be seen as a generalization of the argument used previously. The theorem we can prove is much stronger and much more satisfactory since it is complementary to the stability result we prove afterwards.

Theorem 5.2. *There exists a constant $\epsilon_1 > 0$ such that for any $\delta > 0$ and any $s \geq 0$ there exists a solution f with $\|f(0) - \mu\|_{H^s} \leq \delta$ but such that:*

$$\|f(T_\delta) - \mu\|_{L^2} \geq \epsilon_1 \quad (5.17)$$

with $T_\delta = O(|\log \delta|)$. In particular, μ is unstable with respect to the L^2 norm.

Remark 5.2. Of course, T_δ depends also on s .

Proof. Basically the idea of Grenier is to construct a better approximation of the nonlinear equation than the “usual” solution of the linearized equation. Indeed instead of showing that $f - \mu$ is closed to of a well chosen eigenfunction, we show that it is closed to something like

$$f_{app}^N = \delta u_1 + \sum_{i=2}^N \delta^i u_i \quad (5.18)$$

where $u_1 = Re^{Re \lambda t}$ (R being an eigenvector associated to an eigenvalue λ with maximal real part $Re \lambda$). Note that for any $s > 0$, $\|u_1\|_{H^s} \leq Ce^{Re \lambda t}$.

f_{app} is constructed in order to have the following approximation:

$$\partial_t f_{app} + L f_{app} + E_{app}^\perp \cdot \nabla_x f_{app} = R_{app} \quad (5.19)$$

with $\|R_{app}\|_{H^{L-2N-1}} \leq C\delta^{N+1} \exp((N+1) Re \lambda t)$ and $E_{app} = \nabla \Delta^{-1}(f_{app}^1 + f_{app}^2)$.

We first have to revisit the previous corollary of Weyl’s theorem.

Lemma 5.2. *Let $s \geq 0$ and $Y = \{y_1, y_2 \in H^s([0, 1] \times \mathbb{R}/\mathbb{Z})^2, \int (y_1 + y_2) dx = 0\}$. Let L be the linear operator defined by:*

$$g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \in Y \mapsto L \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} -1/2v^2 \partial_{x_2} g^+ - E_2 \\ E_2 \end{pmatrix} \quad (5.20)$$

with $E_2 = -\partial_{x_2} V$, $-\Delta V = (g^+ + g^-)$ and with Dirichlet boundary conditions on $x_1 = 0, 1$. Then there exists an eigenvalue λ with a non-vanishing and maximal real part associated to a C^∞ eigenvector. Furthermore for any $\gamma > Re \lambda$, there is a constant $C(\gamma, s)$ such that for all $s \geq 0$:

$$\|e^{-tL}\|_{H^s \rightarrow H^s} \leq C(\gamma, s) e^{t\gamma} \quad (5.21)$$

Let $N \in \mathbb{N}^*$ to be chosen later and choose any $M > 0$ such that $M > 2N + 1$. Choose also $\theta < 1$ such that $\frac{1}{2}\theta \geq \frac{\theta^2}{1-\theta}$ and define T_δ such that $\theta = \delta \exp(Re \lambda T_\delta)$.

Now we can construct the u_j by recursion. We will show that for all $j \geq 1$, $\int (u_j^1 + u_j^2) dx = 0$ and $\|u_j\|_{H^{M-j}} \leq C \exp(j Re \lambda t)$.

Suppose we have u_j for $j \leq k$. Then define u_{k+1} as the solution of the linear equation:

$$\partial_t u_{k+1} + L u_{k+1} + \sum_{j=1}^k E_j^\perp \cdot \nabla_x u_{k+1-j} + E_{k+1-j}^\perp \cdot \nabla_x u_j = 0 \quad (5.22)$$

with $E_j = \nabla \Delta^{-1}(u_j^1 + u_j^2)$ and $u_{k+1}(0, x) = 0$ as initial condition. Intuitively, u_{k+1} is chosen in order to "kill" the non-linear interaction between the previous terms of the expansion.

Thanks to lemma 5.2 with $\gamma \in]\operatorname{Re} \lambda, 2 \operatorname{Re} \lambda[$, we get the following estimate:

$$\begin{aligned} \|u_{k+1}\|_{H^{M-(k+1)}} &\leq \int_0^t \|e^{L(t-s)} \left(\sum_{j=1}^k E_j^\perp \cdot \nabla_x u_{k+1-j} + E_{k+1-j}^\perp \cdot \nabla_x u_j \right)\|_{H^{L-(k+1)}} ds \\ &\leq C \int_0^t e^{\gamma(t-s)} \left(\sum_{j=1}^k \|E_j^\perp\|_{H^{M-(k+1)}} \|u_{k+1-j}\|_{H^{M-k}} + \|E_{k+1-j}^\perp\|_{H^{M-(k+1)}} \|u_j\|_{H^{M-k}} \right) ds \\ &\leq C \int_0^t e^{\gamma(t-s)} \exp((k+1) \operatorname{Re} \lambda s) ds \\ &\leq C \exp((k+1) \operatorname{Re} \lambda t) \end{aligned}$$

Note also that we clearly have $\int (u_{k+1}^1 + u_{k+1}^2) dx = 0$ (indeed, $\frac{d}{dt} \int (u_{k+1}^1 + u_{k+1}^2) dx = 0$). Now we can see that:

$$\partial_t f_{app} + L f_{app} + E_{app}^\perp \cdot \nabla_x f_{app} = R_{app} \quad (5.23)$$

with $R_{app} = \sum_{2N \geq j+j' > N} \delta^{j+j'} E_j^\perp \cdot \nabla_x u_{j'}$. Hence the following estimate follows:

$$\|R_{app}\|_{H^{M-2N-1}} \leq C_N \delta^{N+1} \exp((N+1) \operatorname{Re} \lambda t) \quad (5.24)$$

(notice indeed that for all $t \leq T_\delta$, $\delta \exp(\operatorname{Re} \lambda t) \leq \theta < 1$.) C_N means that this constant depends on N .

Now we consider a solution f such that $f(0) - \mu = f_{app}(0)$; the equation satisfied by $w = f - \mu - f_{app}$ is the following:

$$\partial_t w + \begin{pmatrix} -1/2v^2 \partial_{x_2} w \\ 0 \end{pmatrix} + E_w^\perp \cdot \nabla_x w + E_{app}^\perp \cdot \nabla_x w + E_w^\perp \cdot \nabla_x f_{app} = -E_w^\perp \cdot \nabla_x \mu - R_{app} \quad (5.25)$$

with $E_w = \nabla \Delta^{-1}(f - \mu - f_{app})$

Now, we want to estimate $\|w\|_{L^2}$. We multiply by w and integrate with respect to x :

$$\begin{aligned} \frac{d}{dt} \|w\|_{L^2}^2 &\leq \int |E_w^\perp \cdot \nabla_x f_{app} w| dx + \int |E_w^\perp \cdot \nabla_x \mu w| dx + \|R_{app}\|_{L^2} \|w\|_{L^2} \\ &\leq (\|\nabla_x f_{app}\|_{L^\infty} + \|\nabla_x \mu\|_{L^\infty}) \|E_w^\perp\|_{L^2} \|w\|_{L^2} + \frac{1}{2} \|w\|_{L^2}^2 + \frac{1}{2} \|R_{app}\|_{L^2}^2 \\ &\leq C \left((1 + \|\nabla_x f_{app}\|_{L^\infty}) \|w\|_{L^2}^2 + \|R_{app}\|_{L^2}^2 \right) \\ &\leq C \left((1 + \|\nabla_x f_{app}\|_{L^\infty}) \|w\|_{L^2}^2 + C_N \delta^{2(N+1)} \exp(2(N+1) \operatorname{Re} \lambda t) \right) \end{aligned}$$

But for $\alpha > 0$ such that $2 + \alpha < M - N$ and for $t \leq T_\delta$:

$$\begin{aligned} \|\nabla_x f_{app}\|_{L^\infty} &\leq \|f_{app}\|_{H^{2+\alpha}} \\ &\leq \sum_{i=1}^N \delta^i \|u_i\|_{H^{2+\alpha}} \leq \sum_{i=1}^N \delta^i \|u_i\|_{H^{L-i}} \\ &\leq \sum_{i=1}^N \delta^i \exp(i \operatorname{Re} \lambda t) \\ &\leq \sum_{i=1}^N \theta^i \leq \frac{\theta}{1-\theta} \leq \frac{1}{2} \end{aligned}$$

Now choose N such that:

$$N + 1 > \frac{3C}{4 \operatorname{Re} \lambda} \quad (5.26)$$

By Gronwall's lemma we then get:

$$\|f - \mu - f_{app}\|_{L^2} = \|w\|_{L^2} \leq C_N \delta^{N+1} \exp((N+1) \operatorname{Re} \lambda t) \leq C_N \theta^{N+1} \quad (5.27)$$

On the other hand we have for $t = T_\delta$:

$$\begin{aligned} \|f_{app}\|_{L^2} &\geq \delta \|u_1\|_{L^2} - \sum_{i=2}^N \delta^i \|u_i\|_{L^2} \\ &\geq \delta \exp(\operatorname{Re} \lambda T_\delta) - \sum_{i=2}^N \delta^i \exp(i \operatorname{Re} \lambda T_\delta) \\ &= \theta - \sum_{i=2}^N \theta^i \\ &\geq \frac{1}{2} \theta \end{aligned}$$

Finally we have, for $t = T_\delta$:

$$\begin{aligned} \|f - \mu\|_{L^2} &\geq \|f_{app}\|_{L^2} - \|f - \mu - f_{app}\|_{L^2} \\ &\geq \frac{1}{2} \theta - C_N \delta^{N+1} \exp((N+1) \operatorname{Re} \lambda t) \geq \frac{1}{2} \theta - C_N \theta^{N+1} \\ &\geq \frac{1}{4} \theta = \epsilon_1 > 0 \end{aligned}$$

if θ is chosen small enough with respect to N .

□

6 Nonlinear stability in the “good curvature” region

Let us now consider the case of the “good curvature” region, where the equilibrium is given by:

$$\mu^{good}(x_1) = \begin{pmatrix} x_1 \\ 1 - x_1 \end{pmatrix}$$

Indeed we can take $L = 1$ since we have seen in the linear study that the size of the box does not matter.

For the sake of readability, we write μ instead of μ^{good} in the following, except in the ambiguous cases.

Let $f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix}$ a solution of the nonlinear transport equation (3.6). We remind that the equation satisfied by $(f - \mu)$ states in this case:

$$\partial_t(f - \mu) + \begin{pmatrix} -1/2v^2 \partial_{x_2}(f^+ - \mu^+) \\ 0 \end{pmatrix} + E^\perp \cdot \nabla_x(f - \mu) = \begin{pmatrix} -E_2 \\ E_2 \end{pmatrix} \quad (6.1)$$

and $E = -\nabla_x V$ with $-\Delta V = f^+ + f^- - \mu^+ - \mu^-$ and with the Dirichlet conditions on the boundaries $x_1 = 0$ and $x_1 = 1$.

The idea to show stability is to obtain a good energy estimate (Lyapunov functional).

Theorem 6.1. For any initial data $f_0 \in L^2$, the solution f of (3.6) satisfies:

$$\mathcal{E}(t) = \|f - \mu\|_{L^2}^2 + \frac{2}{v^2} \int |\nabla V|^2 dx \leq \mathcal{E}(0) \quad (6.2)$$

with $\|f - \mu\|_{L^2}^2 = \|f^+ - \mu^+\|_{L^2}^2 + \|f^- - \mu^-\|_{L^2}^2$.

As an immediate consequence of this theorem, we obtain L^2 stability:

Corollary 6.1. The equilibrium μ^{good} is nonlinearly stable in the L^2 norm.

Proof of the corollary. Thanks to Theorem (6.1) and to the Poisson equation:

$$\begin{aligned} \|f - \mu\|_{L^2}^2 &\leq \|f - \mu\|_{L^2}^2 + \frac{2}{v^2} \int |\nabla V|^2 dx \\ &\leq \|f(0) - \mu\|_{L^2}^2 + \frac{2}{v^2} \int |\nabla V(0)|^2 dx \\ &\leq \|f(0) - \mu\|_{L^2}^2 + C \frac{2}{v^2} \|f(0) - \mu\|_{L^2}^2 \end{aligned} \quad (6.3)$$

This means that for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\|f(0) - \mu\|_{L^2} \leq \delta$ then for any $t \geq 0$, $\|f - \mu\|_{L^2} \leq \epsilon$. In other words, μ^{good} is nonlinearly stable in the L^2 norm. \square

Remark 6.1. In the ‘‘bad-curvature’’ region, in a box of size L we can show that the following quantity is conserved:

$$\|f - \mu\|_{L^2}^2 - \frac{2}{Lv^2} \int |\nabla V|^2 dx \quad (6.4)$$

We can easily see in Fourier variables that:

$$\int |\nabla V|^2 dx \leq \frac{2L^2}{\pi^2} \|f - \mu\|_{L^2}^2 \quad (6.5)$$

So we get:

$$\begin{aligned} \|f - \mu\|_{L^2}^2 &\leq \mathcal{E}(0) + \frac{2}{Lv^2} \int |\nabla V|^2 dx \\ &\leq \mathcal{E}(0) + \frac{2}{Lv^2} \frac{2L^2}{\pi^2} \|f - \mu\|_{L^2}^2 \end{aligned}$$

Hence,

$$\left(1 - \frac{4L}{\pi^2 v^2}\right) \|f - \mu\|_{L^2}^2 \leq \|f(0) - \mu\|_{L^2}^2 \quad (6.6)$$

So there is L^2 nonlinear stability, provided that

$$v^2 > \frac{4L}{\pi^2} \quad (6.7)$$

Otherwise, we can not deduce anything.

Note also that in the linear discussion, there was stability provided that

$$v^2 > 5 \frac{L}{8\pi^2} \quad (6.8)$$

We do not know what happens for $v^2 \in [5 \frac{L}{8\pi^2}, \frac{4L}{\pi^2}]$ in the nonlinear case.

Remark 6.2. Let us add that the explicit form of the equilibria is crucial in the proof of the theorem. It would not work similarly if we had taken an equilibrium of the form:

$$\mu(x_1) = \begin{pmatrix} \Phi(x_1) \\ 1 - \Phi(x_1) \end{pmatrix}$$

with Φ a smooth function. In this case it should be maybe more relevant to use the general Lyapunov functionals method of Arnold [2].

Likewise one can notice that the proof would have not worked if we had chosen any other boundary condition than Dirichlet.

Remark 6.3. We observe that in the Lyapunov functional, the first term is nothing but the enstrophy in bidimensional fluid dynamics, while the second term can be interpreted as the (rescaled) kinetic energy of the fluid.

Proof of Theorem 6.1. We start by taking the scalar product with $(f - \mu)$ in the transport equation and integrate with respect to x which entails:

$$\frac{d}{dt} \|f - \mu\|_{L^2}^2 = \int -E_2(f^+ - \mu^+) dx + \int E_2(f^- - \mu^-) dx \quad (6.9)$$

Indeed, thanks to the periodicity with respect to x_2 , we first have:

$$\begin{aligned} \int \partial_{x_2}(f^+ - \mu^+)(f^+ - \mu^+) dx &= \int \frac{1}{2} \partial_{x_2}(f^+ - \mu^+)^2 dx \\ &= 0 \end{aligned}$$

In the same fashion, with Green's formula, we can show that $\int E^\perp \cdot \nabla_x(f - \mu)(f - \mu) dx = 0$ (we take advantage here of the fact: $\operatorname{div} E^\perp = 0$):

$$\begin{aligned} \int E^\perp \cdot \nabla_x(f - \mu)(f - \mu) dx &= \frac{1}{2} \int E^\perp \cdot \nabla_x(f - \mu)^2 dx \\ &= \frac{1}{2} \int \operatorname{div}(E^\perp (f - \mu)^2) dx \\ &= \frac{1}{2} \left(\int_{x_1=0} E^\perp (f - \mu)^2 \cdot (-e_1) dx_2 + \int_{x_1=1} E^\perp (f - \mu)^2 \cdot e_1 dx_2 \right) \\ &= \frac{1}{2} \int_{x_1=0} -E_2(f - \mu)^2 dx_2 + \frac{1}{2} \int_{x_1=1} E_2(f - \mu)^2 dx_2 \\ &= 0 \end{aligned}$$

because $E_2 = -\partial_{x_2} V = 0$ on $x_1 = 0, 1$.

We now give a technical lemma which will help for the following of the proof.

Lemma 6.1.

$$\frac{d}{dt} \|f^+ - \mu^+\|_{L^2}^2 = \frac{d}{dt} \|f^- - \mu^-\|_{L^2}^2$$

Proof. In order to see this, one can simply compute:

$$\begin{aligned} \int E_2((f^- - \mu^-) - (f^+ - \mu^+)) dx &= \int E_2(f^+ + f^- - \mu^+ - \mu^- - 2(f^+ - \mu^+)) dx \\ &= \int E_2(\Delta V - 2(f^+ - \mu^+)) dx \\ &= -2 \int E_2(f^+ - \mu^+) dx \end{aligned}$$

Indeed, thanks to periodicity with respect to x_2 and to the Dirichlet boundary conditions on $x_1 = 0, 1$, we get:

$$\begin{aligned}
\int \partial_{x_2} V \Delta V dx &= - \int V \partial_{x_2} \Delta V dx + \underbrace{\int \operatorname{div}(V \Delta V e_2) dx}_{=0} \\
&= \int \nabla V \partial_{x_2} \nabla V dx - \underbrace{\int \operatorname{div}(V \nabla \partial_{x_2} V) dx}_{=0} \\
&= \int \partial_{x_2} \left(\frac{|\nabla V|^2}{2} \right) dx \\
&= 0
\end{aligned}$$

If we make the same calculation, we can also see that:

$$\int E_2 ((f^- - \mu^-) - (f^+ - \mu^+)) dx = 2 \int E_2 (f^- - \mu^-) dx$$

□

Let us go back to the main calculation. Thanks to this lemma we get:

$$\begin{aligned}
\frac{d}{dt} \|f - \mu\|_{L^2}^2 &= \int -E_2 (f^+ - \mu^+) dx + \int E_2 (f^- - \mu^-) \\
&= -2 \int E_2 (f^+ - \mu^+) dx \\
&= -2 \int V \partial_{x_2} (f^+ - \mu^+) dx + \underbrace{\int \operatorname{div}(V (f^+ - \mu^+) e_2) dx}_{=0} \\
&= \frac{4}{v^2} \int -V \left(\partial_t (f^+ - \mu^+) + E^\perp \cdot \nabla_x (f^+ - \mu^+) + E_2 \right) dx \\
&= \frac{4}{v^2} \int V \left(-\partial_t (f^+ + f^- - \mu^+ - \mu^-) - E^\perp \cdot \nabla_x (f^+ + f^- - \mu^+ - \mu^-) \right) dx \\
&= \frac{4}{v^2} \int V \left(\partial_t \Delta V - E^\perp \cdot \nabla_x \Delta V \right) dx
\end{aligned}$$

where we have plugged in the equation satisfied by $\partial_{x_2} (f^+ - \mu^+)$ and by $(f^- - \mu^-)$.

To conclude the proof, we compute:

$$\int V \partial_t \Delta V dx = -\frac{d}{dt} \frac{1}{2} \left(\int |\nabla V|^2 dx \right) \quad (6.10)$$

and

$$\begin{aligned}
\int V E^\perp \cdot \nabla \Delta V dx &= \int V \operatorname{div}(E^\perp \Delta V) dx \\
&= \underbrace{- \int \nabla V \cdot E^\perp \Delta V dx}_{=0} + \underbrace{\int \operatorname{div}(V E^\perp \Delta V) dx}_{=0}
\end{aligned}$$

The first term is equal to zero since $E \cdot E^\perp = 0$, the second one thanks to the Dirichlet boundary condition on $x_1 = 0, 1$ and to the periodicity with respect to x_2 . □

7 Conclusion

We have finally managed to provide a mathematical explanation of stability in the “good curvature” region and instability in the “bad curvature” region with our nonlinear model. In our analysis we have pointed out a strange phenomenon that tells us that there is nonlinear stability even in the bad curvature region, provided that the temperature gradient is large enough (in other terms, if there is enough heating, there is good confinement : this is the H-mode).

We would like to point out that there remain many interesting open questions, mainly related to the modeling we have made.

Problem 1 Does the gyroaverage (which was neglected in Approximation **A1**) play a role in the stability/instability issue ? The expected answer should be negative: one may think of the gyrofluid version of system (3.6) and lead the same analysis than in this paper.

Problem 2 Another problem that naturally comes into mind is to generalize this work to a kinetic model. In this case, the main difficulty would be to define the plasma equilibria and then study their linear stability. Grenier’s method can be generalized quite straightforwardly to the kinetic case. For linear instability, a generalized Lyapunov functional method may still work.

Problem 3 Another issue is the choice of realistic boundary conditions for the Poisson equation. Maybe a Neumann condition would be more relevant (perfect conductor assumption); but in this case the explicit construction for the nonlinear stability proof would not work.

Problem 4 It would be also interesting to lead the analysis with other modelings of the equilibria than the one used in this paper. For instance the ”bi-temperature” assumption seems very rough. We may choose a smooth transition between the cold and the hot part, for instance by considering the following system:

$$\left\{ \begin{array}{l} \partial_t f - \frac{1}{2}v^2(x_1)\partial_{x_2}f + E^\perp \cdot \nabla_x f = 0 \\ E = -\nabla_x V \\ -\Delta_x V = f - 1 \\ V = 0 \text{ on } x_1 = 0, L \\ f|_{t=0} = f_0 \text{ with } \int f_0 dx = 1 \end{array} \right. \quad (7.1)$$

where $v(x_1)$ a smooth monotonic function satisfying $v(0) = v_-$ (resp. v_+) and $v(L) = v_+$ (resp. v_-), with $v_- < v_+$.

Problem 5 The more challenging and maybe interesting problem would be to prove overall confinement with a twisting magnetic field (as it has been mentioned in the introduction). In this case, the accurate parameter to consider is the so-called safety factor, which stands for the number of times the magnetic field lines twist around the torus the long way for each time they twist around the short way. In “real” tokamaks, it has to be chosen with precaution in order to get good confinement properties (see [13]). But it seems to be much more complicated, since one has to deal with many drifts due to the geometry of the magnetic field.

Some of these questions should be addressed in a forthcoming work.

Finally let us conclude by mentioning that the analysis of confinement provided in this paper is very naive since it is now well known that there is a loss of confinement in tokamak plasmas, referred to as anomalous transport. Many models have been proposed and intensively studied to justify these phenomena: we intend to study some of them from a mathematical viewpoint in forthcoming works.

References

- [1] G. Allaire, *Homogenization and Two-scale Convergence*, SIAM J. Math. Anal. **XXIII** (1992), no. 6, 1482–1518.
- [2] V.I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Annales de l'Institut Fourier **16** (1996), 319–361.
- [3] J.-Y. Chemin, *On some recent results in fluid mechanics*, Journées Equations aux dérivées partielles (1994), 1–45.
- [4] B. Desjardins and E. Grenier, *On nonlinear Rayleigh-Taylor instabilities*, Acta Math. Sin. (Engl. Ser.) **22** (2006), no. 4, 1007–1016.
- [5] E. Frénod and E. Sonnendrücker, *The Finite Larmor Radius Approximation*, SIAM J. Math. Anal. **32** (2001), no. 6, 1227–1247.
- [6] V. Grandgirard et al, *Global full-f gyrokinetic simulations of plasma turbulence*, Plasma Phys. Control. Fusion **49** (2007), 173–182.
- [7] E. Grenier, *Nonlinear instability of Euler and Prandtl equations*, Comm. Pure Appl. Math. **53** (2000), 1067–1091.
- [8] Y. Guo and W.A. Strauss, *Nonlinear instability of double-humped equilibria*, Annales de l'I.H.P., section C **12** (1995), no. 3, 339–352.
- [9] H.J. Hwang and Y. Guo, *On the dynamical Rayleigh-Taylor instability*, Arch. Ration. Mech. Anal. **167** (2003), no. 3, 235–253.
- [10] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), no. 3, 608–623.
- [11] O. Penrose, *Electrostatic instability of a uniform non-Maxwellian plasma*, Phys. Fluids **3** (1960), 258–265.
- [12] F. Wagner et al., *Regime of Improved Confinement and High Beta in Neutral-Beam-Heated Divertor Discharges of the ASDEX Tokamak*, Phys. Rev. Lett. **49** (1982), no. 19, 1408 – 1412.
- [13] J. Wesson, *Tokamaks*, Clarendon Press-Oxford (2004).