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An elementary proof of an inequality of Maz'ya involving L^1 vector fields

Pierre Bousquet*, Petru Mironescu†

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Abstract

We give a short elementary proof of the inequality

$$\|D(-\Delta)^{-1}\mathbf{f}\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}), \mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n), 1 \leq q < \frac{n}{n-1},$$

essentially established by Maz'ya [4].

1 Introduction

For any $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$, we denote by $\mathbf{u} := (-\Delta)^{-1}\mathbf{f}$ the Newtonian (logarithmic for $n = 2$) potential of \mathbf{f} :

$$\mathbf{u}(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\mathbf{f}(y) dy,$$

where Γ is the fundamental solution of $-\Delta$:

$$\Gamma(x) := \begin{cases} \frac{-1}{2\pi} \ln |x| & \text{if } n = 2 \\ \frac{1}{|S^{n-1}|(n-2)} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3. \end{cases}$$

One of the main results in [3] states that

$$\|D\mathbf{u}\|_{L^{n'}(\mathbb{R}^n)} \leq C(\|\mathbf{f}\|_{L^1} + \|\operatorname{div} \mathbf{f}\|_{W^{-2,n'}}) \quad (1)$$

where $n' = n/(n-1)$.

In [4], Maz'ya established the following family of estimates related to (1)

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Theorem 0 Let $1 \leq q < n'$ and $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Assume that $\int_{\mathbb{R}^n} \mathbf{f} = \mathbf{0}$. Let $h := \operatorname{div} \mathbf{f}$.

i)] If $q > 1$ and $\nabla(-\Delta)^{-1}h \in L^1$, then

$$\|D\mathbf{u}\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1}).$$

ii)] If $q = 1$ and $(-\Delta)^{-1/2}h \in \mathcal{H}^1$ (where \mathcal{H}^1 denotes the Hardy space), then

$$\|D\mathbf{u}\|_{L^1(|x|^{-1}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|(-\Delta)^{-1/2}h\|_{\mathcal{H}^1}).$$

These estimates partly solve Open Problem 1 in [3].

The aim of this note is to unify the two statements of Theorem 0 and to present a proof both shorter and more elementary than the original one in [4]. Our result is

Theorem 1 Let $1 \leq q < n'$ and $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $h := \operatorname{div} \mathbf{f}$. If $\nabla(-\Delta)^{-1}h \in L^1$, then

$$\|D\mathbf{u}\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1}).$$

Remarks. i) In Theorem 0, it is required that $\int_{\mathbb{R}^n} \mathbf{f} = \mathbf{0}$. In fact, this equality is implied by the assumptions $\mathbf{f} \in L^1$ and $\nabla(-\Delta)^{-1}h \in L^1$. ii) If $(-\Delta)^{-1/2}h \in H$, then $\nabla(-\Delta)^{-1}h \in L^1$, but the converse is false. Thus, when $q = 1$, Theorem 1 requires a weaker assumption than Theorem 0.

We start by proving i). The Fourier transform of $\partial_j(-\Delta)^{-1}h$ is

$$F_j(\xi) = -\sum_k \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k(\xi) = -\sum_k \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k(0) + o(1) \text{ as } \xi \rightarrow 0.$$

The continuity of F_j at the origin implies $\widehat{f}_j(0) = 0$, i. e., $\int f_j = 0$.

We next briefly justify ii). If $g := (-\Delta)^{-1/2}h \in \mathcal{H}^1$, then the Riesz transforms of g satisfy $R_j g \in L^1$, $1 \leq j \leq n$, so that $\nabla(-\Delta)^{-1}h = \iota(R_1 g, \dots, R_n g) \in L^1$. In order to see that the converse is false, pick a temperate distribution g such that $R_j g \in L^1$, $1 \leq j \leq n$, but $g \notin L^1$. Such a g exists, see [5], 6.16, p. 184 and the references therein. If $\mathbf{f} := -\iota(R_1 g, \dots, R_n g) \in L^1$, then $(-\Delta)^{-1/2} \operatorname{div} \mathbf{f} = g \notin L^1$, while $\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} = -\mathbf{f} \in L^1$.

2 Proof of Theorem 1

Let $\rho_0 \in C_c^\infty(\mathbb{R}^+)$ be such that $0 \leq \rho_0 \leq 1$ and

$$\rho_0(r) = \begin{cases} 1 & \text{if } r \leq 1/4 \\ 0 & \text{if } r \geq 1/2 \end{cases}.$$

We introduce $\rho(y, x) = \rho_0(|y|/|x|)$ for $(y, x) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. For $1 \leq k \leq n$, we have

$$\partial_{x_k} \mathbf{u}(x) = c_n \int_{\mathbb{R}^n} \frac{x_k - y_k}{|x - y|^n} \mathbf{f}(y) dy = I_1(x) + I_2(x),$$

where

$$I_1(x) = c_n \int_{\mathbb{R}^n} \rho(y, x) \frac{x_k - y_k}{|x - y|^n} \mathbf{f}(y) dy, \quad I_2(x) = c_n \int_{\mathbb{R}^n} (1 - \rho(y, x)) \frac{x_k - y_k}{|x - y|^n} \mathbf{f}(y) dy.$$

We estimate $\|I_2\|_{L^q(|x|^{n(q-1)-q} dx)}$ using the following straightforward consequence of Hölder's inequality

$$\left\| x \mapsto \int f(y) g(x, y) dy \right\|_{L^q} \leq \|f\|_{L^1} \sup_y \|g(\cdot, y)\|_{L^q(dx)}. \quad (2)$$

We have

$$|I_2(x)| \leq c \int_{|y| \geq |x|/4} |\mathbf{f}(y)| \frac{dy}{|x - y|^{n-1}}$$

so that, by (2),

$$\|I_2\|_{L^q(|x|^{n(q-1)-q} dx)} \leq c \int |\mathbf{f}(y)| dy \sup_{y \neq 0} \left\{ \int_{|x| \leq 4|y|} \frac{|x|^{n(q-1)-q}}{|x - y|^{(n-1)q}} \right\}^{1/q}.$$

The quantity $\int_{|x| \leq 4|y|} \frac{|x|^{n(q-1)-q}}{|x - y|^{(n-1)q}} dx$ is finite and does not depend on $y \neq 0$ (since it depends only on the norm of y and is homogeneous of degree 0). This implies that

$$\|I_2\|_{L^q(|x|^{n(q-1)-q} dx)} \leq c \int |\mathbf{f}(y)| dy.$$

In order to estimate $\|I_1\|_{L^q(|x|^{n(q-1)-q} dx)}$, we note that for $|y| \leq |x|/2$ we have

$$\left| \frac{x_k - y_k}{|x - y|^n} - \frac{x_k}{|x|^n} \right| \leq c \frac{|y|}{|x|^n}.$$

Thus

$$|I_1(x)| \leq c \left\{ \frac{1}{|x|^n} \int_{|y| \leq |x|/2} |\mathbf{f}(y)| |y| dy + J(x) \right\}$$

where

$$J(x) := \frac{1}{|x|^{n-1}} \left| \int_{\mathbb{R}^n} \rho(y, x) \mathbf{f}(y) dy \right|.$$

Using (2), we obtain

$$\begin{aligned} \|I_1\|_{L^q(|x|^{n(q-1)-q} dx)} &\leq c \int |\mathbf{f}(y)| |y| dy \sup_{y \neq 0} \left\{ \int_{|x| \geq 2|y|} \frac{dx}{|x|^{n+q}} \right\}^{1/q} \\ &+ c \|J\|_{L^q(|x|^{n(q-1)-q} dx)} \leq c \|\mathbf{f}\|_{L^1} + c \|J\|_{L^q(|x|^{n(q-1)-q} dx)}. \end{aligned}$$

It then remains to estimate $\|J\|_{L^q(|x|^{n(q-1)-q}dx)}$.

To start with, we assume, in addition to the hypotheses of Theorem 1, that $\mathbf{f} \in C^\infty$. Then we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \operatorname{div} (y_1 \rho(y, x) \mathbf{f}(y)) \, dy \\ &= \int_{\mathbb{R}^n} \left[\rho(y, x) f_1(y) + \frac{y_1}{|y||x|} \rho'_0 \left(\frac{|y|}{|x|} \right) \sum_i y_i f_i(y) + y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \right] \, dy. \end{aligned}$$

Thus

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \rho(y, x) f_1(y) \, dy \right| &\leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} |\mathbf{f}(y)| \, dy \\ &+ \left| \int_{\mathbb{R}^n} y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \, dy \right|. \end{aligned} \quad (3)$$

We claim that, with $h = \operatorname{div} \mathbf{f}$, we have

$$\left| \int_{\mathbb{R}^n} y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \, dy \right| \leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} |\nabla(-\Delta)^{-1} h(y)| \, dy. \quad (4)$$

Indeed, let $\mathbf{k} := \nabla(y_1 \rho(y, x))$ and

$$\mathbf{l} := \left(-\frac{\partial}{\partial y_2}(y_2 \rho(y, x)), \frac{\partial}{\partial y_1}(y_2 \rho(y, x)), 0, \dots, 0 \right).$$

Then $\operatorname{div} \mathbf{k} = \Delta(y_1 \rho(y, x))$, $\operatorname{div} \mathbf{l} = 0$ and

$$(\mathbf{k} + \mathbf{l})_i = \begin{cases} y_1 \frac{\partial}{\partial y_1} \rho(y, x) - y_2 \frac{\partial}{\partial y_2} \rho(y, x) & \text{if } i = 1, \\ y_1 \frac{\partial}{\partial y_2} \rho(y, x) + y_2 \frac{\partial}{\partial y_1} \rho(y, x) & \text{if } i = 2, \\ y_1 \frac{\partial}{\partial y_i} \rho(y, x) & \text{if } i \geq 3. \end{cases}$$

Thus $|\mathbf{k} + \mathbf{l}| \leq c \frac{|y|}{|x|} \mathbf{1}_{|y| \leq |x|/2}$.

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \, dy &= - \int_{\mathbb{R}^n} \operatorname{div} (\mathbf{k} + \mathbf{l}) (-\Delta)^{-1} (\operatorname{div} \mathbf{f}) \\ &= \int_{\mathbb{R}^n} (\mathbf{k} + \mathbf{l}) \nabla (-\Delta)^{-1} (\operatorname{div} \mathbf{f}). \end{aligned}$$

By taking the absolute values in the above identity, we find that (4) holds under the additional assumption that \mathbf{f} is smooth. The general case follows by noting that $(-\Delta)^{-1} \operatorname{div}$ and $\nabla(-\Delta)^{-1} \operatorname{div}$ commute with convolution of vector fields with a scalar mollifier. Applying (4) to $\mathbf{f} * \rho_\varepsilon$, where ρ is a compactly supported mollifier, and letting $\varepsilon \rightarrow 0$, we find that (4) holds for all \mathbf{f} .

Now, (3) and (4) imply that

$$\left| \int_{\mathbb{R}^n} \rho(y, x) f_1(y) dy \right| \leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} (|\mathbf{f}(y)| + |\nabla(-\Delta)^{-1}h(y)|) dy.$$

The same is true with \mathbf{f} instead of f_1 on the left hand side. Using again (2), it follows that

$$\|J\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c \{ \|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1} \}. \quad (5)$$

This completes the proof of Theorem 1. □

References

- [1] J. Bourgain, H. Brezis, *On the equation $\operatorname{div} Y=f$ and application to control of phases*, J. Amer. Math. Soc. 16 (2002) 393–426.
- [2] J. Bourgain, H. Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris 338 (2004) 539–543.
- [3] J. Bourgain, H. Brezis, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. 9 (2007) 277–315.
- [4] V. Maz'ya, *Estimates for differential operators of vector analysis involving L^1 -norm*, arXiv:0808.0414v2.
- [5] E.M. Stein, “Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals”, Princeton University Press, 1993.