



HAL
open science

An elementary proof of an inequality of Maz'ya involving L^1 -vector fields

Pierre Bousquet, Petru Mironescu

► **To cite this version:**

Pierre Bousquet, Petru Mironescu. An elementary proof of an inequality of Maz'ya involving L^1 -vector fields. Contemporary mathematics, 2011, 540 (Nonlinear partial differential equations), pp.59-63. 10.1090/conm/540/10659 . hal-00425043

HAL Id: hal-00425043

<https://hal.science/hal-00425043>

Submitted on 19 Oct 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

An elementary proof of an inequality of Maz'ya involving L^1 vector fields

Pierre Bousquet*, Petru Mironescu†

October 16 2009

Abstract

We give a short elementary proof of the inequality

$$\|D(-\Delta)^{-1}\mathbf{f}\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f}\|_{L^1}), \mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n), 1 \leq q < \frac{n}{n-1},$$

essentially established by Maz'ya [4].

1 Introduction

For any $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$, we denote by $\mathbf{u} := (-\Delta)^{-1}\mathbf{f}$ the Newtonian (logarithmic for $n = 2$) potential of \mathbf{f} :

$$\mathbf{u}(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\mathbf{f}(y) dy,$$

where Γ is the fundamental solution of $-\Delta$:

$$\Gamma(x) := \begin{cases} \frac{-1}{2\pi} \ln |x| & \text{if } n = 2 \\ \frac{1}{|S^{n-1}|(n-2)} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3. \end{cases}$$

One of the main results in [3] states that

$$\|D\mathbf{u}\|_{L^{n'}(\mathbb{R}^n)} \leq C(\|\mathbf{f}\|_{L^1} + \|\operatorname{div} \mathbf{f}\|_{W^{-2,n'}}) \quad (1)$$

where $n' = n/(n-1)$.

In [4], Maz'ya established the following family of estimates related to (1)

*Laboratoire d'analyse, topologie, probabilités UMR6632, Université Aix-Marseille 1, CMI 39, rue Frédéric Joliot Curie 13453 Marseille Cedex 13, France. bousquet@cmi.univ-mrs.fr

†Université de Lyon; Université Lyon 1; INSA de Lyon, F-69621; Ecole Centrale de Lyon; CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France. mironescu@math.univ-lyon1.fr

Theorem 0 Let $1 \leq q < n'$ and $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Assume that $\int_{\mathbb{R}^n} \mathbf{f} = \mathbf{0}$. Let $h := \operatorname{div} \mathbf{f}$.

i)] If $q > 1$ and $\nabla(-\Delta)^{-1}h \in L^1$, then

$$\|D\mathbf{u}\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1}).$$

ii)] If $q = 1$ and $(-\Delta)^{-1/2}h \in \mathcal{H}^1$ (where \mathcal{H}^1 denotes the Hardy space), then

$$\|D\mathbf{u}\|_{L^1(|x|^{-1}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|(-\Delta)^{-1/2}h\|_{\mathcal{H}^1}).$$

These estimates partly solve Open Problem 1 in [3].

The aim of this note is to unify the two statements of Theorem 0 and to present a proof both shorter and more elementary than the original one in [4]. Our result is

Theorem 1 Let $1 \leq q < n'$ and $\mathbf{f} \in L^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $h := \operatorname{div} \mathbf{f}$. If $\nabla(-\Delta)^{-1}h \in L^1$, then

$$\|D\mathbf{u}\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c(\|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1}).$$

Remarks. i) In Theorem 0, it is required that $\int_{\mathbb{R}^n} \mathbf{f} = \mathbf{0}$. In fact, this equality is implied by the assumptions $\mathbf{f} \in L^1$ and $\nabla(-\Delta)^{-1}h \in L^1$. ii) If $(-\Delta)^{-1/2}h \in H$, then $\nabla(-\Delta)^{-1}h \in L^1$, but the converse is false. Thus, when $q = 1$, Theorem 1 requires a weaker assumption than Theorem 0.

We start by proving i). The Fourier transform of $\partial_j(-\Delta)^{-1}h$ is

$$F_j(\xi) = -\sum_k \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k(\xi) = -\sum_k \frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}_k(0) + o(1) \text{ as } \xi \rightarrow 0.$$

The continuity of F_j at the origin implies $\widehat{f}_j(0) = 0$, i. e., $\int f_j = 0$.

We next briefly justify ii). If $g := (-\Delta)^{-1/2}h \in \mathcal{H}^1$, then the Riesz transforms of g satisfy $R_j g \in L^1$, $1 \leq j \leq n$, so that $\nabla(-\Delta)^{-1}h = \iota(R_1 g, \dots, R_n g) \in L^1$. In order to see that the converse is false, pick a temperate distribution g such that $R_j g \in L^1$, $1 \leq j \leq n$, but $g \notin L^1$. Such a g exists, see [5], 6.16, p. 184 and the references therein. If $\mathbf{f} := -\iota(R_1 g, \dots, R_n g) \in L^1$, then $(-\Delta)^{-1/2} \operatorname{div} \mathbf{f} = g \notin L^1$, while $\nabla(-\Delta)^{-1} \operatorname{div} \mathbf{f} = -\mathbf{f} \in L^1$.

2 Proof of Theorem 1

Let $\rho_0 \in C_c^\infty(\mathbb{R}^+)$ be such that $0 \leq \rho_0 \leq 1$ and

$$\rho_0(r) = \begin{cases} 1 & \text{if } r \leq 1/4 \\ 0 & \text{if } r \geq 1/2 \end{cases}.$$

We introduce $\rho(y, x) = \rho_0(|y|/|x|)$ for $(y, x) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. For $1 \leq k \leq n$, we have

$$\partial_{x_k} \mathbf{u}(x) = c_n \int_{\mathbb{R}^n} \frac{x_k - y_k}{|x - y|^n} \mathbf{f}(y) dy = I_1(x) + I_2(x),$$

where

$$I_1(x) = c_n \int_{\mathbb{R}^n} \rho(y, x) \frac{x_k - y_k}{|x - y|^n} \mathbf{f}(y) dy, \quad I_2(x) = c_n \int_{\mathbb{R}^n} (1 - \rho(y, x)) \frac{x_k - y_k}{|x - y|^n} \mathbf{f}(y) dy.$$

We estimate $\|I_2\|_{L^q(|x|^{n(q-1)-q} dx)}$ using the following straightforward consequence of Hölder's inequality

$$\left\| x \mapsto \int f(y)g(x, y) dy \right\|_{L^q} \leq \|f\|_{L^1} \sup_y \|g(\cdot, y)\|_{L^q(dx)}. \quad (2)$$

We have

$$|I_2(x)| \leq c \int_{|y| \geq |x|/4} |\mathbf{f}(y)| \frac{dy}{|x - y|^{n-1}}$$

so that, by (2),

$$\|I_2\|_{L^q(|x|^{n(q-1)-q} dx)} \leq c \int |\mathbf{f}(y)| dy \sup_{y \neq 0} \left\{ \int_{|x| \leq 4|y|} \frac{|x|^{n(q-1)-q}}{|x - y|^{(n-1)q}} \right\}^{1/q}.$$

The quantity $\int_{|x| \leq 4|y|} \frac{|x|^{n(q-1)-q}}{|x - y|^{(n-1)q}} dx$ is finite and does not depend on $y \neq 0$ (since it depends only on the norm of y and is homogeneous of degree 0). This implies that

$$\|I_2\|_{L^q(|x|^{n(q-1)-q} dx)} \leq c \int |\mathbf{f}(y)| dy.$$

In order to estimate $\|I_1\|_{L^q(|x|^{n(q-1)-q} dx)}$, we note that for $|y| \leq |x|/2$ we have

$$\left| \frac{x_k - y_k}{|x - y|^n} - \frac{x_k}{|x|^n} \right| \leq c \frac{|y|}{|x|^n}.$$

Thus

$$|I_1(x)| \leq c \left\{ \frac{1}{|x|^n} \int_{|y| \leq |x|/2} |\mathbf{f}(y)| |y| dy + J(x) \right\}$$

where

$$J(x) := \frac{1}{|x|^{n-1}} \left| \int_{\mathbb{R}^n} \rho(y, x) \mathbf{f}(y) dy \right|.$$

Using (2), we obtain

$$\begin{aligned} \|I_1\|_{L^q(|x|^{n(q-1)-q} dx)} &\leq c \int |\mathbf{f}(y)| |y| dy \sup_{y \neq 0} \left\{ \int_{|x| \geq 2|y|} \frac{dx}{|x|^{n+q}} \right\}^{1/q} \\ &+ c \|J\|_{L^q(|x|^{n(q-1)-q} dx)} \leq c \|\mathbf{f}\|_{L^1} + c \|J\|_{L^q(|x|^{n(q-1)-q} dx)}. \end{aligned}$$

It then remains to estimate $\|J\|_{L^q(|x|^{n(q-1)-q}dx)}$.

To start with, we assume, in addition to the hypotheses of Theorem 1, that $\mathbf{f} \in C^\infty$. Then we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \operatorname{div} (y_1 \rho(y, x) \mathbf{f}(y)) \, dy \\ &= \int_{\mathbb{R}^n} \left[\rho(y, x) f_1(y) + \frac{y_1}{|y||x|} \rho'_0 \left(\frac{|y|}{|x|} \right) \sum_i y_i f_i(y) + y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \right] \, dy. \end{aligned}$$

Thus

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \rho(y, x) f_1(y) \, dy \right| &\leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} |\mathbf{f}(y)| \, dy \\ &+ \left| \int_{\mathbb{R}^n} y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \, dy \right|. \end{aligned} \quad (3)$$

We claim that, with $h = \operatorname{div} \mathbf{f}$, we have

$$\left| \int_{\mathbb{R}^n} y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \, dy \right| \leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} |\nabla(-\Delta)^{-1} h(y)| \, dy. \quad (4)$$

Indeed, let $\mathbf{k} := \nabla(y_1 \rho(y, x))$ and

$$\mathbf{l} := \left(-\frac{\partial}{\partial y_2}(y_2 \rho(y, x)), \frac{\partial}{\partial y_1}(y_2 \rho(y, x)), 0, \dots, 0 \right).$$

Then $\operatorname{div} \mathbf{k} = \Delta(y_1 \rho(y, x))$, $\operatorname{div} \mathbf{l} = 0$ and

$$(\mathbf{k} + \mathbf{l})_i = \begin{cases} y_1 \frac{\partial}{\partial y_1} \rho(y, x) - y_2 \frac{\partial}{\partial y_2} \rho(y, x) & \text{if } i = 1, \\ y_1 \frac{\partial}{\partial y_2} \rho(y, x) + y_2 \frac{\partial}{\partial y_1} \rho(y, x) & \text{if } i = 2, \\ y_1 \frac{\partial}{\partial y_i} \rho(y, x) & \text{if } i \geq 3. \end{cases}$$

Thus $|\mathbf{k} + \mathbf{l}| \leq c \frac{|y|}{|x|} \mathbf{1}_{|y| \leq |x|/2}$.

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} y_1 \rho(y, x) \operatorname{div} \mathbf{f}(y) \, dy &= - \int_{\mathbb{R}^n} \operatorname{div} (\mathbf{k} + \mathbf{l}) (-\Delta)^{-1} (\operatorname{div} \mathbf{f}) \\ &= \int_{\mathbb{R}^n} (\mathbf{k} + \mathbf{l}) \nabla (-\Delta)^{-1} (\operatorname{div} \mathbf{f}). \end{aligned}$$

By taking the absolute values in the above identity, we find that (4) holds under the additional assumption that \mathbf{f} is smooth. The general case follows by noting that $(-\Delta)^{-1} \operatorname{div}$ and $\nabla(-\Delta)^{-1} \operatorname{div}$ commute with convolution of vector fields with a scalar mollifier. Applying (4) to $\mathbf{f} * \rho_\varepsilon$, where ρ is a compactly supported mollifier, and letting $\varepsilon \rightarrow 0$, we find that (4) holds for all \mathbf{f} .

Now, (3) and (4) imply that

$$\left| \int_{\mathbb{R}^n} \rho(y, x) f_1(y) dy \right| \leq c \int_{|y| \leq |x|/2} \frac{|y|}{|x|} (|\mathbf{f}(y)| + |\nabla(-\Delta)^{-1}h(y)|) dy.$$

The same is true with \mathbf{f} instead of f_1 on the left hand side. Using again (2), it follows that

$$\|J\|_{L^q(|x|^{n(q-1)-q}dx)} \leq c \{ \|\mathbf{f}\|_{L^1} + \|\nabla(-\Delta)^{-1}h\|_{L^1} \}. \quad (5)$$

This completes the proof of Theorem 1. □

References

- [1] J. Bourgain, H. Brezis, *On the equation $\operatorname{div} Y=f$ and application to control of phases*, J. Amer. Math. Soc. 16 (2002) 393–426.
- [2] J. Bourgain, H. Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris 338 (2004) 539–543.
- [3] J. Bourgain, H. Brezis, *New estimates for elliptic equations and Hodge type systems*, J. Eur. Math. Soc. 9 (2007) 277–315.
- [4] V. Maz'ya, *Estimates for differential operators of vector analysis involving L^1 -norm*, arXiv:0808.0414v2.
- [5] E.M. Stein, “Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals”, Princeton University Press, 1993.