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Ladder Theorem for a Leray alpha model of turbulence

Hani Ali*

Abstract

In this paper, we study the Modified Leray alpha model with periodic boundary conditions. We show that when the initial data are infinitely differentiable then the unique solution are infinitely differentiable in space and time. Furthermore, this regular solution verifies a sequence of energy inequalities that is called "ladder inequalities".

MSC:76B03; 76F05; 76D05; 35Q30.

Keywords: Turbulence models; Regularity; Navier-Stokes equations

1 Introduction

We consider in this paper the ML-α model of turbulence introduced in [6]:

$$\left\{\begin{array}{ll} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{u}} - \nu\Delta\mathbf{u} + \nabla p = \overline{\mathbf{f}} & \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ -\alpha^2\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}} + \nabla\pi = \mathbf{u} & \text{in } \mathbb{T}_3, \\ \nabla\cdot\mathbf{u} = \nabla\cdot\overline{\mathbf{u}} = 0, \ \oint_{\mathbb{T}_3}\mathbf{u} = \oint_{\mathbb{T}_3}\overline{\mathbf{u}} = 0, \ \mathbf{u}_{t=0} = \overline{\mathbf{u}_0}. \end{array}\right. \tag{1}$$

The boundary conditions are periodic boundary conditions. Therefore we consider these equations on the three dimensional torus $\mathbb{T}_3 = \left(\mathbb{R}^3/\mathcal{T}_3\right)$ where $\mathcal{T}_3 = 2\pi\mathbb{Z}^3/L$, $\mathbf{x} \in \mathbb{T}_3$, and $t \in]0, +\infty[$. The unkowns are the velocity vector field $\mathbf{u}$ and the scalar pressure p. The viscosity ν, the initial velocity vector field $\mathbf{u}_0$, the external force $\mathbf{f}$ with $\nabla\cdot\mathbf{f} = 0$ are given.
This model has been first studied in [6], where the authors prove the global existence and uniqueness of the solution. It is easily seen that when $\alpha = 0$, eqs. (1) reduce to the usual Navier Stokes equations for incompressible fluids.
Assuming that $\mathbf{f} \in C^\infty$, any C^∞ solution to the Navier Stokes equations verifies formally what is called the ladder inequality [4]. That means, for any C^∞ solution $(\mathbf{u}, p)$ to the (NSE), the velocity part $\mathbf{u}$ satisfies the following relation between its higher derivatives,

$$\begin{array}{l} \dfrac{1}{2}\dfrac{d}{dt}H_N \leq -\nu H_{N+1} + C_N H_N \left\|\nabla\mathbf{u}\right\|_\infty + H_N^{1/2}{\Phi_N}^{1/2}, \\ \text{where } H_N = \displaystyle\int_{\mathbb{T}_3}\left|\nabla^N\mathbf{u}\right|^2 d\mathbf{x} \text{ and } \Phi_N = \int_{\mathbb{T}_3}|\nabla^N\mathbf{f}|^2 d\mathbf{x}. \end{array} \tag{2}$$

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This ladder inequalities are used to study the singularities of solutions to the Navier Stokes equations see [7]. The ladder inequalties are generalized in [8] to other equations based on the Navier stokes equations such as Navier Stokes-alpha model [5] and Leray alpha model [3].
We aim to study in this paper ladder inequalities for model (1). We first show a regularity result for (1).

Theorem 1.1 *Assume* $\boldsymbol{f} \in C^\infty([0,T]\times\mathbb{T}_3)^3$ *and* $\boldsymbol{u}_0 \in C^\infty(\mathbb{T}_3)^3$. *Let* $(\boldsymbol{u}^\alpha, p^\alpha)$ *be the solution to problem (1). Then the solution is* C^∞ *in space and time.*

In the whole paper, $\alpha > 0$ is given and we assume that the initial data is C^∞. We prove the following result.

Theorem 1.2 *The unique solution to eqs. (1) satisfies the ladder inequalites,*

$$\begin{aligned}\frac{1}{2}(\frac{d}{dt}\overline{H_N} + \alpha^2\frac{d}{dt}\overline{H_{N+1}}) \leq -\nu(\overline{H_{N+1}} + \alpha^2\overline{H_{N+2}}) \\ +C_N\left\|\nabla\overline{\boldsymbol{u}}\right\|_\infty(\overline{H_N} + \alpha^2\overline{H_{N+1}}) + \overline{H_N}^{1/2}{\Phi_N}^{1/2},\end{aligned} \tag{3}$$

where

$$\overline{H_N} = \int_{\mathbb{T}_3}\left|\nabla^N\overline{\boldsymbol{u}}\right|^2 d\boldsymbol{x}. \tag{4}$$

The paper is organized as follows: In section 2, we start by summarizing some helpfuls results about existence and uniqueness for this ML$-\alpha$ model. In section 3, we prove Theorem 1.1 thanks to a general regularity result. In section 4, we prove Theorem 1.2. We stress that for all $N \in \mathbb{N}$ fixed, inequality (3) goes to inequality (2) when $\alpha \to 0$, at least formally.

2 Existence, unicity and convergence results

Results in [6] and [5] can be summarised as follows:

Theorem 2.1 *Assume* $\boldsymbol{f} \in L^2(\mathbb{T}_3)^3$ *and* $\boldsymbol{u}_0 \in H_{-1}(\mathbb{T}_3)^3$. *Then for any* $T > 0$, *(1) has a unique distributional solution* $(\boldsymbol{u}^\alpha, p^\alpha)$ *such that* $\boldsymbol{u}^\alpha \in L^\infty((0,T], H^{-1}(\mathbb{T}_3)^3)\cap L^2([0,T], L^2(\mathbb{T}_3)^3)$. *Furthermore, if* $\boldsymbol{u}_0 \in L^2(\mathbb{T}_3)^3$ *then* $\boldsymbol{u}^\alpha \in L^\infty((0,T], L^2(\mathbb{T}_3)^3)$.

The proof is based on the following energy inequality that is obtained by taking the inner product of (1) with $\overline{\mathbf{u}}$,

$$\frac{1}{2}(\frac{d}{dt}\|\overline{\mathbf{u}}\|_{L^2}^2 + \alpha^2\frac{d}{dt}\|\nabla\overline{\mathbf{u}}\|_{L^2}^2) + \nu(\|\nabla\overline{\mathbf{u}}\|_{L^2}^2 + \alpha^2\|\Delta\overline{\mathbf{u}}\|_{L^2}^2) \leq \|\mathbf{f}\|_{L^2}\|\overline{\mathbf{u}}\|_{L^2}. \tag{5}$$

Remark 2.1 *Note that the pressure may be reconstructed from* $\mathbf{u}^\alpha$ *and* $\overline{\mathbf{u}}$ *by solving the elliptic equation*

$$\Delta p^\alpha = \nabla\cdot((\mathbf{u}^\alpha\cdot\nabla)\overline{\mathbf{u}}).$$

One concludes from the classical elliptic theory that $p^\alpha \in L^1([0,T], L^2(\mathbb{T}_3)^3)$.

We recall that we can extract subsequences of solution that converge as $\alpha \to 0$ to a weak solution of the Navier Stokes equations. This result is proved in [5] for the Navier Stokes-alpha. By the same arguments used there, we can deduce a similarly result for the other alpha models [3],[6],[2],[1].

Corollary 2.1 *We have* $\overline{\mathbf{u}} \in L^2([0,T], L^\infty(\mathbb{T}_3))$. *Thus there exists a constant* $M > 0$ *such that* $\int_0^t \|\overline{\boldsymbol{u}}\|_{L^\infty}^2 \leq M$ *for all* $t \in [0,T]$.

3 Proof of Theorem 1.1

Theorem 1.1 is a consequence of proposition 3.1.

Proposition 3.1 *Assume* $\boldsymbol{f} \in L^2([0,T], \boldsymbol{H}^{m-1}(\mathbb{T}_3))$ *and* $\boldsymbol{u}_0 \in \boldsymbol{H}^{m-2}(\mathbb{T}_3)$, $m \geq 1$, *then* $\boldsymbol{u}^\alpha \in L^\infty([0,T], \boldsymbol{H}^m(\mathbb{T}_3)) \cap L^2([0,T], \boldsymbol{H}^{m+1}(\mathbb{T}_3))$ *and* $p^\alpha \in L^2([0,T], \boldsymbol{H}^m(\mathbb{T}_3))$.

So far as we know, the following result is original.

Lemma 3.1 *For* $\boldsymbol{u}_0 \in H^{-1}(\mathbb{T}_3)^3$ *and* $\boldsymbol{f} \in H^{-1}(\mathbb{T}_3)^3$, *eqs. (1) have a unique solution* $(\boldsymbol{u}^\alpha, p^\alpha)$ *such that* $\boldsymbol{u}^\alpha \in L^\infty([0,T], L^2(\mathbb{T}_3)^3) \cap L^2([0,T], H^1(\mathbb{T}_3)^3)$.

Proof of Lemma 3.1. We show formal a priori estimates for the solution established in the above Theorem. These estimates can be obtained rigorously using the Galerkin procedure.
We take the inner product of (1) with $\mathbf{u}^\alpha$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^\alpha(t,\mathbf{x})\|_{L^2}^2 + \nu\|\nabla\mathbf{u}^\alpha(t,\mathbf{x})\|_{L^2}^2 \leq \|\nabla^{-1}\mathbf{f}\|_{L^2}\|\nabla\mathbf{u}^\alpha\|_{L^2} + |((\mathbf{u}^\alpha\cdot\nabla)\overline{\mathbf{u}},\mathbf{u}^\alpha)|. \tag{6}$$

Integration by parts and Cauchy-Schwarz inequality yield to

$$|((\mathbf{u}^\alpha\cdot\nabla)\overline{\mathbf{u}},\mathbf{u}^\alpha)| \leq \|\mathbf{u}^\alpha\otimes\overline{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{u}^\alpha\|_{L^2} \tag{7}$$

and by Young's inequality, we obtain

$$\begin{aligned} \|\nabla^{-1}\mathbf{f}\|_{L^2}\|\nabla\mathbf{u}^\alpha\|_{L^2} &\leq \frac{1}{\nu}\|\nabla^{-1}\mathbf{f}\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\mathbf{u}^\alpha\|_{L^2}^2, \\ |((\mathbf{u}^\alpha\cdot\nabla)\overline{\mathbf{u}},\mathbf{u}^\alpha)| &\leq \frac{1}{\nu}\|\mathbf{u}^\alpha\otimes\overline{\mathbf{u}}\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\mathbf{u}^\alpha\|_{L^2}^2. \end{aligned} \tag{8}$$

From the above inequalities we get

$$\begin{aligned} \frac{d}{dt}\|\mathbf{u}^\alpha(t,\mathbf{x})\|_{L^2}^2 + \nu\|\nabla\mathbf{u}^\alpha(t,\mathbf{x})\|_{L^2}^2 &\leq \frac{2}{\nu}\|\nabla^{-1}\mathbf{f}\|_{L^2}^2 + \frac{2}{\nu}\|\mathbf{u}^\alpha\overline{\mathbf{u}}\|_{L^2}^2 \\ &\leq \frac{2}{\nu}\|\nabla^{-1}\mathbf{f}\|_{L^2}^2 + \frac{2}{\nu}\|\mathbf{u}^\alpha\|_{L^2}^2\|\overline{\mathbf{u}}\|_{L^\infty}^2. \end{aligned} \tag{9}$$

Using Gronwall's inequality, since $\|\overline{\mathbf{u}}\|_{L^\infty}^2 \in L^1([0,T])$ (Corollary 2.1) we conclude that

$$\|\mathbf{u}^\alpha(t,\mathbf{x})\|_{L^\infty([0,T],L^2)}^2 \leq K_1,$$

where K_1 is given by

$$K_1 = \|\mathbf{u}_0^\alpha\|_{L^2}^2 \exp\frac{2}{\nu}M + \frac{2}{\nu}\|\nabla^{-1}\mathbf{f}\|_{L^2}^2\int_0^t \exp\frac{2}{\nu}M dt',$$

where M is defined in Corollary (2.1). Furthermore, for every $T > 0$ we have from (9),

$$\|\mathbf{u}^\alpha(T,\mathbf{x})\|_{L^2}^2 + \nu\int_0^T \|\nabla\mathbf{u}^\alpha(t,\mathbf{x})\|_{L^2}^2 dt \leq \|\mathbf{u}_0^\alpha\|_{L^2}^2 + \frac{2}{\nu}\|\nabla^{-1}\mathbf{f}\|_{L^2}^2 T + \frac{2}{\nu}K_1 M. \tag{10}$$

Thus $\mathbf{u}^\alpha \in L^2([0,T], H^1(\mathbb{T}_3)^3)$ for all $T > 0$.

With the same assumption in the inital data as in Theorem 2.1 , we can find the following result:

Lemma 3.2 *Assume* $\boldsymbol{f} \in L^2(\mathbb{T}_3)^3$ *and* $\boldsymbol{u}_0 \in H^{-1}(\mathbb{T}_3)^3$ *. Then for any* $T > 0$*, eqs. (1) have a unique regular solution* $(\boldsymbol{u}^\alpha, p^\alpha)$ *such that* $\boldsymbol{u}^\alpha \in C([0,T], H^1(\mathbb{T}_3)^3) \cap L^2([0,T], H^2(\mathbb{T}_3)^3)$, $\dfrac{d\boldsymbol{u}^\alpha}{dt} \in L^2([0,T], L^2(\mathbb{T}_3)^3)$ *and* $p^\alpha \in L^2([0,T], H^1(\mathbb{T}_3)^3)$.

Proof of Lemma 3.2 It is easily checked that since $\mathbf{u}^\alpha \in L^\infty([0,T], L^2(\mathbb{T}_3)^3) \cap L^2([0,T], H^1(\mathbb{T}_3)^3)$, then $\overline{\mathbf{u}} \in L^\infty([0,T], H^2(\mathbb{T}_3)^3) \cap L^2([0,T], H^3(\mathbb{T}_3)^3)$. Consequently, by Sobolev injection Theorem, we deduce that $\overline{\mathbf{u}} \in L^\infty([0,T], L^\infty(\mathbb{T}_3)^3)$ and $\nabla\overline{\mathbf{u}} \in L^2([0,T], L^\infty(\mathbb{T}_3)^3)$.
Therefore,

$$(\mathbf{u}^\alpha \cdot \nabla)\overline{\mathbf{u}} \in L^2([0,T], L^2(\mathbb{T}_3)^3).$$

Now, for fixed t, we can take $\partial_t \mathbf{u}^\alpha$ as a test function in (1) and the procedure is the same as the one in [9]. Once we obtain that $\mathbf{u}^\alpha \in L^\infty([0,T], H^1(\mathbb{T}_3)^3) \cap L^2([0,T], H^2(\mathbb{T}_3)^3) \cap H^1([0,T], L^2(\mathbb{T}_3)^3)$ and $p^\alpha \in L^2([0,T], H^1(\mathbb{T}_3)^3)$. Interpolating between $L^2([0,T], H^2(\mathbb{T}_3)^3)$ and $H^1([0,T], L^2(\mathbb{T}_3)^3)$ yields to $\mathbf{u}^\alpha \in C([0,T], H^1(\mathbb{T}_3)^3)$.

Proof of Proposition 3.1. We proceed by induction. The case $m = 1$ follows from Lemma 3.2.
Assume that for any $k = 1, ..., m-1$, if $\mathbf{f} \in L^2([0,T], H^{k-1}(\mathbb{T}_3)^3)$ and $\mathbf{u}_0 \in H^{k-2}(\mathbb{T}_3)^3$ then $\mathbf{u}^\alpha \in L^\infty([0,T], H^k(\mathbb{T}_3)^3) \cap L^2([0,T], H^{k+1}(\mathbb{T}_3)^3)$ holds.
It remains to prove that when $k = m$, $\mathbf{f} \in L^2([0,T], H^{m-1}(\mathbb{T}_3)^3)$ and $\mathbf{u}_0 \in H^{m-2}(\mathbb{T}_3)^3$ that $\mathbf{u}^\alpha \in L^\infty([0,T], H^m(\mathbb{T}_3)^3) \cap L^2([0,T], H^{m+1}(\mathbb{T}_3)^3)$.
It is easily checked that for $\mathbf{u}^\alpha \in L^\infty([0,T], H^k(\mathbb{T}_3)^3) \cap L^2([0,T], H^{k+1}(\mathbb{T}_3)^3)$,
$\overline{\mathbf{u}} \in L^\infty([0,T], H^{k+2}(\mathbb{T}_3)^3) \cap L^2([0,T], H^{k+3}(\mathbb{T}_3)^3)$. Consequently, by Sobolev injection Theorem, we deduce that $\nabla^k\overline{\mathbf{u}} \in L^\infty([0,T], L^\infty(\mathbb{T}_3)^3)$, and $\nabla^{k+1}\overline{\mathbf{u}} \in L^2([0,T], L^\infty(\mathbb{T}_3)^3)$.
Therefore, after using Leibniz Formula

$$\nabla^{m-1}\left((\mathbf{u}^\alpha \cdot \nabla)\overline{\mathbf{u}}\right) = \sum_{k=0}^{m-1} C_{m-1}^k \nabla^k \mathbf{u}^\alpha \nabla^{m-k}\overline{\mathbf{u}}, \tag{11}$$

since

$$\nabla^k \mathbf{u}^\alpha \in L^\infty([0,T], L^2(\mathbb{T}_3)^3)$$

and

$$\nabla^{k+1}\overline{\mathbf{u}} \in L^2([0,T], L^\infty(\mathbb{T}_3)^3),$$

for any $k = 1, ..., m-1$.
We deduce that

$$\nabla^{m-1}\left((\mathbf{u}^\alpha \cdot \nabla)\overline{\mathbf{u}}\right) \in L^2([0,T], L^2(\mathbb{T}_3)^3). \tag{12}$$

From (1) we have

$$\frac{\partial \nabla^{m-1}\mathbf{u}^\alpha}{\partial t} + \nabla^{m-1}\left((\mathbf{u}^\alpha \cdot \nabla)\overline{\mathbf{u}}\right) - \nu\nabla^{m-1}\Delta\mathbf{u}^\alpha + \nabla^{m-1}\nabla p^\alpha = \nabla^{m-1}\overline{\mathbf{f}} \tag{13}$$

Now, for fixed t, we can take $\partial_t \nabla^{m-1}\mathbf{u}^\alpha$ as a test function in (13) and the procedure is the same as the one in [9]. One obtains that $\mathbf{u}^\alpha \in L^\infty([0,T], H^m(\mathbb{T}_3)^3) \cap L^2([0,T], H^{m+1}(\mathbb{T}_3)^3)$ and $p^\alpha \in L^2([0,T], H^m(\mathbb{T}_3)^3)$. This finishes the proof of proposition 3.1.

4 Ladder Inequality for the ML-alpha model

Proof of theorem 1.2. We first write the equation satisfied by $(\overline{\mathbf{u}}, p)$. Since

$$-\alpha^2\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}} = \mathbf{u},$$

we get

$$\frac{\partial(\overline{\mathbf{u}} - \alpha^2\Delta\overline{\mathbf{u}})}{\partial t} + ((\overline{\mathbf{u}} - \alpha^2\Delta\overline{\mathbf{u}})\cdot\nabla)(\overline{\mathbf{u}}) - \nu\Delta(\overline{\mathbf{u}} - \alpha^2\Delta\overline{\mathbf{u}}) + \nabla p = \overline{\mathbf{f}}. \tag{14}$$

Thanks to Theorem 1.1, we can write that

$$\frac{d}{dt}\int_{\mathbb{T}_3}\left|\nabla^N\overline{\mathbf{u}}\right|^2 d\mathbf{x} + \alpha^2\frac{d}{dt}\int_{\mathbb{T}_3}\left|\nabla^{N+1}\overline{\mathbf{u}}\right|^2 d\mathbf{x} = \nu\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N\Delta\overline{\mathbf{u}}d\mathbf{x} - \nu\alpha^2\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N\Delta\Delta\overline{\mathbf{u}}d\mathbf{x}$$

$$+\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N((\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x} - \alpha^2\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N((\Delta\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x} + \int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N\overline{\mathbf{f}}d\mathbf{x}.$$

After integrations by parts, we find

$$\begin{aligned}\frac{1}{2}(\frac{d}{dt}\overline{H_N} + \alpha^2\frac{d}{dt}\overline{H_{N+1}}) \leq -\nu(\overline{H_{N+1}} + \alpha^2\overline{H_{N+2}}) + \left|\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N((\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x}\right| \\ +\alpha^2\left|\int_{\mathbb{T}_3}\nabla^{N+1}\overline{\mathbf{u}}\nabla^{N-1}((\Delta\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x}\right| + \left|\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N\overline{\mathbf{f}}d\mathbf{x}\right|.\end{aligned} \tag{15}$$

We claim that the nonlinear first term $\left|\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N((\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x}\right|$ is estimated with the Gagliardo-Nirenberg inequality [4] by $c_N\left\|\nabla\overline{\mathbf{u}}\right\|_\infty\overline{H_N}$, where $c_0 = 0$ and $c_N \leq c2^N$. Indeed, the nonlinear first term $\left|\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N((\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x}\right|$ is found to satisfy

$$\left|\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N((\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x}\right| \leq 2^N\overline{H_N}^{1/2}\sum_{l=1}^{N}\left\|\nabla^l\overline{\mathbf{u}}\right\|_{L^p}\left\|\nabla^{N+1-l}\overline{\mathbf{u}}\right\|_{L^q},$$

where p and q satisfy $1/p + 1/q = 1/2$ according to the Hölder inequality. Now we use the two Gagliardo-Nirenberg inequalities

$$\left\|\nabla^l\overline{\mathbf{u}}\right\|_{L^p} \leq c_1\left\|\nabla^N\overline{\mathbf{u}}\right\|_{L^2}^a\left\|\nabla\overline{\mathbf{u}}\right\|_\infty^{1-a},$$

$$\left\|\nabla^{N+1-l}\overline{\mathbf{u}}\right\|_{L^q} \leq c_2\left\|\nabla^N\overline{\mathbf{u}}\right\|_{L^2}^b\left\|\nabla\overline{\mathbf{u}}\right\|_\infty^{1-b}.$$

Where a anb b must satisfy

$$\frac{1}{p} = \frac{l-1}{3} + a\left(\frac{1}{2} - \frac{N-1}{3}\right),$$

$$\frac{1}{q} = \frac{N-l}{3} + b\left(\frac{1}{2} - \frac{N-1}{3}\right).$$

Since $1/p + 1/q = 1/2$ we deduce that $a + b = 1$. Thus we obtain

$$\left|\int_{\mathbb{T}_3}\nabla^N\overline{\mathbf{u}}\nabla^N((\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x}\right| \leq c_N\left\|\nabla\overline{\mathbf{u}}\right\|_\infty\overline{H_N}. \tag{16}$$

In the same way, we can estimate the nonlinear second term with Gagliardo-Nirenberg inequality in order to have

$$\alpha^2 \Big| \int_{\mathbb{T}_3} \nabla^{N+1}\overline{\mathbf{u}}\nabla^{N-1}((\Delta\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}})d\mathbf{x}\Big| \leq c_N^{'}\alpha^2 \left\|\nabla\overline{\mathbf{u}}\right\|_\infty \overline{H_{N+1}}, \tag{17}$$

where $c_N^{'} \leq c2^N$.
The result (3) then follows.

Remark 4.1 *We note that, as $\alpha \to 0$, $\overline{H_N} \to H_N$. Thus we find the inequality (2).*

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