Singular limit of a two-phase flow problem in porous medium as the air viscosity tends to zero

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Abstract

In this paper we consider a two-phase flow problem in porous media and study its singular limit as the viscosity of the air tends to zero; more precisely, we prove the convergence of subsequences to solutions of a generalized Richards model.

1 Introduction

Hydrologists have studied air-water flow in soils, mainly using the so-called Richards approximation. At least two hypotheses are physically required for this model to be applicable: the water pressure in the saturated region must be larger than the atmospheric pressure and all the unsaturated regions must have a boundary connected to the surface. However, in many situations, these hypotheses are not satisfied and a more general two-phase flow model must be considered. This work explores the limit of this general model as the viscosity of the air tends to zero, which is one of the hypotheses required in the Richards model. To that purpose we prove the existence of a weak solution of the two-phase flow problem and prove estimates which are uniform in the air viscosity. In this paper, we assume that the air and water phases are incompressible and immiscible. The geometric domain is supposed to be horizontal, homogeneous and isotropic. Our starting point is the following two-phase flow model, which one can deduce from Darcy's law

$$(\mathcal{TP}) \begin{cases} u_t - div(k_w(u)\nabla(p)) = s_w \\ (1-u)_t - div(\frac{1}{\mu}k_a(u)\nabla(p + p_c(u))) = s_a, \end{cases}$$

*This work was supported by the GNR MoMaS (PACEN/CNRS, ANDRA, BRGM, CEA, EdF, IRSN), France.

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where u and p are respectively the saturation and the pressure of the water phase, k_w and k_a are respectively the relative permeabilities of the water and the air phase, μ is the ratio between the viscosity of the air phase and that of the water phase, p_c is the capillary pressure, s_w is an internal source term for the water phase and s_a is an internal source term for the air phase; these source terms are used to represent exchanges with the outside. We suppose in particular that the physical functions k_w, k_a and p_c only depend on the saturation u of the water phase, and that $k_w(1) = k_a(0) = 1$. The aim of this paper is the study of the limit of the two-phase flow problem as $\mu \downarrow 0$.

The classical Richards model as formulated by the engineers is given by

$$(\mathcal{R}) \begin{cases} u_t - div(k_w(u)\nabla p) = s_w \\ u = p_c^{-1}(p_{atm} - p). \end{cases}$$

where the properties of capillary pressure $p_c = p_c(u)$ are describes in hypothesis (H_8) below. For the existence and uniqueness of the solution of Richards model together with suitable initial and boundary conditions as well as qualitative properties of the solution and methods for numerical approximations we refer to [1], [6], [10], [11]. In this article, we will show that the singular limit as $\mu \downarrow 0$ of the two phase flow problem $(\mathcal{TP})$ has the form

$$(\mathcal{FBP}) \begin{cases} u_t - div(k_w(u)\nabla p) = s_w \\ u = 1 \text{ or } \nabla(p + p_c(u)) = 0 \text{ a.e. in } \Omega \times (0,T). \end{cases}$$

We remark that a solution of $(\mathcal{R})$ with $u > 0$ satisfies $(\mathcal{FBP})$.

This paper is organized as follows. In Section 2 we present a complete mathematical formulation of the problem, and state the main mathematical results, which include a precise formulation of the singular limit problem. We give a sequence of regularized problems in Section 3, and prove the existence of a classical solution. In Section 4 we present a priori estimates, which are uniform in an extra regularization parameter δ and in the air viscosity μ. In Section 5, we let $\delta \downarrow 0$ and prove that the solution converges to a solution of the two phase flow problem. We study its limiting behavior as the air viscosity μ tends to zero in Section 6. Finally in Section 7 we propose a finite volume algorithm in a one dimensional context and present a variety of numerical solutions.

2 Mathematical formulation and main results

We consider the two-phase flow problem

$$(S^\mu)\begin{cases} u_t = div\Big(k_w(u)\nabla p\Big) + f^\mu(c)\overline{s} - f^\mu(u)\underline{s}, & \text{in } Q_T, \quad (2.1)\\ (1-u)_t = div\Big(\dfrac{1}{\mu}k_a(u)\nabla(p+p_c(u))\Big) & \\ \qquad\qquad + (1-f^\mu(c))\overline{s} - (1-f^\mu(u))\underline{s}, & \text{in } Q_T, \quad (2.2)\\ \displaystyle\int_\Omega p(x,t)dx = 0, & \text{for } t\in(0,T), \quad (2.3)\\ \nabla p.n = 0, & \text{on } \partial\Omega\times(0,T), \quad (2.4)\\ \nabla(p+p_c(u)).n = 0, & \text{on } \partial\Omega\times(0,T), \quad (2.5)\\ u(x,0) = u_0(x), & \text{for } x\in\Omega, \quad (2.6) \end{cases}$$

where T is a positive constant, $Q_T := \Omega\times(0,T)$ and where we suppose that

- (H_1) Ω is a smooth bounded domain of $\mathbf{R}^N$ where the space dimension N is arbitrary,
- (H_2) $u_m \in (0,1)$,
- (H_3) $c\in L^\infty(\Omega\times(0,T))$ and $u_m \le c \le 1$,
- (H_4) $u_0\in L^\infty(\Omega)$ and $u_m\le u_0\le 1$,
- (H_5) $\overline{s}\in L^2(\Omega)$, $\overline{s}\ge 0$, $\underline{s}\in L^2(\Omega)$, $\underline{s}\ge 0$ and $\displaystyle\int_\Omega(\overline{s}(x)-\underline{s}(x))dx = 0$,
- (H_6) $k_w\in C^2([0,1])$, $k_w'\ge 0$, $k_w(0)=0$, $k_w(1)=1$ and $k_w(u_m)>0$,
- (H_7) $k_a\in C^2([0,1])$, $k_a'\le 0$, $k_a(1)=0$, $k_a(0)=1$ and $k_a(s)>0$ for all $s\in[0,1)$,
- (H_8) $p_c\in C^0([0,1])\cup C^3([0,1))$, $p_c'<0$ and $\sup_{s\in[0,1)}(-k_a(s)p_c'(s))<+\infty$,
- (H_9) $\mu\in(0,1]$.

In this model, u and p are respectively the saturation and the pressure of the water phase, k_w and k_a are respectively the mobilities of the water phase and the mobility of the non-water phase and p_c is the capillary pressure. We assume in particular that the permeability functions k_w, k_a and the capillary pressure p_c only depend on the saturation u of the water phase. Here, we suppose that the flow of the water phase in the reservoir is driven by an injection term $f^\mu(c)\overline{s}$ and an extraction term $f^\mu(u)\underline{s}$ where $\overline{s}$ and $\underline{s}$ are given space dependent functions, c is the saturation of the injected fluid; if $c=1$, only water will be injected, if $c=0$, only air will be injected, whereas a mixture of water and air will be injected if $0<c<1$. The function f^μ is the fractional flow of the water phase, namely

$$f^\mu(s) = \frac{k_w(s)}{M^\mu(s)},\ \text{with } M^\mu(s) = k_w(s) + \frac{1}{\mu}k_a(s). \quad (2.7)$$

In particular, we remark that

$$f^\mu(s) \text{ is non decreasing.} \quad (2.8)$$

Next we introduce a set of notations, which will be useful in the sequel.

$$g(s) = -\int_0^s k_a(\tau)p_c'(\tau)d\tau, \tag{2.9}$$

$$\zeta(s) = \int_0^s \sqrt{k_a(\tau)}p_c'(\tau)d\tau, \tag{2.10}$$

$$\mathcal{Q}^\mu(s) = \int_0^s f^\mu(\tau)p_c'(\tau)d\tau, \tag{2.11}$$

and

$$\mathcal{R}^\mu(s) = \int_0^s \frac{k_a(\tau)}{k_a(\tau)+\mu k_w(\tau)}p_c'(\tau)d\tau, \tag{2.12}$$

for all $s \in [0,1]$. This implies in particular that

$$\mathcal{R}^\mu(s) + \mathcal{Q}^\mu(s) = p_c(s) - p_c(0), \text{ for all } s \in [0,1]. \tag{2.13}$$

Definition 2.1 *The pair (u^μ, p^μ) is a weak solution of Problem (S^μ) if*

$$u^\mu \in L^\infty(\Omega\times(0,T)), \qquad \text{with } 0 \le u^\mu \le 1 \text{ in } Q_T,$$

$$p^\mu \in L^2(0,T;H^1(\Omega)), \qquad \int_\Omega p^\mu(x,t)dx = 0 \text{ for almost every } t \in (0,T),$$

$$g(u^\mu) \in L^2(0,T;H^1(\Omega)),$$

with

$$\begin{aligned}\int_0^T\int_\Omega u^\mu\varphi_t dxdt = &\int_0^T\int_\Omega k_w(u^\mu)\nabla p^\mu.\nabla\varphi dxdt - \int_0^T\int_\Omega \Big(f^\mu(c)\overline{s} - f^\mu(u^\mu)\underline{s}\Big)\varphi dxdt\\ &-\int_\Omega u_0(x)\varphi(x,0)dx,\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\int_0^T\int_\Omega \Big(1-u^\mu\Big)\varphi_t dxdt = \int_0^T\int_\Omega \frac{1}{\mu}k_a(u^\mu)\Big(\nabla p^\mu + \nabla p_c(u^\mu)\Big).\nabla\varphi dxdt\\ &-\int_0^T\int_\Omega \Big((1-f^\mu(c))\overline{s} - (1-f^\mu(u^\mu))\underline{s}\Big)\varphi dxdt - \int_\Omega \Big(1-u_0(x)\Big)\varphi(x,0)dx,\end{aligned} \tag{2.15}$$

for all φ in $\mathcal{C} := \{w \in W_2^{2,1}(Q_T), w(.,T) = 0 \text{ in } \Omega\}$.

Our first result, which we prove in Section 3, is the following

Theorem 2.2 *Suppose that the hypotheses* $(H_1)-(H_9)$ *are satisfied, then there exists a weak solution* (u^μ, p^μ) *of Problem* (S^μ).

Next we define the discontinuous function χ by

$$\chi(s) := \begin{cases} 0 & \text{if } s \in [0,1) \\ 1 & \text{if } s = 1, \end{cases}$$

as well as the graph

$$H(s) := \begin{cases} 0 & \text{if } s \in [0,1) \\ [0,1] & \text{if } s = 1. \end{cases}$$

The main goal of this paper is to prove the following convergence result,

Theorem 2.3 *Suppose that the hypotheses* $(H_1)-(H_9)$ *are satisfied, then there exists a subsequence* $((u^{\mu_n}, p^{\mu_n}))_{n\in N}$ *of weak solutions of Problem* (S^{μ_n}) *and functions* u, p, $\hat{f}$ *such that*

$$u \in L^\infty(Q_T), \quad 0 \le u \le 1 \text{ in } Q_T,$$

$$\hat{f} \in L^\infty(Q_T), \quad 0 \le \hat{f} \le 1 \text{ in } Q_T,$$

$$p \in L^2(0,T; H^1(\Omega)),$$

$$k_a(u)\nabla p_c(u) \in L^2(\Omega \times (0,T)),$$

and

$$(u^{\mu_n})_{n\in N} \text{ tends to } u \text{ strongly in } L^2(Q_T),$$
$$(p^{\mu_n})_{n\in N} \text{ tends to } p \text{ weakly in } L^2(0,T; H^1(\Omega)),$$

as μ_n *tends to zero and*

$$\begin{aligned} \int_0^T \int_\Omega u\varphi_t dxdt = & \int_0^T \int_\Omega k_w(u)\nabla p.\nabla\varphi dxdt - \int_0^T \int_\Omega \Big(\chi(c)\overline{s} - \hat{f}\underline{s}\Big)\varphi dxdt \\ & - \int_\Omega u_0(x)\varphi(x,0)dx, \end{aligned} \tag{2.16}$$

for all $\varphi \in \mathcal{C}$, *where* $\hat{f}(x,t) \in H(u(x,t))$ *for* $(x,t) \in Q_T$. *Moreover we also have that*

$$\int_0^T \int_\Omega \Big[k_a(u)\Big]^2 \Big[\nabla p + \nabla p_c(u)\Big]^2 dxdt = 0 \tag{2.17}$$

and

$$\int_\Omega p(x,t)dx = 0, \text{ for almost every } t \in (0,T). \tag{2.18}$$

Formally, u satisfies the following limit problem

$$\begin{cases} u_t = div\Big(k_w(u)\nabla p\Big) + \chi(c)\bar{s} - \hat{f}\underline{s}, & \text{in } Q_T, \\ \nabla u.n = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{for } x \in \Omega. \end{cases}$$

More precisely the following corollary holds

Corollary 2.4 *Suppose that $u < 1$ in $\mathcal{O} = \cup_{t\in[\tau,T]}\Omega_t$, where $\tau > 0$ and Ω_t, for $t \in [\tau, T]$, are smooth subdomains of Ω and $\mathcal{O}$ is a smooth domain of $\Omega \times [\tau, T]$ and that $u = 1$ in $Q_T \setminus \overline{\mathcal{O}}$ then*

$$p(x,t) = -p_c(u(x,t)) + constant(t), \ \textit{for all } (x,t) \in \mathcal{O}$$

and u satisfies

$$\begin{cases} u_t = -div\Big(k_w(u)\nabla p_c(u)\Big) + \chi(c)\bar{s}, & \textit{in } \mathcal{O}, \\ \dfrac{\partial u}{\partial n} = 0, & \textit{on } \partial\mathcal{O} \cap \Big(\partial\Omega \times (0,T)\Big), \\ u = 1, & \textit{elsewhere on } \partial\mathcal{O}, \\ u(x,0) = u_0(x), & \textit{for } x \in \Omega. \end{cases}$$

Finally we remark that another form of the limit problem involves a parabolic equation, which is close to the standard Richards equation. Indeed if we set $\phi(s) := p_c(0) - p_c(s)$ and denote by β the inverse function of ϕ, the function $v := \phi(u)$ is a weak solution of the problem

$$\begin{cases} \beta(v)_t = div\Big(k_w(\beta(v))\nabla v\Big) + \chi(c)\bar{s} - \hat{f}\underline{s}, & \text{in } Q_T, \\ \nabla\beta(v).n = 0, & \text{on } \partial\Omega \times (0,T), \\ \beta(v)(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases}$$

with $\hat{f} \in H(\beta(v))$.

3 Existence of a solution of an approximate problem (S_δ^μ) of Problem (S^μ)

Let δ be an arbitrary positive constant. In order to prove the existence of a solution of Problem (S^μ) we introduce a sequence of regularized problems (S_δ^μ), namely

$$(S_\delta^\mu)\begin{cases} u_t = div\Big(k_w(u)\nabla p\Big) + f^\mu(c_\delta)\overline{s}_\delta & \\ \qquad - f^\mu(u)\Big(\underline{s}_\delta + \fint_\Omega(\overline{s}_\delta - \underline{s}_\delta)dx\Big), & \text{in } \Omega\times(0,T), \quad (3.1)\\ (1-u)_t = div\Big(\dfrac{1}{\mu}k_a(u)\nabla(p+p_c(u))\Big) + \Big(1-f^\mu(c_\delta)\Big)\overline{s}_\delta & \\ \qquad - \Big(1-f^\mu(u)\Big)\Big(\underline{s}_\delta + \fint_\Omega(\overline{s}_\delta - \underline{s}_\delta)dx\Big), & \text{in } \Omega\times(0,T), \quad (3.2)\\ \displaystyle\int_\Omega p(x,t)dx = 0, & \text{for } t\in(0,T), \quad (3.3)\\ \nabla p.n = 0, & \text{on } \partial\Omega\times(0,T), \quad (3.4)\\ \nabla(p+p_c(u)).n = 0, & \text{on } \partial\Omega\times(0,T), \quad (3.5)\\ u(x,0) = u_0^\delta(x), & \text{for } x\in\Omega, \quad (3.6)\end{cases}$$

where u_0^δ, c_δ, $\overline{s}_\delta$ and $\underline{s}_\delta$ are smooth functions such that u_0^δ tends to u_0 in $L^2(\Omega)$ and c_δ, $\overline{s}_\delta$ and $\underline{s}_\delta$ tend respectively to c, $\overline{s}$ and $\underline{s}$ in $L^2(Q_T)$, as $\delta\downarrow 0$. In particular we suppose that there exists a positive constant C such that

$$\underline{s}_\delta \geq 0, \quad \overline{s}_\delta \geq 0 \text{ and } \int_\Omega \underline{s}_\delta^2 + \int_\Omega \overline{s}_\delta^2 \leq C. \tag{3.7}$$

Moreover we suppose that u_0^δ, c_δ satisfy

$$0 < u_m \leq u_0^\delta \leq 1-\delta < 1 \quad \text{in } \Omega \tag{3.8}$$

and

$$0 < u_m \leq c_\delta \leq 1-\delta < 1 \quad \text{in } Q_T. \tag{3.9}$$

Adding up (3.1) and (3.2) we deduce the equation

$$-div\Big(M^\mu(u)\nabla p + \frac{1}{\mu}k_a(u)\nabla(p_c(u))\Big) = \overline{s}_\delta - \underline{s}_\delta - \fint_\Omega(\overline{s}_\delta - \underline{s}_\delta)dx. \tag{3.10}$$

We formulate below an equivalent form of Problem (S_δ^μ). To that purpose we define the global pressure, $\mathcal{P}$, by

$$\mathcal{P} := p + \mathcal{R}^\mu(u) = p + \int_0^u \frac{k_a(\tau)}{k_a(\tau)+\mu k_w(\tau)}p_c'(\tau)d\tau,$$

so that (3.10) gives

$$-div\Big(M^{\mu}(u)\nabla\mathcal{P}\Big)=\overline{s}_{\delta}-\underline{s}_{\delta}-\fint_{\Omega}(\overline{s}_{\delta}-\underline{s}_{\delta})dx. \tag{3.11}$$

We rewrite the equation (3.1) of Problem (S_{δ}^{μ}) as

$$u_t=\Delta\psi^{\mu}(u)+div\Big(f^{\mu}(u)M^{\mu}(u)\nabla\mathcal{P}\Big)+f^{\mu}(c_{\delta})\overline{s}_{\delta}-f^{\mu}(u)\Big(\underline{s}_{\delta}+\fint_{\Omega}(\overline{s}_{\delta}-\underline{s}_{\delta})dx\Big), \tag{3.12}$$

where

$$\psi^{\mu}(s)=-\frac{1}{\mu}\int_0^s\frac{k_a(\tau)k_w(\tau)}{M^{\mu}(\tau)}p_c'(\tau)d\tau \tag{3.13}$$

is continuous on $[0,1]$ and differentiable on $[0,1)$. Multiplying (3.11) by $f^{\mu}(u_{\delta}^{\mu})$ and adding the result to (3.12) we deduce that

$$u_t=\Delta\psi^{\mu}(u)+M^{\mu}(u)\nabla f^{\mu}(u).\nabla\mathcal{P}+\Big(f^{\mu}(c_{\delta})-f^{\mu}(u)\Big)\overline{s}_{\delta}.$$

This yields a problem equivalent to (S_{δ}^{μ}), namely

$$(\tilde{S}_{\delta}^{\mu})\begin{cases}
u_t=\Delta\psi^{\mu}(u)+M^{\mu}(u)\nabla f^{\mu}(u).\nabla\mathcal{P}+[f^{\mu}(c_{\delta})-f^{\mu}(u)]\overline{s}_{\delta}, & \text{in } \Omega\times(0,T), \quad (3.14)\\
-div\Big(M^{\mu}(u)\nabla\mathcal{P}\Big)=\overline{s}_{\delta}-\underline{s}_{\delta}-\fint_{\Omega}(\overline{s}_{\delta}-\underline{s}_{\delta})dx, & \text{in } \Omega\times(0,T), \quad (3.15)\\
\displaystyle\int_{\Omega}\mathcal{P}(x,t)dx=\int_{\Omega}\mathcal{R}^{\mu}(u(x,t))dx, & \text{for } t\in(0,T), \quad (3.16)\\
\nabla\mathcal{P}.n=0, & \text{on } \partial\Omega\times(0,T), \quad (3.17)\\
\nabla\psi^{\mu}(u).n=0, & \text{on } \partial\Omega\times(0,T), \quad (3.18)\\
u(x,0)=u_0^{\delta}(x), & \text{for } x\in\Omega. \quad (3.19)
\end{cases}$$

In order to prove the existence of a smooth solution of (S_{δ}^{μ}), we introduce the set

$$\mathcal{K}:=\{u\in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T),\quad u_m\le u\le 1-\delta\},$$

where $\alpha\in(0,1)$, and we prove the following result

Lemma 3.1 *Assume $(H_1)-(H_9)$ then there exists $(u_{\delta}^{\mu},\mathcal{P}_{\delta}^{\mu})$ solution of $(\tilde{S}_{\delta}^{\mu})$ such that*

$$u_{\delta}^{\mu}\in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q}_T),\quad u_m\le u_{\delta}^{\mu}\le 1-\delta$$

and $\mathcal{P}_{\delta}^{\mu}$, $\nabla\mathcal{P}_{\delta}^{\mu}\in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T)$ and $\Delta\mathcal{P}_{\delta}^{\mu}\in C^{\alpha,\frac{\alpha}{2}}(Q_T)$.

Proof: Let T^1 be the map defined for all $V \in \mathcal{K}$ by $T^1(V) = W$, where W is the unique solution of the elliptic problem,

$$(Q_V^1)\begin{cases} -div\Big(M^\mu(V)\nabla W\Big) = \overline{s}_\delta - \underline{s}_\delta - \fint_\Omega (\overline{s}_\delta - \underline{s}_\delta)dx, & \text{in } \Omega\times(0,T),\\ \displaystyle\int_\Omega W(x,t)dx = \int_\Omega \mathcal{R}^\mu(V(x,t))dx, & \text{for } t\in(0,T),\\ \nabla W.n = 0, & \text{on } \partial\Omega\times(0,T). \end{cases}$$

By standard theory of elliptic system (see [7] Theorem 3.2 p 137), we have that

$$|W|_{Q_T}^{1+\alpha,\frac{1+\alpha}{2}} + |\nabla W|_{Q_T}^{1+\alpha,\frac{1+\alpha}{2}} \leq D_1|V|_{Q_T}^{1+\alpha,\frac{1+\alpha}{2}} + D_2. \tag{3.20}$$

For W solution of (Q_V^1) we consider T^2 defined by $T^2(W) = \hat{V}$, where $\hat{V}$ is the solution of the parabolic problem,

$$(Q_W^2)\begin{cases} \hat{V}_t = \Delta\psi_\varepsilon^\mu(\hat{V}) + M^\mu(\hat{V})\nabla f^\mu(\hat{V}).\nabla W + [f^\mu(c_\delta) - f^\mu(\hat{V})]\overline{s}_\delta, & \text{in } \Omega\times(0,T),\\ \nabla\psi_\varepsilon^\mu(\hat{V}).n = 0, & \text{on } \partial\Omega\times(0,T),\\ \hat{V}(x,0) = u_0^\delta(x), & \text{in } \Omega. \end{cases}$$

¿From the standard theory of parabolic equations, we have that

$$|\hat{V}|_{Q_T}^{2+\alpha,\frac{2+\alpha}{2}} \leq D_3\Big(|W|_{Q_T}^{1+\alpha,\frac{1+\alpha}{2}} + |\nabla W|_{Q_T}^{1+\alpha,\frac{1+\alpha}{2}}\Big) + D_4. \tag{3.21}$$

Moreover defining by $\mathcal{L}$ the parabolic operator arising in (Q_W^2), namely

$$\mathcal{L}(\hat{V})(x,t) := \hat{V}_t - \Delta\psi_\varepsilon^\mu(\hat{V}) - M^\mu(\hat{V})\nabla f^\mu(\hat{V}).\nabla W - [f^\mu(c_\delta) - f^\mu(\hat{V})]\overline{s}_\delta,$$

we remark that (2.8), the property (3.9) of c_δ and the fact that, by (3.7), $\overline{s}_\delta$ is positive imply that

$$\mathcal{L}(u_m) \leq 0 \text{ and } \mathcal{L}(1-\delta) \geq 0. \tag{3.22}$$

Setting $T := T^2 \circ T^1$, the inequalities (3.22) ensure that T maps the convex set $\mathcal{K}$ into itself. Moreover we deduce from (3.21) that $T(\mathcal{K})$ is relatively compact in $\mathcal{K}$.
Next, we check that T is continuous. Suppose that a sequence $(V_m)_{m\in N}$ converges to a limit $V \in \mathcal{K}$ in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T)$, as $m \to \infty$. Since $(V_m)_{m\in N}$ is bounded in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T)$, it follows from (3.20) that the sequence $(W_m := T^1(V_m))_{m\in N}$, where W_m is the solution of $(Q_{V_m}^1)$, is bounded in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T)$, so that as $m \to \infty$, W_m converges to the unique solution W of Problem (Q_V^1) in $C^{1+\beta,\frac{1+\beta}{2}}(\overline{Q}_T)$ for all $\beta \in (0,\alpha)$. Moreover $W \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T)$. Further it also follows from (3.20) that $(\nabla W_m)_{m\in N}$ is bounded in $C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}_T)$, so that the solution $\hat{V}_m = T^2(W_m)$ of Problem $(Q_{W_m}^2)$ is bounded in $C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q}_T)$. Since $\hat{V}_m = T^2(W_m) = T(V_m)$, $(T(V_m))_{m\in N}$ converges to the unique solution $\hat{V}$ of Problem (Q_W^2)

in $C^{2+\beta,\frac{2+\beta}{2}}(\overline{Q}_T)$ for all $\beta \in (0,\alpha)$, as $m \to \infty$, so that $\hat{V} = T^2(W) = T^2 \circ T^1(V)$. Therefore we have just proved that $(T^2 \circ T^1(V_m))_{m\in N}$ converges to $T^2 \circ T^1(V)$ in $C^{2+\beta,\frac{2+\beta}{2}}(\overline{Q}_T)$ for all $\beta \in (0,\alpha)$, as $m \to \infty$, which ensures the continuity of the map T. It follows from the Schauder fixed point theorem that there exists a solution $(u_\delta^\mu, \mathcal{P}_\delta^\mu)$ of $(\tilde{S}_\delta^\mu)$ such that

$$u_\delta^\mu \in K \cap C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q}_T) \text{ and } \mathcal{P}_\delta^\mu,\ \nabla\mathcal{P}_\delta^\mu, \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T),\ \Delta\mathcal{P}_\delta^\mu \in C^{\alpha,\frac{\alpha}{2}}(Q_T).$$

This concludes the proof of Lemma 3.1. Moreover we deduce from Lemma 3.1 the existence of a solution of (S_δ^μ), namely

Corollary 3.2 *Assume the hypotheses $(H_1)-(H_9)$ then there exists $(u_\delta^\mu, p_\delta^\mu)$ solution of (S_δ^μ) such that $u_\delta^\mu \in C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{Q}_T)$,*

$$u_m \le u_\delta^\mu(x,t) \le 1-\delta \tag{3.23}$$

and $p_\delta^\mu,\ \nabla p_\delta^\mu \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T),\ \Delta p_\delta^\mu \in C^{\alpha,\frac{\alpha}{2}}(Q_T)$.

4 A priori Estimates

In view of (2.8) and (3.23) we deduce the following bounds

$$0 = f^\mu(0) \le f^\mu(u_\delta^\mu(x,t)) \le 1 = f^\mu(1), \tag{4.1}$$
$$0 = f^\mu(0) \le f^\mu(c_\delta(x,t)) \le 1 = f^\mu(1), \tag{4.2}$$
$$0 < k_w(u_m) \le k_w(u_\delta^\mu(x,t)) \le k_w(1) = 1, \tag{4.3}$$
$$0 = k_a(1) \le k_a(u_\delta^\mu(x,t)) \le k_a(0) = 1, \tag{4.4}$$
$$0 < k_w(u_m) \le M^\mu(u_\delta^\mu), \tag{4.5}$$
$$p_c(1) \le p_c(u_\delta^\mu(x,t)) \le p_c(0), \tag{4.6}$$
$$p_c(1) - p_c(0) \le \mathcal{R}^\mu(u_\delta^\mu(x,t)) \le 0, \tag{4.7}$$
$$p_c(1) - p_c(0) \le \mathcal{Q}^\mu(u_\delta^\mu(x,t)) \le 0, \tag{4.8}$$

for all $(x,t) \in \Omega \times (0,T)$. Next we state some essential a priori estimates.

Lemma 4.1 *Let $(u_\delta^\mu, p_\delta^\mu)$ be a solution of Problem (S_δ^μ). There exists a positive constant C, which only depends on Ω, k_w, k_a and T such that*

$$\int_0^T \int_\Omega k_a(u_\delta^\mu)|\nabla p_\delta^\mu + \nabla p_c(u_\delta^\mu)|^2 dxdt \le C\mu, \tag{4.9}$$

$$\int_0^T \int_\Omega |\nabla p_\delta^\mu|^2 dxdt \le C, \tag{4.10}$$

and

$$0 \leq -\int_0^T \int_\Omega \nabla g(u_\delta^\mu).\nabla p_c(u_\delta^\mu) dxdt \leq C, \tag{4.11}$$

$$\int_0^T \int_\Omega |\nabla \zeta(u_\delta^\mu)|^2 dxdt \leq C, \tag{4.12}$$

$$\int_0^T \int_\Omega |\nabla g(u_\delta^\mu)|^2 dxdt \leq C. \tag{4.13}$$

Proof: We first prove (4.9). Multiplying (3.11) by $\mathcal{P} = p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu)$ and integrating the result on $Q_T = \Omega \times (0,T)$ we obtain

$$\int_{Q_T} M^\mu(u_\delta^\mu)|\nabla(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu))|^2 \leq \frac{1}{h}\int_{Q_T}(\overline{s}_\delta - \underline{s}_\delta)^2 + h\int_{Q_T}(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu))^2, \tag{4.14}$$

for all $h > 0$. Moreover we have by Poincaré-Wirtinger inequality that

$$\int_{Q_T}(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu))^2 \leq C_1\Big[\int_{Q_T}|\nabla(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu))|^2 + \Big(\int_{Q_T} p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu)\Big)^2\Big].$$

Using (3.3) and (4.7), it follows that

$$\int_{Q_T}(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu))^2 \leq C_1\int_{Q_T}|\nabla(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu))|^2 + C_2,$$

which we substitute into (4.14) with $h = \dfrac{k_w(u_m)}{2C_1}$ to deduce, also in view of (3.7) and (4.5), that

$$\int_{Q_T}|\nabla(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu))|^2 \leq C_3 \text{ and } \int_{Q_T}|p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu)|^2 \leq C_3. \tag{4.15}$$

Furthermore multiplying (3.1) by p_δ^μ and (3.2) by $p_\delta^\mu + p_c(u_\delta^\mu)$, adding up both results and integrating on Q_T we obtain

$$-\int_{Q_T}(u_\delta^\mu)_t p_c(u_\delta^\mu) + \int_{Q_T} k_w(u_\delta^\mu)|\nabla p_\delta^\mu|^2 + \frac{1}{\mu}k_a(u_\delta^\mu)|\nabla p_\delta^\mu + \nabla p_c(u_\delta^\mu)|^2 = I, \tag{4.16}$$

where

$$\begin{aligned} I := \ & \int_{Q_T}\Big[f^\mu(c_\delta)\overline{s}_\delta - f^\mu(u_\delta^\mu)\Big(\underline{s}_\delta + \fint_\Omega(\overline{s}_\delta - \underline{s}_\delta)\Big)\Big]p_\delta^\mu dxdt \\ & + \int_{Q_T}\Big[(1-f^\mu(c_\delta))\overline{s}_\delta - \Big(1-f^\mu(u_\delta^\mu)\Big)\Big(\underline{s}_\delta + \fint_\Omega(\overline{s}_\delta - \underline{s}_\delta)\Big)\Big](p_\delta^\mu + p_c(u_\delta^\mu))dxdt. \end{aligned}$$

We check below that first term on the left-hand-side of (4.16) and I are bounded. Denoting by $\mathcal{P}_c$ a primitive of p_c we have that

$$\int_{Q_T} p_c(u_\delta^\mu)(u_\delta^\mu)_t = \int_\Omega\int_0^T \frac{\partial}{\partial t}[\mathcal{P}_c(u_\delta^\mu)].$$

Since $\mathcal{P}_c$ is continuous and u_δ^μ is bounded this gives

$$\Big| \int_{Q_T} p_c(u_\delta^\mu)(u_\delta^\mu)_t dxdt \Big| \leq C_4. \tag{4.17}$$

Moreover we have using (3.3) and (2.13) that

$$\begin{aligned} I = \; & \int_{Q_T} \Big(p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu)\Big)(\overline{s}_\delta - \underline{s}_\delta) dxdt \\ & - \int_{Q_T} \mathcal{R}^\mu(u_\delta^\mu)\Big[f^\mu(c_\delta)\overline{s}_\delta - f^\mu(u_\delta^\mu)\underline{s}_\delta + \Big(1 - f^\mu(u_\delta^\mu)\Big)\Big(\fint_\Omega (\overline{s}_\delta - \underline{s}_\delta)\Big)\Big] dxdt \\ & + \int_{Q_T} \Big[(1 - f^\mu(c_\delta))\overline{s}_\delta - \Big(1 - f^\mu(u_\delta^\mu)\Big)\Big(\underline{s}_\delta + \fint_\Omega (\overline{s}_\delta - \underline{s}_\delta)\Big)\Big]\Big[\mathcal{Q}^\mu(u_\delta^\mu) + p_c(0)\Big] dxdt. \end{aligned} \tag{4.18}$$

In view of (H_5), (3.7), (4.1), (4.2), (4.7) and (4.8) we obtain

$$I \leq C_5 \int_{Q_T} |p_\delta^\mu + \mathcal{R}^\mu(u_\delta^\mu)|^2 + C_6.$$

This together with (4.15) yields $I \leq C_5 C_3 + C_6$. Substituting this into (4.16) and also using (4.17) we obtain that

$$\int_{Q_T} k_w(u_\delta^\mu)|\nabla p_\delta^\mu|^2 + \frac{1}{\mu} k_a(u_\delta^\mu)|\nabla p_\delta^\mu + \nabla p_c(u_\delta^\mu)|^2 dxdt \leq C_7, \tag{4.19}$$

which implies (4.9). In view of (4.3), we also deduce from (4.19) the estimate (4.10). Next we prove (4.11). By the definition (2.9) of g, we obtain from (3.10) that

$$-div\Big(M^\mu(u_\delta^\mu)\nabla p_\delta^\mu\Big) + \frac{1}{\mu}\Delta g(u_\delta^\mu) = \overline{s}_\delta - \underline{s}_\delta - \fint_\Omega (\overline{s}_\delta - \underline{s}_\delta) dx. \tag{4.20}$$

Multiplying (4.20) by $f^\mu(u_\delta^\mu)$ and subtracting the result from (3.1) we deduce that

$$(u_\delta^\mu)_t = \frac{1}{\mu} f^\mu(u_\delta^\mu)\Delta g(u_\delta^\mu) + div\Big(k_w(u_\delta^\mu)\nabla p_\delta^\mu\Big) - f^\mu(u_\delta^\mu) div\Big(M^\mu(u_\delta^\mu)\nabla(p^\mu)\Big) + \overline{s}_\delta[f^\mu(c_\delta) - f^\mu(u_\delta^\mu)]. \tag{4.21}$$

Moreover using the definition (2.7) of f^μ and M^μ we note that

$$\begin{aligned} div\Big(M^\mu(u_\delta^\mu) f^\mu(u_\delta^\mu)\nabla p_\delta^\mu\Big) \; & = div\Big(k_w(u_\delta^\mu)\nabla p_\delta^\mu\Big) \\ & = M^\mu(u_\delta^\mu)\nabla(f^\mu(u_\delta^\mu)).\nabla p_\delta^\mu + f^\mu(u_\delta^\mu) div\Big(M^\mu(u_\delta^\mu)\nabla(p^\mu)\Big), \end{aligned}$$

which we substitute into (4.21) to obtain

$$(u_\delta^\mu)_t - \frac{1}{\mu} f^\mu(u_\delta^\mu)\Delta g(u_\delta^\mu) - M^\mu(u_\delta^\mu)\nabla(f^\mu(u_\delta^\mu)).\nabla p_\delta^\mu = \overline{s}_\delta[f^\mu(c_\delta) - f^\mu(u_\delta^\mu)]. \tag{4.22}$$

We set

$$D^\mu(a) := p_c(a)f^\mu(a) - \mathcal{Q}^\mu(a), \tag{4.23}$$

for all $a \in [0,1]$, so that by the definition (2.11) of $\mathcal{Q}^\mu$ we have $\nabla D^\mu(u_\delta^\mu) = p_c(u_\delta^\mu)\nabla(f^\mu(u_\delta^\mu))$. Substituting this into (4.22), which we have multiplied by $p_c(u_\delta^\mu)$, we deduce that

$$p_c(u_\delta^\mu)(u_\delta^\mu)_t - \frac{1}{\mu}f^\mu(u_\delta^\mu)p_c(u_\delta^\mu)\Delta g(u_\delta^\mu) - M^\mu(u_\delta^\mu)\nabla D^\mu(u_\delta^\mu).\nabla p_\delta^\mu = p_c(u_\delta^\mu)\overline{s}_\delta[f^\mu(c_\delta) - f^\mu(u_\delta^\mu)]. \tag{4.24}$$

Multiplying (4.20) by $D^\mu(u_\delta^\mu)$, adding the result to (4.24) and also using the fact that

$$div\Big(M^\mu(u_\delta^\mu)D^\mu(u_\delta^\mu)\nabla p_\delta^\mu\Big) = M^\mu(u_\delta^\mu)\nabla D^\mu(u_\delta^\mu).\nabla p_\delta^\mu + D^\mu(u_\delta^\mu)div\Big(M^\mu(u_\delta^\mu)\nabla p_\delta^\mu\Big),$$

we deduce that

$$\begin{aligned} p_c(u_\delta^\mu)(u_\delta^\mu)_t - \frac{1}{\mu}\Big(f^\mu(u_\delta^\mu)p_c(u_\delta^\mu) - D^\mu(u_\delta^\mu)\Big)\Delta g(u_\delta^\mu) - div\Big(M^\mu(u_\delta^\mu)D^\mu(u_\delta^\mu)\nabla p_\delta^\mu\Big) \\ = p_c(u_\delta^\mu)\overline{s}_\delta\Big[f^\mu(c_\delta) - f^\mu(u_\delta^\mu)\Big] + D^\mu(u_\delta^\mu)\Big(\overline{s}_\delta - \underline{s}_\delta - \fint_\Omega(\overline{s}_\delta - \underline{s}_\delta)\Big). \end{aligned} \tag{4.25}$$

Integrating (4.25) on Q_T and using the fact that the definition (4.23) of D^μ implies

$$p_c(u_\delta^\mu)f^\mu(u_\delta^\mu) - D^\mu(u_\delta^\mu) = \mathcal{Q}^\mu(u_\delta^\mu),$$

we obtain

$$\int_{Q_T} p_c(u_\delta^\mu)(u_\delta^\mu)_t dxdt - \frac{1}{\mu}\int_{Q_T}\mathcal{Q}^\mu(u_\delta^\mu)\Delta g(u_\delta^\mu)dxdt = J, \tag{4.26}$$

where

$$\begin{aligned} J := \ & \int_{Q_T} p_c(u_\delta^\mu)\overline{s}_\delta[f^\mu(c_\delta) - f^\mu(u_\delta^\mu)]dxdt \\ & + \int_{Q_T}\Big(p_c(u_\delta^\mu)f^\mu(u_\delta^\mu) - \mathcal{Q}^\mu(u_\delta^\mu)\Big)\Big(\overline{s}_\delta - \underline{s}_\delta - \fint_\Omega(\overline{s}_\delta - \underline{s}_\delta)\Big)dxdt. \end{aligned}$$

It follows from (4.1), (4.2), (4.6), (4.8) and (3.7) that $|J| \leq C_8$. Substituting this into (4.26) and also using (4.17) we obtain that

$$0 \leq -\frac{1}{\mu}\int_{Q_T}\nabla\mathcal{Q}^\mu(u_\delta^\mu).\nabla(g(u_\delta^\mu))dxdt \leq C_9. \tag{4.27}$$

Furthermore we remark that

$$\frac{1}{\mu}f^\mu(u_\delta^\mu) \geq \frac{k_w(u_m)}{2},$$

which together with (4.27) and the fact that $\nabla \mathcal{Q}^\mu(u_\delta^\mu) = f^\mu(u_\delta^\mu)\nabla p_c(u_\delta^\mu)$ yields

$$0 \leq -\int_{Q_T} \nabla p_c(u_\delta^\mu)\nabla(g(u_\delta^\mu))dxdt \leq C_{10}. \tag{4.28}$$

By the definition (2.10) of ζ, we have $-\nabla p_c(u_\delta^\mu)\nabla g(u_\delta^\mu) = |\nabla \zeta(u_\delta^\mu)|^2$. This together with (4.28) implies (4.11) and (4.12), which in view of (4.4) gives (4.13). This completes the proof of Lemma 4.1.

In what follows we give estimates of differences of space translates of p_δ^μ and $g(u_\delta^\mu)$. We set for $r \in \mathbf{R}^+$ sufficiently small:

$$\Omega_r = \{x \in \Omega, \quad B(x, 2r) \subset \Omega\}.$$

Lemma 4.2 *Let $(u_\delta^\mu, p_\delta^\mu)$ be a solution of Problem (S_δ^μ); there exists a positive constant C such that*

$$\int_0^T \int_{\Omega_r} \left| p_\delta^\mu(x+\xi, t) - p_\delta^\mu(x,t) \right|^2 (x,t)dxdt \leq C\xi^2 \tag{4.29}$$

and

$$\int_0^T \int_{\Omega_r} \left| g(u_\delta^\mu)(x+\xi, t) - g(u_\delta^\mu)(x,t) \right|^2 dxdt \leq C\xi^2, \tag{4.30}$$

where $\xi \in \mathbf{R}^N$ and $|\xi| \leq 2r$.

Proof: The inequalities (4.29) and (4.30) follow from (4.10) and (4.13) respectively.

Next we estimate differences of time translates of $g(u_\delta^\mu)$.

Lemma 4.3 *Let $(u_\delta^\mu, p_\delta^\mu)$ be a solution of Problem (S_δ^μ) then there exists a positive constant C such that*

$$\int_0^{T-\tau} \int_\Omega [g(u_\delta^\mu)(x, t+\tau) - g(u_\delta^\mu)(x,t)]^2 dxdt \leq C\tau, \tag{4.31}$$

for all $\tau \in (0, T)$.

Proof: We set

$$A(t) := \int_\Omega [g(u_\delta^\mu)(x, t+\tau) - g(u_\delta^\mu)(x,t)]^2 dx.$$

Since g is a non decreasing Lipschitz continuous function with the Lipschitz constant C_g we have that

$$A(t) \quad \leq \quad C_g \int_\Omega [g(u_\delta^\mu(x, t+\tau)) - g(u_\delta^\mu(x,t))][u_\delta^\mu(x, t+\tau) - u_\delta^\mu(x,t)]dx$$

$$
\begin{aligned}
&\leq C_g \int_\Omega [g(u^\mu_\delta(x,t+\tau)) - g(u^\mu_\delta(x,t))] \Big[\int_t^{t+\tau} (u^\mu_\delta)_t(x,\theta)d\theta\Big]\, dx \\
&\leq C_g \int_\Omega \int_t^{t+\tau} \Big[g(u^\mu_\delta(x,t+\tau)) - g(u^\mu_\delta(x,t))\Big] \\
&\quad \Big[div(k_w(u^\mu_\delta)\nabla p^\mu_\delta) + f^\mu(c_\delta)\overline{s}_\delta - f^\mu(u^\mu_\delta)\Big(\underline{s}_\delta + \fint_\Omega (\overline{s}_\delta(y) - \underline{s}_\delta(y))dy\Big)\Big](x,\theta)d\theta dx,
\end{aligned}
$$

where we have used (3.1). Integrating by parts this gives

$$
\begin{aligned}
A(t) \leq C_g \Big\{ &\int_t^{t+\tau} \int_\Omega \Big| k_w(u^\mu_\delta)(x,\theta)\nabla p^\mu_\delta(x,\theta)\nabla g(u^\mu_\delta)(x,t+\tau)\Big| dxd\theta \\
&+ \int_t^{t+\tau} \int_\Omega \Big| k_w(u^\mu_\delta)(x,\theta)\nabla p^\mu_\delta(x,\theta)\nabla g(u^\mu_\delta)(x,t)\Big| dxd\theta \\
&+ \Big| \int_\Omega \Big[g(u^\mu_\delta)(x,t+\tau) - g(u^\mu_\delta)(x,t)\Big] K(x,t,\tau)dx\Big|\Big\}, \qquad (4.32)
\end{aligned}
$$

where

$$
K(x,t,\tau) := \int_t^{t+\tau} \Big(f^\mu(c_\delta(x,\theta))\overline{s}_\delta(x) - f^\mu(u_\delta(x,\theta))\Big[\underline{s}_\delta(x) + \fint_\Omega (\overline{s}_\delta(y) - \underline{s}_\delta(y))dy\Big]\Big) d\theta. \qquad (4.33)
$$

Next we estimate the right hand side of (4.32). Using (4.3) we have that

$$
\begin{aligned}
&\int_t^{t+\tau} \int_\Omega \Big| k_w(u^\mu_\delta)(x,\theta)\nabla p^\mu_\delta(x,\theta)\nabla g(u^\mu_\delta)(x,t+\tau)\Big| dxd\theta \\
&\leq \frac{1}{2}\Big(\int_t^{t+\tau} \int_\Omega |\nabla p^\mu_\delta(x,\theta)|^2 dxd\theta + \int_t^{t+\tau} \int_\Omega |\nabla g(u^\mu_\delta)(x,t+\tau)|^2 dxd\theta\Big) \\
&\leq \frac{1}{2}\Big(\int_t^{t+\tau} \int_\Omega |\nabla p^\mu_\delta(x,\theta)|^2 dxd\theta + \tau \int_\Omega |\nabla g(u^\mu_\delta)(x,t+\tau)|^2 dx\Big). \qquad (4.34)
\end{aligned}
$$

Similarly we have that

$$
\begin{aligned}
&\int_t^{t+\tau} \int_\Omega \Big| k_w(u^\mu_\delta)(x,\theta)\nabla p^\mu_\delta(x,\theta)\nabla g(u^\mu_\delta)(x,t)\Big| dxd\theta \\
&\leq \frac{1}{2}\Big(\int_t^{t+\tau} \int_\Omega |\nabla p^\mu_\delta(x,\theta)|^2 dxd\theta + \tau \int_\Omega |\nabla g(u^\mu_\delta)(x,t)|^2 dx\Big). \qquad (4.35)
\end{aligned}
$$

Moreover using (4.1) and (4.2) we obtain from the definition (4.33) of K that

$$
|K(x,t,\tau)| \leq \int_t^{t+\tau} \Big[|\overline{s}_\delta| + |\underline{s}_\delta| + \fint_\Omega |\overline{s}_\delta - \underline{s}_\delta| dx\Big] d\theta \leq \Big[|\overline{s}_\delta| + |\underline{s}_\delta| + \fint_\Omega |\overline{s}_\delta - \underline{s}_\delta| dx\Big]\tau.
$$

This together with (3.7) and the fact that the function $g(u^\mu_\delta)$ is bounded uniformly on μ and δ yields

$$
\Big| \int_\Omega \Big[g(u^\mu_\delta)(x,t+\tau) - g(u^\mu_\delta)(x,t)\Big] K(x,t,\tau)\Big| dx \leq \tilde{C}\tau. \qquad (4.36)
$$

Substituting (4.34), (4.35) and (4.36) into (4.32) we deduce that

$$\begin{aligned} A(t) &\leq C_g\Big(\int_t^{t+\tau} \int_\Omega |\nabla p_\delta^\mu(x,\theta)|^2 dx d\theta + \frac{\tau}{2} \int_\Omega |\nabla g(u_\delta^\mu)(x,t+\tau)|^2 dx \\ &\quad + \frac{\tau}{2} \int_\Omega |\nabla g(u_\delta^\mu)(x,t)|^2 dx + \tilde{C}\tau \Big), \end{aligned}$$

which we integrate on $[0, T-\tau]$ to obtain

$$\begin{aligned} \int_0^{T-\tau} A(t)dt &\leq C_g\Big(\int_0^{T-\tau} \int_t^{t+\tau} \int_\Omega |\nabla p_\delta^\mu(x,\theta)|^2 dx d\theta dt + \tau \int_0^T \int_\Omega |\nabla g(u_\delta^\mu)|^2 dx dt + \tilde{C}\tau T \Big) \\ &\leq C_g\Big(\tau \int_0^T \int_\Omega |\nabla p_\delta^\mu(x,\theta)|^2 dx d\theta + \tau \int_0^T \int_\Omega |\nabla g(u_\delta^\mu)|^2 dx dt + \tilde{C}\tau T \Big). \end{aligned}$$

In view of (4.10) and (4.13) we deduce (4.31), which completes the proof of Lemma 4.3.

5 Convergence as $\delta \downarrow 0$.

Letting δ tend to 0, we deduce from the estimates given in Lemmas 4.1 and 4.2 the existence of a weak solution of Problem (S^μ). More precisely, we have the following result,

Lemma 5.1 *There exists a weak solution (u^μ, p^μ) of Problem (S^μ), which satisfies*

$$\int_0^T \int_\Omega \Big[k_a(u^\mu)\Big]^2 \Big[\nabla p^\mu + \nabla p_c(u^\mu)\Big]^2 dx dt \leq C\mu, \tag{5.1}$$

$$\int_0^T \int_\Omega |\nabla p^\mu|^2 dx dt \leq C, \tag{5.2}$$

$$\int_0^T \int_\Omega |\nabla g(u^\mu)|^2 dx dt \leq C, \tag{5.3}$$

$$\int_0^T \int_{\Omega_r} [g(u^\mu)(x+\xi,t) - g(u^\mu)(x,t)]^2 dx dt \leq C\xi^2, \tag{5.4}$$

where $\xi \in \mathbf{R}^N$ and $|\xi| \leq 2r$. Moreover the following estimate of differences of time translates holds

$$\int_0^{T-\tau} \int_\Omega [g(u^\mu)(x,t+\tau) - g(u^\mu)(x,t)]^2 dx dt \leq C\tau, \tag{5.5}$$

for all $\tau \in (0,T)$.

Proof: We deduce from (4.10), (4.30) and (4.31) that there exist functions $\hat{g}^\mu$ and p^μ and a subsequence $((u^\mu_{\delta_n}, p^\mu_{\delta_n}))_{n\in N}$ of weak solutions of Problem $(S^\mu_{\delta_n})$ such that

$$\begin{aligned} &(g(u^\mu_{\delta_n}))_{n\in N} \text{ tends to } \hat{g}^\mu \text{ strongly in } L^2(Q_T), \\ &(p^\mu_{\delta_n})_{n\in N} \text{ tends to } p^\mu \text{ weakly in } L^2(0,T;H^1(\Omega)), \end{aligned} \tag{5.6}$$

as δ_n tends to zero. Thus for a subsequence, which we denote again by δ_n, we have that

$$(g(u^\mu_{\delta_n}))_{n\in N} \text{ tends to } \hat{g}^\mu \text{ for almost } (x,t) \in Q_T. \tag{5.7}$$

Using the fact that g is bijective we deduce that

$$(u^\mu_{\delta_n})_{n\in N} \text{ tends to } u^\mu := g^{-1}(\hat{g}^\mu) \text{ strongly in } L^2(Q_T) \text{ and almost everywhere in } Q_T, \tag{5.8}$$

as δ_n tends to zero. Moreover we have in view of (4.13) and (5.6) that $\nabla g(u^\mu_{\delta_n})$ tends to $\nabla g(u^\mu)$ weakly in $L^2(Q_T)$ as $\delta_n \downarrow 0$, so that by the definition (2.9) of g

$$k_a(u^\mu_{\delta_n})\nabla p_c(u^\mu_{\delta_n}) \text{ tends to } k_a(u^\mu)\nabla p_c(u^\mu) \text{ weakly in } L^2(Q_T) \text{ as } \delta_n \downarrow 0. \tag{5.9}$$

Letting δ_n tend to 0 in (3.23) we deduce that

$$u_m \le u^\mu(x,t) \le 1. \tag{5.10}$$

Moreover we deduce from (3.3) that

$$\int_\Omega p^\mu(x,t)dx = 0, \text{ for almost every } t \in (0,T). \tag{5.11}$$

Multiplying (3.1) by $\varphi \in \mathcal{C}$, integrating by parts and letting δ_n tend to 0 we obtain

$$\begin{aligned} \int_0^T \int_\Omega u^\mu \varphi_t dxdt = &\int_0^T \int_\Omega k_w(u^\mu)\nabla p^\mu.\nabla\varphi dxdt - \int_0^T \int_\Omega \Big(f^\mu(c)\overline{s} - f^\mu(u^\mu)\underline{s}\Big)\varphi dxdt \\ &- \int_\Omega u_0(x)\varphi(x,0)dx, \end{aligned} \tag{5.12}$$

where we have used that u_0^δ tends to u_0 in $L^2(\Omega)$ and that c_δ, $\overline{s}_\delta$ and $\underline{s}_\delta$ tend respectively to c, $\overline{s}$ and $\underline{s}$ in $L^2(Q_T)$ as $\delta \downarrow 0$. Similarly, multiplying (3.2) by $\varphi \in \mathcal{C}$, integrating by parts and letting δ_n tend to 0 we deduce that

$$\begin{aligned} &\int_0^T \int_\Omega \Big(1 - u^\mu\Big)\varphi_t dxdt = \frac{1}{\mu}\int_0^T \int_\Omega \Big(k_a(u^\mu)\nabla p^\mu + \nabla g(u^\mu)\Big).\nabla\varphi dxdt \\ &- \int_0^T \int_\Omega \Big((1 - f^\mu(c))\overline{s} - (1 - f^\mu(u^\mu))\underline{s}\Big)\varphi dxdt - \int_\Omega \Big(1 - u_0(x)\Big)\varphi(x,0)dx, \end{aligned} \tag{5.13}$$

which since $\nabla g(u^\mu) = k_a(u^\mu)\nabla p_c(u^\mu)$ coincides with (2.15). Next we prove (5.1). We first check that

$$k_a(u^\mu_{\delta_n})\nabla p^\mu_{\delta_n} \text{ tends to } k_a(u^\mu)\nabla p^\mu \text{ weakly in } L^2(Q_T), \tag{5.14}$$

as δ_n tends to 0. Let $\varphi \in L^2(Q_T)$, we have that

$$\Big| \int_{Q_T} \Big(k_a(u^\mu_{\delta_n}) \nabla p^\mu_{\delta_n} - k_a(u^\mu) \nabla p^\mu \Big) \varphi \; dxdt \Big| \leq |I^1_{\delta_n}| + |I^2_{\delta_n}|, \tag{5.15}$$

where

$$I^1_{\delta_n} := \int_{Q_T} \Big(k_a(u^\mu_{\delta_n}) - k_a(u^\mu) \Big) \nabla p^\mu_{\delta_n} \varphi \; dxdt$$

and

$$I^2_{\delta_n} = \int_{Q_T} k_a(u^\mu) \varphi \Big(\nabla p^\mu_{\delta_n} - \nabla p^\mu \Big) \; dxdt.$$

Using the fact that $\nabla p^\mu_{\delta_n}$ converges to ∇p^μ weakly in $L^2(Q_T)$ as $\delta_n \downarrow 0$, we deduce, since $k_a(u^\mu)\varphi \in L^2(Q_T)$, that

$$|I^2_{\delta_n}| \text{ tends to } 0 \text{ as } \delta_n \downarrow 0. \tag{5.16}$$

Moreover we have by (4.10) that

$$\begin{aligned} |I^1_{\delta_n}| \quad &\leq \Big(\int_{Q_T} \Big| k_a(u^\mu_{\delta_n}) - k_a(u^\mu) \Big|^2 \varphi^2 dxdt \Big)^{1/2} \Big(\int_{Q_T} |\nabla p^\mu_{\delta_n}|^2 dxdt \Big)^{1/2} \\ &\leq C \Big(\int_{Q_T} \Big| k_a(u^\mu_{\delta_n}) - k_a(u^\mu) \Big|^2 \varphi^2 dxdt \Big)^{1/2}. \end{aligned}$$

Since $\Big| k_a(u^\mu_{\delta_n}) - k_a(u^\mu) \Big|^2 \varphi^2 \leq 4\varphi^2$ and since $k_a(u^\mu_{\delta_n})$ tends to $k_a(u^\mu)$ almost everywhere, we deduce from the Dominated Convergence Theorem that $I^1_{\delta_n}$ tends to 0 as $\delta_n \downarrow 0$. This with (5.16) implies (5.14), which with (5.9) gives that

$$k_a(u^\mu_{\delta_n}) \Big[\nabla p^\mu_{\delta_n} + \nabla p_c(u^\mu_{\delta_n}) \Big] \text{ tends to } k_a(u^\mu) \Big[\nabla p^\mu + \nabla p_c(u^\mu) \Big] \text{ weakly in } L^2(Q_T). \tag{5.17}$$

The functional $v \mapsto \displaystyle\int_{Q_T} v^2 dxdt$ is convex and lower semi continuous from $L^2(Q_T)$ to $\overline{R}$ therefore it is also weakly l.s.c. (see [2] Corollary III.8) and thus we deduce from (4.4), (4.9) and (5.17) that

$$\begin{aligned} \int_{Q_T} \Big[k_a(u^\mu) \Big]^2 \Big[\nabla p^\mu + \nabla p_c(u^\mu) \Big]^2 dxdt \quad &\leq \quad \liminf_{\delta_n \downarrow 0} \int_{Q_T} \Big[k_a(u^\mu_{\delta_n}) \Big]^2 \Big[\nabla p^\mu_{\delta_n} + \nabla p_c(u^\mu_{\delta_n}) \Big]^2 dxdt \\ &\leq \quad \liminf_{\delta_n \downarrow 0} \int_{Q_T} k_a(u^\mu_{\delta_n}) \Big[\nabla p^\mu_{\delta_n} + \nabla p_c(u^\mu_{\delta_n}) \Big]^2 dxdt \\ &\leq \quad C\mu, \end{aligned}$$

which coincides with (5.1). Finally, we deduce respectively from (4.10), (4.13), (4.30) and (4.31) the estimates (5.2), (5.3), (5.4) and (5.5). This concludes the proof of Lemma 5.1.

6 Convergence as $\mu \downarrow 0$.

The goal of this section is to prove Theorem 2.3. We first deduce from the estimates (5.2), (5.4) and (5.5) that there exists a couple of functions (u, p) and a subsequence $((u^{\mu_n}, p^{\mu_n}))_{n\in N}$ such that

$$(u^{\mu_n})_{n\in N} \text{ tends to } u \text{ strongly in } L^2(Q_T) \text{ and almost everywhere in } Q_T,$$
$$(p^{\mu_n})_{n\in N} \text{ tends to } p \text{ weakly in } L^2(0,T;H^1(\Omega)),$$

as μ_n tends to zero. Moreover since

$$0 \leq f^{\mu_n}(u^{\mu_n}) \leq 1,$$

there exists a function $\hat{f} \in L^2(Q_T)$ with $0 \leq \hat{f} \leq 1$ and a subsequence $(f^{\mu_{n_m}}(u^{\mu_{n_m}}))_{n_m\in N}$ of $(f^{\mu_n}(u^{\mu_n}))_{n\in N}$ such that $(f^{\mu_{n_m}}(u^{\mu_{n_m}}))_{n_m\in N}$ tends to $\hat{f}$ weakly in $L^2(Q_T)$ as μ_{n_m} tends to zero. Moreover we deduce respectively from (5.10) and (5.11) that $0 \leq u \leq 1$ and that

$$\int_\Omega p(x,t)dx = 0, \text{ for almost every } t \in (0,T),$$

which gives (2.18). As it is done in Section 6 in the proof of (5.1), one can first check that

$$k_a(u^{\mu_{n_m}})(\nabla p^{\mu_{n_m}} + \nabla p_c(u^{\mu_{n_m}})) \text{ tends to } k_a(u)(\nabla p + \nabla p_c(u)) \text{ weakly in } L^2(Q_T),$$

as $\mu_{n_m} \downarrow 0$ and then deduce from (5.1) the estimate (2.17). Furthermore letting μ_{n_m} tends to zero into (5.12) we obtain, since $\lim_{\mu_{n_m}\downarrow 0} f^{\mu_{n_m}}(s) = \chi(s)$ for all $s \in [0,1]$, that

$$\begin{aligned}\int_0^T\int_\Omega u\varphi_t dxdt = &\int_0^T\int_\Omega k_w(u)\nabla p.\nabla\varphi dxdt - \int_0^T\int_\Omega \Big(\chi(c)\overline{s} - \hat{f}\underline{s}\Big)\varphi dxdt \\ &- \int_\Omega u_0(x)\varphi(x,0)dx,\end{aligned}$$

which coincides with (2.16) and concludes the proof of Theorem 2.3.

7 Numerical simulations

7.1 The saturation equation and the numerical algorithm

In this section we present numerical simulations in one space dimension. To that purpose we apply the finite volume method, which we present below. To begin with, we rewrite the equations (2.1) and (2.2) in the case that $\Omega = (0,1)$; this gives for $(x,t) \in (0,1)\times(0,T)$

$$u_t^\mu = \partial_x\Big(k_w(u^\mu)\partial_x p^\mu\Big) + f^\mu(c)\overline{s} - f^\mu(u^\mu)\underline{s}, \tag{7.1}$$

$$(1-u^\mu)_t = \partial_x\Big(\frac{1}{\mu}k_a(u^\mu)\partial_x(p^\mu + p_c(u^\mu))\Big) + (1-f^\mu(c))\overline{s} - (1-f^\mu(u^\mu))\underline{s}. \tag{7.2}$$

Adding up both equations and using the boundary conditions (2.4) and (2.5) we obtain

$$\partial_x p^\mu = -\frac{k_a(u^\mu)}{k_a(u^\mu)+\mu k_w(u^\mu)}\partial_x(p_c(u^\mu)). \tag{7.3}$$

Substituting (7.3) into (7.1) yields

$$u_t^\mu = -\partial_x\Big[f^\mu(u^\mu)\frac{k_a(u^\mu)}{\mu}\partial_x(p_c(u^\mu))\Big] + f^\mu(c)\overline{s} - f^\mu(u^\mu)\underline{s}. \tag{7.4}$$

Moreover we deduce from (7.3) and the definition (2.12) of $\mathcal{R}^\mu$ that $\partial_x p^\mu = -\partial_x(\mathcal{R}^\mu(u^\mu))$, so that in view of (2.3) we have

$$p^\mu(x,t) = -\mathcal{R}^\mu(u^\mu)(x,t) + \int_0^1 \mathcal{R}^\mu(u^\mu)(y,t)dy. \tag{7.5}$$

In the sequel, we compare numerically the solution u^μ of (7.4) with the solution u of the limit equation in the case that $u < 1$, namely

$$u_t = -\partial_x\Big(k_w(u)\partial_x p_c(u)\Big) + \chi(c)\bar{s}. \tag{7.6}$$

We discretize the time evolution equation (7.4) together with the initial condition and the homogeneous Neumann boundary condition. The time explicit finite volume scheme is defined by the following equations in which $\mathcal{K} > 0$ and $\mathcal{J} > 0$ denote respectively the time and the space step.
(i) The discrete initial condition is given for $i \in \{0, ..., [1/\mathcal{J}]\}$ by

$$[U^\mu]_i^0 = u^\mu(i\mathcal{J}, 0). \tag{7.7}$$

(ii) For $i \in \{0, ..., [1/\mathcal{J}]\}$ and for $n \in \{0, ..., [T/\mathcal{K}]\}$ the discrete equation is given by

$$\frac{1}{\mathcal{K}}\Big([U^\mu]_i^{n+1} - [U^\mu]_i^n\Big) \;=\; [F^\mu]_{i+1}^n - [F^\mu]_i^n + f^\mu(C_i^n)\overline{S}_i^n - f^\mu([U^\mu]_i^n)\underline{S}_i^n, \tag{7.8}$$

where

$$[F^\mu]_i^n = -\frac{1}{\mathcal{J}}\Big(p_c([U^\mu]_{i+1}^n) - p_c([U^\mu]_i^n\Big)\frac{k_w([U^\mu]_{i+1}^n)k_a([U^\mu]_i^n)}{\mu k_w([U^\mu]_{i+1}^n) + k_a([U^\mu]_i^n)}.$$

(iii) For $n \in \{0, ..., [T/\mathcal{K}]\}$ the discrete Neumann condition is defined by

$$[F^\mu]_0^n = 0 \text{ and } [F^\mu]_{[1/\mathcal{J}]}^n = 0. \tag{7.9}$$

The numerical scheme (7.7)-(7.9) allows to build an approximate solution, $u_{\mathcal{J},\mathcal{K}} : [0,1] \times [0,T] \to \mathbf{R}$ for all $i \in \{0, ..., [1/\mathcal{J}]\}$ and all $n \in \{0, ..., [T/\mathcal{K}]\}$, which is given by

$$u_{\mathcal{J},\mathcal{K}}(x,t) = u_i^n, \text{ for all } x \in (i\mathcal{J}, (i+1)\mathcal{J}] \text{ and for all } t \in (n\mathcal{K}, (n+1)\mathcal{K}]. \tag{7.10}$$

In order to also compute the pressures, we propose the following discrete equation corresponding to (7.5)

$$[P^\mu]_i^n = -\mathcal{R}([U^\mu]_i^n) + \mathcal{J}\Sigma_{j=1}^{[1/\mathcal{J}]}\mathcal{R}([U^\mu]_j^n). \tag{7.11}$$

Finally, setting $p_g^\mu(x,t) = p^\mu(x,t) + p_c(u^\mu)(x,t)$ we deduce that

$$([P_g^\mu])_i^n = -\mathcal{R}([U^\mu]_i^n + p_c([U^\mu]_i^n) + \mathcal{J}\Sigma_{j=1}^{[1/\mathcal{J}]}\mathcal{R}([U^\mu]_j^n), \tag{7.12}$$

for all $i \in \{0, ..., [1/\mathcal{J}]\}$ and all $n \in \{0, ..., [T/\mathcal{K}]\}$. Similarly we propose a finite volume scheme corresponding to the equation (7.6), namely

$$\frac{1}{\mathcal{K}}\Big(U_i^{n+1} - U_i^n\Big) = F_{i+1}^n - F_i^n + \chi(C_i^n)\overline{S}_i^n, \tag{7.13}$$

where

$$F_i^n = -\frac{1}{\mathcal{J}}\Big(p_c(U_{i+1}^n) - p_c(U_i^n\Big)k_w(U_{i+1}^n),$$

for all $(i,n) \in \{0, ..., [1/\mathcal{J}]\} \times \{0, ..., [T/\mathcal{K}]\}$.

7.2 Numerical tests

For the numerical computation we take $\mu = 10^{-8}$, $p_c(z) = 0,1\sqrt{1-z}$, $k_a(z) = (1-z)^2$, $k_w(z) = \sqrt{z}$ and $\overline{s}(z) = \delta_0(z)$, $\underline{s}(z) = \delta_1(z)$, where δ_a is the Dirac function at the point a. Furthermore u^μ is given by the line with crosses, p_g^μ is given by the lines with diam and the limit function u corresponds to the continuous line.

First test case: The case that $c = 0,7$ and $u_0 = 1$ on $[0,1]$. We obtain at $t = 0,01$ the following pictures

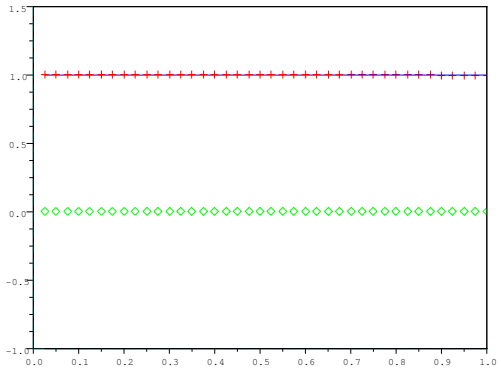

Figure 1 : t=0,01

We note that, for μ small, the functions u and u^μ are very close. Here we only start with water and inject a mixture of water and air. The air immediately invades the whole domain. Figure 1 illustrates the result which we proved in this paper, namely that u^μ tends to the solution u of the limit equation (7.6) as μ tends to 0 and moreover that the pressure p_a^μ is constant. This is indeed the case since $u < 1$.

Second test case: The case that $c = 0,7$ and $u_0(x) = \begin{cases} 0,1 \text{ on } [0,1/3] \\ 0,7 \text{ on } (1/3,1] \end{cases}$. We obtain the following pictures for $t = 0,01$ and for $t = 0,1$ respectively

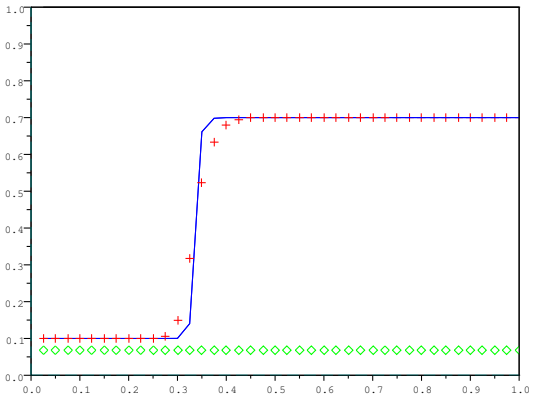

Figure 2 : t=0,01

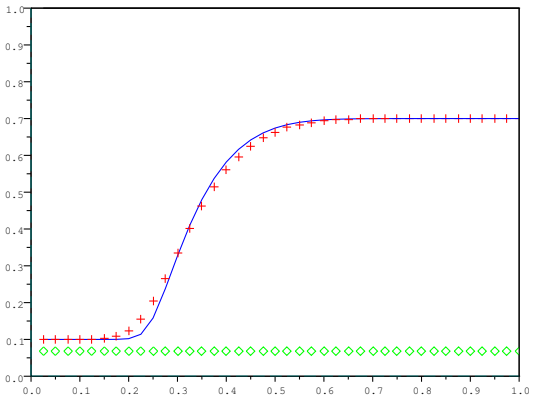

Figure 3 : t=0,1

The injection of a mixture of water and air ($c = 0.7$) takes place in a region of low water saturation. We first remark that both functions u^μ and u evolve very slowly. Here again we have that $u(x,t) < 1$ for all $(x,t) \in (0,1) \times (0,T)$ and we remark that the graphs of the two functions u^μ and u nearly coincide.

Third test case: The case that $c = 1$ and $u_0(x) = \begin{cases} 0,1 \text{ on } [0,1/3] \\ 0,7 \text{ on } (1/3,1] \end{cases}$. We obtain the following pictures for $t = 0,01$ and for $t = 0,1$ respectively

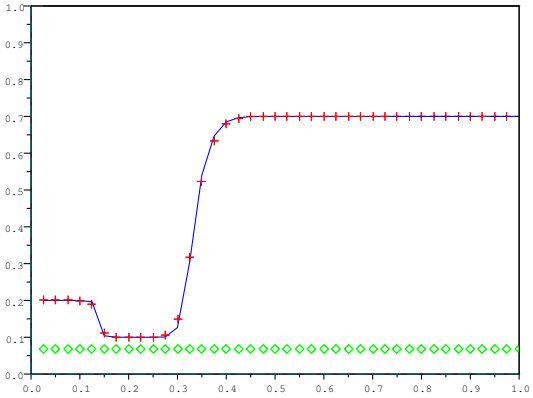

Figure 4 : t=0,01

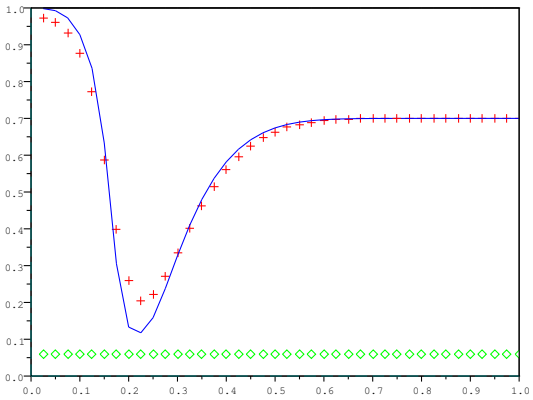

Figure 5 : t=0,1

Here only water is injected; note that the saturation u^μ evolves rather fast.

Acknowledgement

We would like to thank Professor Raphaèle Herbin from the University of Provence for her interest in this work.

References

[1] H. W. Alt and S. Luckhaus, Quasilinear elliptic equations, *Math.-Z.*, **183 (4)**, (1983), pp. 311-341.

[2] H. Brezis, Analyse fonctionnelle Théorie et applications, *Masson*, (1993).

[3] Z. Chen, Degenerate two-phase incompressible flow. I. Existence, uniqueness and regularity of a weak solution, *J. Differential Equations*, **171 (2)**, (2001), pp. 203-232.

[4] C.J. Van Duijn and L.A. Peletier, Nonstationary filtration in partially satured porous media,*Arch. Rat. Mech. Anal.*, **78 (2)**, (1982), pp. 173-198.

[5] R. Eymard, M. Gutnic and D. Hilhorst, The finit volume method for an elliptic-parabolic equation, *Acta Mathematica Universitatis Comenianae*, **67**, (1998), pp. 181-195.

[6] J. Hulshof and N. Wolanski, Monotone flows in n-dimensional partially saturated porous media: Lipschitz-continuity of the interface, *Arch. Rat. Mech. Anal.*, **102 (4)**, (1988), pp. 287-305.

[7] O.A. Ladyhenskaya, N.N. Ural'ceva, *Linear and Quasilinear Elliptic Equations*, American Mathematical Society, (1964).

[8] O.A. Ladyhenskaya, V.A. Solonnikov, N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, (1968).

[9] F. Otto, L^1-concentration and uniqueness for quasilinear elliptic-parabolic equations, *J. Diff. Eq.*, **131**, (1996), pp. 20-38.

[10] I.S. Pop, Error estimates for a time discretization method for the Richard's equation, *Computational Geosciences*, **6 (2)**, (2002), pp. 141-160.

[11] F.A. Radu, I.S. Pop, P. Knabner, Order of convergence estimates for an Euler implicit, mixed finite element discretization of Richards' equation, *SIAM Journal on Numerical Analysis*, **42 (4)**, (2004), pp. 1452-1478.