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Discrete convolution operators in positive characteristic: variations on the Floquet-Bloch Theory

Mikhail Zaidenberg

ABSTRACT. The classical Floquet theory deals with Floquet-Bloch solutions of periodic PDEs; see e.g., [Ku₃]. A discrete version of this theory for difference vector equations on lattices, including the Floquet theory on infinite periodic graphs, was developed by Peter Kuchment [Ku₁, Ku₂]. Here we propose a variation of this theory for matrix convolution operators acting on vector functions on lattices with values in a field of positive characteristic.

1. Introduction

The classical Floquet theory, founded in the work of Hill, Floquet, Lyapunov, and Felix Bloch, deals with different types of periodic ordinary differential equations. Its version for the periodic PDEs [Ku₃] considers the Floquet-Bloch solutions. To describe its discrete analog alluded to e.g., in [Co], we recall that a periodic graph is an infinite graph equipped with a free action of a lattice i.e., a free abelian group Λ of finite rank. Peter Kuchment [Ku₁]-[Ku₂]¹ adopted the Floquet theory for a periodic difference operator Δ (e.g., the Laplace, Schrödinger, Markov operator, etc.) acting in a translation invariant function space on a periodic graph Γ . In this setting one assumes that a lattice Λ acts on Γ with a finite number of orbits. Then the following hold.

- (1) Consider a periodic difference equation $\Delta(f) = 0$ on an infinite periodic graph Γ . There exists a sequence (f_n) of finite linear combinations of Floquet solutions such that every other solution f can be decomposed in a series $f = \sum_{n=1}^{\infty} a_n f_n$ with constant coefficients.
- (2) Every l_2 -solution can be approximated by solutions with finite support.
- (3) A criterion is provided as to when this equation has no solution.

In these notes we propose a version of the discrete Floquet-Bloch theory over an algebraically closed field of positive characteristic. Our approach is purely algebraic and ignores the analytic aspects, while preserving the main features of the classical theory. In some respect, the Floquet-Bloch theory for discrete operators gains a simplification when passing to a positive characteristic. In particular, the spectral theory occurs to be quite elementary. We consider matrix convolution operators acting on vector functions on a lattice Λ . Again, this includes the case of a periodic graph. As a concrete motivation for this work, we address the survey [Za₃].

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¹See also [KuPi], [KuVa].

The translation invariant function spaces usually considered in the Floquet-Bloch theory (e.g., the span of the set of all Floquet functions) occur in our setting to coincide with the space of pluri-periodic vector functions with arbitrary pluri-periods (see 2.5 and 2.6). This, of course, simplifies things. However, in positive characteristic the latter space is not well adopted for applying the Fourier transform². More suitable in this respect is the subspace $\mathcal{F}_p(\Lambda, \mathbb{K})$ consisting of all pluri-periodic vector functions on Λ with values in the base field \mathbb{K} whose period sublattices $\Lambda' \subseteq \Lambda$ are of finite indices coprime to p . So it is strongly characteristic-dependent. The role of the Bloch functions is played now by the *elementary functions*. Such a function is a product of a constant vector and a character $\Lambda \rightarrow \mathbb{K}^\times$ with values in the multiplicative group \mathbb{K}^\times of \mathbb{K} .

The content of the paper is as follows. In Section 2 we recall necessary preliminaries on the harmonic analysis on lattices in positive characteristic, introduce our principal function space, and define on it the Fourier transform. The rest of the paper is devoted to the spectral analysis of convolution operators. In Section 3 we treat the scalar case, where such operators are semi-simple (i.e., diagonalizable), see Propositions 3.2 and 3.5. The latter does not hold any more in the vector case studied in Section 4. Given a convolution operator Δ acting on vector functions, we construct in Theorems 4.2 and 4.7 a Jordan basis of Δ consisting of elementary functions. In Subsection 4.3 we deduce the existence of a solution with values in a finite field, assuming that such a solution exists with values in its algebraic closure. In concluding Section 5 we provide some examples.

2. Preliminaries

2.1. Harmonic analysis on lattices. Here we recall some generalities; see e.g., [Ni, Za₂] for details.

Given a prime integer $p > 0$, we consider the finite Galois field $\mathbb{F}_p = \text{GF}(p)$ and its algebraic closure $\mathbb{K} = \mathbb{K}_p$. Recall that the multiplicative group \mathbb{K}^\times of \mathbb{K} is a torsion group. Moreover, every finite subgroup of \mathbb{K}^\times is cyclic of order coprime to p , and every cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order coprime to p is isomorphic to a unique subgroup of \mathbb{K}^\times ; see e.g., [LN].

2.1. Lattices and dual tori. Consider a lattice Λ of rank $s > 0$. By a character of Λ over \mathbb{K} we mean a representation $\chi : \Lambda \rightarrow \text{GL}(1, \mathbb{K}) \cong \mathbb{K}^\times$. The image $\chi(\Lambda) \subseteq \mathbb{K}^\times$ is a cyclic subgroup of order coprime to p . The dual group Λ^\vee of Λ over \mathbb{K} is the group of all characters $\Lambda \rightarrow \mathbb{K}^\times$. It is isomorphic to the algebraic torus $\mathbb{T}^s := (\mathbb{K}^\times)^s$. By the duality theorem $(\Lambda^\vee)^\vee \cong \Lambda$. Indeed, the natural duality pairing $\Lambda \times \Lambda^\vee \rightarrow \mathbb{K}^\times$ is non-degenerate.

More concretely, given a base $\mathcal{V} = (v_1, \dots, v_s)$ of Λ one can identify Λ with the standard integer lattice \mathbb{Z}^s by sending $v = \sum_{i=1}^s \lambda_i v_i \in \Lambda$ to $\lambda(v) := (\lambda_1, \dots, \lambda_s) \in \mathbb{Z}^s$. The pairing now becomes

$$\mathbb{Z}^s \times \mathbb{T}^s \rightarrow \mathbb{K}^\times, \quad (\lambda, z) \mapsto z^\lambda = \prod_{i=1}^s z_i^{\lambda_i}, \quad \text{where } z = (z_1, \dots, z_s) \in \mathbb{T}^s.$$

Fixing $z \in \mathbb{T}^s$ gives a character $\mathbb{Z}^s \rightarrow \mathbb{K}^\times$, $\lambda \mapsto z^\lambda$, and every character $\mathbb{Z}^s \rightarrow \mathbb{K}^\times$ arises in this way. On the other hand, every character of the torus \mathbb{T}^s with values in \mathbb{K}^\times (i.e., a rational representation $\mathbb{T}^s \rightarrow \text{GL}_1(\mathbb{K}) = \mathbb{K}^\times$) is given by a Laurent monomial $z \mapsto z^\lambda$. This yields an isomorphism $(\Lambda^\vee)^\vee \cong \Lambda$.

2.2. Convolution operators. Consider the group algebra $\mathbb{K}[\Lambda]$ over \mathbb{K} i.e., the algebra of functions $\Lambda \rightarrow \mathbb{K}$ with finite support endowed with the convolution

$$(f, g) \mapsto f * g, \quad \text{where } (f * g)(u) = \sum_{v \in \Lambda} f(v)g(u - v).$$

²We avoid using the Floquet transform as in [Ku₁], [Ku₂], and use instead the Fourier transform.

A convolution operator

$$\Delta_a \in \text{End}(\mathbb{K}[\Lambda]), \quad \Delta_a(f) := a * f, \quad \text{where } a \in \mathbb{K}[\Lambda],$$

can be written as

$$\Delta_a = \sum_{v \in \Lambda} a(v) \tau_{-v},$$

where τ_v denotes the shift by v : $\tau_v(f)(u) = f(u + v)$. In particular, Δ_a commutes with shifts: $[\Delta_a, \tau_v] = 0 \forall v \in \Lambda$. In fact, any endomorphism $\Delta \in \text{End}_{\mathbb{K}}(\mathbb{K}[\Lambda])$ commuting with shifts is a convolution operator with kernel $a = \Delta(\delta_0)$, where $\delta_v \in \mathbb{K}[\Lambda]$ stands for the delta-function concentrated on $v \in \Lambda$ i.e.,

$$\delta_v(v) = 1 \quad \text{and} \quad \delta_v(u) = 0 \quad \forall u \neq v.$$

In particular, if $a = \delta_v$ then $\Delta_a = \tau_{-v}$; indeed $\tau_{-v}(\delta_0) = \delta_v$. The \mathbb{K} -algebra of convolution operators is naturally isomorphic to the group algebra $\mathbb{K}[\Lambda]$.

2.3. The Fourier transform. The Fourier transform sends the group ring $\mathbb{K}[\Lambda]$ isomorphically onto the coordinate ring of the torus $\mathcal{O}(\Lambda^\vee)$. Given a base \mathcal{V} of Λ and the identifications $\Lambda \cong \mathbb{Z}^s$ and $\Lambda^\vee \cong \mathbb{T}^s$ as above, the Fourier transform is given by

$$F : f \longmapsto \hat{f}(z) = \sum_{\lambda \in \mathbb{Z}^s} f(\lambda) z^\lambda.$$

It sends $\mathbb{K}[\Lambda] \cong \mathbb{K}[\mathbb{Z}^s]$ isomorphically onto the algebra of Laurent polynomials

$$\mathcal{O}(\Lambda^\vee) \cong \mathcal{O}(\mathbb{T}^s) = \mathbb{K}[z, z^{-1}] = \mathbb{K}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}].$$

The inverse Fourier transform $F^{-1} : \mathbb{K}[z, z^{-1}] \rightarrow \mathbb{K}[\mathbb{Z}^s]$ sends a Laurent polynomial $\sum_{\lambda \in \mathbb{Z}^s} f(\lambda) z^\lambda$ into its coefficient function $f \in \mathbb{K}[\mathbb{Z}^s]$. In particular, the delta-function $\delta_v \in \mathbb{K}[\Lambda]$ is sent to the Laurent monomial $\hat{\delta}_v = z^{\lambda(v)} \in \mathcal{O}(\mathbb{T}^s)$. The equality $\delta_v * \delta_w = \delta_{v+w}$ in $\mathbb{K}[\Lambda]$ amounts to $z^{\lambda(v)} z^{\lambda(w)} = z^{\lambda(v)+\lambda(w)}$ in $\mathcal{O}(\mathbb{T}^s)$.

2.2. The Fourier transform in translation invariant function spaces.

2.4. Consider the space $\mathcal{F} = \mathcal{F}(\Lambda, \mathbb{K})$ of all \mathbb{K} -valued functions on Λ . This is a $\mathbb{K}[\Lambda]$ -module with respect to the natural action $\mathbb{K}[\Lambda] \times \mathcal{F} \rightarrow \mathcal{F}$, $(a, f) \longmapsto a * f = \Delta_a(f)$. A subspace $E \subseteq \mathcal{F}$ is a $\mathbb{K}[\Lambda]$ -submodule if and only if it is translation invariant i.e., $\tau_v(E) = E$ for all $v \in \Lambda$. A one-dimensional subspace of \mathcal{F} is a $\mathbb{K}[\Lambda]$ -submodule if and only if it is spanned by a character $\chi \in \Lambda^\vee$. The following function spaces are translation invariant. They arise naturally in the framework of the Floquet-Kuchment Theory.

- EXAMPLES 2.5. (i) The subspace $\mathbb{K}[\Lambda] \subseteq \mathcal{F}$ of all functions with finite support.
(ii) Given a sublattice $\Lambda' \subseteq \Lambda$, we let $\mathcal{F}_{\Lambda'}$ denote the subspace of all Λ' -periodic functions i.e.,

$$\mathcal{F}_{\Lambda'} = \{f \in \mathcal{F} \mid \Lambda(f) \supseteq \Lambda'\},$$

where

$$\Lambda(f) := \{v \in \Lambda \mid \tau_v(f) = f\}$$

stands for the period lattice of f .

- (iii) Given an isomorphism $\Lambda \cong \mathbb{Z}^s$ we define the subspace $E_1 \subseteq \mathcal{F}$ of all traces of polynomial functions i.e.,

$$E_1 = \{q|_{\mathbb{Z}^s} \in \mathcal{F} \mid q \in \mathbb{K}[x_1, \dots, x_s]\}.$$

- (iv) The subspace $E_2 \subseteq \mathcal{F}$ of all pluri-periodic functions,

$$E_2 = \bigcup_{\Lambda' \subseteq \Lambda, \text{ind}_\Lambda \Lambda' < \infty} \mathcal{F}_{\Lambda'},$$

is clearly translation invariant.

- (v) The subspace $E_3 \subseteq \mathcal{F}$ of all polynomially periodic functions. Given an isomorphism $\Lambda \cong \mathbb{Z}^s$, a function $g \in \mathcal{F}$ is called *polynomially periodic* if there exists a sublattice $\Lambda' \subseteq \Lambda$ of finite index and a set of polynomials $p_{\bar{v}} \in \mathbb{K}[x_1, \dots, x_s]$, where $\bar{v} \in \Lambda/\Lambda'$, such that

$$g(v) = p_{\bar{v}}(v) \quad \text{if } v \in \bar{v}.$$

- (vi) The subspace $E_4 \subseteq \mathcal{F}$ of all *Floquet functions*:

$$E_4 = \left\{ f = \sum_{i=1}^n \chi_i g_i \in \mathcal{F} \mid \chi_i \in \Lambda^\vee = \text{Char}(\Lambda, \mathbb{K}^\times), g_i \in E_3 \right\}.$$

- (vii) The subspace $E_5 \subseteq \mathcal{F}$ spanned by all Λ' -automorphic functions on Λ , where Λ' runs over the set of all finite index sublattices of Λ . Given a sublattice $\Lambda' \subseteq \Lambda$, a function $f \in \mathcal{F}$ is called Λ' -*automorphic* if, for some character χ of Λ' ,

$$(\tau_u f)(v) = \chi(u) f(v) \quad \forall u \in \Lambda', \quad \forall v \in \Lambda.$$

- (viii) The subspace $E_6 \subseteq \mathcal{F}$ of all finite-valued functions on Λ , etc.

The following proposition shows that among the function spaces E_1, \dots, E_6 as defined above, there are exactly three different. In particular, in positive characteristic the Floquet functions are the same as the pluri-periodic functions.

PROPOSITION 2.6. *We have*

$$E_1 \subsetneq E_2 = E_3 = E_4 = E_5 \subsetneq E_6.$$

PROOF. We observe that $(x + p\lambda)^n = x^n$ for every monomial $x^n = \prod_{i=1}^s x_i^{n_i}$ regarded as a function in $x \in \mathbb{Z}^s = \Lambda$ and for every $\lambda \in \mathbb{Z}^s$, where $p = \text{Char}(\mathbb{K})$. Hence every polynomial function on Λ is Λ' -periodic, where $\Lambda' := p\Lambda$. Therefore

$$E_1 \subseteq \mathcal{F}_{\Lambda'} \subseteq E_2.$$

By a similar reason, $E_3, E_4 \subseteq E_2$. For instance, if $g \in E_3$ is as in (v) then $p\Lambda' \subseteq \Lambda(g)$. Indeed, for every $v \in \Lambda$ and $v' \in \Lambda'$ we have $v + pv' + \Lambda' = v + \Lambda'$. Hence

$$g(v + pv') = p_{\overline{v+pv'}}(v + pv') = p_{\overline{v}}(v) = p_{\bar{v}}(v) = g(v).$$

Thus $E_3 \subseteq E_2$.

It is easily seen that the sum and the product of pluri-periodic functions are again pluri-periodic. Since every character $\chi \in \Lambda^\vee$ is Λ' -periodic, where the sublattice $\Lambda' := \ker(\chi)$ is of finite index in Λ , the inclusion $E_4 \subseteq E_2$ holds. The inclusions $E_2 \subseteq E_3 \subseteq E_4$ are evident. It follows that $E_1 \subseteq E_2 = E_3 = E_4 \subseteq E_6$.

Every pluri-periodic function $f \in E_2$ (and in particular, every character of Λ) is $\Lambda(f)$ -automorphic. Hence $E_2 \subseteq E_5$. Let furthermore $f \in E_5$ be a Λ' -automorphic function as in (viii), where $\Lambda' \subseteq \Lambda$ is a finite index sublattice. Since the period lattice $\Lambda(f)$ contains the finite index sublattice $\Lambda'' := \ker(\chi) \subseteq \Lambda' \subseteq \Lambda$, f is pluri-periodic. Thus $E_2 = E_5$ and so $E_1 \subseteq E_2 = E_3 = E_4 = E_5 \subseteq E_6$.

It remains to show that $E_1 \neq E_2 \neq E_6$. The latter assertion is easy and we leave it to the reader. As for the former, it is enough to consider the case of a lattice $\Lambda \cong \mathbb{Z}$ of rank 1. In this case, whenever $p \geq 3$, a non-constant 2-periodic function $f \in E_1$ cannot be a trace of a polynomial. In other words, a 2-periodic trace of a polynomial $q \in \mathbb{K}[x]$ on \mathbb{Z} must be constant. Indeed, since $q(1) = q(1-p) = q(0)$ we have $q|2\mathbb{Z} = q|(2\mathbb{Z} + 1)$. In the remaining case $p = 2$, a 3-periodic function f on \mathbb{Z} with $f(0) = f(1) = 0$ and $f(2) = 1$ cannot be a trace of a polynomial over \mathbb{K} , since $f(0) \neq f(2)$. Thus indeed $E_1 \neq E_2$. \square

2.3. Extending the Fourier transform.

2.7. In positive characteristic, the function space $E_2 (= E_3 = E_4 = E_5)$ of all pluri-periodic functions on Λ is not quite appropriate for applying the Fourier transform. More suitable is the subspace $\mathcal{F}_p(\Lambda, \mathbb{K}) \subseteq \mathcal{F}$ spanned by all characters of Λ with values in \mathbb{K}^\times . It is definitely translation invariant. In Proposition 2.17 below we characterize this subspace in terms of the period lattices of its members. This description depends essentially on the characteristic p . To define the Fourier transform on $\mathcal{F}_p(\Lambda, \mathbb{K})$ we need some preparations.

2.8. Let $\Lambda' \subseteq \Lambda$ be a sublattice of finite index coprime to p . Then Λ' is p -saturated i.e., $pv \in \Lambda'$ implies that $v \in \Lambda'$. Or, equivalently, the quotient group $G = \Lambda/\Lambda'$ (of order $\text{ind}_\Lambda(\Lambda')$ coprime to p) has no p -torsion. Clearly, the intersection of two p -saturated sublattices is again p -saturated. If Λ' is p -saturated and every sublattice $\Lambda'' \supseteq \Lambda'$ is, since $\text{ind}_\Lambda \Lambda'' | \text{ind}_\Lambda \Lambda'$.

If Λ' is p -saturated then the dual group G^\vee is isomorphic to the group of \mathbb{K}^\times -characters $\text{Char}(G, \mathbb{K}^\times)$. Otherwise $\text{Char}(G, \mathbb{K}^\times) \cong G^\vee/G^\vee(p)$, where $G^\vee(p) \subseteq G^\vee$ is the Sylow p -component (cf. [Za₂, §1]).

2.9. Suppose that Λ' is p -saturated. Let $\pi : \Lambda \rightarrow G = \Lambda/\Lambda'$ be the canonical surjection. The dual group $G^\vee \cong \text{Char}(G, \mathbb{K}^\times)$ of the finite group $G = \Lambda/\Lambda'$ can be realized as the subgroup

$$\Lambda'^\perp := \{\chi \in \Lambda^\vee \mid \Lambda' \subseteq \ker \chi\}$$

of the dual torus Λ^\vee via the pullback

$$\pi^\vee : G^\vee \xrightarrow{\cong} \Lambda'^\perp, \quad G^\vee \ni \bar{\chi} \mapsto \pi^\vee(\bar{\chi}) := \bar{\chi} \circ \pi \in \Lambda'^\perp.$$

2.10. Suppose as before that $\text{ord}(G)$ is coprime to p . The Fourier transform

$$\mathcal{F}(G, \mathbb{K}) \ni \bar{f} \mapsto \hat{f} = \sum_{g \in G} \hat{f}(g) \bar{\chi}(g) \in \mathcal{F}(G^\vee, \mathbb{K}),$$

yields an isomorphism

$$\bar{F} : \mathcal{F}(G, \mathbb{K}) \xrightarrow{\cong} \mathcal{F}(G^\vee, \mathbb{K}),$$

which sends characters into delta-functions and vice versa. Thus $\hat{\bar{\chi}} = \delta_{\bar{\chi}}$ for every $\bar{\chi} \in G^\vee \cong \text{Char}(G, \mathbb{K}^\times)$. Writing

$$\hat{f} = \sum_{\bar{\chi} \in G^\vee} \hat{f}(\bar{\chi}) \delta_{\bar{\chi}}$$

we obtain

$$\bar{f} = \bar{F}^{-1}(\hat{f}) = \sum_{\bar{\chi} \in G^\vee} \hat{f}(\bar{\chi}) \cdot \bar{\chi}.$$

In particular, $\mathcal{F}(G, \mathbb{K}) = \text{span}(\bar{\chi} \mid \bar{\chi} \in G^\vee)$.

Letting $f := \bar{f} \circ \pi \in \mathcal{F}_{\Lambda'}(\Lambda, \mathbb{K})$ and $\chi := \bar{\chi} \circ \pi \in \Lambda'^\perp$, we get a presentation

$$f = \sum_{\chi \in \Lambda'^\perp} \alpha(\chi) \cdot \chi, \quad \text{where } \alpha(\chi) := \hat{f}(\bar{\chi}).$$

Clearly, every function $f \in \mathcal{F}_{\Lambda'}(\Lambda, \mathbb{K})$ admits such a presentation.

2.11. Every character $\chi \in \Lambda^\vee$ regarded as a function in $\mathcal{F}_p(\Lambda, \mathbb{K})$ is pluri-periodic, with the period lattice $\Lambda(\chi) = \ker(\chi)$. The quotient group

$$G(\chi) := \Lambda/\Lambda(\chi) \cong \chi(\Lambda) \subseteq \mathbb{K}^\times$$

is a finite cyclic group of order $\text{ord}(\chi)$ coprime to p . Hence the period lattice $\Lambda(\chi)$ is p -saturated (see 2.8).

According to Definition 2.7, every function $f \in \mathcal{F}_p(\Lambda, \mathbb{K})$ is a linear combination of characters: $f = \sum_{i=1}^m \alpha_i \chi_i$, where $\alpha_i \in \mathbb{K}$, $\chi_i \in \Lambda^\vee$. Therefore the period lattice

$$\Lambda(f) \supseteq \bigcap_{i=1}^m \Lambda(\chi_i)$$

is also p -saturated.

DEFINITION 2.12. We let $\mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{K})$ denote the space of all functions on Λ^\vee with finite support. It comes equipped with the structure of an $\mathcal{O}(\Lambda^\vee)$ -module. For a function

$$f = \sum_{i=1}^m \alpha_i \chi_i \in \mathcal{F}_p(\Lambda, \mathbb{K})$$

we define its Fourier transform $\widehat{f} = F(f)$ as follows:

$$\widehat{f} = \sum_{i=1}^m \alpha_i \delta_{\chi_i} \in \mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{K}).$$

So $\text{supp}(\widehat{f}) \subseteq \Lambda(f)^\perp$.

PROPOSITION 2.13. (a) *The Fourier transform as defined in 2.12 yields an isomorphism*

$$F : \mathcal{F}_p(\Lambda, \mathbb{K}) \xrightarrow{\cong} \mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{K}).$$

(b) *For every $a \in \mathbb{K}[\Lambda]$ and for every $f \in \mathcal{F}_p(\Lambda, \mathbb{K})$,*

$$(1) \quad \widehat{a * f} = (\widehat{a} \circ \iota) \cdot \widehat{f} = \widehat{a} \cdot (\widehat{f} \circ \iota),$$

where $\iota : \chi \mapsto \chi^{-1}$ is the inversion in the group Λ^\vee .

PROOF. (a) The inverse Fourier transform F^{-1} sends a function φ on Λ^\vee with finite support into a linear combination of characters:

$$\mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{K}) \ni \varphi = \sum_{\chi \in \Lambda^\vee} \varphi(\chi) \cdot \delta_\chi \mapsto F^{-1}(\varphi) = \sum_{\chi \in \text{supp}(\varphi)} \varphi(\chi) \cdot \chi \in \mathcal{F}_p(\Lambda, \mathbb{K}).$$

This shows (a).

Since the Fourier transform is linear both on $\mathbb{K}[\Lambda]$ (spanned by the delta functions) and on $\mathcal{F}_p(\Lambda, \mathbb{K})$ (spanned by the characters of Λ), it is enough to verify (b) in the particular case where $a = \delta_v$ ($v \in \Lambda$) and $f = \chi$ ($\chi \in \Lambda^\vee$). In this case we have

$$\widehat{\delta_v * \chi} = \widehat{\tau_{-v}(\chi)} = \chi^{-1}(v) \cdot \widehat{\chi} = \chi^{-1}(v) \cdot \delta_\chi.$$

On the other hand,

$$\widehat{\delta_v} = \text{eval}_v \in \mathcal{F}(\Lambda^\vee, \mathbb{K}),$$

hence

$$\chi^{-1}(v) = \text{eval}_v(\chi^{-1}) = (\widehat{\delta_v} \circ \iota)(\chi).$$

Now (1) follows. \square

REMARK 2.14. The translation invariant subspaces $\mathbb{K}[\Lambda] =: \mathcal{F}_{\text{fs}}(\Lambda, \mathbb{K})$ and $\mathcal{F}_p(\Lambda, \mathbb{K})$ of \mathcal{F} are transversal. Thus the Fourier transform can be defined on the direct sum $\mathcal{F}_{\text{fs}}(\Lambda, \mathbb{K}) \oplus \mathcal{F}_p(\Lambda, \mathbb{K})$, sending it isomorphically onto the direct sum $\mathcal{O}(\Lambda^\vee) \oplus \mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{K})$.

DEFINITION 2.15. Let $f \in \mathcal{F}$ be a pluri-periodic function with the period lattice $\Lambda(f)$ of finite index (see Example 2.5(iv)). We say that f is p -pure if $\Lambda(f)$ is p -saturated. We let $\mathcal{F}'_p(\Lambda, \mathbb{K})$ denote the space of all pluri-periodic p -pure functions on Λ with values in \mathbb{K} . This is indeed a linear space because $\Lambda(f+g) \supseteq \Lambda(f) \cap \Lambda(g)$ is p -saturated if $\Lambda(f)$ and $\Lambda(g)$ are, see 2.8. Moreover, since $\Lambda(\tau_v(f)) = \Lambda(f) \forall v \in \Lambda$, the space $\mathcal{F}'_p(\Lambda, \mathbb{K})$ is translation invariant and so, is a convolution module over $\mathbb{K}[\Lambda]$.

REMARK 2.16. In Example 2.5 above we have shown that the trace on Λ of a polynomial $q \in \mathbb{K}[x_1, \dots, x_s]$ is Λ' -periodic, where $\Lambda' = p\Lambda = \langle pv_1, \dots, pv_s \rangle$. However, this sublattice of finite index is not p -saturated. Moreover, $q|\Lambda$ is p -pure (in the above sense) if and only if $q|\Lambda$ is a constant function.

The following proposition shows that the subspace $\mathcal{F}'_p(\Lambda, \mathbb{K}) \subseteq \mathcal{F}$ from 2.15 is generated by characters of Λ with values in \mathbb{K}^\times and so, coincides with $\mathcal{F}_p(\Lambda, \mathbb{K}) = \text{span}(\chi \mid \chi \in \Lambda^\vee)$.

PROPOSITION 2.17. $\mathcal{F}'_p(\Lambda, \mathbb{K}) = \mathcal{F}_p(\Lambda, \mathbb{K})$.

PROOF. The inclusion $\mathcal{F}_p(\Lambda, \mathbb{K}) \subseteq \mathcal{F}'_p(\Lambda, \mathbb{K})$ was established already in 2.11. It remains to show the converse inclusion $\mathcal{F}'_p(\Lambda, \mathbb{K}) \subseteq \mathcal{F}_p(\Lambda, \mathbb{K})$. Let f be a pluri-periodic function with period lattice $\Lambda(f) \subseteq \Lambda$ of index coprime to p . Then f is the pullback of a function $\bar{f} : G(f) \rightarrow \mathbb{K}$, where $G(f) := \Lambda/\Lambda(f)$ is a finite abelian group of order coprime with p . Hence the function space $\mathcal{F}(G(f), \mathbb{K})$ is spanned by characters of $G(f)$ with values in \mathbb{K}^\times , see 2.10 or [Za2, Lemma 2.1(a)]. In particular, there is a presentation

$$\bar{f} = \sum_{i=1}^n \alpha_i \bar{\chi}_i, \quad \text{where } \bar{\chi}_i \in \text{Char}(G(f), \mathbb{K}^\times) \quad \text{and} \quad \alpha_i \in \mathbb{K}.$$

Thus $f = \sum_{i=1}^n \alpha_i \chi_i$, where the characters $\chi_i = \bar{\chi}_i \circ \pi \in \Lambda^\vee$ are the pullbacks of $\bar{\chi}_i$, $i = 1, \dots, n$. This yields the desired inclusion. \square

3. Convolution operators: the scalar case

3.1. Δ_a -harmonic functions.

3.1. Let $a \in \mathbb{K}[\Lambda]$. A function $f \in \mathcal{F}$ satisfying the equation

$$\Delta_a(f) = a * f = 0$$

will be called Δ_a -harmonic. Choosing a base of Λ , we identify Λ with \mathbb{Z}^s and Λ^\vee with the torus \mathbb{T}^s . We associate with every convolution operator Δ_a on $\Lambda = \mathbb{Z}^s$ its *symbolic variety* $\Sigma_a = \Sigma_{\Delta_a}$. This is an affine hypersurface in the dual torus $\Lambda^\vee \cong \mathbb{T}^s$ defined as follows:

$$\Sigma_a = \{z \in \mathbb{T}^s \mid \hat{a}(z^{-1}) = 0\},$$

where $\hat{a} = F(a)$ is a Laurent polynomial in $z = (z_1, \dots, z_s)$, see [Za2, 2.7]. Every character $\chi \in \Sigma_a$ viewed as a function on $\Lambda = \mathbb{Z}^s$ is Δ_a -harmonic. Indeed, for $z \in \Sigma_a$ we have

$$\Delta_a(z^\lambda) = \sum_{v \in \Lambda} a(v) \tau_{-v}(z^\lambda) = \sum_{v \in \Lambda} a(v) z^{\lambda-v} = z^\lambda \hat{a}(z^{-1}) = 0.$$

PROPOSITION 3.2. ³

- (a) Every Δ_a -harmonic function $f \in \mathcal{F}_p(\Lambda, \mathbb{K})$ is a linear combination of Δ_a -harmonic characters $\chi \in \Sigma_a$. In other words,

$$\ker(\Delta_a|_{\mathcal{F}_p(\Lambda, \mathbb{K})}) = \text{span}(\chi \mid \chi \in \Sigma_a).$$

³Cf. Theorems 1-3 in [Ku2].

(b) If $a \neq 0$ then a nonzero Δ_a -harmonic function on Λ cannot have finite support.

PROOF. (a) If $a = c\delta_v$ i.e., Δ_a is proportional to a shift, then the only Δ_a -harmonic function in \mathcal{F} is the zero function. Assume further that $a \neq c\delta_v$ i.e., \widehat{a} is not a Laurent monomial and so $\Sigma_a \neq \emptyset$. Let $f \in \mathcal{F}_{\Lambda'}$ be Δ_a -harmonic and Λ' -periodic, where $\Lambda' \subseteq \Lambda$ is a sublattice of a finite index coprime to p . Then f is the pullback of a function $\bar{f} : G = \Lambda/\Lambda' \rightarrow \mathbb{K}$. We let $a_* : G \rightarrow \mathbb{K}$ denote the pushforward function defined by

$$a_*(v + \Lambda') = \sum_{v' \in \Lambda'} a(v + v').$$

This is a unique function in $\mathcal{F}(G, \mathbb{K})$ satisfying the condition $a_* \circ \pi = a * \delta_{\Lambda'}$, where $\pi : \Lambda \rightarrow G$ is the canonical surjection and $\delta_{\Lambda'}$ stands for the characteristic function of $\Lambda' \subseteq \Lambda$ (see [Za2, (7)]). Since f is Δ_a -harmonic, by the Pushforward Lemma 1.1 in [Za2], $\Delta_{a_*}(\bar{f}) = 0$. It follows that \bar{f} is a linear combination of a_* -harmonic characters of G , see [Za2, Corollary 2.2]. Their pullbacks are Δ_a -harmonic characters of Λ , and so f is a linear combination of the latter characters, as stated in (a).

To show (b) suppose that $f \in \mathbb{K}[\Lambda]$ is Δ_a -harmonic. Then $(\widehat{a} \circ \iota) \cdot \widehat{f} = 0$ and so $\widehat{a}, \widehat{f} \in \mathbb{K}[z, z^{-1}]$ are zero divisors. Since the algebra $\mathbb{K}[z, z^{-1}]$ of Laurent polynomials is an integral domain and $\widehat{a} \neq 0$, this implies the equality $f = 0$. \square

The following equivalencies are evident:

$$a = 0 \iff \Sigma_a = \mathbb{T}^s \iff \ker(\Delta_a | \mathcal{F}_p(\Lambda, \mathbb{K})) = \mathcal{F}_p(\Lambda, \mathbb{K}).$$

In the opposite direction, we have the following result.

COROLLARY 3.3. ⁴ There is no nonzero Δ_a -harmonic function on Λ if and only if $\Sigma_a = \emptyset$, if and only if Δ_a is proportional to a shift ⁵.

3.2. The Floquet-Fermi hypersurfaces. This notion is borrowed in [KuVa, Def. 2].

DEFINITION 3.4. Consider the convolution operator Δ_a , where $a \in \mathbb{K}[\Lambda]$. By definition, the Floquet-Fermi hypersurface of Δ_a of level $\mu \in \mathbb{K}$ is

$$\Sigma_{a,\mu} = \{z \in \mathbb{T}^s \mid \widehat{a}(z^{-1}) = \mu\}.$$

Thus $\Sigma_{a,\mu}$ coincides with the symbolic hypersurface $\Sigma_{a-\mu\delta_0}$ of the operator $\Delta_a - \mu\text{Id}$. The eigenspace of $\Delta_a | \mathcal{F}_p(\Lambda, \mathbb{K})$ with eigenvalue μ is

$$E_{a,\mu} = \text{span}(\chi \mid \chi \in \Sigma_{a,\mu}).$$

The following proposition is straightforward. Nevertheless, we provide a short argument.

PROPOSITION 3.5. We have

$$\mathcal{F}_p(\Lambda, \mathbb{K}) = \bigoplus_{\mu \in \mathbb{K}} E_{a,\mu}.$$

In particular, the convolution operator Δ_a on the space $\mathcal{F}_p(\Lambda, \mathbb{K})$ is diagonalizable in the basis of characters.

PROOF. Choosing a base (v_1, \dots, v_s) of the lattice Λ , we identify Λ with \mathbb{Z}^s and Λ^\vee with the torus \mathbb{T}^s . Letting τ_i denote the shift by $-v_i$, the convolution operator Δ_a can be written as a Laurent polynomial in the basic shifts τ_1, \dots, τ_s . Namely,

$$\Delta_a = \sum_{\lambda \in \mathbb{Z}^s} a(\lambda) \tau_{-\lambda} = \sum_{\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{Z}^s} a(\lambda) \prod_{i=1}^s \tau_i^{\lambda_i} = \widehat{a}(\tau_1, \dots, \tau_s).$$

⁴Cf. [Ku1, Cor. 2].

⁵If and only if Δ_a is invertible.

Using the equalities $\tau_i(z^\lambda) = z_i^{-1} \cdot z^\lambda$, for every $z \in \mathbb{K}^\times$ we obtain

$$\Delta_a(z^\lambda) = \widehat{a}(\tau_1, \dots, \tau_s)(z^\lambda) = \widehat{a}(z^{-1}) \cdot z^\lambda.$$

Now both assertions follow. \square

REMARKS 3.6. 1. Here is an alternative approach. According to Proposition 2.13(b), the Fourier transform F sends the operator $\Delta_a|_{\mathcal{F}_p(\Lambda, \mathbb{K})}$ to the operator of multiplication by the function $\widehat{a} \circ \iota$ on the space $\mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{K}) = F(\mathcal{F}_p(\Lambda, \mathbb{K}))$. The latter operator being semi-simple, the convolution operator Δ_a is semi-simple too.

2. Let us identify Λ with \mathbb{Z}^s and Λ^\vee with the torus \mathbb{T}^s . Note that the spectrum of $\Delta_a|_{\mathcal{F}_p(\Lambda, \mathbb{K})}$ coincides with the range of the Laurent polynomial

$$\widehat{a}(z^{-1}) = \sum_{\lambda \in \mathbb{Z}^n} a(\lambda) z^{-\lambda}.$$

Hence

- $\text{spec}(\Delta_a) = \{\mu\}$ if and only if $a = \mu\delta_0$;
- $\text{spec}(\Delta_a) = \mathbb{K}^\times$ if and only if $a = \alpha\delta_v$ for some $\alpha \in \mathbb{K}^\times$ and $v \in \Lambda \setminus \{0\}$;
- $\text{spec}(\Delta_a) = \mathbb{K}$ otherwise.

Moreover, if $a \neq \alpha\delta_v$ then

$$(2) \quad E_{a,\mu} = \text{span}(z^{-\lambda} | \widehat{a}(z) = \mu) = \left\{ \begin{array}{l} f \in \mathcal{F}_p(\Lambda, \mathbb{K}) \mid f(\lambda) = \sum_{z \in \mathbb{T}^s} \alpha(z) z^{-\lambda} : \\ \alpha \in \mathcal{F}_{\text{fs}}(\mathbb{T}^s, \mathbb{K}) \text{ s.t. } \text{supp}(\alpha) \subseteq \widehat{a}^{-1}(\mu) \end{array} \right\}.$$

If $s > 1$ then every level set $\widehat{a}^{-1}(\mu)$, where $\mu \in \text{spec}(\Delta_a)$, is infinite and countable. The latter is due to the fact that \mathbb{K} is a countable algebraically closed field. Hence for every $\mu \in \mathbb{K}$ the eigenspace $E_{a,\mu}$ has a countable basis of characters; in particular $\dim(E_{a,\mu}) = +\infty$. While for $s = 1$, these eigenspaces are of finite dimension.

4. Convolution operators: the vector case

Letting $\mathbb{A}^n = \mathbb{A}_{\mathbb{K}}^n$ be the affine n -space over \mathbb{K} and (e_1, \dots, e_n) be the canonical basis of \mathbb{A}^n , we consider the space $\mathcal{F}^n = \mathcal{F}(\Lambda, \mathbb{A}^n)$ of all \mathbb{K} -valued vector functions on Λ , and its subspaces $\mathbb{K}[\Lambda]^n$ of all vector functions on Λ with finite support and $\mathcal{F}_p(\Lambda, \mathbb{A}^n)$ of all pluri-periodic p -pure vector functions on Λ .

4.1. Let us identify Λ with \mathbb{Z}^s and Λ^\vee with \mathbb{T}^s . The Fourier transform sends a vector function $f \in \mathbb{K}[\Lambda]^n$ into the vector Laurent polynomial

$$\widehat{f}(z) = \sum_{\lambda \in \mathbb{Z}^n} f(\lambda) z^\lambda, \quad z \in \mathbb{T}^s,$$

where $f(\lambda) \in \mathbb{A}^n \forall \lambda \in \mathbb{Z}^s$. In particular, a basic vector delta-function $\delta_{i,\lambda} = \delta_\lambda \cdot e_i \in \mathbb{K}[\Lambda]^n$, where $\lambda \in \Lambda = \mathbb{Z}^s$, is sent to the vector Laurent monomial $\widehat{\delta_{i,\lambda}} = z^\lambda \cdot e_i$ ($i = 1, \dots, n$). A pluri-periodic p -pure vector function

$$f = \sum_{i=1}^m \alpha_i \chi_i \in \mathcal{F}_p(\Lambda, \mathbb{A}^n),$$

where $\alpha_i \in \mathbb{A}^n$ and $\chi_i \in \Lambda^\vee$ (see Proposition 2.17), is sent to the vector function on Λ^\vee with finite support

$$\widehat{f} = \sum_{i=1}^m \alpha_i \delta_{\chi_i} \in \mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{A}^n).$$

4.1. Δ -harmonic vector functions. Consider an endomorphism $\Delta \in \text{End}(\mathbb{K}[\Lambda]^n)$ commuting with shifts on Λ . Letting $a_i = \Delta(\delta_{i,0}) = (a_{i1}, \dots, a_{in}) \in \mathbb{K}[\Lambda]^n$, $i = 1, \dots, n$, we get a square matrix $A = (a_{ij})$ of order n with entries in $\mathbb{K}[\Lambda]$. The images of the remaining basic vector functions $\delta_{i,v}$ can be obtained as suitable shifts of the vectors a_i :

$$\Delta(\delta_{i,v}) = \Delta(\tau_{-v}\delta_{i,0}) = \tau_{-v}(\Delta(\delta_{i,0})) = \tau_{-v}(a_i).$$

If $f = (f_1, \dots, f_n) \in \mathbb{K}[\Lambda]^n$ and $g = \Delta(f) = (g_1, \dots, g_n) \in \mathbb{K}[\Lambda]^n$, then $g = A * f$ i.e.,

$$(3) \quad g_i = \sum_{j=1}^n a_{ij} * f_j, \quad i = 1, \dots, n.$$

In other words, Δ acts on $\mathbb{K}[\Lambda]^n$ as a square matrix of convolution operators $\Delta_A = (\Delta_{a_{ij}})$. The latter matrix convolution operator can be applied to an arbitrary vector function $f = (f_1, \dots, f_n) \in \mathcal{F}(\Lambda, \mathbb{A}^n) = \mathcal{F}^n$. This yields an extension of Δ to \mathcal{F}^n , which we denote by the same letter. The extended endomorphism $\Delta \in \text{End}(\mathcal{F}^n)$ leaves invariant any translation invariant subspace. In particular, it acts on the subspace $\mathcal{F}_p(\Lambda, \mathbb{A}^n) \subseteq \mathcal{F}^n$.

The Fourier transform sends A into a matrix of Laurent polynomials $\widehat{A} = (\widehat{a}_{ij})$. By Proposition 2.13(b), the relation $g = A * f$, where $f, g \in \mathcal{F}_p(\Lambda, \mathbb{A}^n)$, is transformed into

$$\widehat{g} = (\widehat{A} \circ \iota) \cdot \widehat{f},$$

where $\widehat{f}, \widehat{g} \in \mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{A}^n)$ (here ι stands as before for the inversion $\chi \mapsto \chi^{-1}$ in Λ^\vee).

The following theorem provides an algebraic analog of Theorems 1-3 in [K \mathbf{u}_1] (cf. also Theorem 8 in [K \mathbf{u}_2]), with a similar proof.

THEOREM 4.2. *Let $\Delta \in \text{End}(\mathbb{K}[\Lambda]^n)$ be an endomorphism commuting with shifts. Consider the equation*

$$(4) \quad \Delta(f) = 0,$$

where Δ has been extended to the space \mathcal{F}^n via (3). Then the following hold.

- (a) *There exists a non-zero solution $f \in \mathbb{K}[\Lambda]^n$ of (4) if and only if $\det(\widehat{A}) = 0$.*
- (b) *Let $\Lambda' \subseteq \Lambda$ be a sublattice of finite index coprime to p . There exists a Λ' -periodic solution $f \in \mathcal{F}_{\Lambda'}(\Lambda, \mathbb{A}^n)$ of (4) if and only if there exists a character $\chi \in \Lambda^\vee$ such that $\Lambda' \subseteq \ker(\chi)$ (i.e., $\chi \in (\Lambda')^\perp$) and $\det(\widehat{A}(\chi^{-1})) = 0$. Every such solution can be written as*

$$(5) \quad f = \sum_{i=1}^m \chi_i \cdot u_i,$$

where the characters $\chi_i \in \Lambda^\vee$ are such that $\det(\widehat{A}(\chi_i^{-1})) = 0$, $\Lambda' \subseteq \ker(\chi_i)$, and $u_i \in \ker(\widehat{A}(\chi_i^{-1}))$ for all $i = 1, \dots, m$. Vice versa, any vector function as in (5) is a Λ' -periodic solution of (4).

PROOF. (a) Choosing a base \mathcal{V} of Λ we identify Λ with \mathbb{Z}^s and Λ^\vee with \mathbb{T}^s . Assume that the equation

$$(6) \quad (\widehat{A} \circ \iota) \cdot \widehat{f} = 0$$

admits a rational solution $\widehat{f} \in [\mathbb{K}(z)]^n = [\mathbb{K}(z_1, \dots, z_s)]^n$. Multiplying \widehat{f} by a suitable polynomial $q \in \mathbb{K}[z]$ we can get a polynomial solution of (6). Consequently (6) admits a solution in vector Laurent polynomials. In turn, (4) admits a solution with finite support. Thus we can treat (6) as a linear system over the rational function field $\mathbb{K}(z)$. Now (a) follows by the usual linear algebra argument.

(b) Consider the finite abelian group $G = \Lambda/\Lambda'$ of order coprime to p . Every Λ' -periodic vector function $f \in \mathcal{F}_{\Lambda'}(\Lambda, \mathbb{A}^n)$ is the pullback of a vector function $\bar{f} \in \mathcal{F}(G, \mathbb{A}^n)$. The image $g = A_* f$ being also Λ' -periodic, it is the pullback of a function $\bar{g} \in \mathcal{F}(G, \mathbb{A}^n)$. The endomorphism $\Delta_* : \bar{f} \mapsto \bar{g}$ is given by the pushforward matrix $A_* = (a_{ij*})$, where $a_{ij*} \in \mathcal{F}(G, \mathbb{K}) \forall (i, j)$. The latter acts via

$$\bar{g}_i = \sum_{j=1}^n a_{ij*} * \bar{f}_j, \quad i = 1, \dots, n$$

(cf. the Pushforward Lemma 1.1 in [Za₂]). Applying the Fourier transform yields a matrix function $\widehat{A}_* = (\widehat{a_{ij*}})$ on the dual group G^\vee acting on vector functions on G^\vee via the usual matrix multiplication. Thus (4) leads to the equation

$$(7) \quad (\widehat{A}_* \circ \iota) \cdot \widehat{f} = 0.$$

Since $p \nmid \text{ord } G$, the dual group G^\vee can be embedded in the dual torus Λ^\vee as the subgroup $\pi^\vee(G^\vee) = \Lambda'^\perp$, see 2.8. Moreover, according to Definition 2.12, $\pi_*^\vee(\widehat{f}) = \widehat{f}|_{\Lambda'^\perp}$ for every $f \in \mathcal{F}_{\Lambda'}(\Lambda, \mathbb{K})$ (cf. [Za₂, Lemma 2.9]).

If $f = (f_1, \dots, f_n)$ is a nonzero Λ' -periodic solution of (4) then $\widehat{f} = (\widehat{f}_1, \dots, \widehat{f}_n)$ is a nonzero solution of (6). In turn $\pi_*^\vee(\widehat{f}) := \widehat{f}|_{\Lambda'^\perp}$ yields a nonzero solution of (7). Hence \widehat{f} does not vanish at some point $\chi \in \Lambda'^\perp$. Because of (7) we have $(\det \widehat{A})(\chi^{-1}) = (\det \widehat{A}_*)(\chi^{-1}) = 0$.

Conversely, if $(\det \widehat{A})(\chi^{-1}) = (\det \widehat{A}_*)(\chi^{-1}) = 0$ at some point $\chi \in \Lambda'^\perp$ then (7) admits a solution $\delta_\chi \cdot u$, where $u \in \ker(\widehat{A}_*(\chi^{-1}))$. The vector function $u \cdot \delta_\chi$ is the Fourier transform of $u \cdot \chi \in \mathcal{F}_{\Lambda'}(\Lambda, \mathbb{K})$. The latter is a Λ' -periodic (elementary) solution of (4), as required. This shows the first assertion in (b). The proof of the second one is similar, and we leave it to the reader. \square

DEFINITION 4.3. Consider as before an endomorphism $\Delta \in \text{End}(\mathbb{K}[\Lambda]^n)$ commuting with shifts, where $n \in \mathbb{N}$. Similarly to the scalar case $n = 1$, we call the hypersurface

$$\Sigma_\Delta := \{z \in \Lambda^\vee \mid \det(\widehat{A}(z^{-1})) = 0\}$$

the *symbolic variety* of the operator Δ (cf. 3.1). A point $z \in \Sigma_\Delta$ corresponding to a character $\chi = z^\lambda$ of $\Lambda \cong \mathbb{Z}^s$ is called a *multiplier* of (7). In case (b) of the theorem, the multipliers belong to the finite subgroup $\Lambda'^\perp \subseteq \Lambda^\vee$. More precisely, they run over the intersection $\Sigma_\Delta \cap \Lambda'^\perp$. Thus every Λ' -periodic solution f of (4) as in (5) can be written as

$$f(\lambda) = \sum_{z \in \Sigma_\Delta \cap \Lambda'^\perp} z^\lambda \cdot u(z).$$

The vector functions on $\Lambda = \mathbb{Z}^s$ of the form $f(\lambda) = z^\lambda \cdot u(z)$, where $z \in \mathbb{T}^s$, will be called *elementary*. These functions are analogs in positive characteristic of the classical Bloch functions.

COROLLARY 4.4. *Under the assumptions of Theorem 4.2, the equation (4) admits a nonzero solution $f \in \mathcal{F}_p(\Lambda, \mathbb{K})$ if and only if it admits such an elementary solution, if and only if $\Sigma_\Delta \neq \emptyset$. Moreover,*

$$\ker(\Delta) = \text{span} \left(z^\lambda \cdot u \mid z \in \Sigma_\Delta, u \in \ker(\widehat{A}(z^{-1})) \right).$$

REMARK 4.5. When Λ' runs over the set of all sublattices of Λ of finite indices coprime to p , the finite subgroups Λ'^\perp exhaust the whole dual torus $\Lambda^\vee \cong \mathbb{T}^s$. Given $\Delta \in \text{End}(\mathbb{K}[\Lambda]^n)$ commuting with shifts, it would be interesting to find the counting function $\text{card}(\Sigma_\Delta \cap \Lambda'^\perp)$, or a suitable analog of the Weil zeta-function.

4.2. The Jordan form of matrix convolution operators. Every square matrix over an algebraically closed field \mathbb{K} admits a Jordan basis [Bou, Ch. VII, §5.2, Prop. 5]. We construct below such a basis for any matrix convolution operator.

DEFINITION 4.6. Given an operator $\Delta \in \text{End}(\mathbb{K}[\Lambda])$ commuting with shifts, for every $\mu \in \mathbb{K}$ we consider its *Floquet-Fermi hypersurface*

$$\Sigma_{\Delta, \mu} = \{z \in \mathbb{T}^s \mid \det(\widehat{A}(z^{-1}) - \mu \cdot I_n) = 0\}.$$

We consider also the corresponding eigenspace

$$E_{\Delta, \mu} = \ker((\Delta - \mu \cdot \text{Id})|_{\mathcal{F}_p(\Lambda, \mathbb{K})})$$

and the generalized eigenspace

$$E_{\Delta}^{(\mu)} = \ker((\Delta - \mu \cdot \text{Id})^n|_{\mathcal{F}_p(\Lambda, \mathbb{K})}).$$

The proof of the following theorem is straightforward.

THEOREM 4.7. *Let $\Delta \in \text{End}(\mathbb{K}[\Lambda])$ be an endomorphism commuting with shifts, extended via (3) to the space $\mathcal{F}_p(\Lambda, \mathbb{K})$ as a matrix convolution operator. Then the following hold.*

(a) *There is a decomposition*

$$\mathcal{F}_p(\Lambda, \mathbb{A}^n) = \bigoplus_{\mu \in \mathbb{K}} E_{\Delta}^{(\mu)}.$$

(b) *Every generalized eigenspace $E_{\Delta}^{(\mu)} \subseteq \mathcal{F}_p(\Lambda, \mathbb{K})$ is generated by elementary vector-functions. Namely,*

$$\begin{aligned} E_{\Delta}^{(\mu)} &= \text{span}(z^\lambda \cdot u \mid z \in \Sigma_{\Delta, \mu}, u \in \ker(A(z^{-1}) - \mu \cdot I_n)^n) \\ &= \left\{ f \in \mathcal{F}_p(\Lambda, \mathbb{A}^n) \mid f(\lambda) = \sum_{z \in \Sigma_{\Delta, \mu}} \alpha(z) z^\lambda, \text{ where } \alpha \in \mathcal{F}_{\text{fs}}(\Lambda^\vee, \mathbb{A}^n) \text{ is s.t.} \right. \\ &\quad \left. \alpha(z) \in \ker(\widehat{A}(z^{-1}) - \mu \cdot I_n)^n \ \forall z \in \text{supp}(\alpha) \right\}. \end{aligned}$$

(c) *The induced endomorphism $\Delta \in \text{End}(\mathcal{F}_p(\Lambda, \mathbb{K}))$ admits a Jordan basis consisting of elementary vector functions. More precisely, fixing for every $z \in \mathbb{T}^n$ a Jordan basis $(u_1(z), \dots, u_n(z))$ in \mathbb{A}^n for $\widehat{A}(z^{-1})$, we obtain a Jordan basis $(z^\lambda \cdot u_i(z) \mid z \in \mathbb{T}^n)$ for Δ in $\mathcal{F}_p(\Lambda, \mathbb{A}^n)$.*

REMARK 4.8. À priori, the choice of the vector functions $u_i(z)$ in (c) does not suppose any specific dependence on $z = (z_1, \dots, z_s)$. However, they can be chosen as (non-rational, in general) algebraic vector functions in z .

4.3. Convolution equations over finite fields. Since the equation (4) is linear, taking suitable traces we can obtain solutions with values in a finite field. We discuss below this matter in more detail.

Given a square matrix function $A = (a_{ij})$ of order n with entries $a_{ij} \in \mathbb{K}[\Lambda]$, we consider as before the matrix convolution equation

$$(8) \quad A * f = 0, \quad \text{where } f = (f_1, \dots, f_n) \in \mathcal{F}_p(\Lambda, \mathbb{A}^n).$$

The proof of the following proposition follows the lines of the proof of Theorem 4.4 in [Za₂].

PROPOSITION 4.9. *Consider a finite Galois field $GF(q) = \mathbb{F}_q \subset \mathbb{K}$, where $q = p^\sigma$. Suppose that $a_{ij} \in GF(q)[\Lambda] \ \forall i, j = 1, \dots, n$. If (8) admits a nonzero solution $f \in \mathcal{F}_p(\Lambda, \mathbb{A}^n)$, then it admits such a solution $\tilde{f} \in \mathcal{F}_p(\Lambda, \mathbb{A}^n)$ with values in the field $GF(q)$.*

PROOF. The equation (8) is equivalent to the system

$$\sum_{j=1}^n a_{ij} * f_j = 0, \quad i = 1, \dots, n.$$

Since $GF(q)$ is the subfield of invariants of the Frobenius automorphism $z \mapsto z^q$ acting on \mathbb{K} , applying this automorphism yields the equalities

$$\sum_{j=1}^n a_{ij} * f_j^q = 0, \quad i = 1, \dots, n.$$

That is, $f^q := (f_1^q, \dots, f_n^q) \in \mathcal{F}_p(\Lambda, \mathbb{A}^n)$ is again a solution of (8).

The function f_j being pluri-periodic its image $f_j(\Lambda) \subseteq \mathbb{K}$ is finite. Let $GF(q(f))$, where $q(f) = p^{r(f)}$, be the smallest subfield of \mathbb{K} containing the field $GF(q)$ and all the images $f_j(\Lambda)$, $j = 1, \dots, n$. We let $\text{Tr}(f) = (\text{Tr}(f_1), \dots, \text{Tr}(f_n))$, where

$$\text{Tr}(f_j) := \text{Tr}_{GF(q(f)):GF(q)}(f_j) := f_j + f_j^q + \dots + f_j^{q^{r(f)-1}} \in \mathcal{F}_p(G, \mathbb{K}).$$

The trace $\text{Tr}(f)$ being fixed by the Frobenius automorphism, it represents a solution with values in the field $GF(q)$. It remains to show that there exists such a nonzero solution.

Let $\Delta \in \text{End}(\mathcal{F}_p(\Lambda, \mathbb{A}^n))$ denote the matrix convolution operator $\Delta(f) = A * f$. By Corollary 4.4, $\ker(\Delta)$ is spanned by the elementary solutions $f = z^\lambda \cdot u$, where $z \in \Sigma_\Delta \subseteq \mathbb{T}^s$. By our assumption $\ker(\Delta) \neq (0)$, hence $\Sigma_\Delta \neq \emptyset$. Choosing $z \in \Sigma_\Delta$ we let

$$r(z) = \min \left\{ r \mid z^\lambda, (z^q)^\lambda, \dots, (z^{q^{r-1}})^\lambda \text{ are all distinct} \right\}.$$

The entries of the matrix $\widehat{A}(z^{-1})$ being Laurent polynomials with coefficients in the field $GF(q)$, there exists a vector $u \in \ker(\widehat{A}(z^{-1}))$, $u \neq 0$, with coordinates in the field $GF(q^{r(z)})$. Thus $f = z^\lambda \cdot u$ is a nonzero elementary solution with values in the field $GF(q^{r(z)})$. Let $j \in \{1, \dots, n\}$ be such that $u_j \neq 0$. Then $f_j = u_j \cdot z^\lambda \neq 0$. Moreover,

$$\text{Tr}(f_j) = u_j \cdot z^\lambda + u_j^q \cdot (z^q)^\lambda + \dots + u_j^{q^{r(f)-1}} \cdot (z^{q^{r(f)-1}})^\lambda$$

is a non-trivial linear combination of pairwise distinct characters. Therefore, $\text{Tr}(f)$ is a nonzero elementary solution with values in the field $GF(q)$, as required. \square

REMARK 4.10. By Theorem 4.4 in [Za₂], in the scalar case $\ker(\Delta) \cap \mathcal{F}_p(\Lambda, GF(q))$ is spanned by suitable traces of characters (i.e., of elementary solutions).

5. Examples and applications

The constructions in 5.1-5.2 below are borrowed from [Ku₁].

5.1. Let $\Lambda = \mathbb{Z}^s$ be a free abelian group acting on a countable set V with a finite quotient V/Λ of cardinality n . Choosing a fundamental domain $\Omega = \{v_1, \dots, v_n\} \subseteq V$ (i.e., a set of representatives of the orbits of Λ) we obtain a disjoint partition

$$V = \bigsqcup_{v \in \Lambda} \tau_v(\Omega).$$

According to this partition, every function $f \in \mathcal{F}(V, \mathbb{K})$ yields a vector function $\text{fragm}(f) \in \mathcal{F}(\Lambda, \mathbb{A}_{\mathbb{K}}^n)$. This provides an isomorphism of ‘fragmentation’

$$\text{fragm} : \mathcal{F}(V, \mathbb{K}) \xrightarrow{\cong} \mathcal{F}(\Lambda, \mathbb{A}^n).$$

Any endomorphism $\Delta \in \text{End}(\mathcal{F}(V, \mathbb{K}))$ commuting with the induced Λ -action on the space $\mathcal{F}(V, \mathbb{K})$ amounts to an endomorphism $\text{Fragm}(\Delta) \in \text{End}(\mathcal{F}(\Lambda, \mathbb{A}^n))$ commuting with shifts. This yields a representation

$$\text{Fragm} : \text{End}_\Lambda(\mathcal{F}(V, \mathbb{K})) \rightarrow \text{End}_\Lambda(\mathcal{F}(\Lambda, \mathbb{A}^n)),$$

where End_Λ stands for the set of endomorphisms commuting with the action of Λ . In 5.2 and 5.3 below we provide two particular occasions of this construction.

5.2. Covering graphs. Let Γ be a finite connected graph with n vertices. Consider the maximal abelian covering $\tilde{\Gamma} \rightarrow \Gamma$, which corresponds to the commutator subgroup of the fundamental group $\pi_1(\Gamma)$. The deck transformation group (i.e., the Galois group) of the covering $\tilde{\Gamma} \rightarrow \Gamma$ is a free abelian group $H_1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^s$, where $s = b_1(\Gamma)$. It acts freely on $\tilde{\Gamma}$ with quotient $\Gamma = \tilde{\Gamma}/\mathbb{Z}^s$. Letting V be the set of vertices of the periodic graph $\tilde{\Gamma}$. Every convolution operator $\Delta \in \text{End}(\mathcal{F}(V, \mathbb{K}))^6$ gives rise to an operator $\text{Fragm}(\Delta) \in \text{End}_\Lambda(\mathcal{F}(\mathbb{Z}^s, \mathbb{A}^n))$ commuting with shifts. This yields as before a representation of the algebra of convolution operators on $\tilde{\Gamma}$. The same construction works for any abelian covering of Γ .

To give further examples, let us consider the following toy model.

5.3. Fragmentation: a toy model. Let Λ be a sublattice of finite index n in a bigger lattice $\tilde{\Lambda}$. Consider the action of Λ on the set $V = \tilde{\Lambda}$ by shifts. Then any endomorphism $\Delta \in \text{End}_\Lambda(\mathcal{F}(\tilde{\Lambda}, \mathbb{K}))$ commuting with shifts by Λ gives rise to an endomorphism $\text{Fragm}_\Lambda(\Delta) \in \text{End}_\Lambda(\mathcal{F}(\Lambda, \mathbb{A}^n))$ commuting with shifts. In particular, for different choices of Λ this yields a collection of representations

$$\text{Fragm}_\Lambda : \mathbb{K}[\tilde{\Lambda}] \rightarrow \text{End}_\Lambda(\mathcal{F}(\Lambda, \mathbb{A}^n))$$

of the convolution algebra on $\tilde{\Lambda}$.

New representations appear when restricting the morphism of fragmentation to a translation invariant subspace e.g., to the subspace $\mathcal{F}_p(\Lambda, \mathbb{A}^n) \subseteq \mathcal{F}(\Lambda, \mathbb{A}^n)$. We need the following simple lemma.

LEMMA 5.4. *In the notation as before, for any n coprime to p we have*

$$(9) \quad \text{fragm}_\Lambda : \mathcal{F}_p(\tilde{\Lambda}, \mathbb{K}) \xrightarrow{\cong} \mathcal{F}_p(\Lambda, \mathbb{K})^n = \mathcal{F}_p(\Lambda, \mathbb{A}^n).$$

PROOF. For every $f = (f_1, \dots, f_n) \in \mathcal{F}_p(\Lambda, \mathbb{K})^n$, the finite index sublattice

$$\bigcap_{i=1}^n \Lambda(f_i) \subseteq \Lambda(f)$$

of Λ is p -saturated. Therefore, by 2.8, $\Lambda(f)$ is p -saturated as well. This gives the inclusion $\mathcal{F}_p(\Lambda, \mathbb{K})^n \subseteq \mathcal{F}_p(\Lambda, \mathbb{A}^n)$. The opposite inclusion being evident, this shows the last equality in (9).

Furthermore, letting $f = \text{fragm}_\Lambda(\tilde{f})$, where $\tilde{f} \in \mathcal{F}_p(\tilde{\Lambda}, \mathbb{K})$, it is easily seen that the sublattice of finite index

$$\Lambda(f) \supseteq \Lambda \cap \tilde{\Lambda}(\tilde{f})$$

is p -saturated. Thus fragm_Λ sends $\mathcal{F}_p(\tilde{\Lambda}, \mathbb{K})$ injectively to $\mathcal{F}_p(\Lambda, \mathbb{K})^n$. In fact, this is a bijection. Indeed, let (v_1, \dots, v_n) be a set of representatives of the cosets of Λ in $\tilde{\Lambda}$. For a vector function $f = (f_1, \dots, f_n) \in \mathcal{F}_p(\Lambda, \mathbb{K})^n$ we let

$$\tilde{f}(v) = f_i(v - v_i) \quad \text{if } v - v_i \in \Lambda.$$

It is easily seen that $f = \text{fragm}_\Lambda(\tilde{f})$. Moreover, $\tilde{\Lambda}(\tilde{f}) \supseteq \Lambda(f)$, where $\Lambda(f)$ is p -saturated in $\tilde{\Lambda}$. Hence $\tilde{\Lambda}(\tilde{f})$ is also p -saturated and so fragm_Λ yields an isomorphism as in (9). \square

⁶This applies in particular to the Markov and the Laplace operators on $\tilde{\Gamma}$ properly adopted to the case of positive characteristic.

In 5.5-5.8 below we apply our toy model 5.3 in different concrete settings.

EXAMPLE 5.5. Consider a lattice $\tilde{\Lambda} = \mathbb{Z}$ of rank $s = 1$ and the shift $\tau = \tau_{-1}$ on $\tilde{\Lambda}$. By Proposition 3.5 the induced action of τ on the space $\mathcal{F}_p(\tilde{\Lambda}, \mathbb{K}) = \text{span}(z^\lambda \mid z \in \mathbb{K}^\times)$ can be diagonalized in the basis of characters. More precisely, $\tau(z^\lambda) = z^{-1} \cdot z^\lambda$, so $\text{spec}(\tau) = \mathbb{K}^\times$ and

$$\mathcal{F}_p(\tilde{\Lambda}, \mathbb{K}) = \bigoplus_{\mu \in \mathbb{K}^\times} E_{\tau, \mu}, \quad \text{where } E_{\tau, \mu} = \mathbb{K} \cdot \mu^{-\lambda}.$$

The same is true for any convolution operator Δ_a on $\tilde{\Lambda}$, where $a \in \mathbb{K}[\tilde{\Lambda}]$. Indeed, we have $\Delta_a = \widehat{a}(\tau)$ and so

$$\Delta_a(z^\lambda) = \widehat{a}(z^{-1}) \cdot z^\lambda \quad \forall z \in \mathbb{K}^\times.$$

Consider further a sublattice $\Lambda = n\mathbb{Z}$ of $\tilde{\Lambda} = \mathbb{Z}$ and the fragmentation $\text{Fragm}(\tau) = \text{Fragm}_n(\tau) \in \text{End}_\Lambda(\mathcal{F}(\Lambda, \mathbb{A}^n))$. Since $\tau(\delta_k) = \delta_{k+1} \forall k \in \mathbb{Z} = \tilde{\Lambda}$, for the basic vector delta-functions $\delta_{i, \lambda} = \delta_\lambda \cdot e_i$ on Λ we have

$$\text{Fragm}(\tau)(\delta_{0, k}) = \delta_{0, k+1} \text{ for } k = 0, \dots, n-2, \text{ and } \text{Fragm}(\tau)(\delta_{0, n-1}) = \delta_{1, 0}.$$

The corresponding matrix of Laurent polynomials $\widehat{A}(z) := \widehat{A_{\text{Fragm}(\tau)}}(z)$ and its inverse matrix $\widehat{A}(z)^{-1} = \widehat{A_{\text{Fragm}(\tau^{-1})}}(z)$ are

$$\widehat{A}(z) = \begin{pmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \text{ resp. } \widehat{A}(z)^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ z^{-1} & 0 & \dots & 0 & 0 \end{pmatrix} = \widehat{A}^t(z^{-1}).$$

Thus

$$\det(xI_n - \widehat{A}(z)) = x^n - z = \prod_{i=1}^n (x - \xi_i),$$

where

$$\{\xi_1, \dots, \xi_n\} = \{\sqrt[n]{z}\} = \text{spec}(\widehat{A}(z)) \subseteq \mathbb{K}^\times.$$

Suppose that $p \nmid n$. Then for $z \in \mathbb{K}^\times$ we have $\xi_i \neq \xi_j$ if $i \neq j$. So the matrix $\widehat{A}(z^{-1})$ with eigenvalues $\xi_1^{-1}, \dots, \xi_n^{-1}$ can be diagonalized in the basis of eigenvectors

$$(10) \quad v_i = \begin{pmatrix} 1 \\ \xi_i \\ \vdots \\ \xi_i^{n-1} \end{pmatrix}, \quad i = 1, \dots, n,$$

where

$$(11) \quad \widehat{A}(z^{-1})(v_i) = \xi_i^{-1} v_i \quad \forall i = 1, \dots, n.$$

The collection of elementary vector functions

$$(12) \quad (\lambda \mapsto v_i \cdot z^\lambda \mid z \in \mathbb{K}^\times, i = 1, \dots, n)$$

form a diagonalizing basis for the endomorphism $\text{Fragm}_n(\tau) \in \text{End}_\Lambda(\mathcal{F}_p(\Lambda, \mathbb{A}^n))$ (cf. Proposition 4.7(c)). Actually $v_i \cdot z^\lambda = \text{fragm}_n(\xi_i^\lambda)$, where $\lambda \mapsto \xi_i^\lambda$ is a character of $\tilde{\Lambda} = \mathbb{Z}$.

In conclusion, the isomorphism of fragmentation fragm_n as in (9) sends the basis of characters in $\mathbb{F}_p(\tilde{\Lambda}, \mathbb{K})$ to a basis of elementary functions (12), which is diagonalizing for the endomorphism $\text{Fragm}_n(\tau) \in \text{End}_\Lambda(\mathcal{F}_p(\Lambda, \mathbb{A}^n))$.

EXAMPLE 5.6. In the same setting as in 5.5, let us consider an endomorphism

$$\Delta_{n,\varphi} = \varphi(\text{Fragm}_n(\tau)) = \text{Fragm}_n(\varphi(\tau)) \in \text{End}_\Lambda(\mathcal{F}(\Lambda, \mathbb{A}^n))$$

commuting with shifts, where $\varphi \in \mathbb{K}[t, t^{-1}]$ is a Laurent polynomial. The matrix of Laurent polynomials that corresponds to $\Delta_{n,\varphi}$ is

$$\widehat{A}_{\Delta_{n,\varphi}} = \varphi(\widehat{A}) = \varphi(\widehat{A_{\text{Fragm}_n(\tau)}}) = \widehat{A_{\text{Fragm}_n(\varphi(\tau))}},$$

where \widehat{A} is as in 5.5. Recall that the equation $\Delta_{n,\varphi}(f) = 0$ for $f \in \mathcal{F}_p(\Lambda, \mathbb{A}^n)$ is equivalent to

$$\varphi(\widehat{A}(z^{-1}))(\widehat{f}(z)) = 0 \quad \forall z \in \mathbb{K}^\times.$$

By the Spectral Mapping Theorem, for every $z \in \mathbb{K}^\times$ we have

$$\text{spec}(\varphi(\widehat{A}(z^{-1}))) = \varphi(\text{spec}(\widehat{A}(z^{-1}))) = \{\varphi(\xi_1^{-1}), \dots, \varphi(\xi_n^{-1})\},$$

where as before $\{\xi_1, \dots, \xi_n\} = \{\sqrt[n]{z}\}$. The basis (12) is diagonalizing for $\Delta_{n,\varphi}$. Namely,

$$\Delta_{n,\varphi}(v_i \cdot z^\lambda) = \varphi(\xi_i^{-1}) \cdot v_i \cdot z^\lambda.$$

Indeed,

$$\begin{aligned} A_{\varphi(\text{Fragm}_n(\tau))}(z^{-1}) \cdot \widehat{v_i \cdot z^\lambda} &= A_{\varphi(\text{Fragm}_n(\tau))}(z^{-1}) \cdot v_i \cdot \delta_z = \varphi(\widehat{A}(z^{-1}))(v_i) \cdot \delta_z \\ &= \varphi(\xi_i^{-1}) \cdot v_i \cdot \delta_z = \varphi(\xi_i^{-1}) \cdot \widehat{v_i \cdot z^\lambda}. \end{aligned}$$

The Floquet-Fermi levels of $\Delta_{n,\varphi}$ in $\mathbb{T}^1 = \mathbb{K}^\times$ are

$$\Sigma_{\Delta_{n,\varphi}, \mu} = \{\xi^n \in \mathbb{K}^\times \mid \varphi(\xi^{-1}) = \mu\} = \iota(\varphi^{-1}(\mu))^n = (\varphi^{-1}(\mu))^{-n}.$$

In particular, the symbolic variety of $\Delta_{n,\varphi}$ is

$$\Sigma_{\Delta_{n,\varphi}} = \{\xi^n \in \mathbb{K}^\times \mid \varphi(\xi^{-1}) = 0\}.$$

Letting $\varphi = q/z^\beta$, where $q = \prod_{j=1}^d (t - \alpha_j)^{k_j} \in \mathbb{K}[t]$, $q(0) \neq 0$, we have

$$\begin{aligned} 0 \in \text{spec}(\varphi(\widehat{A})(z^{-1})) &\iff \exists i : \varphi(\xi_i^{-1}) = 0 \iff \exists i : q(\xi_i^{-1}) = 0 \\ &\iff \exists i : \xi_i = \alpha_j^{-1} \text{ for some } j \in \{1, \dots, d\} \iff z \in \{\alpha_1^{-n}, \dots, \alpha_d^{-n}\}, \end{aligned}$$

where as before $\xi_i^n = z \in \mathbb{K}^\times$, $i = 1, \dots, n$. Note that the sequence $\alpha_j, \alpha_j^2, \dots$ is periodic with period $d_j = \text{ord}_{\mathbb{K}^\times}(\alpha_j)$ coprime to p .

The kernel $\ker(\Delta_{n,\varphi})$ is spanned by the elementary vector functions $v_i \cdot z^\lambda$ as in (12) with $z \in \{\alpha_1^{-n}, \dots, \alpha_d^{-n}\}$. Choose $l \in \{0, \dots, d_j - 1\}$ such that $l \equiv n \pmod{d_j}$. Letting $z_{jl} = \alpha_j^{-l} = \xi_i^l$, such an elementary vector function is proportional to the vector function

$$f_{jl} = z_{jl}^\lambda \cdot v_i = (\dots, z_{jl}^{-1} v_i, v_i, z_{jl} v_i, z_{jl}^2 v_i, \dots) \in \mathcal{F}_p(\Lambda, \mathbb{A}^n),$$

where $v_i \in \mathbb{A}^n$ is an eigenvector of the matrix $\widehat{A}(z_{jl}^{-1})$ as in (10) with the eigenvalue $\alpha_j = \xi_i^{-1}$. Actually $f_{jl} = \text{fragm}_n(\alpha_j^{-\lambda})$, where the character $\alpha_j^{-\lambda} = \xi_i^\lambda : \widehat{\Lambda} \rightarrow \mathbb{K}^\times$ of order d_j (equal to its period) does not depend on n .

Let us illustrate the latter issue on a concrete example.

EXAMPLE 5.7. Suppose that $p \neq 3$. Consider the equation

$$(13) \quad \Delta_{n,\varphi}(f) = \varphi(\text{Fragm}_n(\tau))(f) = \text{Fragm}_n(\varphi(\tau))(f) = 0,$$

where $\varphi(t) = t + 1 + t^{-1}$. The roots $\alpha_1 = \omega$ and $\alpha_2 = \omega^{-1}$ of φ , where $\omega \in \mathbb{K}^\times$ is a primitive cubic root of unity, have multiplicative orders $d_1 = d_2 = 3$. There exists an elementary solution $z^\lambda \cdot v$ of

(13) if and only if $z \in \{\omega^n, \omega^{-n}\}$, respectively $\xi \in \{\omega^{-1}, \omega\}$. Assuming that $n \equiv 0 \pmod{3}$ we obtain $l = 0$ and $z = 1$. Letting $\xi = \omega^{-1}$ yields an elementary solution

$$f = (\dots, v, v, v, \dots), \quad \text{where } v = \begin{pmatrix} 1 \\ \omega^{-1} \\ \omega \\ 1 \\ \vdots \\ \omega^{-1} \\ \omega \end{pmatrix} \in \mathbb{A}^n.$$

This amounts to a 3-periodic solution

$$\tilde{f} = (\dots, 1, \omega^{-1}, \omega, 1, \omega^{-1}, \omega, \dots) \in \mathcal{F}_p(\tilde{\Lambda}, \mathbb{K})$$

of the scalar equation

$$(14) \quad \Delta_a(f) = \varphi(\tau)(f) = 0 \quad \text{with } a = \delta_{-1} + \delta_0 + \delta_1 \in \mathbb{K}[\tilde{\Lambda}],$$

where as before $\tilde{\Lambda} = \mathbb{Z}$. The latter solution does not depend on n . Any other elementary solution can be obtained from this one by a shift and the inversion $\omega \mapsto \omega^{-1}$. Actually every elementary solution of (14) is proportional either to $\tilde{f}(\lambda)$, or to $\tilde{f}(\lambda^{-1})$. Thus the subspace of all Δ_a -harmonic functions

$$\ker(\Delta_a) = \text{span}(\tilde{f}, \tilde{f} \circ \iota) \subseteq \mathcal{F}_p(\tilde{\Lambda}, \mathbb{K})$$

is two-dimensional. It is easy to check that the same holds for every $p \neq 3$ and $n \not\equiv 0 \pmod{p}$.

EXAMPLE 5.8. For any $p \neq 2$, consider the lattice $\tilde{\Lambda} = \mathbb{Z}^2$, its sublattice $\Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \subseteq \tilde{\Lambda}$ of index 4, and the commuting basic shifts $\tau_1 = \tau_{(-1,0)}$ and $\tau_2 = \tau_{(0,-1)}$ on $\tilde{\Lambda}$. In the standard basis $(e_{00}, e_{10}, e_{01}, e_{11})$ in \mathbb{A}^4 the corresponding matrices of Laurent polynomials $\widehat{A}_i := \widehat{A_{\text{Fragm}(\tau_i)}}$, $i = 1, 2$, are as follows:

$$\widehat{A}_1(z^{-1}) = \begin{pmatrix} 0 & z_1^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1^{-1} \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{resp.}, \quad \widehat{A}_2(z^{-1}) = \begin{pmatrix} 0 & 0 & z_2^{-1} & 0 \\ 0 & 0 & 0 & z_2^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In the common basis of eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ x \\ y \\ xy \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ x \\ -y \\ xy \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ -x \\ y \\ xy \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ -x \\ -y \\ xy \end{pmatrix}$$

these commuting matrices are reduced to the diagonal form

$$\text{diag}(x^{-1}, -x^{-1}, x^{-1}, -x^{-1}) \quad \text{resp.} \quad \text{diag}(y^{-1}, y^{-1}, -y^{-1}, -y^{-1}),$$

where $x^2 = z_1$ and $y^2 = z_2$.

Consider further an endomorphism ρ commuting with shifts,

$$\rho := \varphi(\text{Fragm}(\tau_1), \text{Fragm}(\tau_2)) = \text{Fragm}(\varphi(\tau_1, \tau_2)) \in \text{End}_\Lambda(\mathcal{F}(\Lambda, \mathbb{A}^4)),$$

where $\varphi \in \mathbb{K}[z_1, z_2, z_1^{-1}, z_2^{-1}]$ is a Laurent polynomial. The same argument as in 5.6 above shows that the basis of elementary vector functions

$$(\lambda \mapsto v_j(z) \cdot z^\lambda \mid z \in \mathbb{T}^2, j = 1, 2, 3, 4)$$

is diagonalizing for ρ . More precisely, letting

$$\varphi_1(x, y) = \varphi(x, y), \quad \varphi_2(x, y) = \varphi(-x, y), \quad \varphi_3(x, y) = \varphi(x, -y), \quad \varphi_4(x, y) = \varphi(-x, -y),$$

by the Spectral Mapping Theorem we get

$$\rho(v_j \cdot z^\lambda) = \varphi_j(x^{-1}, y^{-1}) \cdot v_j \cdot z^\lambda$$

(cf. Proposition 4.7(c)). The Floquet-Fermi level- μ set of ρ is

$$\Sigma_{\rho, \mu} = \{z = (x^2, y^2) \in \mathbb{T}^2 \mid \exists j \in \{1, 2, 3, 4\} : \varphi_j(x^{-1}, y^{-1}) = \mu\}$$

(see 4.6). The eigenspace of ρ with eigenvalue μ is

$$E_{\rho, \mu} = \text{span}(v_j \cdot z^\lambda \mid z = (x^2, y^2) \in \mathbb{T}^2 : \varphi_j(x^{-1}, y^{-1}) = \mu, j = 1, 2, 3, 4).$$

Moreover,

$$\mathcal{F}_\rho(\Lambda, \mathbb{A}^4) = \bigoplus_{\mu \in \mathbb{K}} E_{\rho, \mu}.$$

In particular, the symbolic hypersurface of ρ is

$$\Sigma_\rho = \{z = (x^2, y^2) \in \mathbb{T}^2 \mid \exists j \in \{1, 2, 3, 4\} : \varphi_j(x^{-1}, y^{-1}) = 0\}.$$

The space $E_{\rho, 0}$ of all ρ -harmonic vector functions is

$$\ker(\rho) = \text{span}(v_j \cdot z^\lambda \mid z = (x^2, y^2) \in \mathbb{T}^2 : \varphi_j(x^{-1}, y^{-1}) = 0, j = 1, 2, 3, 4).$$

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