Stability of multi antipeakon-peakons profile

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Abstract. The Camassa-Holm equation possesses well-known peaked solitary waves that can travel to both directions. The positive ones travel to the right and are called peakon whereas the negative ones travel to the left and are called antipeakons. Their orbital stability has been established by Constantin and Strauss in [20]. In [28] we have proven the stability of trains of peakons. Here, we continue this study by extending the stability result to the case of ordered trains of anti-peakons and peakons.

1 Introduction

The Camassa-Holm equation (C-H),

$$u_t - u_{txx} = -3uu_x + 2u_xu_{xx} + uu_{xxx}, \quad (t,x) \in I\!R^2, \tag{1}$$

can be derived as a model for the propagation of unidirectional shalow water waves over a flat bottom by writing the Green-Naghdi equations in Lie-Poisson Hamiltonian form and then making an asymptotic expansion which keeps the Hamiltonian structure ([7], [31]). Note that the Green-Naghdi equations arise as approximations to the governing equations for shallow-water medium-amplitude regime which captures more nonlinear effects than the classical shallow-water small amplitude KdV regime and thus can accommodate models for breaking waves (cf. [1], [16], [12]). The Camassa-Holm equation was also found independently by Dai [22] as a model for nonlinear waves in cylindrical hyperelastic rods and was, actually, first discovered by the method of recursive operator by Fokas and Fuchsteiner [29] as an example of bi-Hamiltonian equation. Let us also mention that it has also a geometric derivation as a re-expression of geodesic flow on the diffeomorphism

group on the line (cf. [32], [33]) and that this framework is instrumental in showing that the Least Action Principle holds for this equation (cf. [9], [17]).

(C-H) is completely integrable (see [7],[8], [10] and [15]). It possesses among others the following invariants

$$E(v) = \int_{I\!R} v^2(x) + v_x^2(x)\, dx \text{ and } F(v) = \int_{I\!R} v^3(x) + v(x)v_x^2(x)\, dx \qquad (2)$$

and can be written in Hamiltonian form as

$$\partial_t E'(u) = -\partial_x F'(u) \quad . \qquad (3)$$

Camassa and Holm [7] exhibited peaked solitary waves solutions to (C-H) that are given by

$$u(t,x) = \varphi_c(x-ct) = c\varphi(x-ct) = ce^{-|x-ct|},\ c \in I\!R.$$

They are called peakon whenever $c > 0$ and antipeakon whenever $c < 0$. Let us point out here that the feature of the peakons that their profile is smooth, except at the crest where it is continuous but the lateral tangents differ, is similar to that of the waves of greatest height, i.e. traveling waves of largest possible amplitude which are solutions to the governing equations for water waves (cf. [11], [14] and [37]). Note that (C-H) has to be rewriten as

$$u_t + uu_x + (1-\partial_x^2)^{-1}\partial_x(u^2 + u_x^2/2) = 0\ . \qquad (4)$$

to give a meaning to these solutions. Their stability seems not to enter the general framework developed for instance in [4], [30]. However, Constantin and Strauss [20] succeeded in proving their orbital stability by a direct approach. In [28] we combined the general strategy initiated in [34](note that due to the reasons mentioned above, the general method of [34] is not directly applicable here), a monotonicity result proved in [27] on the part of the energy $E(\cdot)$ at the right of a localized solution traveling to the right and localized versions of the estimates established in [20] to derive the stability of ordered trains of peakons. In this work we pursue this study by proving the stability of ordered trains of anti-peakons and peakons. The main new ingredient is a monotonicity result on the part of the functional $E(\cdot) - \lambda F(\cdot)$, $\lambda \geq 0$, at the right of a localized solution traveling to the right. It is worth noticing that the sign of λ plays a crucial role in our analysis.

Before stating the main result let us introduce the function space where we will define the flow of the equation. For I a finite or infinite interval of

$\mathbb{R}$, we denote by $Y(I)$ the function space[1]

$$Y(I) := \left\{ u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})),\ u_x \in L^\infty(I; BV(\mathbb{R})) \right\}. \quad (5)$$

In [18], [23] and [36] (see also [35]) the following existence and uniqueness result for this class of initial data is derived.

Theorem 1.1 *Let $u_0 \in H^1(\mathbb{R})$ with $m_0 := u_0 - u_{0,xx} \in \mathcal{M}(\mathbb{R})$ then there exists $T = T(\|m_0\|_{\mathcal{M}}) > 0$ and a unique solution $u \in Y([-T,T])$ to (C-H) with initial data u_0. The functionals $E(\cdot)$ and $F(\cdot)$ are constant along the trajectory and if m_0 is such that* [2] $\text{supp } m_0^- \subset]-\infty, x_0]$ *and* $\text{supp } m_0^+ \subset [x_0, +\infty[$ *for some $x_0 \in \mathbb{R}$ then u exists for all positive times and belongs to $Y([0,T])$ for all $T > 0$.*
Moreover, let $\{u_{0,n}\} \subset H^1(\mathbb{R})$ such that $u_{0,n} \to u_0$ in $H^1(\mathbb{R})$ with $\{m_{0,n} := u_{0,n} - \partial_x^2 u_{0,n}\}$ bounded in $\mathcal{M}(\mathbb{R})$, $\text{supp } m_{0,n}^- \subset]-\infty, x_{0,n}]$ and $\text{supp } m_{0,n}^+ \subset [x_{0,n}, +\infty[$ for some sequence $\{x_{0,n}\} \subset \mathbb{R}$. Then, for all $T > 0$,

$$u_n \longrightarrow u \text{ in } C([0,T]; H^1(\mathbb{R}))\ . \quad (6)$$

Let us emphasize that the global existence result when the negative part of m_0 lies completely to the left of its positive part is proven in [36] and that the last assertion of the above theorem is not explicitly contained in this paper. However, following the same arguments as those developed in these works (see for instance Section 5 of [35]), one can prove that there exists a subsequence $\{u_{n_k}\}$ of solutions of (1) that converges in $C([0,T]; H^1(\mathbb{R}))$ to some solution v of (1) belonging to $Y([0,T[)$. Since u_{0,n_k} converges to u_0 in H^1, it follows that $v(0) = u_0$ and thus $v = u$ by uniqueness. This ensures that the whole sequence $\{u_n\}$ converges to u in $C([0,T]; H^1(\mathbb{R}))$ and concludes the proof of the last assertion.

Remark 1.1 *It is worth pointing out that recently, in [5] and [6], Bressan and Constantin have constructed global conservative and dissipative solutions of the Camassa-Holm equation for any initial data in $H^1(\mathbb{R})$. However, even if for the conservative solutions, $E(\cdot)$ and $F(\cdot)$ are conserved quantities, these solutions are not known to be continuous with values in $H^1(\mathbb{R})$. Therefore even one single peakon is not known to be orbitally stable in this class of solutions. For this reason we will work in the class of solutions constructed in Theorem 1.1.*

[1] $W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation

[2] $\mathcal{M}(\mathbb{R})$ is the space of Radon measures on $\mathbb{R}$ with bounded total variation. For $m_0 \in \mathcal{M}(\mathbb{R})$ we denote respectively by m_0^- and m_0^+ its positive and negative part.

We are now ready to state our main result.

Theorem 1.2 *Let be given N non vanishing velocities $c_1 < .. < c_k < 0 < c_{k+1} < .. < c_N$. There exist γ_0, $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $u \in Y([0,T[)$, with $0 < T \leq \infty$, is a solution of (C-H) with initial data u_0 satisfying*

$$\|u_0 - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - z_j^0)\|_{H^1} \leq \varepsilon^2 \tag{7}$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_j^0 - z_{j-1}^0 \geq L$, with $L > L_0$, then there exist $x_1(t), .., x_N(t)$ such that

$$\sup_{[0,T[} \|u(t,\cdot) - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - x_j(t))\|_{H^1} \leq A(\sqrt{\varepsilon} + L^{-1/8}) \; . \tag{8}$$

Moreover there exists C^1-functions $\tilde{x}_1, .., \tilde{x}_N$ such that, $\forall j \in \{1, .., N\}$,

$$|x_j(t) - \tilde{x}_j(t)| = O(1) \text{ and } \frac{d}{dt}\tilde{x}_j(t) = c_j + O(\varepsilon^{1/4}) + O(L^{-\frac{1}{16}}), \forall t \in [0,T[\; . \tag{9}$$

Remark 1.2 *We do not know how to prove the monotonicity result in Lemma 3.1, and thus Theorem 1.2, for solutions that are only in $C([0,T[; H^1(I\!R))$ which is the hypothesis required for the stability of a single peakon (cf. [20]). Note anyway that there exists no well-posedness result in the class $C([0,T[; H^1(I\!R))$ for general initial data in $H^1(I\!R)$. On the other hand, according to Theorem 1.1 above, $u \in Y([0,T[)$ as soon as $u_0 \in H^1(I\!R)$ and $(1 - \partial_x^2)u_0$ is a Radon measure with bounded variations.*

Remark 1.3 *Note that under the hypotheses of Theorem 1.2,*

$$\sum_{j=1}^{N} \varphi_{c_j}(\cdot - z_j^0)$$

belongs to the class $v \in H^1(I\!R)$ with $m := v - v_{xx} \in \mathcal{M}(I\!R)$, $\text{supp } m^- \subset]-\infty, x_0]$ and $\text{supp } m^+ \subset [x_0, +\infty[$ for some $x_0 \in I\!R$. Therefore, in view of Theorem 1.1, Theorem 1.2 leads to the orbital stability (for positive times) of such ordered sum of antipeakons and peakons with respect to H^1-perturbations that keep the initial data in this same class.

As discovered by Camassa and Holm [7], (C-H) possesses also special solutions called multipeakons given by

$$u(t,x) = \sum_{i=1}^{N} p_j(t) e^{-|x-q_j(t)|} ,$$

where $(p_j(t), q_j(t))$ satisfy the differential system :

$$\left\{ \begin{array}{l} \dot{q}_i = \sum_{j=1}^{N} p_j e^{-|q_i - q_j|} \\ \dot{p}_i = \sum_{j=1}^{N} p_i p_j \mathrm{sgn}(q_i - q_j) e^{-|q_i - q_j|} . \end{array} \right. \tag{10}$$

In [3] (see also [2] and [7]), the asymptotic behavior of the multipeakons is studied. In particular, the limits as t tends to $+\infty$ and $-\infty$ of $p_i(t)$ and $\dot{q}_i(t)$ are determined. Combining these asymptotics with the preceding theorem and the continuity with respect to initial data stated in Theorem 1.1 we get the following result on the stability for positive times of the variety $\mathcal{N}_{N,k}$ of $H^1(\mathbb{R})$ defined for $N \geq 1$ and $0 \leq k \leq N$ by

$$\mathcal{N}_{N,k} := \left\{ v = \sum_{i=1}^{N} p_j e^{-|\cdot - q_j|},\ (p_1, .., p_N) \in (\mathbb{R}_-^*)^k \times (\mathbb{R}_+^*)^{N-k},\ q_1 < q_2 < .. < q_N \right\} .$$

Corollary 1.1 *Let be given k negative real numbers $p_1^0, .., p_k^0$, $N-k$ positive real numbers $p_{k+1}^0, .., p_N^0$ and N real numbers $q_1^0 < .. < q_N^0$. For any $B > 0$ and any $\gamma > 0$ there exists $\alpha > 0$ such that if $u_0 \in H^1(\mathbb{R})$ is such that $m_0 := u_0 - u_{0,xx} \in \mathcal{M}(\mathbb{R})$ with* supp $m_0^- \subset]-\infty, x_0]$ *and* supp $m_0^+ \subset [x_0, +\infty[$ *for some $x_0 \in \mathbb{R}$, and satisfies*

$$\|m_0\|_{\mathcal{M}} \leq B \quad \text{and} \quad \|u_0 - \sum_{j=1}^{N} p_j^0 \exp(\cdot - q_j^0)\|_{H^1} \leq \alpha \tag{11}$$

then

$$\forall t \in \mathbb{R}_+, \quad \inf_{P \in \mathbb{R}_-^k \times \mathbb{R}_+^{N-k}, Q \in \mathbb{R}^N} \|u(t,\cdot) - \sum_{j=1}^{N} p_j \exp(\cdot - q_j)\|_{H^1} \leq \gamma . \tag{12}$$

Moreover, there exists $T > 0$ such that

$$\forall t \geq T, \quad \inf_{Q \in \mathcal{G}} \|u(t,\cdot) - \sum_{j=1}^{N} \lambda_j \exp(\cdot - q_j)\|_{H^1} \leq \gamma \tag{13}$$

where $\mathcal{G} := \{Q \in \mathbb{R}^N, q_1 < q_2 < .. < q_N\}$ and $\lambda_1 < .. < \lambda_N$ are the eigenvalues of the matrix $\left(p_j^0 e^{-|q_i^0 - q_j^0|/2} \right)_{1 \leq i,j \leq N}$.

Remark 1.4 *Again, note that for* $(p_1^0,..,p_N^0) \in (\mathbb{R}_-^*)^k \times (\mathbb{R}_+^*)^{N-k}$ *and* $q_1^0 < .. < q_N^0$,

$$\sum_{j=1}^{N} p_j^0 \exp(\cdot - q_j^0)$$

belongs to the class $v \in H^1(\mathbb{R})$ *with* $m := v - v_{xx} \in \mathcal{M}(\mathbb{R})$, $\text{supp } m^- \subset]-\infty, x_0]$ *and* $\text{supp } m^+ \subset [x_0, +\infty[$ *for some* $x_0 \in \mathbb{R}$. *Corollary 1.1 thus ensures that the variety* $\mathcal{N}_{N,k}$ *is stable with respect to* H^1*-perturbations that keeps the initial data in this same class.*

This paper is organized as follows. In the next section we sketch the main points of the proof of Theorem 1.2 whereas the complete proof is given in Section 3. After having controlled the distance between the different bumps of the solution we establish the new monotonicity result and state local versions of estimates involved in the stability of a single peakon. Finally, the proof of Theorem 1.2 is completed in Subsection 3.4.

2 Sketch of the proof

Our proof as in [34] combined the stability of a single peakon and a monotonicity result for functionals related to the conservation laws. Recall that the stability proof of Constantin and Strauss (cf. [20]) is principally based on the following lemma of [20].

Lemma 2.1 *For any* $u \in H^1(\mathbb{R})$, $c \in \mathbb{R}$ *and* $\xi \in \mathbb{R}$,

$$E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - \xi)\|_{H^1}^2 + 4c(u(\xi) - c). \tag{14}$$

For any $u \in H^1(\mathbb{R})$, *let* $M = \max_{x\in\mathbb{R}}\{u(x)\}$, *then*

$$ME(u) - F(u) \geq \frac{2}{3}M^3. \tag{15}$$

Indeed, with this lemma at hand, let $u \in C([0,T[; H^1(\mathbb{R}))$ be a solution of (1) with $\|u(0) - \varphi_c\|_{H^1} \leqslant \varepsilon^2$ and let $\xi(t) \in \mathbb{R}$ be such that $u(t,\xi(t)) = \max_{\mathbb{R}} u(t,\cdot)$. Assuming that $u(t)$ is sufficiently H^1-close to $\{r \in \mathbb{R}, \varphi(\cdot - r)\}$, setting $\delta = c - u(t,\xi(t))$, and using that $E(u(t)) = E(u_0) = 2c^2 + O(\varepsilon^2)$ and $F(u(t)) = F(u_0) = \frac{4}{3}c^3 + O(\varepsilon^2)$, (15) leads to

$$\delta^2(c - \delta/3) \leq O(\varepsilon^2)) \Longrightarrow \delta \lesssim \varepsilon$$

and then (14) yields

$$\|u(t) - \varphi_c(\cdot - \xi(t))\|_{H^1} \lesssim \sqrt{\varepsilon}\,. \tag{16}$$

This proves the stability result. At this point, a crucial remark is that, instead of using the conservation of E and F, we can only use that, for any fixed $\lambda \geq 0$, $E(\cdot) - \lambda F(\cdot)$ is non increasing. Indeed, for $M = \max_{x \in \mathbb{R}}\{u(t,x)\} = u(t, \xi(t))$ and $\lambda = 1/M$, (15) then implies

$$ME(u_0) - F(u_0) \geq \frac{2}{3}M^3$$

and, for $\lambda = 0$, (14) implies

$$E(u_0) - E(\varphi_c) \geq \|u - \varphi_c(\cdot - \xi(t))\|_{H^1}^2 + 4c(u(\xi(t)) - c).$$

This leads to (16) exactly as above.
Now, in [28] it is established that (14) and (15) almost still hold if one replaces $E(\cdot)$ and $F(\cdot)$ by their localized version, $E_j(\cdot)$ and $F_j(\cdot)$, around the jth bump. Therefore to prove our result it will somehow suffices to prove that the functionals $E_j(\cdot) + \lambda F_j(\cdot)$ are almost decreasing.

One of the very important discovering of the works of Martel-Merle is that for one dimensional dispersive equations with a linear group that travels to the left, the part of the energy at the right of a localized solution traveling to the right is almost decreasing. In [34] it is noticed that this holds also for the part of the energy at the right of each bump for solutions that are close to the sum of solitary waves traveling to the right. In this paper we will use that, for a fixed $\lambda \geq 0$ and $j \geq k+1$, if we call by $I_j = \sum_{q=j}^{N}(E_q - \lambda F_q)$ the part of the functionals $E(\cdot) - \lambda F(\cdot)$ that is at the right of the (j-1)th bumps, then $I_j(\cdot)$ is almost decreasing in time. Since $I_N = E_N - \lambda F_N$, we infer from above that the Nth bump of the solution stays H^1-close to a translation of φ_{c_N}. Then, since $I_{N-1} = E_{N-1} - \lambda F_{N-1} + I_N$ and I_{N-1} is almost decreasing, we obtain that $E_{N-1} - \lambda F_{N-1}$ is also almost decreasing which leads to the stability result for the $(N-1)th$ bump. Iterating this process until $j = k+1$, we obtain that each bump moving to the right remains close to the orbit of the suitable peakon. Finally, since (C-H) is invariant by the change of unknown $u(t,x) \to -u(t,-x)$, this also ensures that each bump moving to the left remains close to the orbit of the suitable antipeakon. This leads to the desired result since the total energy is conserved.

Actually we will not proceed exactly that way since by using such iterative process one loses some power of ε at each step. More precisely this iterative scheme would prove Theorem 1.2 but with ε^β with $\beta = 4^{1/2-\max(q,N-q)}$

instead of $\varepsilon^{1/2}$ in (8). To derive the desired power of ε we will rather sum all the contributions of bumps that are traveling in the same direction and use Abel's summation argument to get the stability of all these bumps in the same time.

3 Stability of multipeakons

For $\alpha > 0$ and $L > 0$ we define the following neighborhood of all the sums of N antipeakons and peakons of speed $c_1, .., c_N$ with spatial shifts x_j that satisfied $x_j - x_{j-1} \geq L$.

$$U(\alpha, L) = \left\{ u \in H^1(I\!R), \inf_{x_j - x_{j-1} > L} \|u - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - x_j)\|_{H^1} < \alpha \right\}. \quad (17)$$

By the continuity of the map $t \mapsto u(t)$ from $[0, T[$ into $H^1(I\!R)$, to prove the first part of Theorem 1.2 it suffices to prove that there exist $A > 0$, $\varepsilon_0 > 0$ and $L_0 > 0$ such that $\forall L > L_0$ and $0 < \varepsilon < \varepsilon_0$, if u_0 satisfies (7) and if for some $0 < t_0 < T$,

$$u(t) \in U\left(A(\sqrt{\varepsilon} + L^{-1/8}), L/2\right) \text{ for all } t \in [0, t_0] \quad (18)$$

then

$$u(t_0) \in U\left(\frac{A}{2}(\sqrt{\varepsilon} + L^{-1/8}), \frac{2L}{3}\right). \quad (19)$$

Therefore, in the sequel of this section we will assume (18) for some $0 < \varepsilon < \varepsilon_0$ and $L > L_0$, with A, ε_0 and L_0 to be specified later, and we will prove (19).

3.1 Control of the distance between the peakons

In this subsection we want to prove that the different bumps of u that are individualy close to a peakon or an antipeakon get away from each others as time is increasing. This is crucial in our analysis since we do not know how to manage strong interactions. The following lemma is principally proven in [28].

Lemma 3.1 *Let u_0 satisfying (7). There exist $\alpha_0 > 0$, $L_0 > 0$ and $C_0 > 0$ such that for all $0 < \alpha < \alpha_0$ and $0 < L_0 < L$ if $u \in U(\alpha, L/2)$ on $[0, t_0]$ for*

some $0 < t_0 < T$ then there exist C^1-functions $\tilde{x}_1, .., \tilde{x}_N$ defined on $[0, t_0]$ such that $\forall t \in [0, t_0]$,

$$\frac{d}{dt}\tilde{x}_i(t) = c_i + O(\sqrt{\alpha}) + O(L^{-1}),\ i = 1, .., N\,, \tag{20}$$

$$\|u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{H^1} = O(\sqrt{\alpha})\,, \tag{21}$$

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq 3L/4 + (c_i - c_{i-1})t/2, \quad i = 2, .., N. \tag{22}$$

Moreover, for $i = 1, .., N$, it holds

$$|x_i(t) - \tilde{x}_i(t)| = O(1), \tag{23}$$

where $x_i(t) \in [\tilde{x}_i(t) - L/4, \tilde{x}_i(t) + L/4]$ is any point such that

$$|u(t, x_i(t))| = \max_{[\tilde{x}_i(t) - L/4, \tilde{x}_i(t) + L/4]} |u(t)|. \tag{24}$$

Proof. We only sketch the proof and refer to [28] for details. The strategy is to use a modulation argument to construct N C^1-functions $t \mapsto \tilde{x}_i(t)$, $i = 1, .., N$ on $[0, t_0]$ satisfying the following orthogonality conditions :

$$\int_{I\!R} \Big(u(t, \cdot) - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - \tilde{x}_j(t))\Big) \partial_x \varphi_{c_i}(\cdot - \tilde{x}_i(t))\, dx = 0\,. \tag{25}$$

Moreover, setting

$$R_Z(\cdot) = \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i) \tag{26}$$

for any $Z = (z_1, .., z_N) \in I\!R^N$, one can check that

$$\|u(t) - R_{\tilde{X}(t)}\|_{H^1} \lesssim C_0\sqrt{\alpha}\,, \quad \forall t \in [0, t_0]\,. \tag{27}$$

To prove that the speed of $\tilde{x}_i$ stays close to c_i, we set

$$R_j(t) = \varphi_{c_j}(\cdot - \tilde{x}_j(t)) \text{ and } v(t) = u(t) - \sum_{i=1}^{N} R_j(t) = u(t, \cdot) - R_{\tilde{X}(t)}\,.$$

and differentiate (25) with respect to time to get

$$\int_{I\!R} v_t \partial_x R_i = \dot{\tilde{x}}_i \,\langle \partial_x^2 R_i\,,\, v\rangle_{H^{-1}, H^1}\,,$$

and thus

$$\Big|\int_{I\!R} v_t \partial_x R_i\Big| \leq |\dot{\tilde{x}}_i| O(\|v\|_{H^1}) \leq |\dot{\tilde{x}}_i - c_i| O(\|v\|_{H^1}) + O(\|v\|_{H^1}) \,. \qquad (28)$$

Substituting u by $v + \sum_{j=1}^N R_j$ in (4) and using that R_j satisfies

$$\partial_t R_j + (\dot{\tilde{x}}_j - c_j)\partial_x R_j + R_j \partial_x R_j + (1-\partial_x^2)^{-1}\partial_x[R_j^2 + (\partial_x R_j)^2/2] = 0 \,,$$

we infer that v satisfies on $[0, t_0]$,

$$\begin{aligned} v_t - \sum_{j=1}^N (\dot{\tilde{x}}_j - c_j)\partial_x R_j = &-\frac{1}{2}\partial_x\Big[(v + \sum_{j=1}^N R_j)^2 - \sum_{j=1}^N R_j^2\Big] \\ &-(1-\partial_x^2)^{-1}\partial_x\Big[(v + \sum_{j=1}^N R_j)^2 - \sum_{j=1}^N R_j^2 + \frac{1}{2}(v_x + \sum_{j=1}^N \partial_x R_j)^2 - \frac{1}{2}\sum_{j=1}^N (\partial_x R_j)^2\Big] \,. \end{aligned}$$

Taking the L^2-scalar product with $\partial_x R_i$, integrating by parts, using the decay of R_j and its first derivative, (27) and (28), we find

$$|\dot{\tilde{x}}_i - c_i|\Big(\|\partial_x R_i\|_{L^2}^2 + O(\sqrt{\alpha})\Big) \leq O(\sqrt{\alpha}) + O(e^{-L/8}) \,. \qquad (29)$$

Taking α_0 small enough and L_0 large enough we get $|\dot{\tilde{x}}_i - c_i| \leq (c_i - c_{i-1})/4$ and thus, for all $0 < \alpha < \alpha_0$ and $L \geq L_0 > 3C_0\varepsilon$, it follows from (7), (27) and (29) that

$$\tilde{x}_j(t) - \tilde{x}_{j-1}(t) > L - C_0\varepsilon + (c_j - c_{j-1})t/2, \quad \forall t \in [0, t_0] \,. \qquad (30)$$

which yields (22).
Finally from (27) and the continuous embedding of $H^1(I\!R)$ into $L^\infty(I\!R)$, we infer that

$$u(t,x) = R_{\tilde{X}(t)}(x) + O(\sqrt{\alpha}), \quad \forall x \in I\!R \,.$$

Applying this formula with $x = \tilde{x}_i$ and taking advantage of (22), we obtain

$$|u(t, \tilde{x}_i)| = |c_i| + O(\sqrt{\alpha}) + O(e^{-L/4}) \geq 3|c_i|/4 \,.$$

On the other hand, for $x \in [\tilde{x}_i(t) - L/4, \tilde{x}_i(t) + L/4]\backslash]\tilde{x}_i(t) - 2, \tilde{x}_i(t) + 2[$, we get

$$|u(t,x)| \leq |c_i|e^{-2} + O(\sqrt{\alpha}) + O(e^{-L/4}) \leq |c_i|/2 \,.$$

This ensures that x_i belongs to $[\tilde{x}_i - 2, \tilde{x}_i + 2]$.

3.2 Monotonicity property

Thanks to the preceding lemma, for $\varepsilon_0 > 0$ small enough and $L_0 > 0$ large enough, one can construct C^1-functions $\tilde{x}_1, .., \tilde{x}_N$ defined on $[0, t_0]$ such that (20)-(23) are satisfied. In this subsection we state the almost monotonicity of functionals that are very close to the $E(\cdot) - \lambda F(\cdot)$ at the right of the ith bump, $i = k, .., N-1$ of u. The proof follows the same lines as in Lemma 4.2 in [27] but is more delicate since we have also to deal with the functional F. Moreover, F generates a term (J_4 in (41)) that we are not able to estimate in a suitable way but which fortunately is of the good sign.

Let Ψ be a C^∞ function such that $0 < \Psi \leq 1$, $\Psi' > 0$ on $I\!R$, $|\Psi'''| \leq 10|\Psi'|$ on $[-1, 1]$,

$$\Psi(x) = \begin{cases} e^{-|x|} & x < -1 \\ 1 - e^{-|x|} & x > 1 \end{cases} . \tag{31}$$

Setting $\Psi_K = \Psi(\cdot/K)$, we introduce for $j \in \{q, .., N\}$ and $\lambda \geq 0$,

$$I_{j,\lambda}(t) = I_{j,\lambda,K}(t, u(t)) = \int_{I\!R} \Big((u^2(t) + u_x^2(t)) - \lambda(u^3(t) + u u_x^2(t)) \Big) \Psi_{j,K}(t)\, dx\,,$$

where $\Psi_{j,K}(t, x) = \Psi_K(x - y_j(t))$ with $y_j(t)$, $j = k+1, .., N$, defined by

$$y_{k+1}(t) = \tilde{x}_{k+1}(0) + c_{k+1}t/2 - L/4$$

$$\text{and } y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \quad i = k+2, .., N. \tag{32}$$

Finally, we set

$$\sigma_0 = \frac{1}{4} \min\Big(c_{k+1}, c_{k+2} - c_{k+1}, .., c_N - c_{N-1} \Big)\,. \tag{33}$$

Proposition 3.1 *Let $u \in Y([0, T[)$ be a solution of (C-H) satisfying (21) on $[0, t_0]$. There exist $\alpha_0 > 0$ and $L_0 > 0$ only depending on c_{k+1} and c_N such that if $0 < \alpha < \alpha_0$ and $L \geq L_0$ then for any $4 \leq K \lesssim L^{1/2}$ and $0 \leq \lambda \leq 2/c_{k+1}$,*

$$I_{j,\lambda,K}(t) - I_{j,\lambda,K}(0) \leq O(e^{-\frac{\sigma_0 L}{8K}}), \quad \forall j \in \{k+1, .., N\}, \quad \forall t \in [0, t_0]\,. \tag{34}$$

Proof. Let us assume that u is smooth since the case $u \in Y([0, T[)$ follows by modifying slightly the arguments (see Remark 3.2 of [26]).

Lemma 3.2

$$\frac{d}{dt}\int_{I\!R}(u^2+u_x^2)g\,dx = \int_{I\!R}(u^3+4uu_x^2)g'\,dx - \int_{I\!R}u^3g'''\,dx - 2\int_{I\!R}uhg'\,dx. \tag{35}$$

and

$$\frac{d}{dt}\int_{I\!R}(u^3+uu_x^2)g\,dx = \int_{I\!R}(u^4/4+u^2u_x^2)g'\,dx + \int_{I\!R}u^2hg'\,dx + \int_{I\!R}(h^2-h_x^2)g'\,dx. \tag{36}$$

where $h := (1-\partial_x^2)^{-1}(u^2+u_x^2/2)$.

Proof. Since (35) is proven in [28] we concentrate on the proof of (36).

$$\begin{aligned}\frac{d}{dt}\int_{I\!R}(u^3+uu_x^2)g &= 3\int_{I\!R}u_tu^2g + 2\int_{I\!R}u_{tx}u_xug + \int_{I\!R}u_tu_x^2g\\ &= 2\int_{I\!R}u_t(u^2+u_x^2/2)g + \int_{I\!R}u_tu^2g - \int_{I\!R}u_{txx}u^2g - \int_{I\!R}u_{tx}u^2g'\\ &= 2\int_{I\!R}u_t(u^2+u_x^2/2)g + \int_{I\!R}(u_t-u_{txx})u^2g - \int_{I\!R}u_{tx}u^2g'\\ &= I_1+I_2+I_3\,.\end{aligned} \tag{37}$$

Setting $h := (1-\partial_x^2)^{-1}(u^2+u_x^2/2)$ and using the equation we get

$$\begin{aligned}I_1 &= -2\int_{I\!R}uu_x(u^2+u_x^2/2)g - 2\int_{I\!R}gh_x(1-\partial_x^2)h\\ &= -2\int_{I\!R}u^3u_xg - \int_{I\!R}uu_x^3g - 2\int_{I\!R}hh_xg + 2\int_{I\!R}h_xh_{xx}g\\ &= \frac{1}{2}\int_{I\!R}u^4g' - \int_{I\!R}uu_x^3g + \int_{I\!R}(h^2-h_x^2)g'\,.\end{aligned} \tag{38}$$

In the same way,

$$\begin{aligned}I_2 &= -3\int_{I\!R}u^3u_xg - \frac{1}{2}\int_{I\!R}\partial_x(u_x^2)u^2g + \frac{1}{2}\int_{I\!R}\partial_x^3(u^2)u^2g\\ &= \frac{3}{4}\int_{I\!R}u^4g' - \frac{1}{2}\int_{I\!R}\partial_x(u_x^2)u^2g - \frac{1}{2}\int_{I\!R}\partial_x^2(u^2)\partial_x(u^2)g - \frac{1}{2}\int_{I\!R}\partial_x^2(u^2)u^2g'\\ &= \frac{3}{4}\int_{I\!R}u^4g' + \int_{I\!R}uu_x^3g + \frac{1}{2}\int_{I\!R}u_x^2u^2g' + \frac{1}{4}\int_{I\!R}[\partial_x(u^2)]^2g' + \int_{I\!R}\partial_x(u^2)uu_xg'\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{I\!R}\partial_x(u^2)u^2g'' \\
&= \frac{3}{4}\int_{I\!R}u^4g' + \int_{I\!R}uu_x^3g + \frac{1}{2}\int_{I\!R}u_x^2u^2g' + \int_{I\!R}u^2u_x^2g' + 2\int_{I\!R}u_x^2u^2g' + \int_{I\!R}u^3u_xg'' \\
&= \frac{3}{4}\int_{I\!R}u^4g' - \frac{1}{4}\int_{I\!R}u^4g''' + \frac{7}{2}\int_{I\!R}u_x^2u^2g' + \int_{I\!R}uu_x^3g\,. \qquad (39)
\end{aligned}
$$

At this stage it is worth noticing that the terms $\int_{I\!R}uu_x^3g$ cancels with the one in I_1. Finally,

$$
\begin{aligned}
I_3 &= \int_{I\!R}\partial_x(uu_x)u^2g' + \int_{I\!R}g'u^2\partial_x^2h \\
&= -2\int_{I\!R}u^2u_x^2g' - \int_{I\!R}u^3u_xg'' - \int_{I\!R}u^2(u^2+u_x^2/2)g' + \int_{I\!R}u^2hg' \\
&= -2\int_{I\!R}u^2u_x^2g' + \frac{1}{4}\int_{I\!R}u^4g''' - \int_{I\!R}u^4g' - \frac{1}{2}\int_{I\!R}u^2u_x^2g' + \int_{I\!R}u^2hg' \\
&= -\frac{5}{2}\int_{I\!R}u^2u_x^2g' + \frac{1}{4}\int_{I\!R}u^4g''' - \int_{I\!R}u^4g' + \int_{I\!R}u^2hg' \qquad (40)
\end{aligned}
$$

where we used that $\partial_x^2(I-\partial_x^2)^{-1} = -I + (I-\partial_x^2)^{-1}$. Gathering (37)-(40), (36) follows. □

Applying (35)-(36) with $g = \Psi_{j,K}$, $j \geq k+1$, one gets

$$
\begin{aligned}
\frac{d}{dt}I_{j,\lambda,K} &:= \frac{d}{dt}\int_{I\!R}\Psi_{j,K}[(u^2+u_x^2) - \lambda(u^3+uu_x^2)]\,dx \\
&= -\dot{y_j}\int_{I\!R}\Psi'_{j,K}(u^2+u_x^2) \\
&\quad + \int_{I\!R}\Psi'_{j,K}\Big[[(u^3+4uu_x^2) - \lambda\Big(\dot{y_j}(u^3+uu_x^2) - (u^4/4+u^2u_x^2)\Big)\Big]\,dx \\
&\quad - \int_{I\!R}\Psi'''_{j,K}u^3\,dx - \int_{I\!R}\Psi'_{j,K}(2u+\lambda u^2)h\,dx \\
&\quad -\lambda\int_{I\!R}\Psi'_{j,K}(h^2-h_x^2)\,dx \\
&= -\dot{y_j}\int_{I\!R}\Psi'_{j,K}(u^2+u_x^2) + J_1 + J_2 + J_3 + J_4 \\
&\leq -\frac{c_{k+1}}{2}\int_{I\!R}\Psi'_{j,K}(u^2+u_x^2) + J_1 + J_2 + J_3 + J_4\,. \qquad (41)
\end{aligned}
$$

We claim that $J_4 \leq 0$ and that for $i \in \{1,2,3\}$, it holds

$$
J_i \leq \frac{c_{k+1}}{8}\int_{I\!R}\Psi'_{j,K}(u^2+u_x^2) + \frac{C}{K}\,e^{-\frac{1}{K}(\sigma_0 t + L/8)}\,. \qquad (42)
$$

To handle with J_1 we divide $\mathbb{R}$ into two regions D_j and D_j^c with

$$D_j = [\tilde{x}_{j-1}(t) + L/4, \tilde{x}_j(t) - L/4]$$

First since from (22), for $x \in D_j^c$,

$$|x - y_j(t)| \geq \frac{\tilde{x}_j(t) - \tilde{x}_{j-1}(t)}{2} - L/4 \geq \frac{c_j - c_{j-1}}{2}\, t + L/8\,,$$

we infer from the definition of Ψ in Section 3.2 that

$$\Big| \int_{D_j^c} \Psi'_{j,K} \Big[[(u^3 + 4uu_x^2) - \lambda \Big(y_j(u^3 + uu_x^2) - (u^4/4 + u^2 u_x^2) \Big) \Big]\, dx \Big|$$

$$\leq \frac{C}{K}\,(1 + 2\lambda c_N)(\|u_0\|_{H^1}^3 + \|u_0\|_{H^1}^4) e^{-\frac{1}{K}(\sigma_0 t + L/8)}\ .$$

On the other hand, on D_j we notice, according to (21), that

$$\begin{aligned} \|u(t)\|_{L^\infty_{D_j}} &\leq \sum_{i=1}^N \|\varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty}(D_j) + \|u - \sum_{i=1}^N \varphi_{c_i}(\cdot - \tilde{x}_i(t))\|_{L^\infty(D_j)} \\ &\leq C\, e^{-L/8} + O(\sqrt{\alpha})\ . \end{aligned} \tag{43}$$

Therefore, for α small enough and L large enough it holds

$$J_1 \leq \frac{c_{k+1}}{8} \int_{\mathbb{R}} \Psi'_{j,K}(u^2 + u_x^2) + \frac{C}{K}\, e^{-\frac{1}{K}(\sigma_0 t + L/8)}\ .$$

Since J_2 can be handled in exactly the same way, it remains to treat J_3. For this, we first notice as above that

$$\begin{aligned} -\int_{D_j^c} &(2u + \lambda u^2)\Psi'_{j,K}(1 - \partial_x^2)^{-1}(u^2 + u_x^2/2) \\ &\leq (2 + \lambda\|u\|_\infty)\|u\|_\infty \sup_{x \in D_j^c} |\Psi'_{j,K}(x - y_j(t))| \int_{\mathbb{R}} e^{-|x|} * (u^2 + u_x^2/2)\, dx \\ &\leq \frac{C}{K}\|u_0\|_{H^1}^3\, e^{-\frac{1}{K}(\sigma_0 t + L/8)}\ , \end{aligned} \tag{44}$$

since

$$\forall f \in L^1(\mathbb{R}), \quad (1 - \partial_x^2)^{-1} f = \frac{1}{2} e^{-|x|} * f\ . \tag{45}$$

Now in the region D_j, noticing that $\Psi'_{j,K}$ and $u^2+u_x^2/2$ are non-negative, we get

$$\begin{aligned}
&-\int_{D_j}(2+\lambda u)u\Psi'_{j,K}(1-\partial_x^2)^{-1}(u^2+u_x^2/2)\\
&\quad\leq\ (2+\lambda\|u(t)\|_{L^\infty(D_j)})\|u(t)\|_{L^\infty(D_j)}\int_{D_j}\Psi'_{j,K}((1-\partial_x^2)^{-1}(2u^2+u_x^2)\\
&\quad\leq\ (2+\lambda\|u(t)\|_{L^\infty(D_j)})\|u(t)\|_{L^\infty(D_j)}\int_{I\!R}(2u^2+u_x^2)(1-\partial_x^2)^{-1}\Psi'_{j,K} \quad (46)
\end{aligned}$$

On the other hand, from the definition of Ψ in Section 3.2 and (45) we infer that for $K\geq 4$,

$$(1-\partial_x^2)\Psi'_{j,K}\geq(1-\frac{10}{K^2})\Psi'_{j,K}\Rightarrow(1-\partial_x^2)^{-1}\Psi'_{j,K}\leq(1-\frac{10}{K^2})^{-1}\Psi'_{j,K}\ .$$

Therefore, taking $K\geq 4$ and using (43) we deduce for α small enough and L large enough that

$$-\int_{D_j}(2u+\lambda u^2)\Psi'_K(1-\partial_x^2)^{-1}(u^2+u_x^2/2)\leq\frac{c_q}{8}\int_{I\!R}(u^2+u_x^2/2)\Psi'_K\ . \quad (47)$$

This completes the proof of (42). It remains to prove that J_4 is non positive. Recall that $h=(I-\partial_x^2)^{-1}v$ with $v:=u^2+u_x^2/2\geq 0$. Therefore, following [13], it holds

$$\begin{aligned}
h(x) &= \frac{1}{2}e^{-|\cdot|}*v(\cdot)\\
&= \frac{1}{2}e^{-x}\int_{-\infty}^{x}e^y v(y)\,dy+\frac{1}{2}e^x\int_{-\infty}^{x}e^{-y}v(y)\,dy
\end{aligned}$$

and

$$h'(x)=-\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^y v(y)\,dy+\frac{1}{2}e^x\int_{-\infty}^{x}e^{-y}v(y)\,dy$$

which clearly ensures that $h^2\geq h_x^2$. Since $\Psi'_{j,K}\geq 0$ and $\lambda\geq 0$, this leads to the non positivity of $J_4=-\lambda\int_{I\!R}\Psi'_{j,K}(h^2-h_x^2)\,dx$.

Gathering (41) and (42) we infer that

$$\frac{d}{dt}\int_{I\!R}\Psi_{j,K}[u^2+u_x^2-\lambda(u^3+uu_x^2)]\,dx\leq-\frac{c_1}{8}\int_{I\!R}\Psi'_{j,K}(u^2+u_x^2)+\frac{C}{K}(1+\|u_0\|_{H^1}^4)\,e^{-\frac{1}{K}(\sigma_0 t+L/8)}\ .$$

Integrating this inequality between 0 and t, (34) follows.

3.3 Localized estimates

We define the function $\Phi_i = \Phi_i(t,x)$, $i = k+1,..,N$, by $\Phi_N = \Psi_{N,K} = \Psi_K(\cdot - y_N(t))$ and for $i = k+1,..,N-1$

$$\Phi_i = \Psi_{i,K} - \Psi_{i+1,K} = \Psi_K(\cdot - y_i(t)) - \Psi_K(\cdot - y_{i+1}(t)) \ , \tag{48}$$

where $\Psi_{i,K}$ and the y_i's are defined in Section 3.2. It is easy to check that the Φ_i's are positive functions and that $\sum_{i=k+1}^{N} \Phi_i \equiv \Psi_{k+1,K}$. We will take $L/K > 4$ so that (31) ensures that Φ_i satisfies for $i \in \{k+1,..,N\}$,

$$|1 - \Phi_i| \leq 2e^{-\frac{L}{8K}} \text{ on }]y_i + L/8, y_{i+1} - L/8[\tag{49}$$

and

$$|\Phi_i| \leq 2e^{-\frac{L}{8K}} \text{ on }]y_i - L/8, y_{i+1} + L/8[^c \ , \tag{50}$$

where we set $y_{N+1} := +\infty$.
It is worth noticing that, somehow, $\Phi_i(t)$ takes care of only the ith bump of $u(t)$. We will use the following localized version of E and F defined for $i \in \{k+1,..,N\}$, by

$$E_i^t(u) = \int_{I\!R} \Phi_i(t)(u^2 + u_x^2) \text{ and } F_i^t(u) = \int_{I\!R} \Phi_i(t)(u^3 + uu_x^2) \ . \tag{51}$$

Please note that henceforth we take $K = L^{1/2}/8$.
The following lemma gives a localized version of (15). Note that the functionals E_i and F_i do not depend on time in the statement below since we fix $y_{k+1} < .. < y_{N+1} = +\infty$.

Lemma 3.3 *Let be given $u \in H^1(I\!R)$ with $\|u\|_{H^1} = \|u_0\|_{H^1}$ and $N-k$ real numbers $y_{k+1} < .. < y_N$ with $y_i - y_{i-1} \geq 2L/3$. For $i = k+1,..,N$, set $J_i :=]y_i - L/4, y_{i+1} + L/4[$ with $y_{N+1} = +\infty$, and assume that there exist $x_i \in]y_i + L/4, y_{i+1} - L/4[$ such that $u(x_i) = \max_{J_i} u := M_i > 0$. Then, defining the functional E_i's and F_i's as in (48)-(51), it holds*

$$F_i(u) \leqslant M_i E_i(u) - \frac{2}{3} M_i^3 + \|u_0\|_{H^1}^3 O(L^{-1/2}), \quad i \in \{k+1,..,N\} \, . \tag{52}$$

and for any $x_1 < .. < x_k$ with $x_k < y_{k+1} - L/4$, setting $X := (x_1,..,x_N) \in I\!R^N$, it holds

$$E_i(u) - E(\varphi_{c_i}) = E_i(u - R_X) + 4c_i(M_i - c_i) + \|u_0\|_{H^1}^2 O(L^{-1/2}), \quad i \in \{k+1,..,N\}, \tag{53}$$

where R_X is defined in (26).

Proof. Let $i \in \{k+1,..,N\}$ be fixed. Following [20], we introduce the function g defined by

$$g(x) = \begin{cases} u(x) - u_x(x) & \text{for } x < x_i \\ u(x) + u_x(x) & \text{for } x > x_i \end{cases} .$$

Integrating by parts we compute

$$\begin{aligned} \int u g^2 \Phi_i &= \int_{-\infty}^{x_i} (u^3 + u u_x^2 - 2u^2 u_x)\Phi_i + \int_{x_i}^{+\infty} (u^3 + u u_x^2 + 2u^2 u_x)\Phi_i \\ &= F_i(u) - \frac{4}{3} u(x_i)^3 \Phi_i(x_i) + \frac{2}{3}\int_{-\infty}^{x_i} u^3 \Phi_i' - \frac{2}{3}\int_{x_i}^{+\infty} u^3 \Phi_i' . \quad (54) \end{aligned}$$

Recall that we take $K = \sqrt{L}/8$ and thus $|\Phi'| \leq C/K = O(L^{-1/2})$. Moreover, since $x_i \in]y_i + L/4, y_{i+1} - L/4[$, it follows from (49) that $\Phi_i(x_i) = 1 + O(e^{-L^{1/2}})$ and thus

$$\int u g^2 \Phi_i = F_i(u) - \frac{4}{3} M_i^3 + \|u\|_{H^1}^3 O(L^{-1/2}) . \quad (55)$$

On the other hand, with (50) at hand,

$$\begin{aligned} \int u g^2 \Phi_i &\leq M_i \int_{J_i} g^2 \Phi_i + \int_{J_i^c} |u| g^2 \Phi_i \\ &\leq M_i \int_{-\infty}^{+\infty} g^2 \Phi_i + \|u\|_{L^\infty(\mathbb{R})} \int_{J_i^c} g^2 \Phi_i \\ &\leq M_i \Big(E_i(u) - 2\int_{-\infty}^{x_i} u u_x \Phi_i + 2\int_{x_i}^{+\infty} u u_x \Phi_i \Big) + \|u\|_{H^1}^3 \sup_{x \in J_i^c} |\Phi_i(x)| \\ &\leq M_i E_i(u) - 2M_i^3 + \|u\|_{H^1}^3 O(L^{-1/2}) . \quad (56) \end{aligned}$$

This proves (52). To prove (53), we use the relation between φ an its derivative and integrate by parts, to get

$$\begin{aligned} E_i(u - R_X) &= E_i(u) + E_i(R_X) - 2\int \Phi_i \Big(u\, \varphi_{c_i}(\cdot - x_i) + u_x\, \partial_x \varphi_{c_i}(\cdot - x_i) \Big) \\ &= E_i(u) + E_i(R_X) - 2\int \Phi_i u\, \varphi_{c_i}(\cdot - x_i) \\ &\quad + 2\int_{x_i}^{+\infty} \Phi_i u_x\, \varphi_{c_i}(\cdot - x_i) - 2\int_{-\infty}^{x_i} \Phi_i u_x\, \varphi_{c_i}(\cdot - x_i) \\ &= E_i(u) + E_i(R_X) - 2\int \Phi_i u\, \varphi_{c_i}(\cdot - x_i) + 2\int \Phi_i' u\, \varphi_{c_i}(\cdot - x_i) \end{aligned}$$

$$
\begin{aligned}
&+2\int_{z_i}^{+\infty}\Phi_i u_x\,\varphi_{c_i}(\cdot-x_i)-2\int_{-\infty}^{z_i}\Phi_i u_x\,\varphi_{c_i}(\cdot-x_i)\\
&=E_i(u)+E_i(R_X)-4c_iu(x_i)\Phi_i(x_i)+2\int\Phi_i' u\,\varphi_{c_i}(\cdot-x_i)\\
&\quad-2\int_{x_i}^{+\infty}\Phi_i' u\,\varphi_{c_i}(\cdot-x_i)+2\int_{-\infty}^{x_i}\Phi_i' u\,\varphi_{c_i}(\cdot-x_i)\,.
\end{aligned}
$$

From (49)-(50), it is easy to check that $E_i(R_X)=E(\varphi_{c_i})+O(e^{-\sqrt{L}/8})$. Since $C/K=O(L^{-1/2})$ and, in view of (49), $\Phi_i(x_i)=1+O(e^{-L^{1/2}})$, it follows that

$$
E_i(u)+E_i(\varphi_{c_i})=E_i(u-R_X)+4c_iM_i+\|u\|_{H^1}^2O(L^{-1/2})\,.
$$

This yields the result by using that $E(\varphi_{c_i})=2c_i^2$.

3.4 End of the proof of Theorem 1.2

Proposition 3.2 *There exists constants $C,C'>0$ independent of A such that*

$$
I_{k+1,0}\Big(t_0,u(t_0)-R_{X(t_0)}\Big)=\sum_{i=k+1}^{N}E_i^{t_0}\Big(u(t_0)-R_{X(t_0)}\Big)\le C(\varepsilon+L^{-1/4}) \quad (57)
$$

and

$$
I_{k+1,0}(t_0)=\sum_{i=k+1}^{N}E_i^{t_0}(u(t_0))=\sum_{i=k+1}^{N}E(\varphi_{c_i})+O(\varepsilon+L^{-1/4})\,. \qquad (58)
$$

with $|O(x)|\le C'x$, $\forall x\in\mathbb{R}_+^*$.

Proof. First it is worth noticing that according to Lemma 3.1, $u(t_0)$, $(y_{k+1}(t_0),..,y_{N+1})$, constructed in (32), and $X(t_0)=(x_1(t_0),..,x_N(t_0))$, constructed in (24), satisfy the hypotheses of Lemma 3.3. Indeed, by construction for $i\in\{k+1,..,N\}$, $x_i\in[\tilde{x}_i(t_0)-L/4,\tilde{x}_i(t_0)+L/4,]\subset]y_i(t_0)+L/4,y_{i+1}(t_0)-L/4[$ and it is easy to check that $|u(t_0)|\le O(e^{-\sqrt{L}})+O(\alpha)<3c_i/4\le|u(x_i)|$ on $]y_i(t_0)-L/4,y_{i+1}(t_0)+L/4[\backslash[\tilde{x}_i(t_0)-L/4,\tilde{x}_i(t_0)+L/4]$ so that

$$
0<u(t_0,x_i(t_0))=\max_{]y_i(t_0)-L/4,y_{i+1}(t_0)+L/4[}u(t_0)\,.
$$

Therefore, setting $M_i = u(t_0, x_i(t_0))$, $\delta_i = c_i - M_i$ and taking the sum over $i = k+1, .., N$ of (52) one gets :

$$\sum_{i=k+1}^{N} \Big(M_i E_i^{t_0}(u(t_0)) - F_i^{t_0}(u(t_0)) \Big) \geq -\frac{2}{3} \sum_{i=k+1}^{N} M_i^3 + O(L^{-1/2})$$

Note that by (21) and the continuous embedding of $H^1(I\!R)$ into $L^\infty(I\!R)$, $M_i = c_i + O(\sqrt{\alpha}) + O(e^{-L/8})$, and thus

$$0 < M_{k+1} < \cdot\cdot < M_N \text{ and } \delta_i < c_i/2, \, \forall i \in \{k+1, .., N\} \, . \tag{59}$$

We set $\Delta_0^{t_0} F_i(u) = F_i^{t_0}(u(t_0)) - F^0(u(0))$, $\Delta_0^{t_0} E(u) = E^{t_0}(u(t_0)) - E^0(u(0))$, $\Delta_0^{t_0} I_{i,\lambda}(u) = I_{i,\lambda}(t_0, u(t_0)) - I_{i,\lambda}(0, u(0))$. Using the Abel transformation and the monotonicity estimate (34) (note that $0 \leq 1/M_i \leq 2/c_{k+1}$ for $i \in \{k+1, .., N\}$), we get

$$\sum_{i=k+1}^{N} M_i \Big(\Delta_0^{t_0} E(u) - \frac{1}{M_i} \Delta_0^{t_0} F(u) \Big) = \sum_{i=k+1}^{N} (M_i - M_{i-1}) \Delta_0^t I_{i,1/M_i} \leqslant O(e^{-\sigma_0 \sqrt{L}})$$

and thus

$$\sum_{i=k+1}^{N} \Big(M_i E_i^0(u_0) - F_i^0(u_0)) \Big) \geq -\frac{2}{3} \sum_{i=k+1}^{N} M_i^3 + O(L^{-1/2}) \, . \tag{60}$$

By (7), the exponential decay of the φ_{c_i}'s and the Φ_i's, and the definition of E_i and F_i, it is easy to check that

$$|E_i^0(u_0) - E(\varphi_{c_i})| + |F_i^0(u_0) - F(\varphi_{c_i})| \leq O(\varepsilon^2) + O(e^{-\sqrt{L}}), \; \forall i \in \{1, .., N\} \, . \tag{61}$$

Injecting this in (60), taking advantage of (59) and using that $E(\varphi_{c_i}) = 2c_i^2$ and $F(\varphi_{c_i}) = 4c_i^3/3$, we obtain

$$\begin{aligned} \sum_{i=k+1}^{N} (c_i \delta_i^2 - \frac{1}{3} \delta_i^3) \;\; &= \;\; \sum_{i=k+1}^{N} \delta_i^2 (c_i - \frac{1}{3} \delta_i) \leqslant O(\varepsilon^2 + L^{-1/2}) \\ &\Longrightarrow \sum_{i=k+1}^{N} \delta_i^2 = O(\varepsilon^2 + L^{-1/2}). \end{aligned} \tag{62}$$

On the other hand, summing (53) for $i = k+1, .., N$ one gets

$$I_{k+1,0}(t_0) - \sum_{i=k+1}^{N} E(\varphi_{c_i}) = \sum_{i=k+1}^{N} E_i^{t_0} \Big(u(t_0) - R_{X(t_0)} \Big) + 4 \sum_{i=k+1}^{N} c_i \delta_i + O(L^{-1/2}) \, . \tag{63}$$

Using (59) and the almost monotonicity of $t \mapsto I_{k+1,0}(t)$, we infer that

$$\sum_{i=k+1}^{N} E_i^{t_0}\Big(u(t_0) - R_{X(t_0)}\Big) \leq I_{k+1,0}(0) - \sum_{i=k+1}^{N} E(\varphi_{c_i}) + O(\varepsilon + L^{-1/4})$$

and (61)-(62) then yield (57). Finally, with (57) at hand, (58) follows directly from (62)-(63). □

Now, it is crucial to note that (C-H) is invariant by the change of unknown $u(t,x) \mapsto -u(t,-x)$. Therefore setting, for any $v \in H^1(\mathbb{R})$,

$$\tilde{I}_{k,0}(t,v) := \int_{\mathbb{R}} \Psi(y_k(t) - x)[v^2(x) + v_x^2(x)]\, dx\,,$$

with

$$y_k(t) = \tilde{x}_k(0) + c_k t/2 + L/4\,,$$

we infer from Proposition 3.2 that

$$\tilde{I}_{k,0}\Big(t_0, u(t_0) - R_{X(t_0)}\Big) \leq C(\varepsilon + L^{-1/4}) \tag{64}$$

and

$$\tilde{I}_{k,0}(t_0, u(t_0)) = \sum_{i=1}^{k} E(\varphi_{c_i}) + O(\varepsilon + L^{-1/4})\,. \tag{65}$$

Hence,

$$\begin{aligned}\tilde{I}_{k,0}(t_0, u(t_0)) + I_{k+1,0}(t_0, u(t_0)) &= \sum_{i=1}^{N} E(\varphi_{c_i}) + O(\varepsilon + L^{-1/4}) \\ &= E(u_0) + O(\varepsilon + L^{-1/4})\,.\end{aligned}$$

Since $E(u(t_0)) = E(u_0)$ we deduce that

$$\int_{\mathbb{R}} \Big[1 - \Psi(y_k(t_0) - x) - \Psi(x - y_{k+1}(t_0))\Big][u^2(t_0,x) + u_x^2(t_0,x)]\, dx = O(\varepsilon + L^{-1/4})\,.$$

Therefore, since $|1 - \Psi(y_k(t_0) - x) - \Psi(x - y_{k+1}(t_0))| \leq O(e^{-\sqrt{L}})$ for $x \in \mathbb{R}\backslash]y_k - L/4, y_{k+1} + L/4[$ and by the exponentional decay of φ, (20) and (22),

$$\int_{y_k - L/4}^{y_{k+1} + L/4} |R_{X(t_0)}|^2 + |\partial_x R_{X(t_0)})|^2 \leq O(e^{-\sqrt{L}/4})\,,$$

it follows that

$$\int_{I\!R}\Big[1-\Psi(y_k(t_0)-\cdot)-\Psi(\cdot-y_{k+1}(t_0))\Big][(u(t_0)-R_{X(t_0}})^2+(u_x(t_0)-\partial_x R_{X(t_0)})^2] = O(\varepsilon+L^{-1/4})\ . \tag{66}$$

Combining (57), (64) and (66) we infer that

$$E(u(t_0) - R_{X(t_0)}) = O(\varepsilon + L^{-1/4})$$

which concludes the proof of (19) since, according to Proposition 3.2, $|O(x)| \leq C|x|$ for some constant $C > 0$ independent of A. This proves (8) whereas (9) follows from (20) and (23).

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References

[1] B. Alvarez-Samaniego and D. Lannes, Large time existence for 3D water-waves and asymptotics, *Invent. Math. 171 (2009), 165–186.*

[2] R. Beals, D.H. Sattinger and J. Szmigielski, Multi-peakons and the classical moment problem, *Adv. Math. 154 (2000), no. 2, 229–257.*

[3] R. Beals, D.H. Sattinger and J. Szmigielski, Multipeakons and a theorem of Stieltjes, *Inverse Problems. 15 (1999), no. 1, L1–L4.*

[4] T. B. Benjamin, *The stability of solitary waves. Proc. Roy. Soc. London Ser. A* 328 *(*1972*),* 153 − −183.

[5] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Rational Mech. Anal.* **187** *(2007), 215–239.*

[6] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Analysis and Applications* **5** *(2007), 1–27.*

[7] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. rev. Lett.* **71** *(1993), 1661–1664.*

[8] R. Camassa, D. Holm and J. Hyman, An new integrable shallow water equation, *Adv. Appl. Mech.* **31** *(1994)*

[9] A. Constantin , Existence of permanent and breaking waves for a shallow water equations: a geometric approach, *Ann. Inst. Fourier* **50** *(2000), 321-362.*

[10] A. Constantin , On the scattering problem for the Camassa-Holm equation, *Proc. Roy. Soc. London Ser. A .* **457** *(2001), 953-970.*

[11] A. Constantin , The trajectories of particles in Stolkes waves, *Invent. Math.* **166** *(2006), 523-535.*

[12] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica* **181** *(1998), 229–243.*

[13] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Annali Sc. Norm. Sup. Pisa* **26** *(1998), 303–328.*

[14] A. Constantin and J. Escher, Particle trajectories in solitary waves, *Bull. Amer. Math. Soc.* **44** *(2007), 423–431.*

[15] A. Constantin, V. Gerdjikov and R. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inverse problems* **22** *(2006), 2197-2207.*

[16] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi Equations *Arch. Rat. Mech. Anal.,* **192** *(2009), 165–186.*

[17] A. Constantin and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* **78** *(2003), 787-804.*

[18] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, *Comm. Math. Phys.* **211** *(2000), 45–61.*

[19] A. Constantin and L. Molinet, Orbital stability of solitary waves for a shallow water equation, *Physica D* **157** *(2001), 75-89.*

[20] A. Constantin and W. Strauss, Stability of peakons, *Commun. Pure Appl. Math.* **53** *(2000), 603-610.*

[21] A. Constantin and W. Strauss, Stability of the Camassa-Holm solitons, *J. Nonlinear Sci.* **12** *(2002), 415-422.*

[22] H.-H. Dai, Model equations for nonlinear dispersive waves in compressible Mooney-Rivlin rod, *Acta Mech.* **127** *(1998), 293–308.*

[23] R. Danchin, A few remarks on the Camassa-Holm equation, *Diff. Int. Equ.* **14** *(2001), 953–980*

[24] K. El Dika, Smoothing effect of the generalized BBM equation for localized solutions moving to the right. *Discr. Cont. Dyn. Syst.* **12** *(2005), 973-982.*

[25] K. El Dika, Asymptotic stability of solitary waves for the Benjamin-Bona-Mahony, *Discr. Cont. Dyn. Syst.* **13** *(2005), 583-622.*

[26] K. El Dika and Y. Martel, Stability of N solitary waves for the generalized BBM equations, *Dyn. Partial Differ. Equ.* **1** *(2004), 401-437.*

[27] K. El Dika and L. Molinet, Exponential decay of H^1-localized solutions and stability of the train of N solitary waves for the Camassa-Holm equation, *Phil. Trans. R. Soc. A.* **365** *(2007), 2313–2331*

[28] K. El Dika and L. Molinet, Stability of multipeakons, *Annales I.H.P. (2009), In Press*

[29] A. S. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformation and hereditary symmetries, *Physica D* **4** *(1981), 47–66.*

[30] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry, *J. Funct. Anal.* **74** *(1987), 160-197.*

[31] R.S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** *(2002), 63–82.*

[32] B. Kolev, Lie groups and mechanics: an introduction, *J. Nonlinear Math. Phys.* **11** *(2004), 480–498.*

[33] B. Kolev, Poisson brackets in hydrodynamics, *Discrete Contin. Dyn. Syst.* **19** *(2007), 555–574.*

[34] Y. Martel, F. Merle and T-p. Tsai Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. *Comm. Math. Phys.* **231** *(2002), 347–373.*

[35] L. Molinet On well-posedness results for Camassa-Holm equation on the line: a survey. *J. Nonlinear Math. Phys.* **11** *(2004), 521–533.*

[36] E. Wahlén Global existence of weak solutions to the Camassa-Holm equation, *I.M.R.N. (2006), 1–12.*

[37] J.F. Toland Stokes waves, *Topol. Methods Nonlinear Anal.* **7** *(1996), 1–48.*