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Mimicking the marginal distributions of a semimartingale

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Abstract

We show that the flow of marginal distributions of a discontinuous semimartingale X can be matched by a Markov process whose infinitesimal generator is expressed in terms of the local characteristics of X . Our results extend a “mimicking theorem” of Gyöngy (1986) to discontinuous semimartingales. We use this result to derive a partial integro-differential equation for the one-dimensional distributions of a semimartingale, extending the Kolmogorov forward equation to a non-Markovian setting.

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1 Introduction

Stochastic processes with path-dependent / non-Markovian dynamics used in various fields such as physics and mathematical finance present challenges for computation, simulation and estimation. In some applications where one is interested in the marginal distributions of such processes, such as option pricing or Monte Carlo simulation of densities, the complexity of the model can be greatly reduced by considering a low-dimensional Markovian model with the same marginal distributions. Given a process X , a Markov process Y is said to *mimick* X if X and Y have the same marginal distributions:

$$\forall t \geq 0, \quad X_t \stackrel{d}{=} Y_t \tag{1}$$

Y is called a *Markovian projection* of X . The existence, uniqueness and computation of such Markovian projections has been explored for queues [4], Ito processes [12] and marked point processes [6] and has various applications such as the derivation of forward equations for option pricing [9].

We give here such a Markovian projection result for a (possibly discontinuous) semimartingale [3]. Given a semimartingale X , we give conditions under which there exists a Markov process Y whose flow of marginal distributions is identical to those of X , and give an explicit construction of the Markov process Y as the solution of a martingale problem for an integro-differential operator [2, 18, 21, 22]. Our result can be seen as a generalization to the discontinuous case of a previous result by Gyöngy [12] for continuous semimartingales.

In the martingale case, the Markovian projection problem is related to the problem of constructing martingales with a given flow of marginals, which dates back to Kellerer [17] and has been recently explored by Yor and coauthors [1, 13, 19] using a variety of techniques. In this work we shall focus on the *Markov* property -rather than the martingale property- of the mimicking process. We shall see nevertheless that the Markovian projection preserves the martingale property. Also, whereas the approaches described in [1, 13, 19] use as a starting point the marginal distributions of X , our construction describes the mimicking Markov process Y in terms of the characteristic triplet [15] of the semimartingale X . Our construction thus applies more readily to solutions of stochastic differential equations where the local characteristics are known but not the marginal distributions.

Section 2 presents a Markovian projection result for a \mathbb{R}^d -valued semimartingale given by its local characteristics. Section 3 shows how this result can be applied to processes whose jumps are represented as the integral of a predictable jump amplitude with respect to a Poisson random measure, a representation often used in stochastic differential equations with jumps. We use these results in section 4 to derive a partial integro-differential equation for the one-dimensional distributions of a discontinuous semimartingale, thus extending Kolmogorov's forward equation to a non-Markovian setting. Section 4 applies these results to particular classes of processes: multivariate marked point processes (Section 5.1) and time-changed Lévy processes (Section 5.2).

2 A mimicking theorem for discontinuous semimartingales

Consider, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, an Ito semimartingale given by the decomposition

$$X_t = X_0 + \int_0^t \beta_s^X ds + \int_0^t \delta_s^X dW_s + \int_0^t \int_{\|y\| \leq 1} y \tilde{M}_X(dsdy) + \int_0^t \int_{\|y\| > 1} y M_X(dsdy), \quad (2)$$

where W is a standard \mathbb{R}^n -valued Wiener process, M_X is an integer-valued random measure on $[0, \infty] \times \mathbb{R}^d$ with compensator measure μ_X and $\tilde{M}_X = M_X - \mu_X$ is the compensated measure [15, Ch.II, Sec.1], β^X (resp. δ^X) is an adapted process with values in \mathbb{R}^d (resp. $M_{d \times n}(\mathbb{R})$). We will assume that

Assumption 1. For any $T > 0$, (β^X, δ^X) are bounded on $[0, T]$:

$$\forall T > 0 \exists K_T > 0, \forall t \in [0, T] \|\beta_t^X\| \leq K_T, \|\delta_t^X\| \leq K_T \quad a.s \quad (H1)$$

where $\|\cdot\|$ denotes the Euclidean norm.

Assumption 2. μ_X has a density $m_X(\omega, t, dy)$ with respect to the Lebesgue measure on $[0, \infty[$ and for any $T > 0$,

$$\exists K'_T > 0, \int_{\mathbb{R}^d} (1 \wedge \|y\|^2) m_X(\cdot, t, dy) \leq K'_T < \infty \quad a.s. \quad (H2)$$

Note that (H2) is only slightly stronger than stating that m_X is a Lévy kernel since in that case we already have $\int (1 \wedge \|y\|^2) m_X(\cdot, t, dy) < \infty$.

Let $\Omega_0 = D([0, \infty[, \mathbb{R}^d)$ be the Skorokhod space of right-continuous functions with left limits. Denote by $Y_t(\omega) = \omega(t)$ the canonical process on Ω_0 and \mathcal{F}_t^0 its filtration. Our goal in this section is to construct a law \mathbb{Q} on Ω_0 such that Y is a Markov process under \mathbb{Q} and X and Y have the same one-dimensional distributions: $\forall t \geq 0, X_t \stackrel{d}{=} Y_t$. In order to this, we shall characterize \mathbb{Q} as the solution of a *martingale problem* for an appropriately chosen integro-differential operator L . Recall that a probability measure \mathbb{Q} on Ω_0 is a solution to the martingale problem [23, 14, 11] for $(L, \mathcal{D}(L))$ if for any $f \in \mathcal{D}(L)$, the process

$$f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s) ds$$

is a \mathbb{Q} -martingale. Y then has the Markov property under \mathbb{Q} [14, 11].

2.1 Markovian projection of a semimartingale

Theorem 1 (Markovian projection). Define, for $t \geq 0, z \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d - \{0\})$

$$\begin{aligned} \beta^Y(t, z) &= \mathbb{E}[\beta_t^X | X_{t-} = z]; \\ a^Y(t, z) &= \mathbb{E}[\delta_t^X \delta_t^X | X_{t-} = z]; \\ m_Y(t, A, z) &= \mathbb{E}[m_X(\cdot, t, A) | X_{t-} = z]. \end{aligned} \quad (3)$$

If β^Y, a^Y and $A \rightarrow m_Y(t, A, z)$ are continuous in (t, z) on $[0, \infty[\times \mathbb{R}^d$, there exists a weak solution $((Y_t)_{t \geq 0}, \mathbb{Q}_{X_0})$ for the stochastic differential equation

$$\begin{aligned} Y_t = X_0 &+ \int_0^t \beta^Y(u, Y_u) du + \int_0^t \delta^Y(u, Y_u) dB_u \\ &+ \int_0^t \int_{\|y\| \leq 1} y \tilde{M}_Y(du dy) + \int_0^t \int_{\|y\| > 1} y M_Y(du dy) \end{aligned} \quad (4)$$

where (B_t) is an n -dimensional Brownian motion, M_Y is an integer-valued random measure on $[0, \infty[\times \mathbb{R}^d$ with compensator $m_Y(t, dy, Y_{t-}) dt$, \tilde{M}_Y the associated compensated random measure and $\delta^Y : [0, \infty[\times \mathbb{R}^d \mapsto M_{d \times n}(\mathbb{R})$ is a continuous function such that

$${}^t\delta^Y(t, z)\delta^Y(t, z) = a^Y(t, z)$$

Under \mathbb{Q}_{X_0} , Y is a Markov process with infinitesimal generator L given, for $f \in C_0^\infty([0, \infty[\times \mathbb{R}^d)$, by

$$\begin{aligned} Lf(t, x) = \beta^Y \cdot \nabla f(t, x) &+ \sum_{i,j=1}^d \frac{a_{i,j}^Y(t, x)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) \\ &+ \int_{\mathbb{R}^d} [f(t, x+y) - f(t, x) - y \cdot \nabla f(t, x)] m_Y(t, dy, x) \end{aligned} \quad (5)$$

and the marginal distributions of Y mimic those of X :

$$\forall t \geq 0, Y_t \stackrel{d}{=} X_t.$$

We will call (Y, \mathbb{Q}_{X_0}) the *Markovian projection* of X .

Proof. First, we observe that m_Y is a Lévy kernel : for any (z, t)

$$\int_{\mathbb{R}^d} (1 \wedge \|y\|^2) m_Y(t, dy, z) = \mathbb{E} \left[\int_{\mathbb{R}^d} 1 \wedge \|y\|^2 m_X(t, dy) | X_{t-} = z \right] < \infty \text{ a.s.}$$

where we applied Fubini's theorem using (H_2) . Assumptions (H_1) , (H_2) imply that β^Y, a^Y are bounded with respect to z , uniformly in $t \in [0, T]$. Together with the assumption of continuity of the functions β^Y, a^Y and m_Y with respect to (t, z) on $[0, T] \times \mathbb{R}^d$, these conditions imply the existence of a solution the martingale problem for the operator L defined by (5) on $C_0^\infty([0, T] \times \mathbb{R}^d)$ [21, Theorem 2.2]. Let $((Y_t)_{t \geq 0}, \mathbb{Q}_{X_0})$ be such a solution. (Y, \mathbb{Q}_{X_0}) is then a weak solution to (4) on the canonical space. To show that Y_T and X_T have the same marginal distributions for all $T \geq 0$, we will prove that for all bounded continuous functions f , and for all $T \geq 0$:

$$\mathbb{E}^{\mathbb{P}} [f(X_T)] = \mathbb{E}^{\mathbb{Q}_{X_0}} [f(Y_T)]$$

First, let f be a \mathcal{C}^2 function with compact support on \mathbb{R}^d . Itô's formula yields

$$\begin{aligned}
f(X_T) &= f(X_0) + \sum_{i=1}^d \int_0^T \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{t-}) dX_t^i + \frac{1}{2} \int_0^T \text{tr}[\nabla^2 f(X_{t-}) a_t^X] dt \\
&+ \sum_{t \leq T} \left[f(X_{t-} + \Delta X_t) - f(X_{t-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(X_{t-}) \Delta X_t^i \right] \\
&= f(X_0) + \int_0^T \nabla f(X_{t-}) \cdot \beta_t^X dt + \int_0^T \nabla f(X_{t-}) \cdot \delta_t^X dW_t \\
&+ \frac{1}{2} \int_0^T \text{tr}[\nabla^2 f(X_{t-}) a_t^X] dt \\
&+ \int_0^T \int_{\|y\| \leq 1} (\nabla f(X_{t-}) \cdot y) \tilde{M}_X(dt dy) + \int_0^T \int_{\|y\| > 1} (\nabla f(X_{t-}) \cdot y) M_X(dt dy) \\
&+ \int_0^T \int_{\mathbb{R}^d} (f(X_{t-} + y) - f(X_{t-}) - y \cdot \nabla f(X_{t-})) M_X(dt dy)
\end{aligned}$$

where $a_t^X = {}^t \delta_t^X \cdot \delta_t^X$. We note that

- $\int_0^T \int_{\|y\| \leq 1} (\nabla f(X_{t-}) \cdot y) \tilde{M}_X(dt dy)$ is well defined since $|\nabla f|$ is bounded and $y \rightarrow y^2$ is integrable with respect to μ_X ; thanks to (H_2) this integral is a martingale;
- $\int_0^T \int_{\|y\| > 1} (\nabla f(X_{t-}) \cdot y) M_X(dt dy) < \infty$ a.s. since $|\nabla f|$ is bounded;
- $\int_0^T \int_{\mathbb{R}^d} (f(X_{t-} + y) - f(X_{t-}) - y \cdot \nabla f(X_{t-})) M_X(dt dy)$ is well defined since:

$$|f(X_{t-} + y) - f(X_{t-}) - y \cdot \nabla f(X_{t-})| \leq \|\nabla f\| \frac{\|y\|^2}{2}$$

and $y \rightarrow y^2$ is integrable with respect to μ_X ;

- Since $\nabla f(X_{t-})$ and δ_t^X are uniformly bounded on $[0, T]$, $\int_0^T \nabla f(X_{t-}) \cdot \delta_t^X dW_t$ is a martingale.

$$\begin{aligned}
f(X_T) &= f(X_0) + \int_0^T \nabla f(X_{t-}) \cdot \beta_t^X dt + \int_0^T \nabla f(X_{t-}) \cdot \delta_t^X dW_t \\
&+ \frac{1}{2} \int_0^T \text{tr}[\nabla^2 f(X_{t-}) a_t^X] dt \\
&+ \int_0^T \int_{\|y\| \leq 1} (\nabla f(X_{t-}) \cdot y) \tilde{M}_X(dt dy) \\
&+ \int_0^T \int_{\mathbb{R}^d} (f(X_{t-} + y) - f(X_{t-}) - \mathbf{1}_{\{\|y\| \leq 1\}} y \cdot \nabla f(X_{t-})) M_X(dt dy)
\end{aligned}$$

Taking expectations and given the above remarks, we obtain

$$\begin{aligned}
\mathbb{E}[f(X_T)] &= f(X_0) + \mathbb{E} \left[\int_0^T \nabla f(X_{t-}) \cdot \beta_t^X dt \right] + \mathbb{E} \left[\frac{1}{2} \int_0^T \text{tr}[\nabla^2 f(X_{t-}) a_t^X] dt \right] \\
&+ \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (f(X_{t-} + y) - f(X_{t-}) - 1_{\{\|y\| \leq 1\}} y \cdot \nabla f(X_{t-})) M_X(dt dy) \right] \\
&= f(X_0) + \mathbb{E} \left[\int_0^T \nabla f(X_{t-}) \cdot \beta_t^X dt \right] + \mathbb{E} \left[\frac{1}{2} \int_0^T \text{tr}[\nabla^2 f(X_{t-}) a_t^X] dt \right] \\
&+ \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (f(X_{t-} + y) - f(X_{t-}) - 1_{\{\|y\| \leq 1\}} y \cdot \nabla f(X_{t-})) m_X(t, dy) dt \right]
\end{aligned}$$

Observing that:

$$\begin{aligned}
\mathbb{E} \left[\int_0^T \nabla f(X_{t-}) \cdot \beta_t^X dt \right] &\leq \|\nabla f\|_\infty \mathbb{E} \left[\int_0^T \|\beta_t^X\| dt \right] < \infty, \\
\mathbb{E} \left[\frac{1}{2} \int_0^T \text{tr}[\nabla^2 f(X_{t-}) a_t^X] dt \right] &\leq \|\nabla^2 f\|_\infty \mathbb{E} \left[\int_0^T \|a_t^X\| dt \right] < \infty, \\
\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} |f(X_{t-} + y) - f(X_{t-}) - 1_{\{\|y\| \leq 1\}} y \cdot \nabla f(X_{t-})| m_X(t, dy) dt \right] \\
&\leq \frac{\|\nabla^2 f\|_\infty}{2} \mathbb{E} \left[\int_0^T \int_{\|y\| \leq 1} \|y\|^2 m_X(t, dy) dt \right] + 2\|f\|_\infty \mathbb{E} \left[\int_0^T \int_{\|y\| > 1} m_X(t, dy) dt \right] < +\infty,
\end{aligned}$$

we may apply Fubini's theorem.

$$\begin{aligned}
\mathbb{E}[f(X_T)] &= f(X_0) + \int_0^T \mathbb{E} [\nabla f(X_{t-}) \cdot \beta_t^X] dt + \frac{1}{2} \int_0^T \mathbb{E} [\text{tr}[\nabla^2 f(X_{t-}) a_t^X]] dt \\
&+ \int_0^T \mathbb{E} \left[\int_{\mathbb{R}^d} (f(X_{t-} + y) - f(X_{t-}) - 1_{\{\|y\| \leq 1\}} y \cdot \nabla f(X_{t-})) m_X(t, dy) \right] dt
\end{aligned}$$

Conditioning on X_{t-} and using the iterated expectation property we obtain

$$\begin{aligned}
\mathbb{E}[f(X_T)] &= f(X_0) + \int_0^T \mathbb{E} [\nabla f(X_{t-}) \cdot \mathbb{E} [\beta_t^X | X_{t-}]] dt + \frac{1}{2} \int_0^T \mathbb{E} [\text{tr}[\nabla^2 f(X_{t-}) \mathbb{E} [a_t^X | X_{t-}]]] dt \\
&+ \int_0^T \mathbb{E} \left[\mathbb{E} \left[\int_{\mathbb{R}^d} (f(X_{t-} + y) - f(X_{t-}) - 1_{\{\|y\| \leq 1\}} y \cdot \nabla f(X_{t-})) m_X(t, dy) | X_{t-} \right] \right] dt \\
&= f(X_0) + \int_0^T \mathbb{E} [\nabla f(X_{t-}) \cdot \beta^Y(t, X_{t-})] dt + \frac{1}{2} \int_0^T \mathbb{E} [\text{tr}[\nabla^2 f(X_{t-}) a^Y(t, X_{t-})]] dt \\
&+ \int_0^T \int_{\mathbb{R}^d} \mathbb{E} [(f(X_{t-} + y) - f(X_{t-}) - 1_{\{\|y\| \leq 1\}} y \cdot \nabla f(X_{t-})) m_Y(t, dy, X_{t-})] dt \\
&= \mathbb{E}[f(Y_T)]
\end{aligned}$$

if one applies also Itô formula to $f(Y_t)$. We leave this part for the reader.

Now consider a continuous bounded function $g : \mathbb{R}^d \rightarrow \mathbb{R}$. There exists a sequence $(f_n)_{n \leq 1}$ such that $f_n : \mathbb{R}^d \rightarrow \mathbb{R}$ is a \mathcal{C}^2 function with compact support and $\|f_n - f\|_\infty \rightarrow 0$. A dominated convergence argument shows that $E[f_n(Y_T)] = E[f_n(X_T)] \rightarrow E[f(X_T)]$ and $E[f_n(Y_T)] \rightarrow E[f(Y_T)]$, which shows our result.

Let us now discuss on the uniqueness of the Markovian projection of X . First, note that L maps $C_0^\infty(\mathbb{R}^d)$ into the set $B(\mathbb{R}^d)$ of bounded functions:

$$\forall f \in C_0^\infty(\mathbb{R}^d), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d,$$

$$\begin{aligned} |Lf(t, x)| &\leq \|\beta^Y\| \|\nabla f\|_\infty + \sum_{i,j=1}^d \frac{\|a_{i,j}^Y\|}{2} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\| \\ &\quad + \int_{\mathbb{R}^d} \|f(t, x+y) - f(t, x) - y \cdot \nabla f(t, x)\| m_Y(t, dy, x) \\ &\leq K_T \|\nabla f\|_\infty + \sum_{i,j=1}^d \frac{K_T^2}{2} \|\nabla^2 f\|_\infty + \int_{\mathbb{R}^d} \|\nabla f\|_\infty \frac{\|y\|^2}{2} m_Y(t, dy, x) \\ &\leq K_T \|\nabla f\|_\infty + \sum_{i,j=1}^d \frac{K_T^2}{2} \|\nabla^2 f\|_\infty + K'_T \|\nabla f\|_\infty \end{aligned}$$

We have shown above that any two solutions Y^1 and Y^2 of the martingale problem have the same marginal distributions. Applying [11, Theorem 4.2], we conclude that Y^1 and Y^2 have the same finite-dimensional distributions. This shows the uniqueness in law of the Markovian projection Y . \square

Remark 2.1. *In the case where there are no jumps ($\mu_X \equiv 0$) we recover a result similar to Gyöngy [12, Theorem 4.6] but under slightly different assumptions. Namely, in [12] δ^X is assumed to verify the ellipticity condition*

$${}^t \delta_t^X \cdot \delta_t^X \geq \alpha I$$

for some $\alpha > 0$, in which case the continuity conditions on δ^Y, β^Y are not necessary. In our case we require continuity of the coefficients but not ellipticity: the diffusion term may be degenerate.

Remark 2.2 (Uniqueness in law). *Generally mimicking theorems [1, 6, 12, 19] aim at constructing a process Y in a certain class (Markov, martingale, ..) whose marginals mimic X , but Theorem 1 also shows that if one restricts Y in the class of Markov processes then Y is also unique in law. Under these conditions we can speak unambiguously of the Markovian projection of X . In particular, this guarantees that if X is already a Markov process, then $Y =^d X$.*

Remark 2.3. *The boundedness assumptions (H1) and (H2) may be relaxed to local boundedness, using localization techniques developed in [21, 23]. In the unbounded case additional conditions are needed to ensure that X does not explode, see [23, Chapter 10].*

2.2 Martingale-preserving property

An important property of the construction of Y in Theorem 1 is that it preserves the (local) martingale property.

Proposition 1 (Martingale preserving property). *Consider a semimartingale X which verifies the assumptions of Theorem 1 and whose large jumps are integrable:*

$$E\left[\int_{|y|>1} |y| \mu_X(dt dy)\right] < \infty$$

If X is a martingale then its Markovian projection Y is also a (local) martingale.

Proof. If X is a martingale then $E[\int_{|y|>1} |y| \mu_X(dt dy)] < \infty$ and the uniqueness of its semimartingale decomposition entails that $\mathbb{P}(\beta_t^X + \int_{\|y\|\geq 1} y m_X(t, dy) dt = 0 \text{ a.e.})=1$ so for any $T > 0$,

$$\mathbb{P}\left(\int_0^T dt [\beta^Y(t, z) + \int_{\|y\|\geq 1} y m_Y(t, z, dy)] = 0\right) = 1.$$

The assumptions on m_X, δ^X then entail that Y , as a sum of an Ito integral and a compensated Poisson integral, is a local martingale. \square

Baker and Yor [1] call a process X a 1-martingale if its one-dimensional (i.e. marginal) distributions can be matched by those of a martingale. Our explicit construction yields a sufficient condition for a semimartingale X to be a 1-martingale:

Corollary 1. *If*

$$\forall t \geq 0, \forall z \in \mathbb{R}^d, \quad \mathbb{E}\left[\beta_t^X + \int_{|y|>1} y \mu_X(dt dy) \mid X_{t-} = z\right] = 0 \quad (6)$$

then X is a 1-martingale.

3 Mimicking an Ito semimartingale

The representation (2) is not the most commonly used in applications, where a process is constructed as the solution to a stochastic differential equation driven by a Brownian motion and a Poisson random measure:

$$Z_t = Z_0 + \int_0^t \beta_s^Z ds + \int_0^t \delta_s^Z dW_s + \int_0^t \int \psi_s(y) \tilde{N}_Z(ds dy) \quad (7)$$

where β^Z and δ^Z are non-anticipative cadlag processes, N_Z is a Poisson random measure on $[0, \infty[\times\mathbb{R}^d$ with intensity $\nu_Z(dy) dt$ where

$$\int_{\mathbb{R}^d} (1 \wedge \|y\|^2) \nu_Z(dy) < \infty, \quad \tilde{N}_Z = N_Z - \nu_Z(dy) dt. \quad (8)$$

and the random jump amplitude $\psi : [0, T_*] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, where \mathcal{P} is the predictable σ -algebra on $[0, T_*] \times \Omega$. We assume

$$\psi_t(\omega, 0) = 0 \quad \forall T > 0, \quad \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} (1 \wedge \|\psi_t(\cdot, y)\|^2) \nu_Z(dy) dt \right] < \infty; \quad (\text{H3})$$

The difference between this representation and (2) is the presence of a random jump amplitude $\psi_t(\omega, \cdot)$ in (7). The relation between these two representations for semimartingales has been discussed in great generality in [10, 16]. Here we give a less general result which suffices for our purpose. The following result expresses Z in the form (2) suitable for applying Theorem 1:

Lemma 1 (Absorbing the jump amplitude in the compensator). *Under (H1) and (H3)*

$$Z_t = Z_0 + \int_0^t \beta_s^Z ds + \int_0^t \delta_s^Z dW_s + \int_0^t \int \psi_s(z) \tilde{N}_Z(ds dz)$$

where N_Z is a Poisson random measure with intensity $\nu_Z(dz)dt$, can be also represented as

$$Z_t = Z_0 + \int_0^t \beta_s^Z ds + \int_0^t \delta_s^Z dW_s + \int_0^t \int y \tilde{M}_Z(ds dy) \quad (9)$$

where M_Z is an integer-valued random measure on $\mathbb{R}^+ \times \mathbb{R}^d$ with compensator $\mu_Z(\omega, dt, dy)$ given by

$$\forall A \in \mathcal{B}(\mathbb{R}^d - \{0\}), \quad \mu_Z(\omega, dt, A) = \nu_Z(\psi_t^{-1}(\omega, A)) dt$$

where $\psi_t^{-1}(\omega, A) = \{z \in \mathbb{R}^d, \psi_t(\omega, z) \in A\}$ denotes the inverse image of A under the partial map.

Proof. The result can be deduced from [10, Théorème 12] but we sketch here the proof for completeness. A Poisson random measure N_Z on $[0, T] \times \mathbb{R}^d$ can be represented as a counting measure for some random sequence (T_n, U_n) with values in $[0, T] \times \mathbb{R}^d$.

$$N_Z = \sum_{n \geq 1} 1_{\{T_n, U_n\}}, \quad (10)$$

Let M_Z be the integer-valued random measure defined by:

$$M_Z = \sum_{n \geq 1} 1_{\{T_n, \psi_{T_n}(\cdot, U_n)\}}. \quad (11)$$

μ_Z , the predictable compensator of M_Z is characterized by the following property [15, Thm 1.8.]: for any positive $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable map $\chi : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ and any $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$,

$$\mathbb{E} \left[\int_0^T \int_A \chi(t, y) M_Z(dt dy) \right] = \mathbb{E} \left[\int_0^T \int_A \chi(t, y) \mu_Z(dt dy) \right] \quad (12)$$

Similarly, for $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$

$$\mathbb{E} \left[\int_0^T \int_B \chi(t, y) N_Z(dt dy) \right] = \mathbb{E} \left[\int_0^T \int_B \chi(t, y) \nu_Z(dy) dt \right]$$

Using formulae (10) and (11):

$$\begin{aligned} \mathbb{E} \left[\int_0^T \int_A \chi(t, y) M_Z(dt dy) \right] &= \mathbb{E} \left[\sum_{n \geq 1} \chi(t, \psi_{T_n}(\cdot, U_n)) \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\psi_t^{-1}(\cdot, A)} \chi(t, \psi_t(\cdot, z)) N_Z(dt dz) \right] \\ &= \mathbb{E} \left[\int_0^T \int_{\psi_t^{-1}(\cdot, A)} \chi(t, \psi_t(\cdot, z)) \nu_Z(dz) dt \right] \end{aligned}$$

Formula (12) and the equalities above lead to:

$$\mathbb{E} \left[\int_0^T \int_A \chi(t, y) \mu_Z(dt dy) \right] = \mathbb{E} \left[\int_0^T \int_{\psi_t^{-1}(\cdot, A)} \chi(t, \psi_t(\cdot, z)) \nu_Z(dz) dt \right]$$

Given ψ is a predictable random function, the uniqueness of the predictable compensator μ_Z (take $\phi \equiv Id$ in [15, Thm 1.8.] entails

$$\mu_Z(\omega, dt, A) = \nu_Z(\psi_t^{-1}(\omega, A)) dt \quad (13)$$

Formula (13) defines a new measure μ_Z which is a Lévy kernel (see (H3))

$$\int_0^T \int 1 \wedge \|y\|^2 \mu_Z(dy dt) = \int_0^T \int 1 \wedge \|\psi_t(\cdot, y)\|^2 \nu_Z(dy) dt < \infty$$

To end the proof, we observe that, due to (H3):

$$\begin{aligned} \int_0^t \int y \tilde{M}_Z(ds dy) &= \int_0^t \int y (M_Z - \mu_Z)(ds dy) \\ &= \int_0^t \int \psi_s(z) (N_Z(ds dz) - \nu_Z(dz) ds) \\ &= \int_0^t \int \psi_s(z) \tilde{N}_Z(ds dz). \end{aligned}$$

□

In the case where $\psi_t(\omega, \cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is invertible and differentiable, we can characterize the density of the compensator μ as follows:

Lemma 2 (Differentiable case). *If the Lévy measure ν_Z has a density $n_Z(z)$ and if $\psi_t(\omega, \cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ has a differentiable inverse i.e. there exists*

$$\phi : [0, T_*] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$$

such that

- ϕ is predictable i.e. $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable.
- $\phi_t(\omega, \cdot) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$
- $\forall t \in [0, T_*], \forall z \in \mathbb{R}^d, \quad \phi_t(\omega, \psi_t(\omega, z)) = z.$

then Z , given in (7), has the representation

$$Z_t = Z_0 + \int_0^t \beta_s^Z ds + \int_0^t \delta_s^Z dW_s + \int_0^t \int y \tilde{M}_Z(ds dy)$$

where M_Z is an integer-valued random measure with compensator

$$\mu_Z(\omega; dt dy) = |\det \nabla \phi_t(\omega, y)| n_Z(\phi_t(\omega, y)) dt dy$$

where $\nabla_y \phi_t$ denotes the Jacobian matrix of $\phi_t(\omega, \cdot)$.

Proof. We recall from the proof of Lemma 1:

$$\mathbb{E} \left[\int_0^T \int_A \chi(t, y) \mu_Z(dt dy) \right] = \mathbb{E} \left[\int_0^T \int_{\psi_t^{-1}(\cdot, A)} \chi(t, \psi_t(\cdot, z)) n_Z(z) dt dz \right]$$

We then proceed to the change of variable $z = \phi_t(\cdot, y)$:

$$\mathbb{E} \left[\int_0^T \int_{\psi_t^{-1}(\cdot, A)} \chi(t, \psi_t(\cdot, z)) n_Z(z) dt dz \right] = \mathbb{E} \left[\int_0^T \int_A \chi(t, y) |\det \nabla \phi_t(\cdot, y)| n_Z(\phi_t(\cdot, y)) dt dy \right]$$

The density appearing in the right hand side is predictable since ϕ does. By uniqueness of the predictable compensator μ_Z , we get:

$$\mu_Z(\omega; dt dy) = |\det \nabla \phi_t(\omega, y)| n_Z(\phi_t(\omega, y)) dt dy.$$

□

To combine Lemma 2 and Theorem 1 we make a further assumption:

Assumption 3. *The Lévy measure ν_Z admits a density $n_Z(y)$ with respect to the Lebesgue measure on \mathbb{R}^d and:*

$$\forall T > 0, \exists K'_T > 0 \quad \int_0^T \int_{\|y\| > 1} 1 \wedge \|\psi_t(\cdot, y)\|^2 n_Z(y) dy dt < K'_T \quad \text{a.s.} \quad (\text{H3b})$$

Theorem 2. Let (Z_t)

$$Z_t = Z_0 + \int_0^t \beta_s^Z ds + \int_0^t \delta_s^Z dW_s + \int_0^t \int \psi_s(y) \tilde{N}_Z(ds dy),$$

where $\psi_t(\omega, \cdot) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is invertible and differentiable with inverse $\phi_t(\omega, \cdot)$, β^Z and δ^Z and satisfies Assumption (H1) and ν_Z Assumption (H3b).

Define

$$\begin{aligned} \beta^Y(t, z) &= \mathbb{E} [\beta_t^Z | Z_{t-} = z]; \\ a^Y(t, z) &= \mathbb{E} [{}^t\delta_t^X \delta_t^X | X_{t-} = z]; \\ m_Y(t, y, z) &= \mathbb{E} [n_Z(\phi_t(y)) |\det \nabla \phi_t(y)| | Z_{t-} = z] \end{aligned} \quad (14)$$

Consider the stochastic differential equation

$$Y_t = Z_0 + \int_0^t \beta^Y(u, Y_u) du + \int_0^t \delta^Y(u, Y_u) dB_u + \int_0^t \int y \tilde{M}_Y(du dy) \quad (15)$$

where (B_t) is a Brownian motion, M_Y is an integer-valued random measure on $[0, T] \times \mathbb{R}$ with compensator $\mu_Y(\omega; dt dy) = n_Y(t, y, Y_{t-}) dt dy$ and \tilde{M}_Y the associated compensated random measure and $\delta^Y : [0, \infty[\times \mathbb{R}^d \mapsto M_{d \times n}(\mathbb{R})$ is a measurable function such that

$${}^t\delta^Y(t, z) \delta^Y(t, z) = a^Y(t, z).$$

If the functions β^Y , a^Y and m_Y are continuous in (t, z) on $[0, T] \times \mathbb{R}^d$, then (15) defines uniquely the law, \mathbb{Q}_{Z_0} of a Markov process $((Y_t)_{t \in [0, T]}, \mathbb{Q}_{Z_0})$, defined as the solution to the martingale problem on $C_0^\infty([0, T] \times \mathbb{R}^d)$ for the operator

$$\begin{aligned} Lf(t, x) &= \beta^Y \cdot \nabla f(t, x) + \sum_{i,j=1}^d \frac{a_{i,j}^Y(t, x)^2}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) \\ &+ \int_{\mathbb{R}^d} [f(t, x + y) - f(t, x) - y \cdot \nabla f(t, x)] n_Y(t, y, x) dy \end{aligned} \quad (16)$$

and $(Y_t)_{t \geq 0}$ has the same marginal distributions as $(Z_t)_{t \geq 0}$:

$$\forall t \geq 0 \ Y_t \stackrel{d}{=} Z_t.$$

Proof. We first use Lemma 2 to obtain the new representation of Z :

$$Z_t = Z_0 + \int_0^t \beta_s^Z ds + \int_0^t \delta_s^Z dW_s + \int_0^t \int y \tilde{M}_Z(ds dy)$$

Then, we observe that :

$$\begin{aligned} \int_0^t \int y \tilde{M}_Z(ds dy) &= \int_0^t \int_{\|y\| \leq 1} y \tilde{M}_Z(ds dy) + \int_0^t \int_{\|y\| > 1} y [M_Z(ds dy) - \mu_Z(ds dy)] \\ &= \int_0^t \int_{\|y\| \leq 1} y \tilde{M}_Z(ds dy) + \int_0^t \int_{\|y\| > 1} y M_Z(ds dy) - \int_0^t \int_{\|y\| > 1} y \mu_Z(ds dy) \end{aligned}$$

where the terms above are well-defined thanks to (H3b). Lemma 2 leads to:

$$\int_0^t \int_{\|y\|>1} y \mu_Z(ds dy) = \int_0^t \int_{\|y\|>1} \|\psi_s(\cdot, y)\|^2 n_Z(y) dy ds$$

Hence:

$$\begin{aligned} Z_t = Z_0 &+ \left[\int_0^t \beta_s^Z ds - \int_0^t \int_{\|y\|>1} \|\psi_s(\cdot, y)\|^2 n_Z(y) dy ds \right] + \int_0^t \delta_s^Z dW_s \\ &+ \int_0^t \int_{\|y\|\leq 1} y \tilde{M}_Z(ds dy) + \int_0^t \int_{\|y\|>1} y M_Z(ds dy) \end{aligned}$$

This representation has the form (2) and (H1) and (H3b) guarantee that the characteristics of Z also verify the boundedness assumptions (H1) and (H2). The result is then obtained by applying Theorem 1 to the semimartingale Z . \square

4 Forward equations for semimartingales

An important property of continuous-time Markov processes is their link with partial (integro-)differential equation (PIDE) which allows to use analytical tools for studying their probabilistic properties. In particular the transition density of a Markov process solves the forward Kolmogorov equation (or Fokker-Planck equation) [22]. The above results allow us to derive a forward PIDE for semimartingales: we show that under the assumptions of Theorem 1, the flow $t \mapsto p_t^X(\cdot)$ of marginal distributions of a semi-martingale X can be represented as the solution to a partial (integro-differential) equation:

Theorem 3 (Kolmogorov Forward equation). *Let X be a semimartingale given by (2) and denote $p_t^X(dx)$ the law of X_t . Under the assumptions of Theorem 1, $t \mapsto p_t^X$ is a weak solution, in the sense of distributions, of the forward equation:*

$$\frac{\partial p_t^X}{\partial t} = L_t^* \cdot p_t^X \quad (17)$$

where L^* is the Fokker-Planck operator, defined as the adjoint of (5):

$$\begin{aligned} \forall g &\in C_0^\infty(\mathbb{R}^d, \mathbb{R}), \\ L_t^* g(x) &= -\nabla [\beta^Y(t, x)g(x)] + \nabla^2 \left[\frac{a^Y(t, x)}{2} g(x) \right] \\ &+ \int_{\mathbb{R}^d} [g(x-z)m_Y(t, z, x-z) - g(x)m_Y(t, z, x) - 1_{\|z\|\leq 1} z \cdot \nabla_x [g(x)m_Y(t, dz, x)]] . \end{aligned} \quad (18)$$

where the coefficients β^Y, a^Y, m_Y are defined as in (3).

Proof. Under the assumptions of Theorem 1, X has the same marginals as Y where $((Y_t)_{t \geq 0}, \mathbb{Q}^{(t_0, x_0)})$ is a solution of the martingale problem on $C_0^\infty([0, \infty[\times \mathbb{R}^d)$

with initial condition $Y_{t_0} = y_0$ for the operator

$$\begin{aligned} L_t f(x) &= \beta^Y(t, x) \nabla f(x) + \text{tr} \left[\frac{a^Y(t, x)}{2} \nabla^2 f(x) \right] \\ &+ \int_{\mathbb{R}} [f(t, x + y) - f(t, x) - 1_{\{|y| \leq 1\}} y \frac{\partial f}{\partial x}(t, x)] m_Y(t, dy, x), \end{aligned} \quad (19)$$

Let $p_{t_0, T}(x_0, dy)$ be the law of Y_T under \mathbb{Q}^{t_0, y_0} . The mimicking property of Y implies that

$$p_{t_0, T}(x_0, dy) = p_T^X(dy)$$

so it suffices to show that $p_{t_0, T}(x_0, \cdot)$ verifies (16). Recall the Chapman-Kolmogorov equation for p :

$$p_{t_0, T}(x_0, dy) = \int_{\mathbb{R}^d} p_{t, T}(z, dy) p_{t_0, t}(x_0, dz).$$

So for $f \in C_0^\infty([0, \infty[\times \mathbb{R}^d)$

$$\frac{1}{T-t} \mathbb{E}^{\mathbb{Q}^{t_0, y_0}} [f(Y_T) - f(Y_t) | Y_{t_0} = y_0] = \int_{\mathbb{R}^d} p_{t_0, t}(x_0, dz) \int_{\mathbb{R}^d} f(y) \frac{p_{t, T}(z, dy) - \epsilon_z(dy)}{T-t}$$

where $\epsilon_z(dy)$ denotes a point mass at y . Since $f \in C_0^\infty([0, \infty[\times \mathbb{R}^d) \subset \text{dom}(L_t)$,

$$\int_{\mathbb{R}^d} f(y) \frac{p_{t, T}(z, dy) - \epsilon_z(dy)}{T-t} \xrightarrow{T \downarrow t} L_t f(z)$$

so

$$\frac{1}{T-t} \mathbb{E}^{\mathbb{Q}^{t_0, y_0}} [f(Y_T) - f(Y_t) | Y_{t_0} = y_0] \xrightarrow{T \downarrow t} \int_{\mathbb{R}^d} p_{t_0, t}(x_0, dz) L_t f(z) = \int_{\mathbb{R}^d} p_t^X(dz) L_t f(z)$$

Viewing p_t^X as an element of the dual of $C_0^\infty(\mathbb{R}^d)$ i.e. a Schwartz distribution, we have

$$\forall f \in C_0^\infty(\mathbb{R}^d), \quad \left\langle \frac{p_T^X - p_t^X}{T-t}, f \right\rangle \xrightarrow{T \rightarrow t} \langle p_t^X, L_t f \rangle = \langle L_t^* p_t^X, f \rangle$$

where L_t^* is the adjoint of L_t and $\langle \cdot, \cdot \rangle$ is the duality product. p^X is thus a weak solution, in the sense of distributions, of

$$\frac{\partial p_T^X}{\partial T} = L_t^* p_T^X$$

We now compute $L_t^* f$ for $f \in C_0^\infty(\mathbb{R}^d)$.

$$\forall f \in C_0^\infty(\mathbb{R}^d) \quad \forall t \geq t_0 \quad f(Y_t) - \int_{t_0}^t L_s f(Y_{s-}) ds$$

is a \mathbb{Q}^{t_0, y_0} -martingale, hence :

$$\mathbb{E}^{\mathbb{Q}^{t_0, y_0}} [f(Y_T) - f(Y_t) | Y_{t_0} = y_0] = \mathbb{E}^{\mathbb{Q}^{t_0, y_0}} \left[\int_t^T L_s f(Y_{s-}) ds \right]$$

Using Fubini

$$\mathbb{E}^{\mathbb{Q}^{t_0, y_0}} [f(Y_T) - f(Y_t) | Y_{t_0} = y_0] = \int_t^T ds \int p_{t_0, s}(y_0, dx) Lf(x)$$

For $z \in \mathbb{R}^d$, define the translation operator τ^z by $\tau^z f(x) = f(x + z)$. Then

$$\begin{aligned} & \int p_{t_0, s}(y_0, dx) L_s f(x) \\ &= \int p_{t_0, s}(y_0, dx) \left[\beta^Y(s, x) \nabla f(x) + \frac{1}{2} \text{tr}[a^Y \nabla^2 f(x)] \right. \\ &+ \int_{|z| > 1} (\tau_z f(x) - f(x)) m_Y(s, dz, x) \\ &+ \left. \int_{|z| \leq 1} (\tau_z f(x) - f(x) - z \frac{\partial f}{\partial x}(t, x)) m_Y(s, dz, x) \right] \\ &= \int \left[-f(x) \frac{\partial}{\partial x} [\beta^Y(s, x) p_{t_0, s}(y_0, dx)] + f(x) \frac{\partial^2 f}{\partial x^2} \left[\frac{a^Y(s, x)}{2} p_{t_0, s}(y_0, dx) \right] \right. \\ &+ \int_{|z| > 1} f(x) (\tau_{-z} p_{t_0, s}(y_0, dx) m_Y(s, dz, x)) - p_{t_0, s}(y_0, dx) m_Y(s, dz, x) \\ &+ \int_{|z| \leq 1} f(x) (\tau_{-z} p_{t_0, s}(y_0, dx) m_Y(s, dz, x)) - p_{t_0, s}(y_0, dx) m_Y(s, dz, x) \\ &\left. - z \frac{\partial}{\partial x} (p_{t_0, s}(y_0, dx) m_Y(s, dz, x)) \right] \end{aligned}$$

Dividing by $T - t$ and taking $T - t \rightarrow 0$, we obtain

$$\frac{1}{T - t} \mathbb{E}^{\mathbb{Q}^{t_0, y_0}} [f(T, Y_T) - f(t, Y_t) | Y_{t_0} = y_0] \rightarrow \int f(T, x) L^* \cdot p_{t_0, T}(y_0, dx)$$

where L^* is given by (18). □

5 Examples

We now give two examples of stochastic models used in finance, where Markovian projections can be characterized in a more explicit manner than in the general results above.

5.1 Marked point processes

We first consider the case of a multivariate marked point process [7] with mark space \mathbb{R}^d . Using the notations of Section 2, let $M_X(dt dx)$ be an integer-valued random measure whose compensator $\mu_X(dt dx; \omega) = m_X(t, dx; \omega) dt$ is assumed to be a *finite* measure whose mass

$$\lambda_t(\omega) = \int_{\mathbb{R}^d - \{0\}} m_X(t, dx; \omega)$$

represents the (random) jump intensity of the point process $N_t = M_X([0, t] \times \mathbb{R}^d)$. Denote by $T_1 \leq T_2 \leq \dots$ the jump times of N . The process

$$X_t = \int_0^t \int_{\mathbb{R}^d - \{0\}} x M_X(dt dx)$$

is then a *marked point process* and may be represented as

$$X_t = \sum_{k=1}^{N_t} Z_k$$

where the “mark” Z_k is distributed according to

$$F_t(dx; \omega) = \frac{m_X(t, dx; \omega)}{\lambda_t(\omega)}$$

Marked point processes form an important subclass of the processes considered in section 2 and the corresponding Markovian projection results are of interest in queuing theory and credit risk modeling. Markovian projection of point processes was first discussed in the case of queues by Brémaud [4, p. 30] under the name of “first order equivalence” of queues. A mimicking theorem for marked point processes with IID marks (i.e. $F_t(dx; \omega) = F(dx)$) is given in [6]. We recover here these results as a special case of Theorem 1, which covers the case of arbitrary (i.e. not necessarily independent) marks:

Proposition 2 (Markovian projection of a marked point process). *There exists a weak solution to the stochastic differential equation*

$$Y_t = \int_0^t \int_{\mathbb{R}^d} x M_Y(dt dx)$$

where $M_Y(dt dx)$ is an integer-valued random measure with compensator $m_Y(t, dx; Y_{t-}) dt$ where

$$\forall A \in \mathcal{B}(\mathbb{R}^d) \quad m_Y(t, A; z) = E[m_X(t, A, \cdot) | X_{t-} = z]$$

The solution is a Markovian marked point process with intensity

$$\lambda^Y(t, z) = E[\lambda_t | X_{t-} = z] \tag{20}$$

and mark distribution $F^Y(t, Y_{t-}, \cdot)$ where

$$F^Y(t, z; dx) = \frac{m_Y(t, dx, z)}{\lambda^Y(t, z)} \tag{21}$$

which mimicks the marginal distributions of X : $\forall t \geq 0, X_t \stackrel{d}{=} Y_t$.

Proof. (Y_t) is built as the Markovian projection of (X_t) defined in Theorem 1. Since m_X is a finite measure, this entails that m^Y is also a finite measure. The intensity $\lambda^Y = m^Y(\mathbb{R}^d - \{0\})$ then satisfies :

$$\begin{aligned}\lambda^Y(t, z) &= \int_{\mathbb{R}^d - \{0\}} m_Y(t, dx; z) \\ &= \mathbb{E} \left[\int_{\mathbb{R}^d - \{0\}} m_X(t, dx, \cdot) | X_{t-} = z \right] \\ &= \mathbb{E}[\lambda_t | X_{t-} = z]\end{aligned}$$

where the Fubini theorem is used using the fact that m_X is a finite measure. This ends the proof. \square

5.2 Time changed Lévy processes

Models based on time-changed Lévy processes have been the focus of much recent work especially in mathematical finance [5]. Let L_t be a Lévy process, (b, σ^2, ν_L) be its characteristic triplet, N_L its jump measure. Define

$$X_t = L_{T_t} \quad T_t = \int_0^t \theta_s ds$$

where (θ_t) is a locally bounded \mathcal{F}_t -adapted positive cadlag process, interpreted as the rate of time change.

Theorem 4 (Markovian projection of time-changed Lévy processes). *Let L be a scalar Lévy process with triplet (b, σ^2, ν_L) and let $X_t = L(\int_0^t \theta_s ds)$ where $\theta_t > 0$ is a positive semimartingale. Define*

$$\alpha(t, x) = E[\theta_t | X_{t-} = x]$$

and suppose that $\alpha(t, z)$ is continuous in (t, z) on $[0, \infty \times \mathbb{R}]$. Then

- (X_t) has the same marginals as (Y_t) , defined as the weak solution of

$$\begin{aligned}Y_t &= X_0 + \int_0^t \sigma \sqrt{\alpha(s, Y_{s-})} dB_s + \int_0^t b\alpha(s, Y_{s-}) ds \\ &\quad + \int_0^t \int_{|z| \leq 1} z \tilde{N}(ds dz) + \int_0^t \int_{|z| > 1} z N(ds dz)\end{aligned}$$

where N is an integer-valued random measure with compensator $\nu(dt dz; \omega) = \alpha(t, Y_{t-}(\omega)) \nu_L(dz) dt$.

- The marginal distribution p_t^X of X_t is a weak solution of the forward equation:

$$\frac{\partial p_t^X}{\partial t} = L_t^* \cdot p_t^X$$

where, L_t^* is given by

$$\begin{aligned} L_t^* g(x) &= -b \frac{\partial}{\partial x} [\alpha(t, x) g(x)] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} [\alpha^2(t, x) g(x)] \\ &+ \int_{\mathbb{R}^d} \nu_L(dz) \left[g(x-z) \alpha(t, x-z) - g(x) \alpha(t, x) - 1_{\|z\| \leq 1} z \cdot \frac{\partial}{\partial x} [g(x) \alpha(t, x)] \right]. \end{aligned}$$

Proof. Consider the Lévy-Ito decomposition of L :

$$L_t = bt + \sigma W_t + \int_0^t \int_{|z| \leq 1} z \tilde{N}_L(ds dz) + \int_0^t \int_{|z| > 1} z N_L(ds dz)$$

X is then expressed as

$$\begin{aligned} X_t &= X_0 + \sigma W(T_t) + bT_t \\ &+ \int_0^{T_t} \int_{|z| \leq 1} z \tilde{N}_L(ds dz) + \int_0^{T_t} \int_{|z| > 1} z N_L(ds dz) \end{aligned}$$

$W(T_t)$ is a continuous martingale with quadratic variation $T_t = \int_0^t \theta_s ds$ so, if B is a Brownian motion independent of Z , we have

$$W(T_t) \stackrel{d}{=} \int_0^t \sqrt{\theta_s} dB_s$$

Hence X_t can be written as :

$$\begin{aligned} X_t &= X_0 + \int_0^t \sigma \sqrt{\theta_s} dB_s + \int_0^t b \theta_s ds \\ &+ \int_0^t \int_{|z| \leq 1} z \theta_s \tilde{N}_L(ds dz) + \int_0^t \int_{|z| > 1} z \theta_s N_L(ds dz) \end{aligned}$$

Using the notations of Theorem 1,

$$\beta_t^X = b \theta_t \quad \delta_t^X = \sigma \sqrt{\theta_t} \quad m_X(t, y) = \theta_t \nu_L(dy).$$

Assumption (H_1) and (H_2) are satisfied, and :

$$\begin{aligned} \beta^Y(t, z) &= \mathbb{E} [\beta_t^X | X_{t-} = z] = b \alpha(t, z); \\ \delta^Y(t, z) &= \mathbb{E} [\delta_t^X | X_{t-} = z] = \sigma \sqrt{\alpha(t, z)}; \\ m_Y(t, y, z) &= \mathbb{E} [m_X(t, y) | X_{t-} = z] = \alpha(t, z) \nu_L(dy). \end{aligned}$$

are all continuous in (t, z) on $[0, \infty[\times \mathbb{R}$, an application of Theorem 1 and Theorem 3 yields the result. \square

The impact of the random time change on the marginals can be captured by making the characteristics state dependent

$$(b\alpha(t, X_{t-}), \sigma^2 \alpha(t, X_{t-}), \alpha(t, X_{t-}) \nu_Z)$$

by introducing the *same* adjustment factor $\alpha(t, X_{t-})$ to the drift, diffusion coefficient and Lévy measure. In particular if $\alpha(t, x)$ is affine in x we get an affine process [8] where the affine dependence of the characteristics with respect to the state are restricted to be colinear, which is rather restrictive. This remark shows that time-changed Lévy processes, which in principle allow for a wide variety of choices for θ and L , may not be as flexible as apparently simpler affine models when it comes to reproducing marginal distributions.

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