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h-cobordism and s-cobordism Theorems: Transfer over Semialgebraic and Nash Categories, Uniform bound and Effectiveness

Demdah Kartoue Mady

Abstract

The h-cobordism theorem is a noted theorem in differential and PL topology. A generalization of the h-cobordism theorem for possibly non simply connected manifolds is the so called s-cobordism theorem. In this paper, we prove semialgebraic and Nash versions of these theorems. That is, starting with semialgebraic or Nash cobordism data, we get a semialgebraic homeomorphism (respectively a Nash diffeomorphism). The main tools used are semialgebraic triangulation and Nash approximation.

One aspect of the algebraic nature of semialgebraic or Nash objects is that one can measure their complexities. We show h and s-cobordism theorems with a uniform bound on the complexity of the semialgebraic homeomorphism (or Nash diffeomorphism) obtained in terms of the complexity of the cobordism data. The uniform bound of semialgebraic h-cobordism cannot be recursive, which gives another example of non effectiveness in real algebraic geometry see [ABB]. Finally we deduce the validity of the semialgebraic and Nash versions of these theorems over any real closed field.

Introduction

The h-cobordism theorem is a classical result in differential and PL topology. In this paper we prove that it holds true in semialgebraic and Nash categories over any real closed field.

Let M be a compact smooth manifold having as boundary a disjoint union of two smooth manifolds M_0 and M_1 such that M_0 and M_1 are both deformation retracts of M . A triplet (M, M_0, M_1) like this is said to be an *h-cobordism*. The h-cobordism theorem states:

Theorem 0.1. (*Smale 1961*)

Let (M, M_0, M_1) be a simply connected smooth h-cobordism. If $\dim M \geq 6$ then M is diffeomorphic to $M_0 \times [0, 1]$.

It is a general procedure to use Tarski-Seidenberg Principle to transfer statements from \mathbb{R} to any real closed field, once uniform bounds are found for the complexity of all the semialgebraic or Nash objects involved in the statements.

To do this, first of all we need a semialgebraic or Nash version of the h-cobordism theorem, that we easily get triangulating our manifold and using an approximation result.

Secondly we have to make precise the meaning of some topological facts in a semialgebraic setting and verify that definitions are consistent.

The uniform bound which is established is the following: the complexity of the semialgebraic homeomorphism $f : M \rightarrow M_0 \times [0, 1]$ can be bounded in terms only of the complexity of the h-cobordism (M, M_0, M_1) . This enables us to translate the semialgebraic h-cobordism theorem to a countable family of first order statements of the theory of real closed fields (one for each complexity of the triplet (M, M_0, M_1)).

Once this is done, we can use Tarski-Seidenberg Principle to transfer the semialgebraic or Nash h-cobordism theorem to any real closed field.

In a similar way we get also the semialgebraic and Nash s-cobordism theorems over any real closed field

It is a natural question to ask whether the uniform bounds that we get are effective or not, that is to ask whether the complexity of the isomorphism $f : M \rightarrow M_0 \times [0, 1]$ is bounded by a recursive function of the complexity of (M, M_0, M_1) .

We prove that this cannot be the case for the h-cobordism theorem. The failure is because we have to recognise whether a semialgebraic set is simply connected or not.

The non effectiveness of the h-cobordism theorem is another exemple of non effectiveness in real algebraic geometry see [ABB].

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1 Semialgebraic and Nash h-cobordism theorems and s-cobordism theorems

We shall agree in this work that every semialgebraic mapping is continuous. A semialgebraic manifold is a semialgebraic subset M of \mathbb{R}^n (or of R^n , where R is a real closed field) equipped with a finite semialgebraic atlas, that is, $M = \cup_{i \in I} U_i$, I finite set, U_i open semialgebraic in M and $\phi_i : U_i \rightarrow \mathbb{R}^d$ a semialgebraic homeomorphism onto an open semialgebraic subset of \mathbb{R}^d .

A Nash manifold is a semialgebraic subset of \mathbb{R}^n (or of R^n) which is also a C^∞ submanifold and is equipped with a finite Nash atlas $\{U_i, \phi_i\}$ where ϕ_i is semialgebraic and C^∞ . For more detail see [S].

Any compact semialgebraic set $S \subset \mathbb{R}^n$ (or R^n) can be triangulated, i.e. there

is a finite simplicial complex K in \mathbb{R}^n (or R^n) and a semialgebraic homeomorphism $h : |K| \rightarrow S$ (where $|K|$ is the union of the simplices of K). Moreover the semialgebraic triangulation can be chosen compatible with a finite family $(T_i)_{i \in I}$ of a semialgebraic subsets of S , which means that each T_i is the image by h of the union of some open simplices (see [BCR] p.217). The semialgebraic triangulation is unique, in the sense that any two compact polyhedra K and L which are semialgebraically homeomorphic are PL homeomorphic (cf.[SY]). Hence any semialgebraic set gets a unique PL structure.

Also we get:

Proposition 1.1. *Every compact semialgebraic manifold is semialgebraically homeomorphic to a PL manifold.*

Proof. Let M be a compact semialgebraic manifold of dimension m . There is a semialgebraic triangulation $h : |K| \rightarrow M$ where K is a finite simplicial complex. We have to check that the polyhedron $|K|$ is a PL manifold. Take $x \in |K|$ and $y = h(x)$. By definition, there is a neighbourhood V of y in M semialgebraically homeomorphic to a open semialgebraic set U in \mathbb{R}^m , that is, there is a semialgebraic chart (V, ϕ) such that $\phi : V \rightarrow U$ is a semialgebraic homeomorphism. Then, there is an open neighbourhood $h^{-1}(V)$ of x in $|K|$ semialgebraically homeomorphic to an open semialgebraic set U of \mathbb{R}^m . There is a closed PL ball $B \subset U$ such that $\phi(y) \in \text{Int} B$. It follows that the set $W = h^{-1} \circ \phi^{-1}(B)$ is a closed and bounded neighbourhood of x in $|K|$. Assuming the triangulation h to be compatible with $\phi^{-1}(B)$, one has that W is a polyhedron. It follows that W and B are semialgebraically homeomorphic. By uniqueness, they are PL homeomorphic. Then $\text{Int}(W)$ and $\text{Int}(B)$ are PL homeomorphic. This shows that $|K|$ is a PL manifold. \square

Definition 1.2. Let (M, M_0, M_1) be a triple of compact semialgebraic manifolds such that: $\partial M = M_0 \cup M_1$ and $M_0 \cap M_1 = \emptyset$. Then, (M, M_0, M_1) is called a *semialgebraic cobordism*.

A semialgebraic cobordism (M, M_0, M_1) is said to be a *semialgebraic h-cobordism* if the inclusions $M_0 \hookrightarrow M$ and $M_1 \hookrightarrow M$ are semialgebraic homotopy equivalences, that is, the deformation retractions are semialgebraic.

Let X be a semialgebraic set defined over a real closed field R . The semialgebraic fundamental group of X can be defined considering semialgebraic loops and semialgebraic homotopies between loops. We write $\pi_1(X, x_0)_{alg}$ with $x_0 \in X$. If $R = \mathbb{R}$ we have:

Proposition 1.3. ([DK], Theorem 6.4, p.271)

Let X be a closed semialgebraic subset of \mathbb{R}^n . Then $\pi_1(X, x_0)_{alg}$ and $\pi_1(X, x_0)$ are isomorphic.

The results just recalled enable us to translate the PL h-cobordism theorem to the semialgebraic category.

Theorem 1.4. *Let (M, M_0, M_1) be a semialgebraic h-cobordism simply connected in \mathbb{R}^n . If $\dim M \geq 6$ then M is semialgebraically homeomorphic to $M_0 \times [0, 1]$.*

Proof. By Proposition 1.1, there is a semialgebraic triangulation $\lambda : |K| \rightarrow M$. We may assume that the triangulation is compatible with the submanifolds M_0 and M_1 , i.e. there are simplicial subcomplexes $K_0, K_1 \subset K$ that $\lambda(|K_0|) = M_0$ and $\lambda(|K_1|) = M_1$. The polyhedra $|K|$, $|K_0|$ and $|K_1|$ are compact PL manifolds (Proposition 1.1). The polyhedra $|K_0|$ and $|K_1|$ are semialgebraic deformation retracts of $|K|$. They are also PL deformation retracts of $|K|$, by PL approximation (cf [H], Lemma 4.2, p. 92). It follows that $(|K|, |K_0|, |K_1|)$ is a simply connected PL h-cobordism. Then by the PL h-cobordism theorem $|K| \stackrel{PL}{\cong} |K_0| \times [0, 1]$, where $\stackrel{PL}{\cong}$ indicates the PL homeomorphism. Since a compact PL manifold is a semialgebraic manifold and a PL homeomorphism between compact PL manifolds is a semialgebraic homeomorphism, it follows easily that M is semialgebraically homeomorphic to $M_0 \times [0, 1]$. This ends the proof. \square

2 Extension of some topological properties

In this section, we want to extend the meaning of some topological properties as semialgebraic simple connectedness and s-homotopy from \mathbb{R} to any real closed field R . This will be useful in the sequel of this paper.

Let R and K be two real closed fields such that K is a real closed extension of R . If X is semialgebraic subset of R^n , we denote by X_K the semialgebraic subset of K^n defined by the same boolean combination of polynomial equation and inequalities as X . Actually by Tarski-Seidenberg Principle, X_K depends only on X and not on its description.

Proposition 2.1. *Let X and Y be two semialgebraic subsets of R^n . The semialgebraic sets X and Y are semialgebraically homeomorphic if and only if X_K and Y_K are semialgebraically homeomorphic.*

Proof. The first implication is obvious.

Conversely, set $X = \{x \in R^n : \phi(a, x)\}$, $Y = \{x \in R^n : \psi(b, x)\}$ where $\phi(a, x)$ and $\psi(b, x)$ are first order formulas of the theory of real closed fields with parameters $a \in R^m$ and $b \in R^{m'}$. Let f be a semialgebraic homeomorphism from X_K onto Y_K . Let $\psi(c, x, y)$ be a first order formula of the theory of real closed field with parameter $c \in K^r$ defining $\Gamma_f = \{(x, y) \in K^n \times K^n : \psi(c, x, y)\}$ the graph of f . One can get a first order formula in the theory of real closed fields $\lambda(a, b, c)$ which says that f is a semialgebraic homeomorphism between X_K and Y_K . So, we have $K \models \lambda(a, b, c)$. Let us observe that: $K \models \exists z \lambda(a, b, z)$ with $a \in R^m$ and $b \in R^{m'}$. By Tarski-Seidenberg Principle, we get: $R \models \exists z \lambda(a, b, z)$. That is, there exists a parameter $c' \in R^r$ which defines a homeomorphism between X and Y . This completes the proof. \square

Next step is to show that the notion of being C^r -Nash manifold can be translated into a first order formula of the theory of real closed fields. This can be done using the fact that a C^r -manifold is locally the graph of a C^r -map.

Proposition 2.2. *Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Then, the statement “ S is a C^r -Nash submanifold of \mathbb{R}^n of dimension m ” can be translated into a first order formula of the theory of real closed fields.*

Proof. Set $x = (x_1, \dots, x_n)$, $y = (x_1, \dots, x_m)$, $z = (x_{m+1}, \dots, x_n)$. We can write a formula $\Phi(x)$ which says that there are positive real numbers ε and η such that $S \cap (B^m(y, \varepsilon) \times B^{n-m}(z, \eta))$ is the graph of a C^r -function from $B^m(y, \varepsilon)$ to \mathbb{R}^{n-m} . Furthermore, for all permutation σ of $\{1, \dots, n\}$, let us indicate by $\Phi_\sigma(x)$ the formula that says the same things for the image of x and S by the permutation σ of the coordinates (in order to get all projections on m coordinates among n). There exists a permutation σ of $\{1, \dots, n\}$ such that $\Phi_\sigma(x)$ is true.

We deduce the following formula: $\forall x \in S \bigvee_\sigma \Phi_\sigma(x)$ which says clearly that S is a C^r -Nash submanifold of dimension m of \mathbb{R}^n . □

Proposition 2.3. *Let S and T be semialgebraic subsets of \mathbb{R}^n such that $T \subset S$. Then, the statement “ S is a C^r -Nash submanifold of \mathbb{R}^n of dimension m , with boundary the set T ” can be translated into a first order formula of the theory of real closed fields.*

Proof. Let $x = (x_1, \dots, x_n)$, $y = (x_1, \dots, x_{m-1})$, $z = (x_{m+1}, \dots, x_n)$. We can write a first order formula of the theory of real closed fields $\Psi(x)$ which says that there are positive real numbers ε , δ and η such that both 1) and 2) below hold.

1. $T \cap (B^{m-1}(y, \varepsilon) \times]x_m - \delta, x_m + \delta[\times B^{n-m}(z, \eta))$ is the graph of a C^r -map $g : B^{m-1}(y, \varepsilon) \longrightarrow \mathbb{R}^{n-m+1}$.

Denote by $\xi : B^{m-1}(y, \varepsilon) \rightarrow \mathbb{R}$ the first component of the map g and by $\Gamma_\xi^+ \subset B^{m-1}(y, \varepsilon) \times \mathbb{R}$, $\Gamma_\xi^- \subset B^{m-1}(y, \varepsilon) \times \mathbb{R}$ its over and undergraph (that is $\Gamma_\xi^+ = \{(u, v) \in B^{m-1} \times \mathbb{R} : v \geq \xi(u)\}$ similarly Γ_ξ^-).

2. $S \cap (B^{m-1}(y, \varepsilon) \times]x_m - \delta, x_m + \delta[\times B^{n-m}(z, \eta))$ is the graph of a semialgebraic C^r -map from either $\Gamma_\xi^+ \cap (B^{m-1}(y, \varepsilon) \times]x_m - \delta, x_m + \delta[)$ to \mathbb{R}^{n-m} , or $\Gamma_\xi^- \cap (B^{m-1}(y, \varepsilon) \times]x_m - \delta, x_m + \delta[)$.

Further, for every permutation σ of $\{1, \dots, n\}$, let us indicate by $\Psi_\sigma(x)$ the formula that says the same thing for the image of x , S and T by the permutation σ of coordinates. We construct $\Psi_\sigma(x)$ following the same idea as in the proof of Proposition 2.2, and we take the conjunction of $\forall x \in S \setminus T \bigvee_\sigma \Phi_\sigma(x)$ and $\forall x \in T \bigvee_\sigma \Psi_\sigma(x)$ with $\Phi_\sigma(x)$ as in the proof of Proposition 2.2. There is a delicate point when we say that we have the graph of a C^r -differentiable function

over something which is not open (the overgraph). But we can take the coordinate map

$$g : B^{m-1} \times \mathbb{R}^+ \rightarrow \Gamma_\xi^+ \\ (x, y) \mapsto (x, y + \xi(x)).$$

which identifies the overgraph with a half-space and compute derivatives of this function with respect to these coordinates, which completes the proof. \square

The following proposition assures that the fundamental group of a semialgebraic set does not change during a real closed extension.

Proposition 2.4. ([DK], Theorem 6.3, p. 270)

Let X be a semialgebraic set in R^n , $x_0 \in X$ and K be a real closed extension of R . The map $k : \pi_1(X, x_0)_{alg} \longrightarrow \pi_1(X_K, x_0)_{alg}$, defined by $k[\gamma] := [\gamma_K]$ is a group isomorphism.

3 Semialgebraic h-cobordism theorem over any real closed field

In this section we prove the existence of a uniform bound on the complexity of the homeomorphism in the semialgebraic h-cobordism theorem and use this bound to transfer the semialgebraic h-cobordism theorem over any real closed field.

Definition 3.1. Let R be a real closed field. A semialgebraic subset of R^n is said of complexity at most (p, q) if it admits a description as follows

$$\bigcup_{i=1}^s \bigcap_{j=1}^{k_i} \{x \in R^n \mid f_{ij}(x) *_{ij} 0\},$$

where $f_{ij} \in R[X_1, \dots, X_n]$, and $*_{ij} \in \{<, >, =\}$, $\sum_{i=1}^s k_i \leq p$, $\deg(f_{ij}) \leq q$ for $i = 1, \dots, s$ and $j = 1, \dots, k_i$.

The *complexity* of a semialgebraic subset S of R^n is the smallest couple (p, q) , with respect to the lexicographic order, such that S admits the description above.

Assume a semialgebraic subset $\mathcal{S}(R) \subset R^k$ is defined for any real closed field R . We say that \mathcal{S} is defined *uniformly* when there is a first order formula of the theory of real closed fields without parameter which describes $\mathcal{S}(R)$ for every real closed field R . In order to check that $\mathcal{S}(R)$ is defined uniformly, it suffices to check that, for any real closed extension $R \subset K$, one has $\mathcal{S}(R)_K = \mathcal{S}(K)$.

Assume a semialgebraic subset $\mathcal{S}(R, n, p, q) \subset R^{\alpha(n, p, q)}$ is defined for every real closed field R and any positive integers n, p, q . Assume moreover that for any n, p, q , $\mathcal{S}(R, n, p, q)$ is uniformly defined by a formula without parameter $\Phi_{n, p, q}$:

Then we say that \mathcal{S} is *effectively* defined if there is an algorithm which, given n, p, q , produces $\Phi_{n,p,q}$. (Technically, using a Gödel numbering of formulas, this means that the function which associates to (n, p, q) the Gödel number of $\Phi_{n,p,q}$ is recursive cf. ([Ma], Chap. VII, § 4, p. 242)). In what follows, we drop the explicit dependence on R and we write $\mathcal{S}(n, p, q)$ instead of $\mathcal{S}(R, n, p, q)$.

Proposition 3.2. *There exist a semialgebraic subset $\mathcal{A}(n, p, q)$ in some affine space $R^{\alpha(n,p,q)}$ and a semialgebraic family $\mathcal{S}(n, p, q) \subset \mathcal{A}(n, p, q) \times R^n$ such that:*
(i) For every $a \in \mathcal{A}(n, p, q)$ the fiber

$$\mathcal{S}_a(n, p, q) = \{x \in R^n : (a, x) \in \mathcal{S}(n, p, q)\}$$

is a semialgebraic subset of complexity at most (p, q) of R^n

(ii) For every semialgebraic subset $S \subset R^n$ of complexity at most (p, q) , there is $a \in \mathcal{A}(n, p, q)$ such that: $S = \mathcal{S}_a(n, p, q)$.

$\mathcal{A}(n, p, q)$ and $\mathcal{S}(n, p, q)$ are defined in a uniform way by first order formulas of the theory of real closed fields without parameters which can be effectively constructed from n, p, q .

Proof. Let us first give a description of the fibers of $\mathcal{S}(n, p, q)$ which allow us to show that their union is semialgebraic set.

We start with a set of p polynomials of degree $\leq q$. Let us call f_0, \dots, f_{p-1} the polynomials. A system of sign conditions over these polynomials is given by an element $\sigma \in \{-1, 0, 1\}^p$. This system of signs condition is satisfied in the set

$$\bigcap_{i=0}^{p-1} \{x \in R^n : \text{sign}(f_i(x)) = \sigma_i\}.$$

We will show that a semialgebraic in R^n of complexity at most (p, q) can be described by a boolean combination of sign conditions over p polynomials in n variables of degree $\leq q$, that is, it can be written in the following form:

$$\bigcup_{\sigma \in \Sigma} \bigcap_{i=0}^{p-1} \{x \in R^n : \text{sign}(f_i(x)) = \sigma_i\},$$

where Σ is a subset of $\{-1, 0, 1\}^p$.

Let us index the subsets of $\{-1, 0, 1\}^p$ by the integers l starting from 0 to $2^{3p} - 1$. Now, we describe the space of parameters. To do it, we introduce the notation f_a to indicate the polynomial in n variables of degree $\leq q$ where the list of the coefficients of the monomials of f ordered with respect to the lexicographic order is $a \in R^N$ with $N = \binom{n+q}{q}$. Consider

$$(a_0, \dots, a_{p-1}, l) \in (R^N)^p \times \{0, \dots, 2^{3p} - 1\}.$$

The semialgebraic set of R^n corresponding to this parameter is

$$\bigcup_{\sigma \in \Sigma[l]} \bigcap_{i=0}^{p-1} \{x \in R^n : \text{sign}(f_{a_i}(x)) = \sigma_i\}.$$

We can then describe $\mathcal{S}(n, p, q)$ by the following formula in $(a_0, \dots, a_{p-1}, l, x) \in (R^N)^p \times \{0, \dots, 2^{3p} - 1\} \times R^n$:

$$\Phi_{n,p,q}(a_0, \dots, a_{p-1}, l, x) = \bigvee_{\sigma \in \Sigma[l]} \left(\bigwedge_{i=0}^{p-1} \text{sign}(f_{a_i}(x)) = \sigma_i \right).$$

So, we have

$$\mathcal{S}(n, p, q) = \bigcup_{l=0}^{2^{3p}-1} \{(a_0, \dots, a_{p-1}, l, x) \in (R^N)^p \times R \times R^n : \Phi_{n,p,q}(a_0, \dots, a_{p-1}, l, x)\}.$$

As defined, $\mathcal{S}(n, p, q)$ is a semialgebraic subset of $(R^N)^p \times R \times R^n$. The set $\mathcal{A}(n, p, q) = \bigcup_{l=0}^{2^{3p}} (R^N)^p \times \{l\} \subset (R^N)^p \times R$ gives us the space of parameters of the semialgebraic subsets of R^n of complexity at most (p, q) . Then, one obtains effectively for any real closed field that the space of parameters $\mathcal{A}(n, p, q)$ is a semialgebraic subset of $(R^N)^p \times R$. \square

If \mathcal{S}_a is the semialgebraic set parametrized by $a \in \mathcal{A}(n, p, q)$ by abuse of notation we will write $\mathcal{S}_a \in \mathcal{A}(n, p, q)$. Let us recall the definition of semialgebraic trivialisation of a semialgebraic map.

Definition 3.3. A continuous semialgebraic map $f : A \longrightarrow B$ is said to be *semialgebraically trivial over a semialgebraic subset $C \subset B$* if there is a semialgebraic set F and a semialgebraic homeomorphism $h : f^{-1}(C) \longrightarrow C \times F$, such that the composition of h with the projection $C \times F \rightarrow C$ is equal to the restriction of f to $f^{-1}(C)$. This is shown by the following commutative diagram:

$$\begin{array}{ccc} A \supset f^{-1}(C) & \xrightarrow{h} & C \times F \\ \downarrow f & & \downarrow \text{pr}_1 \\ B \supset C & \xrightarrow{=} & C \end{array}$$

The homeomorphism h is called a *semialgebraic trivialisation* of f over C . We say that the trivialisation h is compatible with a subset $D \subset A$ if there is a subset $G \subset F$ such that $h(D \cap f^{-1}(C)) = C \times G$.

We can now state Hardt's theorem. A detailed proof which works over any real closed in field can be found in [BCR, p.221].

Theorem 3.4. *Let $A \subset R^n$, $B \subset R^m$ be two semialgebraic sets and $f : A \longrightarrow B$ a semialgebraic map. There is a finite semialgebraic partition of $B = \cup_{i=1}^k B_i$ such that f is semialgebraically trivial over each B_i . Moreover, if A_1, \dots, A_h are finitely many semialgebraic subsets of A , we can ask each trivialisation h_i to be compatible with all A_j .*

Remark 3.5. Let a and b be any two elements of the same B_i then, one gets that $f^{-1}(a)$ and $f^{-1}(b)$ are semialgebraically homeomorphic.

Proposition 3.6. *Given the integers n, p and q , there exists a couple of integers (t, u) such that for every couple of semialgebraic sets of complexity at most (p, q) which are semialgebraically homeomorphic, there is a semialgebraic homeomorphism f between them whose graph $\Gamma_f \in \mathcal{A}(2n, t, u)$.*

Proof. Consider the following projection :

$$\begin{aligned} \Pi : R^{pN+1} \times R^n &\rightarrow R^{pN+1} \\ (a, x) &\mapsto a \end{aligned}$$

with $a \in R^{pN+1}$ and $x \in R^n$. We have that $\mathcal{S}(n, p, q) = \{(a, x) \in \mathcal{A}(n, p, q) \times R^n : x \in \mathcal{S}_a\}$ where \mathcal{S}_a is a semialgebraic subset of R^n parametrized by $a \in \mathcal{A}(n, p, q)$. The set $\mathcal{S}(n, p, q)$ is a semialgebraic subset of $R^{pN+1} \times R^n$ (see the proof of Lemma 3.2). The projection $\Pi|_{\mathcal{S}(n, p, q)} : \mathcal{S}(n, p, q) \rightarrow \mathcal{A}(n, p, q)$ is a semialgebraic map. By the Hardt trivialisation theorem, applied to the semialgebraic map $\Pi|_{\mathcal{S}(n, p, q)}$, there exists a finite semialgebraic partition of $\mathcal{A}(n, p, q)$ in S_i : $\mathcal{A}(n, p, q) = \bigcup_{i=1}^s S_i$ such that for each i , there exists a semialgebraic subset X_i and a semialgebraic homeomorphism h_i such that the following diagram commutes:

$$\begin{array}{ccc} \Pi^{-1}(S_i) & \xrightarrow{h_i} & S_i \times X_i \\ \downarrow \Pi & & \downarrow pr_1 \\ S_i & \xrightarrow{=} & S_i \end{array}$$

As the number of trivialisation homeomorphisms is finite, let us take maximum (u, v) of their complexity. We choose a representative in each S_i , $i \in \{1, \dots, s\}$ and take for X_i the corresponding semialgebraic set. Assume X_{i_1} to be semialgebraically homeomorphic to X_{i_2} for some $i_1, i_2 \in \{1, \dots, s\}$. There is a couple of integers $(t_{i_1 i_2}, u_{i_1 i_2})$ such that there exists a semialgebraic homeomorphism $f : X_{i_1} \longrightarrow X_{i_2}$ whose graph belongs to $\mathcal{A}(2n, t_{i_1 i_2}, u_{i_1 i_2})$. Let X and Y be two semialgebraic sets belonging to $\mathcal{A}(n, p, q)$ such that they are semialgebraically homeomorphic. Then there are i_1 and i_2 such that $X = \mathcal{S}_a$ with $a \in S_{i_1}$, $Y = \mathcal{S}_b$ with $b \in S_{i_2}$ and X_{i_1}, X_{i_2} are semialgebraically homeomorphic by f as before. It follows that X is semialgebraically homeomorphic to X_{i_1} by the trivialization homeomorphism, the same for Y and X_{i_2} . We have more precisely:

$h_{i_1|X} : X \longrightarrow X_{i_1}$ defined by $(a, h_{i_1|X}(x)) = h_{i_1}(a, x)$. We get here that the complexity of this restriction is bounded by (u, v) independently of X . And for Y , we have the semialgebraic homeomorphism $h_{i_2|Y} : Y \longrightarrow X_{i_2}$ defined by $(b, h_{i_2|Y}(x)) = h_{i_2}(b, x)$. Consequently this homeomorphism has a complexity bounded by (u, v) , independently of Y . Hence we get an homeomorphism from X to Y by $g = h_{i_2|Y}^{-1} \circ f \circ h_{i_1|X}$. The complexity of g is bounded by $(t'_{i_1 i_2}, u'_{i_1 i_2})$ independently of X and Y , since it is a composition of semialgebraic homeomorphisms with complexity bounded independently of X and Y , and depends only on i_1 and $i_2 \in \{1, \dots, s\}$. Set

$$E = \{(i, j) \in \{1, \dots, s\}^2 \mid X_i \text{ and } X_j \text{ are semialgebraically homeomorphic}\}.$$

This set is finite. Then, take $(t, u) = \left(\max_{(i,j) \in E} (t'_{ij}), \max_{(i,j) \in E} (u'_{ij}) \right)$. \square

We can define the complexity of a semialgebraic cobordism.

Definition 3.7. Let (M, M_0, M_1) be a semialgebraic cobordism such that the semialgebraic manifolds M , M_0 and M_1 have respective complexities (t, u) , (t_0, u_0) and (t_1, u_1) . The *complexity* of the cobordism (M, M_0, M_1) is

$$(v, w) = (\max(t, t_0, t_1), \max(u, u_0, u_1)).$$

The following theorem gives uniform bound for the h-cobordism theorem.

Theorem 3.8. *Given $n, m \geq 6$, $(p, q) \in \mathbb{N}^2$, there exists $(t, u) = \Psi_{HC}(n, m, p, q)$ in \mathbb{N}^2 such that for all simply connected semialgebraic h-cobordism (M, M_0, M_1) in \mathbb{R}^n of complexity at most (p, q) and $\dim M = m$, there exists a semialgebraic homeomorphism $f : M \rightarrow M_0 \times [0, 1]$ whose graph $\Gamma_f \in \mathcal{A}(2n + 1, t, u)$.*

Proof. To prove the existence of the uniform bound (t, u) , we will first construct a set of parameters of semialgebraic h-cobordisms in \mathbb{R}^n with complexity at most (p, q) and semialgebraically simply connected. We need to translate the fact of being:

“a semialgebraic h-cobordism in \mathbb{R}^n of complexity at most (p, q) simply connected”,

into a first order formula of the theory of real closed fields.

Indeed, the fact that a semialgebraic subset of \mathbb{R}^n is a semialgebraic submanifold of \mathbb{R}^n of dimension m can be said by a first order formula of the theory of real closed fields (see Proposition 2.2 (ii)). Which implies that the set of semialgebraic submanifolds of \mathbb{R}^n of dimension m and with complexity at most (p, q) is a semialgebraic subset of $\mathcal{A}(n, p, q)$. Let us denote it by $\mathcal{B}(n, m, p, q)$. So, it is defined by a first order formula of the theory of real closed fields in a uniform and effective way.

The conditions which must satisfy a triplet of semialgebraic manifolds (M, M_0, M_1) to be a cobordism can be translated to a conjunction of first order formulas of the theory of real closed fields with coefficients in \mathbb{Z} . Then the set of elements $(a, b, c) \in \mathcal{B}(n, m, p, q) \times \mathcal{B}(n, m-1, p, q)^2$ such that (M_a, M_b, M_c) is a cobordism is a semialgebraic subset of \mathbb{R}^{3N+3} . This set parametrizes the semialgebraic cobordisms with complexity at most (p, q) and we denote it by $\mathcal{Cob}(n, m, p, q)$. It is defined uniformly and effectively.

There is a semialgebraic family $\mathcal{C}(n, m, p, q) \subset \mathcal{Cob}(n, m, p, q) \times \mathbb{R}^n$ with two subfamilies $\mathcal{C}_0(n, m, p, q) \subset \mathcal{C}(n, m, p, q)$ and $\mathcal{C}_1(n, m, p, q) \subset \mathcal{C}(n, m, p, q)$ such that:

- For every $b \in \mathcal{Cob}(n, m, p, q)$, the fiber

$$\mathcal{C}_b(n, m, p, q) = \{x \in \mathbb{R}^n : (b, x) \in \mathcal{C}(n, m, p, q)\}$$

is a semialgebraic manifold of \mathbb{R}^n of dimension m of complexity at most (p, q) with boundary the disjoint union of the fiber $\mathcal{C}_{0,b}(n, m, p, q)$ and $\mathcal{C}_{1,b}(n, m, p, q)$.

- For every semialgebraic cobordism (M, M_0, M_1) , $M \subset \mathbb{R}^n$ of dimension m and complexity at most (p, q) , there exists $b \in \mathcal{Cob}(n, m, p, q)$ such that:

$$M = \mathcal{C}_b(n, m, p, q), \quad M_0 = \mathcal{C}_{0,b}(n, m, p, q), \quad M_1 = \mathcal{C}_{1,b}(n, m, p, q).$$

The families $\mathcal{C}(n, m, p, q)$, $\mathcal{C}_0(n, m, p, q)$ and $\mathcal{C}_1(n, m, p, q)$ are defined uniformly and effectively. Consider the projection defined by:

$$\begin{aligned} \Pi : \mathcal{C}(n, m, p, q) &\rightarrow \mathcal{Cob}(n, m, p, q) \\ (a, x) &\mapsto a. \end{aligned}$$

Since Π is a semialgebraic map, by Hardt Theorem, there exists a finite semialgebraic partition of $\mathcal{Cob}(n, m, p, q) = \bigcup_{i=1}^s H_i$, compatible with the subfamilies

$\mathcal{C}_0(n, m, p, q)$ and $\mathcal{C}_1(n, m, p, q)$, such that for all i there exists a semialgebraic homeomorphism of trivialisation $\Pi_i : \Pi^{-1}(H_i) \rightarrow H_i \times C_i$ where $C_i = (C_i, C_{i0}, C_{i1})$ is a semialgebraic h-cobordism. Assume Π_i of complexity at most (t_i, u_i) .

Then, there is $J \subset \{1, \dots, s\}$ such that the union $\mathcal{Hcob}(n, m, p, q) = \bigcup_{j \in J} H_j$

parametrizes the set of simply connected semialgebraic h-cobordisms of complexity at most (p, q) . This set is a semialgebraic.

We lose exactly here the effectiveness because the problem of deciding which semialgebraic cobordisms are simply connected h-cobordisms is not effective (cf. [VKT]).

On the other hand the space of parameters $\mathcal{Hcob}(n, m, p, q)$ is uniformly defined since the property of being semialgebraically simply connected is invariant under

extension of real closed fields (Proposition 2.4). Moreover, over \mathbb{R} semialgebraic simple connectedness is the same as topological simple connectedness (Proposition 1.3).

Let (M, M_0, M_1) be a semialgebraic simply connected h-cobordism with a parameter $a \in \mathcal{Hcob}(n, m, p, q)$, then there exists $j \in J$ such that $a \in H_j$. Hence, $\Pi_j|_M : M \rightarrow C_j$ is a semialgebraic homeomorphism with complexity at most (t_j, u_j) . $C_{j_0} \times [0, 1]$ has a complexity bounded in terms (p, q) in an effective way. Since C_j and $C_{j_0} \times I$ are semialgebraically homeomorphic (Theorem 1.4), then by Proposition 3.6, there exists a couple of integers (v, w) which depends only on n, p, q such that there exists a semialgebraic homeomorphism $f_j : C_j \rightarrow C_{j_0} \times I$ whose graph Γ_f admits a complexity at most (v, w) . So we have the following semialgebraic homeomorphism: $g_j = ((\Pi_j|_{M_0})^{-1} \times id_I) \circ f_j \circ \Pi_j|_M : M \rightarrow M_0 \times I$. We get that there exists a bound on the complexity of g_j , write (t'_j, u'_j) , which depends only on j and not on (M, M_0, M_1) . Take

$$(t, u) = (\max_{j \in J}(t'_j), \max_{j \in J}(u'_j))$$

and this ends the proof. \square

As we pointed out in the proof of the above theorem there is a precise point where we loose effectiveness even if we get uniform bounds. We shall look at this question in the next section.

We give now the semialgebraic h-cobordism theorem over any real closed field. Note that by *compact* we mean *closed* and *bounded*.

Theorem 3.9. *Let (M, M_0, M_1) be a semialgebraically simply connected semialgebraic h-cobordism defined over a real closed field R . If $\dim M \geq 6$, then M is semialgebraically homeomorphic to $M_0 \times [0, 1]$.*

Proof. Fix n the dimension of ambient space, $m \geq 6$ the dimension of semialgebraic h-cobordism and (p, q) a bound on its complexity. By the above Theorem, there exists $(t, u) \in \mathbb{N}^2$ such that the following formula holds:

$$\Phi(n, m, p, q, t, u) :=$$

“For every semialgebraic h-cobordism (M, M_0, M_1) in \mathbb{R}^n of complexity at most (p, q) simply connected, there exists a semialgebraic homeomorphism $f : M \rightarrow M_0 \times [0, 1]$ such that its graph $\Gamma_f \in \mathcal{A}(2n+1, t, u)$.”

We ask for this sentence to be true over any real closed field. We can translate the statement $\Phi(n, m, p, q, t, u)$ into a first order sentence of the theory of real closed fields.

Indeed, the space of parameters of semialgebraic h-cobordisms in \mathbb{R}^n of complexity at most (p, q) and semialgebraically simply connected of dimension m is $\mathcal{Hcob}(n, m, p, q)$ as constructed in the above Theorem. Denote by $\mathcal{G}(n, p, q, t, u)$ the set of $(a, b, f) \in \mathcal{A}(n, p, q)^2 \times \mathcal{A}(2n+1, t, u)$ such that $f : \mathcal{S}_a \rightarrow \mathcal{S}_b \times [0, 1]$ is

a semialgebraic homeomorphism. The conditions that must be satisfied by f to be a semialgebraic homeomorphism, can be translated into a first order formula of the theory of real closed fields with coefficients in \mathbb{Z} in an effective way (see the proof of Proposition 2.1). Consequently $\mathcal{G}(n, p, q, t, u)$ is a semialgebraic set defined by a first order formula of the theory of real closed fields with coefficients in \mathbb{Z} . We can now write the following statement:

$\Phi(n, m, p, q, t, u)$:

$$“\forall (a, b, c) \in \mathcal{Hcob}(n, m, p, q) \exists f \in \mathcal{A}(2n + 1, t, u)(a, b, f) \in \mathcal{G}(n, p, q, t, u)”.$$

The statement $\Phi(n, m, p, q, t, u)$ as defined is a first order sentence of the theory of real closed fields with coefficients in \mathbb{Z} . Since $\mathbb{R} \models \Phi(n, m, p, q, t, u)$, by Tarski-Seidenberg Principle, for any real closed field R , one gets $R \models \Phi(n, m, p, q, t, u)$. \square

4 On non-effectiveness of semialgebraic h-cobordism theorem

We proved the existence of a uniform bound in the semialgebraic h-cobordism theorem. On the other hand one could expect, when working with semi-algebraic and compact PL objects, that bounds should be recursive in the sense of [Ma]. To be more precise, what we mean by effective is the following. A statement is effective if admits a uniform bound which is bounded by a recursive function. It is not always the case. There are examples where uniform bounds exist but are not recursive. An example of this type can be found in [ABB]. Namely:

Let K_{Δ^m} be the standard triangulation of the standard simplex Δ^m . Let $B = |K|$ a PL m -ball with K a finite simplicial complex. By Standard Subdivision of B we mean a simplicial isomorphism $g : K' \rightarrow L$ where $K' \triangleleft K$ and $L \triangleleft K_{\Delta^m}$. Fixing m , the authors proved the following:

There exists $\Psi_{SS}(d)$ depending only on d such that for any finite simplicial complex K with at most d vertices such that $|K|$ is a PL m -ball, there exists $K' \triangleleft K$ with at most $\Psi_{SS}(d)$ vertices, and a simplicial isomorphism of K' with L a subdivision of K_{Δ^m} .

But the preceding uniform bound is not recursive

Theorem 4.1. ([ABB], Corollary 2.18)

For $m \geq 5$, Ψ_{SS} cannot be bounded by a recursive function.

We can now prove the non-effectiveness of the semialgebraic h-cobordism theorem.

Theorem 4.2. For $m \geq 6$, the uniform bound Ψ_{HC} of Theorem 3.8 cannot be bounded by a recursive function.

Proof. Let be B a PL m -ball with $m \geq 6$. Assume B triangulated by a finite simplicial complex $K = \{\sigma_1, \dots, \sigma_n\}$ with at most d vertices. Let τ be an m -dimensional simplex in B such that $|\tau| \cap \partial B = \emptyset$ (subdivide K if necessary). It is clear that $(\overline{B \setminus \tau}, \partial\tau, \partial B)$ is a PL simply connected h-cobordism. The complexity of this h-cobordism is bounded by a recursive function in terms of d .

Assume Ψ_{HC} to be recursive. Then there is a semialgebraic homeomorphism: $h : \overline{B \setminus \tau} \rightarrow \partial\tau \times I$ whose complexity is recursively in terms of d . We can attach τ to $\partial\tau \times I$, identifying $\partial\tau \subset \tau$ with $\partial\tau \times \{0\}$. Then extending h over B by the identity on τ , we get a semialgebraic homeomorphism

$$h' : B \rightarrow \tau \bigcup_{\partial\tau} (\partial\tau \times I)$$

with complexity bounded by a recursive function $\Phi(d)$. Let $\tilde{x} = (x_0, \dots, x_m)$ with $x_i \geq 0$ and $\sum x_i = 1$ be a point of τ with its barycentric coordinates. Let $\tilde{b} = (\frac{1}{m+1}, \dots, \frac{1}{m+1})$ the barycenter of standard m - simplex. Now consider the following PL homeomorphism $g : \tau \bigcup_{\partial\tau} (\partial\tau \times I) \rightarrow \Delta_m$ defined by

$$g(x) = \begin{cases} \frac{1}{2}\tilde{b} + \frac{1}{2}\tilde{x} & \text{if } x \in \tau \\ \frac{1-\lambda}{2}\tilde{b} + \frac{1+\lambda}{2}\tilde{x}, & \text{if } (x, \lambda) \in \partial\tau \times I. \end{cases}$$

So we get a semialgebraic homeomorphism $f : B \rightarrow \Delta_m$, given by $f = g \circ h'$, of complexity bounded by a recursive function $\Theta(d)$ in term of d . Since f is a semialgebraic homeomorphism of complexity bounded by $\Theta(d)$ between two compact simplicial complexes with at most d vertices, the effective semialgebraic Hauptvermutung (see [Co2]), implies that there exists a recursive function χ in terms of $\Theta(d)$ and d (so just in term of d) such that there is a simplicial isomorphism between the subdivisions of these simplicial complexes with at most $\chi(d)$ vertices. This is in contradiction with Theorem 4.1. \square

5 Nash h-cobordism theorem over any real closed field

Now we consider Nash manifolds.

Definition 5.1. Let M, M_0, M_1 be compact Nash manifolds such that: $\partial M = M_0 \cup M_1$ and $M_0 \cap M_1 = \emptyset$. Then, the triplet (M, M_0, M_1) is called a Nash cobordism .

A Nash cobordism (M, M_0, M_1) is said to be a *Nash h-cobordism* if the inclusions $M_0 \hookrightarrow M$ and $M_1 \hookrightarrow M$ are semialgebraic homotopy equivalences, that is, the deformation retractions are semialgebraic.

Theorem 5.2. (Nash h-cobordism theorem)

Let (M, M_0, M_1) be a simply connected Nash h-cobordism. If $\dim M \geq 6$ then M is Nash diffeomorphic to $M_0 \times [0, 1]$.

This theorem is an easy consequence of differentiable h-cobordism theorem quoted in the introduction and of the following Nash approximation theorem:

Theorem 5.3. ([S], Theorem VI.2.2, p.202)

Let L_1, L_2 be compact Nash manifold possibly with boundary, and M_1, M_2 their interior. The following conditions are equivalent.

- (i) L_1 and L_2 are C^1 diffeomorphic.
- (ii) L_1 and L_2 are Nash diffeomorphic.
- (iii) M_1 and M_2 are Nash diffeomorphic.

We state two results which give an analogue of Hardt theorem for Nash manifolds with boundaries.

Theorem 5.4. Let $B \subset R^p$ be a semialgebraic set, let X be a semialgebraic subset of $R^n \times B$ such that for every $b \in B$, X_b is a Nash submanifold of R^n . Then there is a stratification $B = \cup_{i \in I} M^i$ of B into a finite number of Nash manifolds, such that for any $i \in I$, $X_{|M^i}$ is Nash manifold and the projection $X_{|M^i} \rightarrow M^i$ is a submersion. If, moreover, X_b is compact for every $b \in B$, we can ask this submersion to be proper.

Proof. See [CS, Corollary 2.3]. □

Theorem 5.5. Let $M \subset R^{m'}$ be a Nash submanifold of dimension m possibly with boundary ∂M . Let $\varpi : M \rightarrow R^k$, $k > 0$ be a proper onto Nash submersion such that $\varpi_{|\partial M}$ is onto submersion. Then there exists a Nash diffeomorphism $\varphi = (\varphi', \varpi) : (M, \partial M) \rightarrow (M \cap \varpi^{-1}(0), \partial M \cap \varpi^{-1}(0)) \times R^k$.

Proof. See [FKS, Theorem I]. □

We can now formulate a Nash triviality in family of Nash manifolds with boundaries.

Theorem 5.6. Let B be a semialgebraic set and $\Pi : R^n \times B \rightarrow B$ be the projection on B . Let X be a semialgebraic subset of $R^n \times B$ such that for all $b \in B$,

$$X_b = \{x \in R^n : (x, b) \in X\}$$

is a compact Nash manifold in R^n with boundary. Then there exists a finite partition of B in $B = \bigcup_{i \in I} M^i$ where M^i are Nash manifolds, and for each $i \in I$ there is an affine Nash manifold $F^i \subset R^n$ with boundary and a Nash diffeomorphism which trivializes $\Pi_{|X_{|M^i}}$

$$h^i : F^i \times M^i \rightarrow X \cap \Pi^{-1}(M^i)$$

compatible with the boundary.

Proof. Set $Y = \{(x, b) : x \in \partial X_b\}$. By Theorem 5.4, one can prove that there exists a finite Nash partition of B into $B = \bigcup_{i=1}^r B_i$ such that for every i , B_i , $X|_{B_i}$ and $Y|_{B_i}$ are Nash submanifolds and the projections $\pi_{X|_{B_i}} : X|_{B_i} \rightarrow B_i$ and $\pi_{Y|_{B_i}} : Y|_{B_i} \rightarrow B_i$ are proper onto Nash submersions. For any i , there exists a partition of B_i , by [BCR, Proposition 2.9.10, p.57], into $B_i = \bigcup_{j=i}^t S_{i_j}$ such that S_{i_j} is Nash diffeomorphic to $]0, 1[^{k_{i_j}}$. However, it is clear that $]0, 1[^{k_{i_j}}$ is Nash diffeomorphic to $R^{k_{i_j}}$. So we get the proper onto Nash submersions $X|_{S_{i_j}} \rightarrow R^{k_{i_j}}$ and $Y|_{S_{i_j}} \rightarrow R^{k_{i_j}}$. By Theorem 5.5 applied to $X|_{S_{i_j}}$ with its boundary $\partial X|_{S_{i_j}} = Y|_{S_{i_j}}$, there exists a Nash submanifold with boundary F_{i_j} of R^n such that: $(X|_{S_{i_j}}; Y|_{S_{i_j}}) \cong (F_{i_j}; \partial F_{i_j}) \times R^{k_{i_j}}$. This is equivalent to: $(X|_{S_{i_j}}; Y|_{S_{i_j}}) \cong (F_{i_j}; \partial F_{i_j}) \times S_{i_j}$. Which ends the proof. \square

By a result of Ramanakoraisina (cf. [R, Proposition 3.5]), given m, p, q there is an integer l such that for any semialgebraic map f of complexity (p, q) , over an open subset of \mathbb{R}^m , f is Nash if and only if f is C^l . Moreover, the integer l is recursive in terms of m, p, q (cf. [CS, Lemma 5.1]). From this we get the following proposition.

Proposition 5.7. *Let S and T be semialgebraic subsets of \mathbb{R}^n such that $T \subset S$. The following statements can be translated into a first order formula of the theory of real closed fields:*

- (i) “ S is a Nash submanifold of \mathbb{R}^n of dimension m ”
- (ii) “ S is a Nash submanifold of \mathbb{R}^n of dimension m , with boundary the set T ”

Proof. (i) Just use the notification above and the proof of Proposition 2.2.

(ii) Use the notification above and the proof of Proposition 2.3. \square

In a similar way to Proposition 2.1, one has the following result in the Nash case.

Proposition 5.8. *Let R and K be two real closed fields such that K is a real closed extension of R . Let X and Y be two Nash submanifold R^n . The Nash manifolds X and Y are Nash diffeomorphic if and only if X_K and Y_K are Nash diffeomorphic.*

Proof. The proof is similar to the proof of Proposition 2.1 adding the property that a semialgebraic map must satisfy to be a Nash map which can be translated in a first order formula of the theory of real closed fields (cf. [R, Proposition 3.5]). This ends the proof. \square

We prove here the analogue of Proposition 3.6 for Nash manifolds.

Proposition 5.9. *Given the integers n, p and q , there exists a couple of integers (t, u) such that for all couples of Nash submanifolds of R^n of complexity at most (p, q) and Nash diffeomorphic, there exists a Nash diffeomorphism f between them, whose graph Γ_f admits a complexity at most (t, u) .*

Proof. Let us notice that all Nash submanifolds of R^n of complexity at most (p, q) belong to $\mathcal{A}(n, p, q)$. We have shown in the proof of Proposition 3.6 that the space of parameters $\mathcal{A}(n, p, q)$ is a semialgebraic set. The set of parameters of Nash submanifolds of R^n of dimension m of complexity at most (p, q) is included in $\mathcal{A}(n, p, q)$ and is a semialgebraic subset of $\mathcal{A}(n, p, q)$. Indeed, the condition for a semialgebraic subset to be a Nash manifold can be translated uniformly and effectively in a first order formula of the theory of real closed fields (Proposition 5.7.(i)). Let us denote this set by $\mathcal{N}(n, m, p, q)$. By the Nash Triviality theorem mentioned above, there exists a finite Nash stratification $\mathcal{N}(n, m, p, q) = \bigcup_{i=1}^s M^i$ of $\mathcal{N}(n, m, p, q)$ into Nash manifolds M^i such that the Nash manifolds parametrized by points in M^i are Nash diffeomorphic. The remaining part of the proof is similar to the end of the proof of Proposition 3.6. \square

Now, the existence of a uniform bound for Nash h-cobordism Theorem, can be deduced in the same way as for the semialgebraic case

Theorem 5.10. *Given $n, m \geq 6$, $(p, q) \in \mathbb{N}^2$, there exists $(t, u) \in \mathbb{N}^2$ such that for all Nash h-cobordism (M, M_0, M_1) in \mathbb{R}^n of complexity at most (p, q) simply connected and $\dim M = m$, there exists a Nash diffeomorphism $f : M \rightarrow M_0 \times [0, 1]$ such that its graph $\Gamma_f \in \mathcal{A}(2n + 1, t, u)$.*

The validity over any real closed field follows as consequence of the above theorem.

Theorem 5.11. *Let (M, M_0, M_1) be a Nash h-cobordism, semialgebraically simply connected defined over a real closed field R . If $\dim M \geq 6$, then M is Nash diffeomorphic to $M_0 \times [0, 1]$.*

6 Semialgebraic and Nash s-cobordism theorems

We first define the notion of simple homotopy equivalence.

Definition 6.1. Let P and Q be two polyhedra such that $Q \subset P$. Let B^n a PL n -ball. If $P = Q \cup B^n$, $Q \cap B^n \subset \partial B^n$ and $Q \cap B^n$ is a PL $(n - 1)$ -ball, we say that Q is obtained from P by an *elementary collapse*. We denote it by $P \Downarrow Q$. We say also that there is *elementary expansion* of Q in P .

We say that P collapse on Q and we denote it by $P \searrow Q$ if there is a finite sequence of elementary collapses $P = P_0 \Downarrow P_1 \Downarrow \dots \Downarrow P_m = Q$. We will say also that there is an expansion of Q to P and we denote it by $Q \nearrow P$.

Definition 6.2. A polyhedron P is said to be *simply homotopic* to another polyhedron Q if P is obtained from Q by a sequence of collapses and expansions of Q

$$P = P_0 \searrow P_1 \nearrow P_2 \searrow \dots \searrow P_n = Q \text{ rel } Q$$

where $\text{rel } Q$ means that during the collapses and expansions operations Q remains unchanged. We will say: P is *s-homotopic* to Q .

Now we define the notion of semialgebraic simple homotopy equivalence.

Definition 6.3. Let X and Y be two semialgebraic sets such that $Y \subset X$. We say that X *elementary semialgebraically collapses* on Y , and we write $X \Downarrow Y$, if there is a semialgebraic map $f : I^n \rightarrow X$ such that

- $f|_{(0,1] \times I^{n-1}}$ is an embedding
- $f(0 \times I^{n-1}) \subset Y$, $f((0,1] \times I^{n-1}) \cap Y = \emptyset$ and $X = Y \cup f(I^n)$

We say that X semialgebraically collapses on Y , and write $X \searrow Y$, if there is a finite sequence of elementary semialgebraic collapses $X = X_0 \Downarrow X_1 \Downarrow \dots \Downarrow X_m = Y$.

Remark 6.4. A PL elementary collapse of compact polyhedra is in particular a semialgebraic collapse. A kind of converse is proved in next lemma.

In the following definition, the notation “ $\text{rel } Y$ ” means that during the collapse and expansion operations, Y remains unchanged.

Definition 6.5. Let $Y \subset X$ be compact semialgebraic sets. The semialgebraic set X is *semialgebraically s-homotopic* to the semialgebraic set Y if and only if there is a sequence of semialgebraic collapses and expansions of X on $Y \text{ rel } Y$.

Lemma 6.6. *Let $Y \subset X$ be two compact semialgebraic sets and assume $X \Downarrow Y$. Take a triangulation $|K| \rightarrow X$ compatible with Y and put $|K'| = h^{-1}(Y)$. Then $|K| \searrow |K'|$.*

Proof. By hypothesis $X \Downarrow Y$. This means that there is a semialgebraic map $f : I^n \rightarrow X$ satisfying the properties of Definition 6.3. Set $g = f|_{0 \times I^{n-1}}$. We may, consider X as a mapping cylinder M_g of g , setting $X = M_g = I^n \cup Y/(0, x) \sim g(0, x)$. A semialgebraic triangulation $|K|$ of X , compatible with Y and $cl(M_g \setminus Y)$, induces a semialgebraic triangulation $|K'|$ of Y and another $|K''|$ of $cl(M_g \setminus Y)$. This means that $K' \subset K$ and $K'' \subset K$. It follows that $|K| = |K''| \cup |K'|$. We can construct a projection $\pi : |K''| \rightarrow [0, 1]$ such that $\pi^{-1}(0) = |K''| \cap |K'|$, in the following way. Let x and y be in I^n . One defines an equivalence relation “ $x \approx y$ ” if and only if $f(x) = f(y)$. The map induced by f , say f' , defined by

$$\begin{aligned} I^n / \approx &\longrightarrow f(I^n) \\ [t] &\mapsto f'([t]) = f(t) \end{aligned}$$

where $t = (t_1, \dots, t_n) \in I^n$, is a homeomorphism for the quotient topology. The semialgebraic triangulation of X induces a semialgebraic homeomorphism $k : |K''| \longrightarrow f(I^n)$. Let be $p : I \times I^{n-1} \longrightarrow I$ such that $p(t_1, \dots, t_n) = t_1$ for all $(t_1, \dots, t_n) \in I^n$ and $p' : I^n / \approx \longrightarrow I$ the induced continuous map. Consider the map π as follows $\pi = p' \circ f'^{-1} \circ k : |K''| \longrightarrow [0, 1]$. There exists $0 < \epsilon < 1$ such that $\pi^{-1}([0, \epsilon])$ contains all vertices of the simplices that have at least one face in $\pi^{-1}(0)$ making some subdivisions of K'' if necessary. We observe that: $\pi^{-1}([0, \epsilon]) = k^{-1}(f([0, \epsilon] \times I^{n-1}))$. The semialgebraic subset $f([0, \epsilon] \times I^{n-1})$ is a semialgebraic compact neighbourhood of $f(0 \times I^{n-1})$ in the semialgebraic set $f(I^n)$. It follows that $\pi^{-1}([0, \epsilon]) = U$ is a regular neighbourhood $\pi^{-1}(0)$ in $|K''|$, asking that k is compatible with $f([0, \epsilon] \times I^{n-1})$. Since $cl(|K''| \setminus U)$ is PL homeomorphic to I^n , $|K''|$ PL collapses on U . Furthermore, we have that U PL collapses on $\pi^{-1}(0)$, since it is a regular neighbourhood of $\pi^{-1}(0)$ cf. ([RS], Corollary 3.30). We get that $|K|$ PL collapses on $|K'|$. This closes the proof. \square

Corollary 6.7. *Let M and M_0 be two compact semialgebraic manifolds with M_0 a deformation retraction of M . Then, M collapses semialgebraically on M_0 if and only if $|K|$ PL collapses on $|K_0|$, where $|K|$ is a semialgebraic triangulation of M compatible with M_0 and $|K_0|$ the induced triangulation of M_0 .*

Proof. The proof follows using Lemma 6.6 and Definition 6.5. \square

We get now a semialgebraic version of the s-cobordism theorem.

Theorem 6.8. *Let (M, M_0, M_1) be a connected semialgebraic h-cobordism with $\dim M \geq 6$. Then, M and $M_0 \times [0, 1]$ are semialgebraically homeomorphic if and only if M is semialgebraically s-homotopic to M_0 .*

Proof. It is an easy consequence of the classical PL s-cobordism theorem using Corollary 6.7. \square

To close this section, we prove the Nash version of s-cobordism theorem.

Theorem 6.9. *Let (M, M_0, M_1) be a connected Nash h-cobordism with $\dim M \geq 6$. then M and $M_0 \times [0, 1]$ are Nash diffeomorphic if and only if M is semialgebraically s-homotopic to M_0 .*

Proof. \Rightarrow) Assume that M is Nash diffeomorphic to $M_0 \times [0, 1]$. Since M and $M_0 \times [0, 1]$ are Nash manifolds, they are in particular smooth and diffeomorphic. By the smooth s-cobordism theorem, one has M is s-homotopic to M_0 . This implies that M is semialgebraically s-homotopic to M_0 .

\Leftarrow) Conversely, assume that M is semialgebraically s-homotopic to M_0 . This implies in particular that M is s-homotopic to M_0 . By the smooth s-cobordism theorem (cf.[K]), one gets M diffeomorphic to $M_0 \times [0, 1]$. It follows by Theorem 5.3 that M is Nash diffeomorphic to $M_0 \times [0, 1]$. This closes the proof. \square

7 Semialgebraic and Nash s-cobordism theorems over any real closed field

In this section we prove that the semialgebraic and Nash s-cobordism theorems hold over any real closed field. To make short, we will just write “sas-homotopic” instead of “semialgebraically s-homotopic”. Since one implication in the s-cobordism theorem is obvious, in the following we will only consider the other one.

First we see that sas-homotopy is preserved in a real closed extension.

Proposition 7.1. *Let X and Y be two semialgebraic subset of R^r and K a closed real extension of R . Then, $X \Downarrow Y$ algebraically if and only if $X_K \Downarrow Y_K$ semialgebraically.*

Proof. Use Tarski-Seidenberg Principle. □

Proposition 7.2. *Let X and Y be two semialgebraic subset of R^r and K a real closed extension of R . Then, X is sas-homotopic to Y if and only if X_K is sas-homotopic to Y_K .*

Proof. Set: $X = \{x \in R^r : \phi(a, x)\}$, $Y = \{x \in R^r : \psi(b, x)\}$ where $\phi(a, x)$ and $\psi(b, x)$ are first order formulas of the theory of real closed fields with parameters $a \in R^m$ and $b \in R^{m'}$.

\Rightarrow) X is sas-homotopic to Y if there is a sequence of semialgebraic collapses and expansions: $X = X_0 \searrow X_1 \nearrow X_2 \searrow \dots \searrow X_s = Y \text{ rel } Y$. Let i be such that $X_i \searrow X_{i+1}$. It is equivalent to say that there exists a finite sequence of elementary semialgebraic collapses. By the above Proposition, one has $X_{iK} \searrow X_{i+1K}$. If on the other hand $X_i \nearrow X_{i+1}$, it is the same to say that $X_{i+1} \searrow X_i$. We deduce in the same way that X is sas-homotopic to Y .

\Leftarrow) X_K is sas-homotopic Y_K if there exists a sequence of semialgebraic collapses and expansions: $X_K = X_0 \searrow X_1 \nearrow X_2 \searrow \dots \searrow X_s = Y_K \text{ rel } Y_K$. Assume that $X_i = \{x \in K^r : \phi_i(a_i, x)\}$ with $a_i \in K^{m_i}$, where $\phi_i(a_i, x)$ is a first order formula of the theory of real closed fields for $i = 0, \dots, s$ and $\phi_0(a_0, x) = \phi(a, x)$, $a_0 = a$, $m_0 = m$, $\phi_s(a_s, x) = \psi(b, x)$, $a_s = b$ and $m_s = m'$.

Let $i \in \{0, \dots, s\}$ be such that $X_i \searrow X_{i+1}$. This implies that there exists a sequence of elementary semialgebraic collapses $X_i = X_{i0} \Downarrow X_{i1} \Downarrow \dots \Downarrow X_{ik_i} = X_{i+1}$. Set $X_{ij} = \{x \in K^r : \phi_{ij}(a_{ij}, x)\}$. The fact that $X_{ij} \Downarrow X_{i,j+1}$ can be translated in a first order sentence of the theory of real closed fields, denote it by: $\beta_{ij}(a_{ij}, a_{i,j+1}, c_{ij})$ where c_{ij} is a parameter which define the graph of the semialgebraic map which define the elementary semialgebraic collapse. Then the fact that $X_i \searrow X_{i+1}$ can be translated into a first order sentence of the theory of real closed fields as follows: $\bigwedge_{j=0}^{k_i} \beta_{ij}(a_{ij}, a_{i,j+1}, c_{ij})$. Now assume that $X_i \nearrow X_{i+1}$. This is equivalent to $X_{i+1} \searrow X_i$. By the same techniques we construct a first order formula of the theory of real closed fields as follows: $\bigwedge_{j=0}^{k_i} \beta'_{ij}(a_{i+1,j}, a_{i+1,j+1}, c_{i+1,j})$.

Then the fact that $X_K = X_0 \searrow X_1 \nearrow X_2 \searrow \dots \searrow X_s = Y_K \text{ rel } Y_K$ can be translated into a first order sentence of the theory of real closed fields as following:

$$\left(\bigwedge_{i=0}^s \left(\bigwedge_{j=0}^{k_i} \beta_{ij}(a_{ij}, a_{i,j+1}, c_{ij}) \right) \bigvee \left(\bigwedge_{j=0}^{k_{i+1}} \beta'_{ij}(a_{i+1,j}, a_{i+1,j+1}, c_{i+1,j}) \right) \right) \bigwedge \left(\bigwedge_{i,j} \lambda(a_{ij}, b) \right),$$

where $\lambda(a_{ij}, b)$ translate the fact that $Y_k \subset X_{ij}$ and the collapses let fix pointwise Y_k for all i and j . Let us denote this sentence by $\Phi((a_{ij}), (c_{ij}))$, where $a_{00} = a$ and $a_{s k_s} = b$. By hypothesis one gets $K \models \Phi((a_{ij}), (c_{ij}))$. Some parameters are already in R . Take the parameters a_{lt} and c_{uv} which are not. Omitting the parameters defined over R in the formula Φ (to make short), one gets: $K \models \exists y_{lt} \exists z_{uv} \Phi(y_{lt}, z_{uv})$, where the quantification runs over the variables with indices l_t, u_v in Φ . So, by Tarski-Seidenberg Principle, we have: $R \models \exists y_{lt} \exists z_{uv} \Phi(y_{lt}, z_{uv})$. This means that there exists a sequence $X_K = X_0 \searrow X_1 \nearrow X_2 \searrow \dots \searrow X_s = Y_K \text{ rel } Y_K$ with all X_i and semialgebraic collapses defined by first order formulas of the theory of real closed fields with coefficients in R . This implies that X is sas-homotopic to Y . \square

We prove that there is a uniform bound on the complexity of the semialgebraic homeomorphism in term of the complexity of the semialgebraic h-cobordism in the s-cobordism theorem.

Theorem 7.3. *Given $n, m \geq 6$, $(p, q) \in \mathbb{N}^2$, there exists $(t, u) \in \mathbb{N}^2$ such that for all semialgebraic h-cobordism (M, M_0, M_1) in \mathbb{R}^n of complexity at most (p, q) with M sas-homotopic to M_0 and $\dim M = m$, there exists a semialgebraic homeomorphism $f : M \rightarrow M_0 \times I$ such that its graph $\Gamma_f \in \mathcal{A}(2n + 1, t, u)$.*

Proof. We follow the proof of Theorem 3.8, instead of choosing, in the partition of $\text{Cob}(n, m, p, q)$, the semialgebraic h-cobordism semialgebraically simply connected, we select the semialgebraic h-cobordism (M, M_0, M_1) such that M is sas-homotopic to M_0 and connected.

This gives us a semialgebraic set of parameters of the semialgebraic h-cobordism (M, M_0, M_1) in \mathbb{R}^n of complexity at most (p, q) with M sas-homotopic to M_0 , which we denote by $\text{Scob}(n, m, p, q)$. The semialgebraic set $\text{Scob}(n, m, p, q)$ is defined uniformly by Proposition 7.2.

Then we change $\mathcal{Hcob}(n, m, p, q)$ by $\text{Scob}(n, m, p, q)$ in the remaining part of the proof of Theorem 3.8. \square

Here is the semialgebraic s-cobordism theorem over any real closed field.

Theorem 7.4. *Let (M, M_0, M_1) be a semialgebraic h-cobordism defined over a real closed field R such that M sas-homotopic to M_0 . If $\dim M \geq 6$, then M is semialgebraically homeomorphic to $M_0 \times I$.*

Proof. Fixing a bound on the complexity of the semialgebraic h-cobordism, with Theorem 7.3, the proof is the same as the proof of Theorem 3.9. \square

We get in the same way the Nash version of these theorems.

Theorem 7.5. *Given $n, m \geq 6$, $(p, q) \in \mathbb{N}^2$, there exists $(t, u) \in \mathbb{N}^2$ such that for all Nash h -cobordism (M, M_0, M_1) in \mathbb{R}^n of complexity at most (p, q) with M sas-homotopic to M_0 and $\dim M = m$, there exists a Nash diffeomorphism $f : M \rightarrow M_0 \times I$ such that its graph $\Gamma_f \in \mathcal{S}(2n + 1, t, u)$.*

Proof. The proof is the same as the proof of Theorem 7.3. \square

We close this section with the Nash s-cobordism Theorem over any real closed field.

Theorem 7.6. *Let (M, M_0, M_1) be a Nash h -cobordism defined over a real closed field R such that M sas-homotopic to M_0 . If $\dim M \geq 6$, then M is Nash diffeomorphic to $M_0 \times [0, 1]$.*

Proof. Fixing a bound on the complexity of the Nash h -cobordism, with Theorem 7.5, the proof is the same as the proof of Theorem 5.11. \square

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