HAL
open science

Reduction of the chaotic transport of impurities in turbulent magnetized plasmas

Cristel Chandre, Guido Ciraolo, Michel Vittot

To cite this version:

Cristel Chandre, Guido Ciraolo, Michel Vittot. Reduction of the chaotic transport of impurities in turbulent magnetized plasmas. M.A.F. Sanjuan and C. Grebogi. Recent Progress in Controlling Chaos, World Scientific, pp.1, 2010, Stability, Vibration and Control of Systems - Series B. hal-00424256

HAL Id: hal-00424256

https://hal.science/hal-00424256v1

Submitted on 14 Oct 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Chapter 1

Reduction of the chaotic transport of impurities in turbulent magnetized plasmas

C. Chandre[1], G. Ciraolo[2], M. Vittot[1]
[1] *Centre de Physique Théorique , CNRS - Aix-Marseille Universités, Luminy, Case 907, F-13288 Marseille cedex 9, France*
[2] *M2P2, IMT La Jetée, Technopôle de Château Gombert, Marseille cedex 20, F-13451, France*

1.1 Introduction

The control of transport in magnetically confined plasmas is of major importance in the long way to achieve controlled thermonuclear fusion. Two major mechanisms have been proposed for such a turbulent transport: transport governed by the fluctuations of the magnetic field and transport governed by fluctuations of the electric field. There is presently a general consensus to consider, at low plasma pressure, that the latter mechanism agrees with experimental evidence [1]. In the area of transport of trace impurities, i.e. that are sufficiently diluted so as not to modify the electric field pattern, the $\mathbf{E} \times \mathbf{B}$ drift motion of test particles should be the exact transport model. The possibility of reducing and even suppressing chaos (combined with the empirically found states of improved confinement in tokamaks) suggest to investigate the possibility to devise a strategy of control of chaotic transport through some smart perturbations acting at the microscopic level of charged particle motions.
Chaotic transport of particles advected by a turbulent electric field with a strong magnetic field is associated with Hamiltonian dynamical systems under the approximation of the guiding center motion due to $\mathbf{E} \times \mathbf{B}$ drift velocity. For an appropriate choice of turbulent electric field, it has been shown that the resulting diffusive trans-

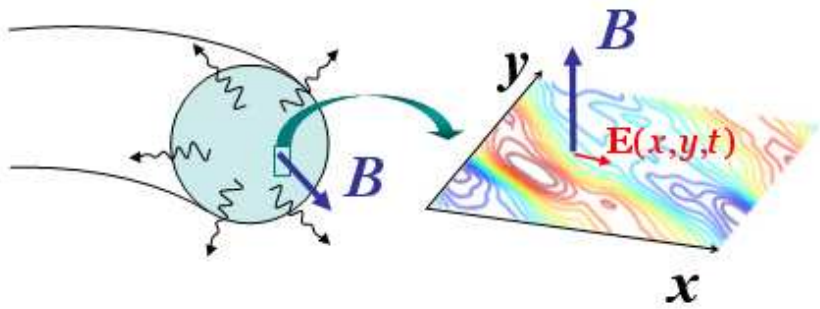

Figure 1.1: The interaction between the electric turbulent field $\mathbf{E}$ and the toroidal magnetic field $\mathbf{B}$ produces the $\mathbf{E} \times \mathbf{B}$ motion which causes a drift of particles from the center of the plasma toward the edge. The guiding center motion of charged test particles on a transversal plane to the toroidal direction can be described by a Hamiltonian formalism with the spatial coordinates (x, y) which represents canonical conjugate variables and with the electrostatic potential V, $\mathbf{E} = -\boldsymbol{\nabla} V$, which plays the role of the Hamiltonian, as described by Eq. (1.1).

port is found to agree with the experimental counterpart [2, 3].

Here we address the turbulent transport of particles governed by the interaction of the electric turbulent potential $\mathbf{E}$ generated by the plasma itself and the strong confining magnetic field $\mathbf{B}$ produced by external coils. The resulting $\mathbf{E} \times \mathbf{B}$ drift motion is perpendicular to the confining magnetic field and causes losses of particles and energy from the inner region to the border of the device, with a consequent decrease of plasma temperature (see Fig. 1.1). The equations of this drift motion for charged test particle, in the guiding center approximation, have a Hamiltonian structure and are given by

$$\dot{\mathbf{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{c}{B^2}\mathbf{E}(\mathbf{x}, t) \times \mathbf{B} = \frac{c}{B}\begin{pmatrix} -\partial_y V(x, y, t) \\ \partial_x V(x, y, t) \end{pmatrix}, \tag{1.1}$$

where $\mathbf{x} = (x, y)$ represents the spatial coordinates of the transversal section to the confining toroidal magnetic field, B the norm of the magnetic field $\mathbf{B}$, c the velocity of light and V is the electric turbulent potential, that is $\mathbf{E} = -\boldsymbol{\nabla} V$. We notice that the resulting dynamics is of Hamiltonian nature with a pair of canonically conjugate variables (x, y) which consists of the position of the guiding center. Since the potential which plays the role of the Hamiltonian is time-dependent, it is expected that the dynamics is chaotic (with one and a half degrees of freedom).

The sensitivity of chaotic systems to small perturbations triggered a strong interdisciplinary effort to control chaos [4, 5, 6, 7, 8, 9, 10, 11]. After the seminal work on optimal control by Pontryagin [12], new and efficient methods were proposed for controlling chaotic systems by nudging targeted trajectories [13, 14, 15, 16, 17, 18]. However, for many body experiments like the magnetic confinement of a plasma or the control of turbulent flows, such methods are hopeless due to the high number of trajectories to deal with simultaneously.

Here the control is performed with the addition of a control term f to the electric potential. It consists as a small and apt modification of the electric potential with relevant effects on the reduction of chaotic particle diffusion. More generally, controlling Hamiltonian systems here means to achieve the goal of suppressing chaos when it is harmful as well as to enhance chaos when it is useful, by driving the dynamical evolution of a system toward a "target behavior" by means of small and apt perturbations. For magnetic confinement devices for controlled thermonuclear fusion, the two possibilities of increasing chaos and suppressing it are of interest. In fact, in the core of a magnetic confinement device the aim is to have a stable and regular dynamics in order to induce as many fusion reactions as possible, that is a high rate of energy production. At the same time, in order to spread heat over a large area, which is fundamental both for a good recycling of energy and for the preservation of plasma facing components, at the edge of the confinement device the problem of draining highly energetic particles requires the capacity of increasing the degree of chaoticity of the dynamics.

For reducing chaos, KAM theory gives a path to integrability and the possibility of controlling Hamiltonian systems by modifying it with an apt and small perturbation which preserves the Hamiltonian structure. In fact, if on one hand it is evident that perturbing an integrable system with a generic perturbation gives rise to a loss of stability and to the break-up of KAM tori, on the other hand the structure of the Hamiltonian system changes on the set of KAM tori with a continuity C^∞ [19] with respect to the amplitude of the perturbation. The idea is that a well selected perturbation instead of producing chaos makes the system integrable or more regular, recovering structures as the KAM tori present in the integrable case.

Starting from these ideas we have addressed the problem of control in Hamiltonian systems. For a wide class of chaotic Hamiltonians, expressed as perturbations of integrable Hamiltonians, that is $H_0 + \varepsilon V$, the aim is to design a control term f such that the dynamics of the controlled Hamiltonian $H_0 + \varepsilon V + f$ has more regular trajectories (e.g. on invariant tori) or less chaotic diffusion than the uncontrolled one. Obviously $f = -\varepsilon V$ is a solution since the resulting Hamiltonian is integrable. However, it is a useless solution since the control is of the same magnitude of the perturbation while, for energetic purposes, the desired control term should be small with respect to the perturbation εV. For example the control term should be of order ε^2 or higher. In many physical situations, the control is only conceivable or interesting in some specific regions of phase space where the Hamiltonian can be

known and/or an implementation is possible. Moreover, it is desirable to control the transport properties without significantly altering the original setup of the system under investigation nor its overall chaotic structure. The possibility of recreating specific regular structures in phase space such as barriers to diffusion, can be used to bound the motion of particles without changing the phase space on both sides of the barrier.

In addition, the construction of the control term has to be robust. In fact, one expects from a KAM theorem also a given robustness with respect to modifications of the well selected and regularizing perturbation. This fact still comes from the previous results that we mentioned above about the transition from integrability to chaos that Hamiltonian systems generically show. In other words, if an apt modification of a chaotic Hamiltonian gives rise to a regularization of the dynamics, with for example the creation of KAM tori, one expects that also some approximations of this apt modification are still able to perform a relevant regularization of the system. This is a fundamental requirement for any kind of control scheme in order to guarantee the stability of the results and in view of possible experimental implementations.

1.2 Control term for Hamiltonian flows

In what follows, we explain a method to compute control terms of order ε^2 where ε is the amplitude of the perturbation, which aims at restoring the stable structures which were present in the case where $\varepsilon = 0$. We notice that such stable structures are present for Hamiltonian systems $H_0 + \varepsilon' V$ where $\varepsilon' < \varepsilon_c < \varepsilon$ (where ε_c is the threshold of break-up of the selected invariant torus), even though they are deformed (by an order ε').

For a function H, let $\{H\}$ be the linear operator such that

$$\{H\}H' = \{H, H'\},$$

for any function H', where $\{\cdot\ ,\cdot\}$ is the Poisson bracket. The time-evolution of a function V following the flow of H is given by

$$\frac{dV}{dt} = \{H\}V,$$

which is formally solved as

$$V(t) = \mathrm{e}^{t\{H\}}V(0),$$

if H is time independent. Let us now consider a given Hamiltonian H_0. The operator $\{H_0\}$ is not invertible since a derivation has always a non-trivial kernel (which is the set of constants of motion). Hence we consider a pseudo-inverse of $\{H_0\}$. We define a linear operator Γ such that

$$\{H_0\}^2\ \Gamma = \{H_0\}, \tag{1.2}$$

i.e.

$$\forall V, \qquad \{H_0, \{H_0, \Gamma V\}\} = \{H_0, V\}.$$

The operator Γ is not unique (see the remark at the end of this section).

We define the *non-resonant* operator $\mathcal{N}$ and the *resonant* operator $\mathcal{R}$ as

$$\mathcal{N} = \{H_0\}\Gamma,$$
$$\mathcal{R} = 1 - \mathcal{N},$$

where the operator 1 is the identity. We notice that the range $\mathrm{Rg}\,\mathcal{R}$ of the operator $\mathcal{R}$ is included in $\mathrm{Ker}\,\{H_0\}$. A consequence is that $\mathcal{R}V$ is constant under the flow of H_0.

Let us now assume that H_0 is integrable with action-angle variables $(\mathbf{A}, \boldsymbol{\varphi}) \in \mathbb{B} \times \mathbb{T}^L$ where $\mathbb{B}$ is an open set of $\mathbb{R}^L$ and $\mathbb{T}^L$ is the L-dimensional torus. Thus $H_0 = H_0(\mathbf{A})$ and the Poisson bracket $\{H, H'\}$ between two elements H and H' of $\mathcal{A}$ is

$$\{H, H'\} = \frac{\partial H}{\partial \mathbf{A}} \cdot \frac{\partial H'}{\partial \boldsymbol{\varphi}} - \frac{\partial H}{\partial \boldsymbol{\varphi}} \cdot \frac{\partial H'}{\partial \mathbf{A}}.$$

The operator $\{H_0\}$ acts on V expanded as follows

$$V = \sum_{\mathbf{k} \in \mathbb{Z}^L} V_{\mathbf{k}}(\mathbf{A}) \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\varphi}},$$

as

$$\{H_0\}V(\mathbf{A}, \boldsymbol{\varphi}) = \sum_{\mathbf{k}} i\boldsymbol{\omega}(\mathbf{A}) \cdot \mathbf{k}\ V_{\mathbf{k}}(\mathbf{A}) \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\varphi}},$$

where

$$\boldsymbol{\omega}(\mathbf{A}) = \frac{\partial H_0}{\partial \mathbf{A}}.$$

A possible choice of Γ is

$$\Gamma V(\mathbf{A}, \boldsymbol{\varphi}) = \sum_{\substack{\mathbf{k} \in \mathbb{Z}^L \\ \omega(\mathbf{A})\cdot\mathbf{k} \neq 0}} \frac{V_{\mathbf{k}}(\mathbf{A})}{i\boldsymbol{\omega}(\mathbf{A}) \cdot \mathbf{k}}\ \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\varphi}}. \tag{1.3}$$

We notice that this choice of Γ commutes with $\{H_0\}$. For a given $V \in \mathcal{A}$, $\mathcal{R}V$ is the resonant part of V and $\mathcal{N}V$ is the non-resonant part:

$$\mathcal{R}V = \sum_{\boldsymbol{\omega}(\mathbf{A})\cdot\mathbf{k}=0} V_{\mathbf{k}}(\mathbf{A}) \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\varphi}}, \tag{1.4}$$

$$\mathcal{N}V = \sum_{\boldsymbol{\omega}(\mathbf{A})\cdot\mathbf{k}\neq 0} V_{\mathbf{k}}(\mathbf{A}) \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\varphi}}. \tag{1.5}$$

From these operators defined from the integrable part H_0, we construct a control term for the perturbed Hamiltonian $H_0 + V$ where $V \in \mathcal{A}$, i.e. f is constructed

such that $H_0 + V + f$ is canonically conjugate to $H_0 + \mathcal{R}V$. We have the following equation

$$\mathrm{e}^{\{\Gamma V\}}(H_0 + V + f) = H_0 + \mathcal{R}V, \tag{1.6}$$

where

$$f = \mathrm{e}^{-\{\Gamma V\}}\mathcal{R}V + \frac{1 - \mathrm{e}^{-\{\Gamma V\}}}{\{\Gamma V\}}\mathcal{N}V - V. \tag{1.7}$$

We notice that the operator $(1 - \mathrm{e}^{-\{\Gamma V\}})/\{\Gamma V\}$ is defined by the expansion

$$\frac{1 - \mathrm{e}^{-\{\Gamma V\}}}{\{\Gamma V\}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\{\Gamma V\}^n.$$

The control term can be expanded in power series as

$$f = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}\{\Gamma V\}^n(n\mathcal{R} + 1)V. \tag{1.8}$$

We notice that if V is of order ϵ, f is of order ϵ^2. Therefore the addition of a well chosen control term f makes the Hamiltonian canonically conjugate to $H_0 + \mathcal{R}V$.

If H_0 is non-resonant then with the addition of a control term f, the Hamiltonian $H_0 + V + f$ is canonically conjugate to the integrable Hamiltonian $H_0 + \mathcal{R}V$ since $\mathcal{R}V$ is only a function of the actions [see Eq. (1.4)]. If H_0 is resonant and $\mathcal{R}V = 0$, the controlled Hamiltonian $H = H_0 + V + f$ is conjugate to H_0. In the case $\mathcal{R}V = 0$, the series (1.8) which gives the expansion of the control term f, can be written as

$$f = \sum_{s=2}^{\infty} f_s, \tag{1.9}$$

where f_s is of order ε^s, where ε is the size of the perturbation V, and is given by the recursion formula

$$f_s = -\frac{1}{s}\{\Gamma V, f_{s-1}\}, \tag{1.10}$$

where $f_1 = V$.

Remark : Non-unicity of Γ– The operator Γ is not unique. Any other choice Γ' satisfies that the range $\mathrm{Rg}(\Gamma' - \Gamma)$ is included into the kernel $\mathrm{Ker}(\{\mathrm{H}_0\}^2)$. It is straightforward to see that adding a constant to Γ does not change the expression of the control term. However, different control terms are obtained by adding a more complex linear operator to Γ. For instance, we choose

$$\Gamma' = \Gamma + \mathcal{R}\mathcal{B},$$

where $\mathcal{B}$ is any linear operator. The operator Γ' satisfies $\{H_0\}^2\Gamma' = \{H_0\}$ since $\{H_0\}\mathcal{R} = 0$. The control term we obtain is given by Eq. (1.8) where Γ is replaced

by Γ'. We notice that the new resonant operator defined as $\mathcal{R}' = 1 - \{H_0\}\Gamma'$ is equal to $\mathcal{R}$ since $\{H_0\}\mathcal{R} = 0$. The new control term is in general different from f. For instance, its leading order is (still of order ε^2)

$$f_2' = f_2 - \frac{1}{2}\{\mathcal{R}\mathcal{B}V, (\mathcal{R}+1)V\}.$$

This freedom in choosing the operator Γ can be used to simplify the control term (for withdrawing some Fourier coefficients) or more generally, to satisfy some additional constraints.

Remark : Higher-order control terms– It is possible to construct higher order control terms, i.e. control terms of order ε^n with $n > 2$, and there are many ways to do so. In this paragraph, we add ε in the control term for bookkeeping purposes. For instance, we notice that $H_0^{(c)} = H + \varepsilon V + \varepsilon^2 f$ is integrable. The perturbed Hamiltonian can be written as

$$H = H_0^{(c)} - \varepsilon^2 f.$$

For simplicity, we assume a non-resonant condition on V, i.e. $\mathcal{R}V = 0$. We define a new operator $\tilde{\Gamma}$ as

$$\tilde{\Gamma} = \mathrm{e}^{-\varepsilon\{\Gamma V\}}\Gamma \mathrm{e}^{\varepsilon\{\Gamma V\}},$$

which is ε-close to Γ. It is straightforward to check that $\tilde{\Gamma}$ satifies

$$\{H_0^{(c)}\}^2\tilde{\Gamma} = \{H_0^{(c)}\},$$

since we have $\{H_0^{(c)}\} = \mathrm{e}^{-\varepsilon\{\Gamma V\}}\{H_0\}\mathrm{e}^{\varepsilon\{\Gamma V\}}$ from Eq. (1.6). By applying again Eq. (1.6) and replacing V by $-\varepsilon^2 f$ and H_0 by $H_0^{(c)}$ (and consequently ε by $-\varepsilon^2$), we have the existence of a control term $\varepsilon^4 g$ which satisfies :

$$H_0 + \varepsilon V + \varepsilon^4 g = \mathrm{e}^{\varepsilon^2\{\tilde{\Gamma} f\}}\left(H_0^{(c)} - \varepsilon^2\tilde{\mathcal{R}}f\right),$$

where $\tilde{\mathcal{R}}$ is the resonant operator associated with $\tilde{\Gamma}$ defined as $\tilde{\mathcal{R}} = 1 - \{H_0^{(c)}\}\tilde{\Gamma}$ (and is ε-close to $\mathcal{R}$). It follows from the commutation of $\{\tilde{\mathcal{R}}V\}$ and $\{H_0^{(c)}\}$ that the controlled Hamiltonian $H_0 + \varepsilon V + \varepsilon^4 g$ is integrable. An expansion in ε of the control term g gives its leading order

$$g_4 = -\frac{\varepsilon^4}{8}\{\Gamma\{\Gamma V, V\}, \{\Gamma V, V\}\}.$$

However, we notice that $\Gamma\{\Gamma V, V\}$ introduces additional small denominators. It is not obvious that, given a value of ε, the control term $\varepsilon^4 g$ is smaller than $\varepsilon^2 f$ (using a standard norm of functions).

1.3 Reduction of chaotic transport for $\mathbf{E} \times \mathbf{B}$ drift in magnetized plasmas

1.3.1 A paradigmatic model of electric potential

We consider the following model of electrostatic potential [20]

$$V(\mathbf{x}, t) = \sum_{\mathbf{k} \in \mathbb{Z}^2} V_{\mathbf{k}} \sin\left[\frac{2\pi}{L}\mathbf{k} \cdot \mathbf{x} + \varphi_{\mathbf{k}} - \omega(\mathbf{k})t\right], \tag{1.11}$$

where $\varphi_{\mathbf{k}}$ are random phases (uniformly distributed) and $V_{\mathbf{k}}$ decrease as a given function of $\|\mathbf{k}\|$, in agreement with experimental data [21], and L is the typical size of the elementary cell as represented in Fig. 1.1. In principle one should use for $\omega(\mathbf{k})$ the dispersion relation for electrostatic drift waves (which are thought to be responsible for the observed turbulence) with a frequency broadening for each $\mathbf{k}$ in order to model the experimentally observed spectrum $S(\mathbf{k}, \omega)$.

Here we consider a quasiperiodic approximation of the turbulent electric potential with a finite number K of frequencies. We assume that $\omega_k \neq 0$ (otherwise, see remark at the end of Sec. 1.3.2). The phases $\varphi_{\mathbf{k}}$ are chosen at random in order to mimic a turbulent field with the reasonable hope that the properties of the realization thus obtained are not significantly different from their average. In addition we take for $\|V_{\mathbf{k}}\|$ a power law in $\|\mathbf{k}\|$ to reproduce the spatial spectral characteristics of the experimental $S(\mathbf{k})$, see Ref. [21]. Thus we consider the following explicit form of the electric potential:

$$V(x, y, t) = \sum_{k=1}^{K} \sum_{m,n=1}^{N} \frac{a_k}{2\pi(n^2 + m^2)^{3/2}} \sin[2\pi(nx + my) + \varphi_{kmn} - 2\pi\omega_k t], \tag{1.12}$$

where φ_{kmn} are random phases.
For example, Fig. 1.2 shows contour plots of the potential (1.12) with only one frequency $\omega_1 = 1$, that is a periodic potential with period 1 in x and y, given by

$$V(x, y, t) = \frac{a}{2\pi} \sum_{\substack{m,n=1 \\ n^2+m^2 \leq N^2}}^{N} \frac{1}{(n^2 + m^2)^{3/2}} \sin\left[2\pi(nx + my) + \varphi_{nm} - 2\pi t\right], \tag{1.13}$$

with $N = 25$.

Since the model is fluctuating in time, the eddies of Fig. 1.2 are rapidly modified in time and where a vortex was initially present, an open line appears, and so on. Two particular properties of the model, anisotropy and propagation have been observed : each image of the potential field shows an elongated structure of the eddies and superposing images obtained at different times a slight propagation in the $y = x$ direction is found. However, this propagation can easily be proved not to disturb the

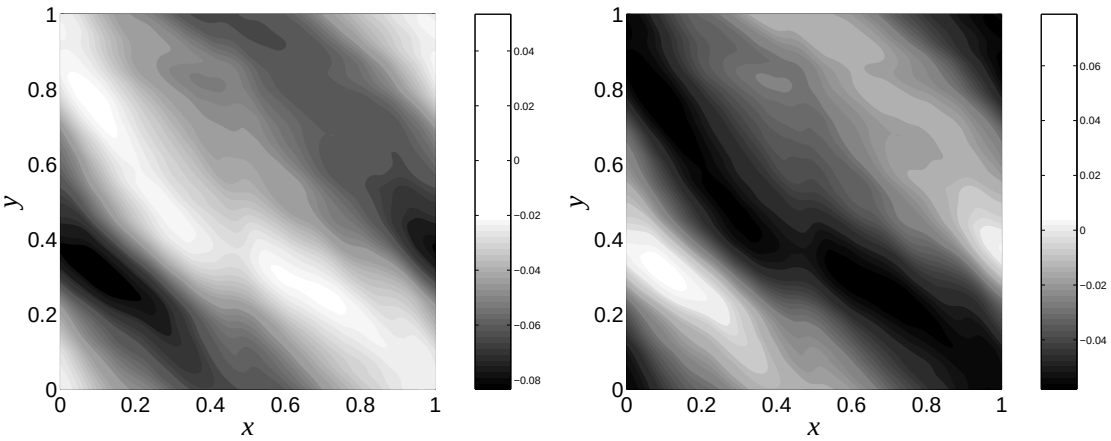

Figure 1.2: Contour plots of $V(x, y, t)$ given by Eq. (1.13) for $a = 1$, $N = 25$, $t = 0$ (left panel) and $t = 1/2$ (right panel), .

diffusive motion of the guiding centers. The property of propagation can be easily understood analytically. In fact, restricting ourselves to the most simplified case of an electric potential given only by a dominant mode ($n = m = 1$) it is immediately evident that at any given time the maxima and minima of the sine are located on the lines $y = -x+$constant. As the amplitudes are decreasing functions of n and m, this structure is essentially preserved also in the case of many waves. The property of anisotropy is an effect of the random phases in producing eddies that are irregular in space.

We notice that there are two typical time scales in the equations of motion: the drift characteristic time τ_d, inversely proportional to the parameter a, and the period of oscillation τ_ω of all the waves that enter the potential. The competition between these two time scales determines what kind of diffusive behaviour is observed [2, 3]. In what follows we consider the case of weak or intermediate chaotic dynamics (coexistence of ordered and chaotic trajectories) which corresponds to the quasi-linear diffusion regime (see Sec. 1.3.5). Whereas in the case of fully developed chaos, that corresponds to the so-called Bohm diffusion regime, one has to introduce a slightly more complicated approach (see remark at the end of Sec. 1.3.2).

1.3.2 Control term for a potential which varies rapidly in time

We consider an electric potential of the form $V(\mathbf{x}, \mathbf{y}, t/\epsilon)$, i.e. such that the time scale of its variation is of order ε.

We consider a time-dependent Hamiltonian system described by the function $V(\mathbf{x}, \mathbf{y}, t/\epsilon)$, with $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2L}$ canonically conjugate variables and t the time. This Hamiltonian system has $L + 1/2$ degrees of freedom.

We map this Hamiltonian system with $L + 1/2$ degrees of freedom to an autonomous Hamiltonian with $L+1$ degrees of freedom by extending the phase space from $(\mathbf{x}, \mathbf{y})$ to $(\mathbf{x}, \mathbf{y}, E, \tau)$ where the new dynamical variable τ evolves as $\tau(t) = t + \tau(0)$ and E is its canonically conjugate variable. The autonomous Hamiltonian is given by

$$H(\mathbf{x}, \mathbf{y}, \tau, E) = E + V(\mathbf{x}, \mathbf{y}, \tau/\epsilon).$$

Rescaling τ by a canonical change of variable,

$$\hat{\tau} = \tau/\epsilon, \qquad \hat{E} = \epsilon E,$$

one obtains

$$H(\mathbf{x}, \mathbf{y}, \tau, E) = E + \epsilon V(\mathbf{x}, \mathbf{y}, \tau), \tag{1.14}$$

where we have renamed $\hat{E} = E$ and $\hat{\tau} = \tau$. In the case of rapidly time-varying potentials, Hamiltonian (1.14) is in the form $H = H_0 + \epsilon V$, that is an integrable Hamiltonian H_0 plus a small perturbation ϵV.

In our case $H_0 = E$, i.e. independent of $\mathbf{x}, \mathbf{y}, \tau$, therefore we have

$$\{H_0\} = \frac{\partial}{\partial \tau}.$$

If we consider a potential $V(\mathbf{x}, \mathbf{y}, \tau)$, in the periodic case we can write

$$V(\mathbf{x}, \mathbf{y}, \tau) = \sum_k V_k(\mathbf{x}, \mathbf{y}) \mathrm{e}^{ik\tau},$$

and the action of Γ, $\mathcal{R}$ and $\mathcal{N}$ on V is given by

$$\Gamma V = \sum_{k \neq 0} \frac{V_k(\mathbf{x}, \mathbf{y})}{ik} \mathrm{e}^{ik\tau}, \tag{1.15}$$

$$\mathcal{R} V = V_0(\mathbf{x}, \mathbf{y}), \tag{1.16}$$

$$\mathcal{N} V = V(\mathbf{x}, \mathbf{y}, \tau) - V_0(\mathbf{x}, \mathbf{y}). \tag{1.17}$$

Otherwise, in the more general case of a non periodic potential, one can write, under suitable hypotheses,

$$V(\mathbf{x}, \mathbf{y}, \tau) = \int_{-\infty}^{+\infty} \hat{V}(\mathbf{x}, \mathbf{y}, k) \mathrm{e}^{ik\tau} dk,$$

and the action of Γ, $\mathcal{R}$ and $\mathcal{N}$ operators on V is given by

$$\Gamma V = PV \int_{-\infty}^{+\infty} \frac{\hat{V}(\mathbf{x}, \mathbf{y}, k)}{ik} \mathrm{e}^{ik\tau} dk, \tag{1.18}$$

$$\mathcal{R} V = \hat{V}(\mathbf{x}, \mathbf{y}, 0), \tag{1.19}$$

and

$$\mathcal{N}V = V(\mathbf{x},\mathbf{y},\tau) - \hat{V}(\mathbf{x},\mathbf{y},0). \tag{1.20}$$

The computation of the control term is now a straightforward application of Eq. (1.8).

Following the previous section, first we map the Hamiltonian system with 1+1/2 degrees of freedom given by Eq. (1.12) to an autonomous Hamiltonian with two degrees of freedom. This is obtained by extending the phase space from (x, y) to (x, y, E, τ) that is considering E the variable conjugate to the new dynamical variable τ. This autonomous Hamiltonian is

$$H(x,y,E,\tau) = E + \sum_{m,n,k} \frac{a_k}{2\pi(n^2+m^2)^{3/2}} \sin[2\pi(nx+my) + \varphi_{kmn} - 2\pi\omega_k\tau]. \tag{1.21}$$

The integrable part of the Hamiltonian from which the operators Γ, $\mathcal{R}$ and $\mathcal{N}$ are constructed is isochronous

$$H_0 = E.$$

We notice that H_0 is resonant (since it does not depend on the action variable x). From the action of Γ and $\mathcal{R}$ operators computed using Eqs. (1.18)-(1.19), we obtain the control term using Eq. (1.8). For instance, the expression of f_2 is

$$\begin{aligned} f_2(x,y,t) = \frac{1}{8\pi} \sum_{\substack{k,m,n\\ k',n',m'}} & \frac{a_k a_{k'}(n'm - nm')}{\omega_k (n^2+m^2)^{3/2}(n'^2+m'^2)^{3/2}} \\ & \times \{\sin[2\pi((n+n')x + (m+m')y) + \varphi_{kmn} + \varphi_{k'm'n'} - 2\pi(\omega_k+\omega_{k'})t] \\ & + \sin[2\pi((n-n')x + (m-m')y) + \varphi_{kmn} - \varphi_{k'm'n'} - 2\pi(\omega_k-\omega_{k'})t]\} \end{aligned} \tag{1.22}$$

Remark: Similar calculations can be done in the case where there is a zero frequency, e.g., $\omega_0 = 0$ and $\omega_k \neq 0$ for $k \neq 0$. The first term of the control term is

$$f_2 = -\frac{1}{2}\{\Gamma V, (\mathcal{R}+1)V\},$$

where

$$\mathcal{R}V = \sum_{m,n} \frac{a_0}{2\pi(n^2+m^2)^{3/2}} \sin[2\pi(nx+my) + \varphi_{0mn}].$$

If we add the exact expression of the control term to $H_0 + V$, the effect on the flow is the confinement of the motion, i.e. the fluctuations of the trajectories of the particles, around their initial positions, are uniformly bounded for any time [22]. In Sec. 1.3.5, we show that truncations of the exact control term f, like for instance f_2 or $f_2 + f_3$, are able to regularize the dynamics and to slow down the diffusion.

We notice that for the particular model (1.13) the partial control term f_2 is independent of time and is given by

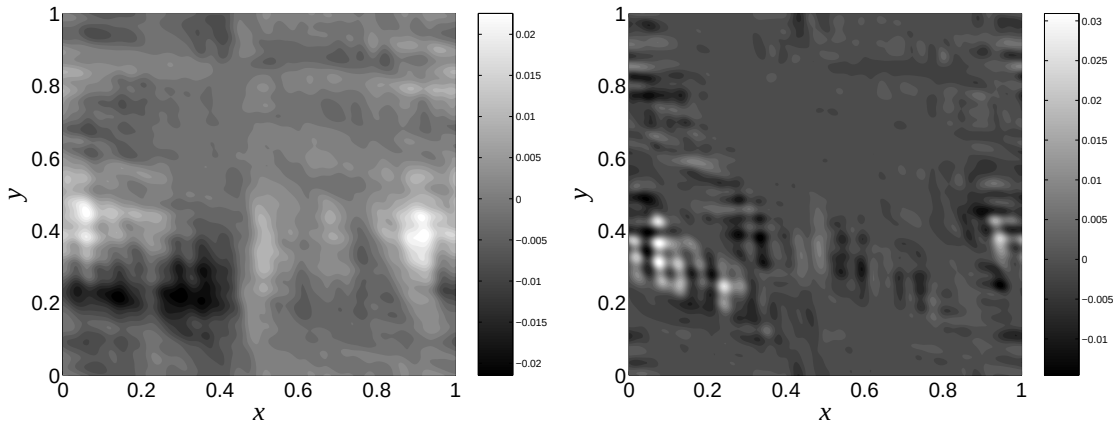

Figure 1.3: Contour plot of f_2 (left panel) given by Eq. (1.23) and f_3 at $t = 0$ for $a = 1$ and $N = 25$.

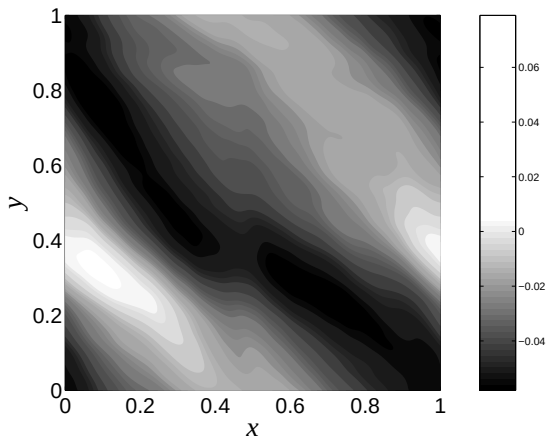

Figure 1.4: Contour plot of $V + f_2$ for $a = 0.8$ and $N = 25$ at $t = 1/2$.

$$f_2(x, y, \tau) = \frac{a^2}{8\pi} \sum_{\substack{n_1, m_1 \\ n_2, m_2}} \frac{n_1 m_2 - n_2 m_1}{(n_1^2 + m_1^2)^{3/2}(n_2^2 + m_2^2)^{3/2}}$$
$$\times \sin[2\pi[(n_1 - n_2)x + (m_1 - m_2)y] + \varphi_{n_1 m_1} - \varphi_{n_2 m_2}]. \tag{1.23}$$

Figure 1.3 depicts a contour plot of f_2 given by Eq. (1.23). Figure 1.4 depicts a contour plot of $V + f_2$ for $a = 0.8$ at $t = 0$. We notice that the controlled potential is a small modification of the potential V since this contour plot looks very similar to the one decicted in Fig. 1.2.

The computation of the other terms of the series (1.9) can be done recursively by using Eq. (1.10). Again, for $a < 1$, $V + f_2 + f_3$ is close to V, meaning that the control in this regime is based on a small modification of the potential.

1.3.3 Properties of the control term

In this section, we state that for a sufficiently small, the exact control term exists and is regular. Then we give estimates of the partial control terms in order to compare the relative sizes of the different terms with respect to the perturbation. The computations are performed for the model (1.13) but can be easily generalized to the more general case (1.12).
Concerning the existence of the control term, we have the following proposition [20] :

Proposition 1 – *If the amplitude a of the potential is sufficiently small, there exists a control term f given by the series (1.9) such that $E+V+f$ is canonically conjugate to E, where V is given by Eq. (1.13).*

We have shown [20] that the exact control term exists for $a \lesssim 7 \times 10^{-3}$. As usual, such estimates are very conservative with respect to realistic values of a. In the numerical study, we consider values of a of order 1.

Concerning the regularity of the control term of the potential (1.13), we notice that each term f_s in the series (1.9) is a trigonometric polynomial with an increasing degree with s. The resulting control term is not smooth but its Fourier coefficients exhibit the same power law mode dependence as V:

Proposition 2 – *All the Fourier coefficients $f^{(s)}_{nmk}$ of the functions f_s of the series (1.9) satisfy:*

$$|f^{(s)}_{nmk}| \leq \frac{a^s C^s}{(n^2+m^2)^{3/2}}, \tag{1.24}$$

for $(n,m) \neq (0,0)$. Consequently, for a sufficiently small, the Fourier coefficients of the control term f given by Eq. (1.9) satisfy:

$$|f_{nmk}| \leq \frac{C_\infty}{(n^2+m^2)^{3/2}},$$

for $(n,m) \neq (0,0)$ and for some constant $C_\infty > 0$.

1.3.4 Magnitude of the control term

A measure of the relative sizes of the control terms is defined by the electric energy density associated with each electric field V, f_2 and f_3. From the potential we get the electric field and hence the motion of the particles. We define an average energy density $\mathcal{E}$ as

$$\mathcal{E} = \frac{1}{8\pi}\langle \ \|\mathbf{E}\|^2 \ \rangle,$$

where $\mathbf{E}(x,y,t) = -\nabla V$. In terms of the particles, it corresponds to the mean value of the kinetic energy $\langle \dot{x}^2 + \dot{y}^2 \rangle$ (up to a multiplicative constant). For $V(x,y,t)$ given

by Eq. (1.13),

$$\mathcal{E} = \frac{a^2}{8\pi} \sum_{\substack{n,m=1 \\ n^2+m^2 \leq N^2}}^{N} \frac{1}{(n^2+m^2)^2}. \tag{1.25}$$

We define the contribution of f_2 and f_3 to the energy density by

$$e_2 = \frac{1}{8\pi} \langle \|\nabla f_2\|^2 \rangle. \tag{1.26}$$

For $N = 25$, these contributions satisfy:

$$\frac{e_2}{\mathcal{E}} \approx 0.1 \times a^2.$$

It means that the control terms f_2 can be considered as small perturbative terms with respect to V when $a < 1$.

Remark on the number of modes in V: In this section, all the computations have been performed with a fixed number of modes $N = 25$ in the potential V given by Eq. (1.13). The question we address in this remark is how the results are modified as we increase N. First we notice that the potential and its electric energy density are bounded with N since

$$|V(x,y,t)| \leq a \sum_{n,m=1}^{\infty} \frac{1}{(n^2+m^2)^{3/2}} < \infty,$$

$$\mathcal{E} \leq \frac{a^2}{8\pi} \sum_{n,m=1}^{\infty} \frac{1}{(n^2+m^2)^2} < \infty.$$

Concerning the partial control term f_2, we see that it is in general unbounded with N. From its explicit form it grows like $N \log N$. Less is known on the control term since it is given by a series whose terms are defined by recursion. However, from Proposition 2, we can show that the value a of existence of the control term decreases like $1/(2N \log N)$. This divergence of the control term comes from the fact that the Fourier coefficients of the potential V are weakly decreasing with the amplitude of the wavenumber.

Therefore, the exact control term might not exist if we increase N by keeping a constant. However for practical purposes the Fourier series of the control term can be truncated to its first terms (the Fourier modes with highest amplitudes). Furthermore in the example we consider as well as for any realistic situation the value of N is bounded by the resolution of the potential. In the case of electrostatic turbulence in plasmas $k\rho_i \sim 1$ determines an upper bound for k, where k is the transverse wave vector related to the indices n, m and ρ_i the ion Larmor radius. The physics corresponds to the averaging effect introduced by the Larmor rotation.

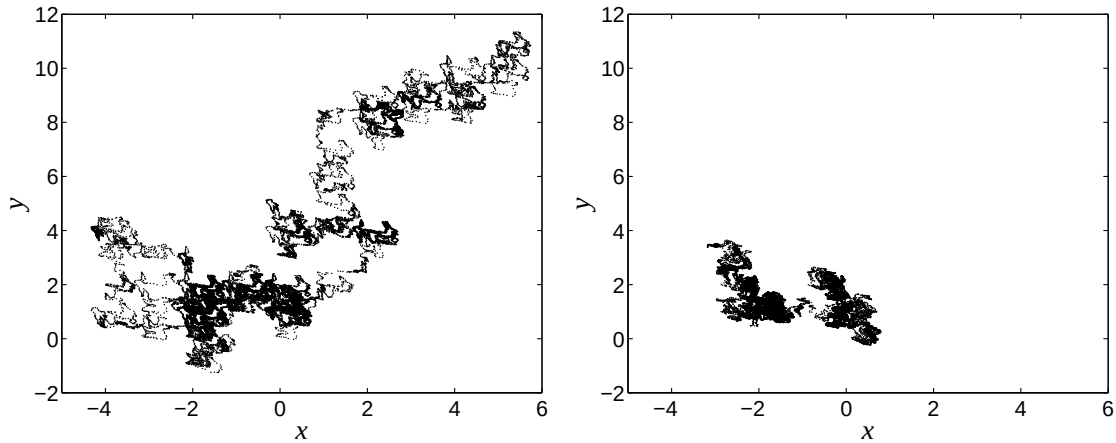

Figure 1.5: Poincaré sections of a trajectory obtained using a generic initial condition for Hamiltonian (1.13) with $a = 0.8$, without control term (upper left panel) and with control term (1.23) (upper right panel).

1.3.5 Efficiency and robustness

With the aid of numerical simulations (see Refs. [2, 3] for more details on the numerics), we check the effectiveness of the control term by comparing the dynamics of particles obtained from the uncontrolled Hamiltonian and from the same Hamiltonian with the control term f_2 and with a more refined control term $f_2 + f_3$.

Diffusion of test particles

The effect of the control terms can first be seen from a few randomly chosen trajectories. We have plotted Poincaré sections (stroboscopic plots of the trajectories of V) on Fig. 1.5 of two trajectories issued from generic initial conditions computed without and with the control term f_2 respectively. Similar pictures are obtained for many other randomly chosen initial conditions. The stabilizing effect of the control term (1.23) is illustrated by such trajectories. The motion remains diffusive but the extension of the phase space explored by the trajectory is reduced.

The dynamics is more clearly seen on a Poincaré section on the $[0, 1]^2$ torus (i.e. by taking x and y modulo 1). Such Poincaré sections are depicted in Fig. 1.6 for V (upper left panel), $V + f_2$ (upper right panel) and $V + f_2 + f_3$ (lower panel). These figures shows that the Poincaré sections are composed of two types of trajectories : ones which are trapped around resonant islands, and diffusive (chaotic) ones which lead to global transport properties. We notice that the number of resonant islands has been drastically increased by the control term. The mechanism of reduction of chaos seems to be that a significant number of periodic orbits has been stabilized by the addition of the control term.

A clear evidence is found for a relevant reduction of the diffusion in presence of

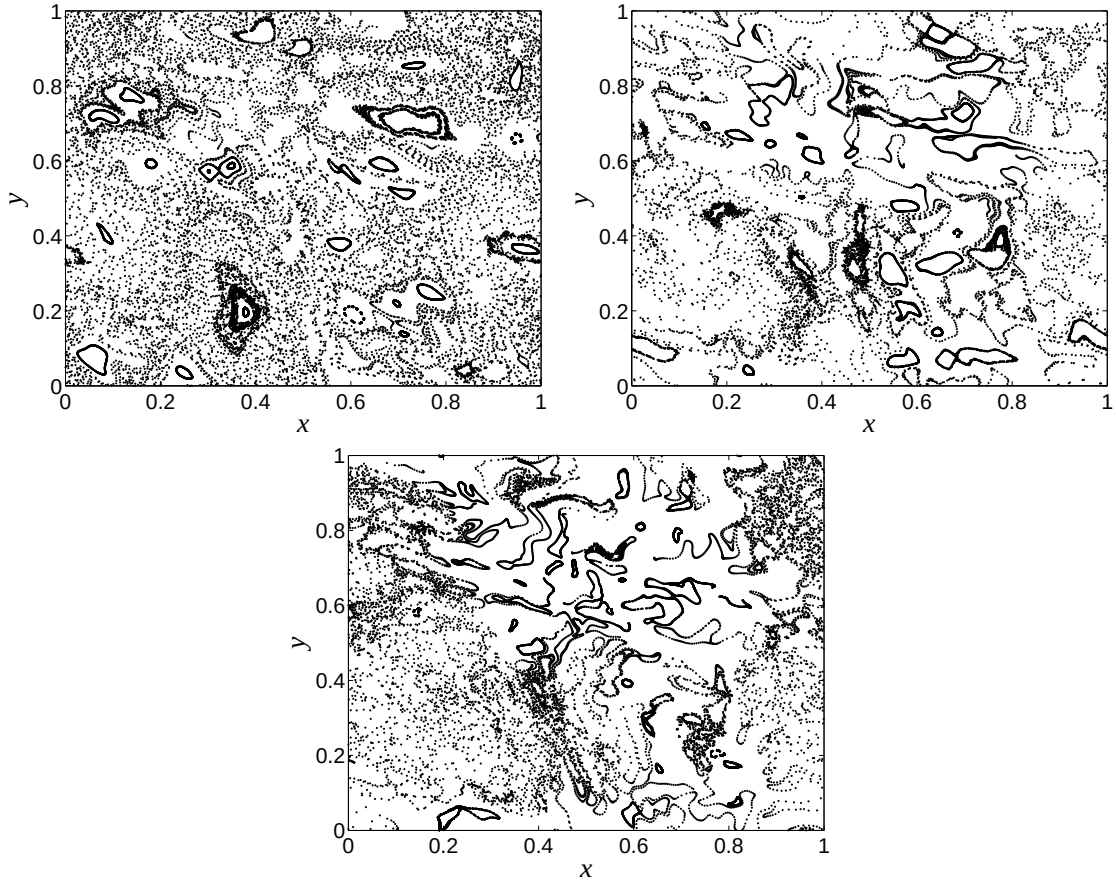

Figure 1.6: Poincaré sections on the $[0,1]^2$ torus for V (upper left panel), $V + f_2$ (upper right panel) and $V + f_2 + f_3$ (lower panel) with $a = 0.6$

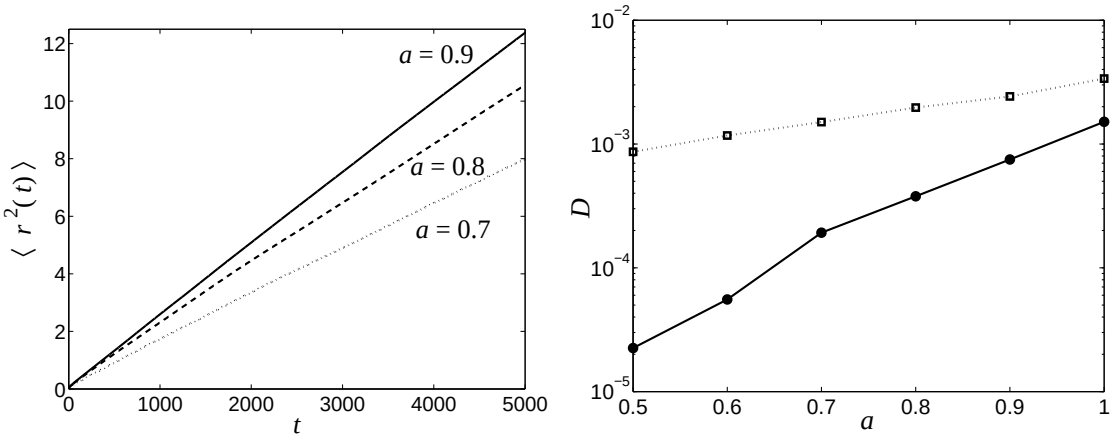

Figure 1.7: On the left panel, mean square displacement $\langle r^2(t)\rangle$ versus time t obtained for Hamiltonian (1.13) with three different values of $a = 0.7$, $a = 0.8$, $a = 0.9$. In the right panel, diffusion coefficient D versus a obtained for Hamiltonian (1.13) (open squares) and Hamiltonian (1.13) plus control term (1.23) (full circles).

the control term (1.22). In order to study the diffusion properties of the system, we have considered a set of $\mathcal{M}$ particles (of order 1000) uniformly distributed at random in the domain $0 \leq x, y \leq 1$ at $t = 0$. We have computed the mean square displacement $\langle r^2(t)\rangle$ as a function of time

$$\langle r^2(t)\rangle = \frac{1}{\mathcal{M}} \sum_{i=1}^{\mathcal{M}} \|\mathbf{x}_i(t) - \mathbf{x}_i(0)\|^2$$

where $\mathbf{x}_i(t) = (x_i(t), y_i(t))$ is the position of the i-th particle at time t obtained by integrating Hamilton's equations with initial conditions $\mathbf{x}_i(0)$. Figure 1.7 (left panels) shows $\langle r^2(t)\rangle$ for three different values of a. For the range of parameters we consider, the behavior of $\langle r^2(t)\rangle$ is always found to be linear in time for t large enough. The corresponding diffusion coefficient is defined as

$$D = \lim_{t\to\infty} \frac{\langle r^2(t)\rangle}{t} \ .$$

Figure 1.7 (right panel) shows the values of D as a function of a with and without control term. It clearly shows a significant decrease of the diffusion coefficient when the control term is added. As expected, the action of the control term gets weaker as a is increased towards the strongly chaotic phase. We notice that the diffusion coefficient is plotted on a log-scale. For $a = 0.7$, the control reduces the diffusion coefficient by a factor approximately equal to 10.

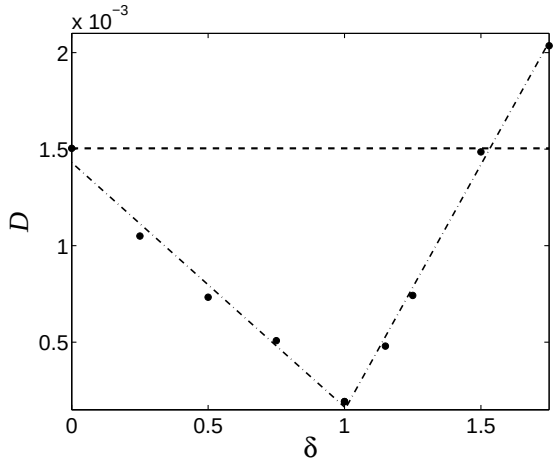

Figure 1.8: Diffusion coefficient D versus the magnitude of the control term $\delta\, f_2$ where f_2 is given by Eq. (1.23) for $a = 0.7$. The horizontal dashed line corresponds to the value of D without control term ($\delta = 0$). The dash-dotted line is a piecewise linear interpolation.

Robustness of the control

In the previous sections, we have seen that a truncation of the series defining the control term by considering the first f_2 or the two first terms $f_2 + f_3$ in the perturbation series in the small parameter a, gives a very efficient control on the chaotic dynamics of the system. This reflects the robustness of the method.
In this section we show that it is possible to use other types of approximations of the control term and still get an efficient control of the dynamics.

We check the robustness of the control by increasing or reducing the amplitude of the control term [20]. We replace f_2 by $\delta \cdot f_2$ and we vary the parameter δ away from its reference value $\delta = 1$. Figure 1.8 shows that both the increase and the reduction of the magnitude of the control term (which is proportional to $\delta \cdot a^2$) result in a loss of efficiency in reducing the diffusion coefficient. The fact that a larger perturbation term – with respect to the computed one – does not work better, also means that the control is not a "brute force" effect.
The interesting result is that one can significantly reduce the amplitude of the control ($\delta < 1$) and still get a reduction of the chaotic diffusion. We notice that the average energy density $e_2(\delta)$ associated with a control term $\delta \cdot f_2$ is equal to $e_2(\delta) = \delta^2 e_2$, where e_2 is given by Eq. (1.26). Therefore, for $\delta = 0.5$ where the energy necessary for the control is one fourth of the optimal control, the diffusion coefficient is significantly smaller than in the uncontrolled case $\delta = 0$ (by nearly a factor 3).

Bibliography

[1] Scott, B.D., 2003, Phys. Plasmas, 10, 963.

[2] Pettini, M., Vulpiani, A., Misguich, J.H., De Leener, M., Orban, J., and Balescu, R., 1988, *Phys. Rev. A*, 38, 344.

[3] Pettini, M., 1989, Non-linear dynamics, G. Turchetti (Ed.), World Scientific, Singapore.

[4] Lichtenberg, A.J. and Lieberman, M.A., 1983, Regular and Stochastic Motion. Applied Mathematical Sciences. Springer-Verlag, New York, 1983.

[5] L.P. Kadanoff. *From Order to Chaos: Essays: Critical, Chaotic and Otherwise.* World Scientific, Singapore, 1998.

[6] G. Chen and X. Dong. *From Chaos to Order.* World Scientific, Singapore, 1998.

[7] D. J. Gauthier. Resource Letter: CC-1: Controlling chaos. *Am. J. Phys.*, 71:750–759, 2003.

[8] W.L. Ditto, S.N. Rauseo, and M.L. Spano. Experimental control of chaos. *Phys. Rev. Lett*, 65:3211–3215, 1990.

[9] V. Petrov, V. Gaspar, J. Masere, and K. Showalter. Controlling chaos in the Belousov-Zhabotinsky reaction. *Nature*, 361:240–243, 1993.

[10] S.J. Schiff, K. Jerger, D.H. Duong, T. Chang, M.L. Spano, and W.L. Ditto. Controlling chaos in the brain. *Nature*, 370:615–620, 1994.

[11] Y. Braiman, J.F. Lindner, and W.L. Ditto. Taming spatiotemporal chaos with disorder. *Nature*, 378:465–467, 1995.

[12] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko. *The Mathematical Theory of Optimal Processes.* Wiley, New York, 1961.

[13] E. Ott, C. Grebogi, and J.A Yorke. Controlling chaos. *Phys. Rev. Lett.*, 64:1196–1199, 1990.

[14] R. Lima and M. Pettini. Suppression of chaos by resonant parametric perturbations. *Phys. Rev. A*, 41:726–733, 1990.

[15] L. Fronzoni, M. Giocondo, and M. Pettini. Experimental evidence of suppression of chaos by resonant parametric perturbations. *Phys. Rev. A*, 43:6483–6487, 1991.

[16] T. Shinbrot, C. Grebogi, E. Ott, and J.A. Yorke. Using small perturbations to control chaos. *Nature*, 363:411–417, 1993.

[17] E. Ott and M. Spano. Controlling chaos. *Phys. Today*, 48:34–40, 1995.

[18] R. Lima and M. Pettini. Parametric resonant control of chaos. *Int. J. Bif. Chaos*, 8:1675–1684, 1998.

[19] J. Poschel. Integrability of Hamiltonian systems on Cantor sets. *Comm. Pure Appl. Math.*, 25:653–695, 1982.

[20] G. Ciraolo, F. Briolle, C. Chandre, E. Floriani, R. Lima, M. Vittot, M. Pettini, C. Figarella, and Ph. Ghendrih. Control of Hamiltonian chaos as a possible tool to control anomalous transport in fusion plasmas. *Phys. Rev. E*, 69:056213, 2004.

[21] A.J. Wootton, H. Matsumoto, K. McGuire, W.A. Peebles, Ch.P. Ritz, P.W. Terry, and S.J. Zweben. Fluctuations and anomalous transport in tokamaks. *Phys. Fluids B*, 2:2879, 1990.

[22] M. Vittot. Perturbation theory and control in classical or quantum mechanics by an inversion formula. *J. Phys. A: Math. Gen.*, 37:6337–6357, 2004.