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AN EXAMPLE OF A TOPOLOGICALLY NON-RIGID FOLIATION OF THE COMPLEX PROJECTIVE PLANE

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ABSTRACT. We give here an explicit example of an algebraic family of foliations of $\mathbb{C}P^2$ which is topologically trivial but not analytically trivial. This example underlines the necessity of some assumptions in Y. Ilyashenko's rigidity theorem.

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1. INTRODUCTION AND PRESENTATION OF THE RESULT

The aim of this article is to provide examples of algebraic foliations of $\mathbb{C}P^2$ which are not topologically rigid. A foliation on $\mathbb{C}P^2$ is defined in an affine chart \mathbb{C}^2 by a differential equation

$$(1.1) \quad P(x, y) y' = Q(x, y)$$

where P and Q are complex polynomials. The space \mathcal{C}_d of all such foliations with P and Q of a fixed degree d is a complex projective space of finite dimension, endowed with the natural topology. A theorem of Yu. S. Ilyashenko [I] states that except maybe for a residual set, all foliations which are topologically conjugate are in fact analytically (thus homographically) conjugate, *i.e.* the generic foliation is topologically rigid. This result was later enhanced by various authors. In [S] A. Scherbakov showed that the set of topologically rigid foliations contains at least the complement of a real analytic set of \mathcal{C}_d . An improvement was given by X. Gómez-Mont and L. Ortíz-Bobadilla [GO], then by L. Neto, P. Sad and P. Scárdua [NSS], showing that the set of all topologically rigid foliation is at least a Zariski-dense open set of \mathcal{C}_d . The argument boils down to proving that non-solvable holonomy representation is the typical behaviour, then applying Nakai's theorem [N] or using other results on density of orbits of (pseudo-)groups of local diffeomorphisms.

Examples of foliations of $\mathbb{C}P^2$ which are not topologically rigid do not abound. First examples of non-rigid foliations of $\mathbb{C}P^2$ can be deduced from the work of N. Ladis [L], where the topological classification of generic homogeneous equations (1.1) is achieved.

We wish to present rather simple foliations which are not rigid. They belong to the class of Liouville-integrable foliations, whose holonomy representation is solvable. The main purpose of this paper is to prove their non-rigidity “by hand”,

by building explicitly homeomorphisms between each members of the following family :

Theorem. *Let $\Omega \subset \mathbb{C}$ be the domain defined by $\Omega := \{\alpha \in \mathbb{C} : |\alpha| < \frac{1}{10}\sqrt{\frac{\pi}{2}}\}$. Each member of the family of linear differential equations in \mathbb{C}^2 :*

$$(1.2) \quad x^3 y' = y + x^2 + \alpha x^3, \quad \alpha \in \Omega$$

induces a foliation of $\mathbb{C}P^2$ which is not topologically rigid. More precisely, they all are topologically conjugate to each other whereas two equations with different α 's are not locally analytically conjugate near $(0, 0)$.

In fact, a part of the second statement is given by P. M. Elizarov's result [E], where he describes the local topological classification of saddle-node equations, and by Martinet-Ramis' one [MR] about their local analytical classification. The cornerstone of the proof here is to build an homeomorphism of \mathbb{C}^2 which extends to the whole of $\mathbb{C}P^2$, which was not possible using Elizarov's purely local construction. For technical reasons and the sake of brevity it was necessary to choose Ω as given above, though the result should be valid for all $\alpha \in \mathbb{C} \setminus \{\pm 1\}$. As a real map, the homeomorphisms introduced here are piece-wise affine but could be chosen C^∞ outside $\{x = 0\}$ by taking small perturbations. Yet according to a rigidity result of S. M. Voronin [V] the homeomorphism cannot be made C^1 in any neighbourhood of this line since otherwise the differential equations would be locally analytically conjugate.

Remark. This family of differential equations is not an unfolding in the sense of J.-F. Mattéi [M] since if there existed some germ of a holomorphic function R such that $x^3 dy - (y + x^2 + \alpha x^3) dx + R(x, y, \alpha) d\alpha$ be integrable (as a 1-form) then one would obtain

$$x^3 \frac{\partial R}{\partial x} + (y + x^2 + \alpha x^3) \frac{\partial R}{\partial y} = (1 + 3x^2) R - x^6.$$

On the one hand this equation admits a unique formal solution [T]. On the other hand so is the case for the differential equation $x^3 f' = (1 + 3x^2) f - x^6$. Taking $R(x, y, \alpha) := f(x)$ thus yields the only possible solution. Unfortunately the latter power series is divergent.

2. LOCAL STUDY OF THE SADDLE-NODE SINGULARITY

Definition 2.1. When we say that two foliations are *locally topologically conjugate* (or simply topologically conjugate) near $(0, 0)$ we mean that there exists an open neighbourhood Δ of $(0, 0)$ and an orientation-preserving homeomorphism $\varphi : \Delta \rightarrow \varphi(\Delta)$ fixing $(0, 0)$ which sends a (trace on Δ of a) leaf of one foliation into a leaf of the other. If moreover φ is an analytic map we say that the vector fields are *locally analytically conjugate*.

In the following we will study foliations \mathcal{F} of $\mathbb{C}P^2$ on subdomains Δ of $\mathbb{C}P^2$ and we will implicitly mean that we consider the restriction of \mathcal{F} to Δ , *i.e.* the foliation whose leaves are the connected components of the trace on Δ of the leaves of \mathcal{F} .

2.1. What is known. We first apply the linear change of variables $(y, \alpha) \mapsto (-i\pi y, \sqrt{\frac{2}{\pi}}\alpha)$ in order to transform the family of equations (1.2) into

$$(2.1) \quad x^3 y' = y - \frac{1}{i\pi} x^2 - \frac{\alpha}{i\sqrt{2\pi}} x^3.$$

This change of variables is performed to simplify the upcoming computations. Let us denote by ω_α the differential 1-form representing (2.1) which, in the affine chart $\mathbb{C}^2 = \{(x, y)\}$, can be written as

$$(2.2) \quad \omega_\alpha(x, y) := \left(y - \frac{1}{i\pi} x^2 - \frac{\alpha}{i\sqrt{2\pi}} x^3 \right) dx - x^3 dy.$$

Such a differential form is integrable and induces a foliation on $\mathbb{C}P^2$, which we denote by \mathcal{F}_α , of saddle-node type at $(0, 0)$, having exactly one separatrix passing through this point (namely $\{x = 0\}$). Notice that all leaves of the foliation are transverse to the fibers of the natural projection

$$\Pi : (x, y) \mapsto x$$

except for the separatrices $\{x = 0\} \cup \{x = \infty\}$.

Let us recall two classical results which our argument is partly based upon.

Theorem 2.2. (Elizarov, [E]) *Let E be the space of all saddle-node foliations given by differential forms $(y + R(x, y)) dx - x^3 dy$, where R is a germ of a holomorphic function at $(0, 0)$ with $R(0, 0) = 0$ and $\frac{\partial R}{\partial y}(0, 0) = 0$. This space splits into E_1 and E_2 according to whether a given foliation has one or two separatrices through $(0, 0)$.*

- (1) *The quotient E_1/t_{op} of local topological equivalence classes has cardinality 2.*
- (2) *Two foliations in E_2 are locally topologically conjugate if, and only if, so are their “weak” holonomies, i.e. the holonomies computed on a transversal $\Pi^{-1}(x_0)$ by lifting through Π a generator of the fundamental group of the second separatrix.*

In fact the differential form (2.2) has such a simple form that it is integrable by quadrature, so its invariant of topological classification can be computed explicitly in terms of α (see the end of [MR] for a similar computation, or [T] for a more general one). It then turns out that when $\alpha^2 \neq 1$ all foliations \mathcal{F}_α , which belong to E_1 , are mutually topologically conjugate. Besides $\mathcal{F}_{\pm 1}$ belong to the other equivalence class.

Theorem 2.3. (Martinet-Ramis, [MR]) *The quotient E/ana of local analytic equivalence classes is in one-to-one correspondence with the space $\mathbb{C} \times (\mathbb{C} \times \mathcal{C}\mathbb{C}\{c\})^2 / \sim$. The invariant of Martinet-Ramis is thus a 5-tuple $\mathcal{M} := (\mu, \tau_0, \varphi_0, \tau_1, \varphi_1)$ where $\mu := \frac{\partial^2 R}{\partial x \partial y}(0, 0)$ is the formal invariant, τ_j are scalars and φ_j are germs of a vanishing holomorphic function at 0, modulo the equivalence relation (with evident notations) : $\mathcal{M} \sim \tilde{\mathcal{M}}$ if and only if $\mu = \tilde{\mu}$, $\tau_j = \lambda \tilde{\tau}_{j+k}$, $\varphi_j(c) = \tilde{\varphi}_{j+k}(\lambda c)$ for some $\lambda \in \mathbb{C}_{\neq 0}$ not depending on $j \in \mathbb{Z}/2$ and for some $k \in \mathbb{Z}/2$.*

Remark 2.4. The space E_2 coincides with the space of foliations such that $\tau_0 = \tau_1 = 0$.

The same computations as above yields that α is an analytic invariant, as we will see in the following section. More precisely one can choose \mathcal{M} as follows :

$$\begin{aligned}\tau_0 &:= 1 + \alpha \\ \tau_1 &:= 1 - \alpha \\ \varphi_j &:= 0.\end{aligned}$$

Hence \mathcal{F}_α and \mathcal{F}_β are always topologically conjugate (under the hypothesis $\alpha, \beta \notin \{-1, 1\}$) whereas they are analytically conjugate if, and only if, $\alpha = \beta$.

Remark 2.5.

- (1) These invariants are not the ‘‘genuine’’ Martinet-Ramis invariants, which are more conventionally seen as gluing maps in the sectorial space of leaves, meaning the diffeomorphisms :

$$\begin{aligned}\psi_j^\infty : c &\mapsto c + \tau_j \\ \psi_j^0 : c &\mapsto c \exp(i\pi\mu + \varphi_j(c)).\end{aligned}$$

In the case where $\chi \in E_2$, i.e. $\psi_j^\infty = Id$, its weak holonomy is analytically conjugate to $\psi_0^0 \circ \psi_1^0$, which is a map tangent to $e^{2i\pi\mu} Id$.

- (2) Elizarov’s topological moduli space of E_1 is the set of all pairs $(\varepsilon_0, \varepsilon_1) \in \{0, 1\}^2 \setminus \{(0, 0)\}$ such that $\varepsilon_j = 0$ if, and only if, $\tau_j = 0$ where $(1, 0)$ and $(0, 1)$ are identified.

We propose here to build explicitly a local topological conjugacy between \mathcal{F}_α and \mathcal{F}_0 when $|\alpha| < \frac{1}{10}$. In fact one could achieve the same construction for any value of α but for the sake of concision we only retain this case. Before doing so we begin with describing the setting for any value of α .

2.2. The sectorial decomposition and induced homomorphisms in the spaces of leaves. We split \mathbb{C}^2 into three parts :

$$\mathbb{C}^2 = \mathcal{V}^+ \cup \mathcal{V}^- \cup \{x = 0\}$$

where the sectors \mathcal{V}^\pm are, as usual, defined by

$$\mathcal{V}^\pm := \left\{ (x, y) : \left| \arg x \mp \frac{\pi}{2} \right| < \frac{3\pi}{4} \right\}.$$

We denote \mathcal{F}_α the foliation induced on $\mathbb{C}P^2$ by ω_α and define \mathcal{F}_α^\pm as the restriction of \mathcal{F}_α to \mathcal{V}^\pm . We let $y_{\alpha, c}^\pm$ be the general solution of the differential equation $\omega_\alpha = 0$ for $c \in \mathbb{C}$:

$$(y_{\alpha, c}^\pm) : x \in \Pi(\mathcal{V}^\pm) \mapsto \exp\left(-\frac{1}{2x^2}\right) \left(c - \int_{\pm 0i}^x \left(\frac{1}{i\pi} + \frac{\alpha}{i\sqrt{2\pi}} z \right) \exp\left(\frac{1}{2z^2}\right) \frac{dz}{z} \right)$$

which are holomorphic functions. The integration here is done over a path linking x to 0 in $\Pi(\mathcal{V}^\pm)$ and tangent to the half-line $\pm i\mathbb{R}_{\geq 0}$ at 0. Notice that the intersection $\mathcal{V}^+ \cap \mathcal{V}^-$ is included in the node-part $\{Re(x^{-2}) > \varepsilon > 0\}$ of the saddle-node singularity of \mathcal{F}_α , meaning that any leaf of \mathcal{F}_α^\pm , or of its restriction to any polydisc Δ centered at $(0, 0)$, accumulates on $(0, 0)$ over these sectors. On the contrary only one leaf accumulates on $(0, 0)$ in the saddle-part $\{Re(x^{-2}) < -\varepsilon < 0\}$. It is the leaf corresponding to $y_{\alpha, 0}^\pm$ and we will call it *the sectorial weak separatrix*. Martinet-Ramis invariants measure how going from one sector \mathcal{V}^\pm to the other changes the value of c while remaining on the same global leaf of \mathcal{F}_α . In the special case we are

considering they simply consist in the Stokes coefficients of the linear differential equation $\omega_\alpha = 0$. One can easily check that

$$\begin{aligned} (\forall \operatorname{Re}(x) < 0) y_{\alpha,c}^+(x) &= y_{\alpha,c+1+\alpha}^-(x) \\ (\forall \operatorname{Re}(x) > 0) y_{\alpha,c}^-(x) &= y_{\alpha,c+1-\alpha}^+(x) \end{aligned}$$

so that

$$\begin{aligned} \tau_0 &= 1 + \alpha \\ \tau_1 &= 1 - \alpha. \end{aligned}$$

Indeed the value of the difference $y_{\alpha,c}^+ - y_{\alpha,c}^-$ can be obtained through Hankel's integral representation of $\frac{1}{\Gamma}$:

$$\int_{\gamma_j} z^a \exp\left(\frac{1}{2z^2}\right) \frac{dz}{z^3} = -\frac{2i\pi}{\Gamma(a/2)} \left(\frac{1}{2}\right)^{a/2} (-1)^{aj}$$

where γ_j is a circle tangent at 0 to $i\mathbb{R}$ centered at $(-1)^j$.

We can assume without loss of generality that, up to changing slightly the aperture of the source and target sectors, $\varphi(\mathcal{V}^\pm \cap \Delta) \subset \mathcal{V}^\pm$. Hence, following the same argument as before, any homeomorphism φ conjugating \mathcal{F}_α and \mathcal{F}_0 on some polydisc Δ induces (unique) homeomorphisms ψ^\pm from the sectorial spaces of leaves of \mathcal{F}_α^\pm to the sectorial space of leaves of \mathcal{F}_0^\pm . Since φ must send sectorial weak separatrices of \mathcal{F}_α onto those of \mathcal{F}_0 the homeomorphisms ψ^\pm shall fix 0 and we derive

$$\begin{aligned} \psi^\pm : \mathbb{C} &\rightarrow \mathbb{C} \\ c &\mapsto \psi^\pm(c) \\ 0 &\mapsto 0 \end{aligned}$$

such that

$$\varphi(\{y = y_{\alpha,c}^\pm(x)\}) \subset \{y = y_{0,\psi^\pm(c)}^\pm(x)\}$$

and ψ^\pm conjugate the actions of $c \mapsto c + \tau_j$. See figure 2.1. They will be called *transverse homeomorphisms* in the sequel as they completely determine the change in the transverse structure of the foliations.

On the converse our aim in the rest of Section 2 is to build special transverse homeomorphisms conjugating the Stokes translations in order that they be realized in the (x, y) -space by a local homeomorphism φ . We will later extend it to the whole $\mathbb{C}P^2$ in Section 3.

2.3. The construction on \mathbb{C}^2 . The strategy to build such a φ consists in the following three steps.

- (1) Finding two transverse homeomorphisms ψ^\pm such that

$$(2.4) \quad \begin{cases} \psi^+(c+1-\alpha) &= \psi^-(c) + 1 \\ \psi^-(c+1+\alpha) &= \psi^+(c) + 1 \\ \psi^+(0) = \psi^-(0) &= 0 \\ \lim_{c \rightarrow 0, \infty} \frac{\psi^\pm(c)}{c} &= 1 \end{cases} .$$

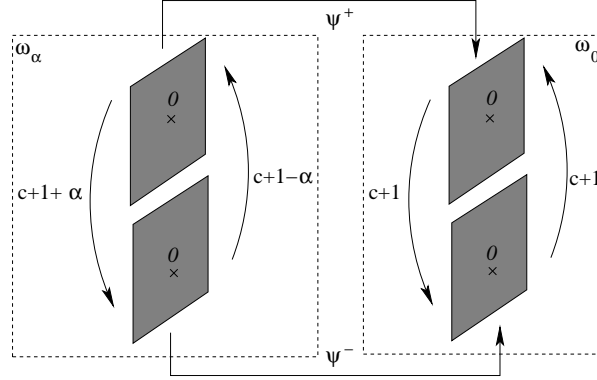


FIGURE 2.1. The induced homeomorphisms ψ^+ and ψ^- between spaces of leaves

Notice the three first conditions are necessarily satisfied by any pair of induced homeomorphisms. Besides the first two relations imply that $\psi^\pm = Id + \eta^\pm$ where η^\pm is 2-periodic.

- (2) Finding a lift φ^\pm of ψ^\pm in the ambient space $\mathcal{V}^\pm \cap \overline{\mathbb{D}} \times \mathbb{C}$ such that $\varphi^+ = \varphi^-$ in $\mathcal{V}^+ \cap \mathcal{V}^-$ (thus defining a homeomorphism φ on $\overline{\mathbb{D}}_{\neq 0} \times \mathbb{C}$).
- (3) Ensuring that φ extends continuously to $\{x = 0\}$. For this we need the last condition in the above system.

To underline that fulfilling these three conditions is tricky we first prove the

Proposition 2.6. *Assume that φ is a local topological conjugacy between \mathcal{F}_α and \mathcal{F}_0 such that φ preserves globally the fibers of Π . Then $\alpha = 0$.*

The meaning of this statement is that, unlike the analytical setting (see [MR]), topological conjugacies between non-analytically conjugate saddle-node foliations cannot be chosen of the form $(x, y) \mapsto (x, Y(x, y))$, nor even of the form $(x, y) \mapsto (X(x), Y(x, y))$, which seriously complicates matters as we will see.

Proof. Let Δ be a polydisc on which φ is realized. For any $\omega \in \mathbb{C}_{\neq 0}$ there exists a sequence $(x_n)_n \subset \mathcal{V}^+$ such that $\exp\left(-\frac{1}{2x_n^2}\right) = \omega$ and $(x_n)_n$ converges towards 0 ; let y be given in order that $(x_n, y)_n \subset \Delta$. By assumption φ takes the form $\varphi(x_n, y) = (X(x_n), Y(x_n, y))$. Since

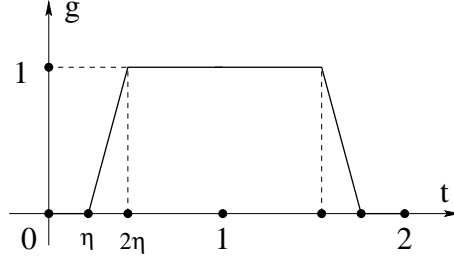
$$(2.5) \quad y_{\alpha, c}^+(x) - y_{\alpha, 0}^+(x) = c \exp\left(-\frac{1}{2x^2}\right)$$

does not depend on α we deduce, by setting $c := y\omega^{-1}$,

$$Y(x_n, y) = y_{0, 0}^+(X(x_n)) + \frac{\psi^+(y\omega^{-1})}{y} (y - y_{\alpha, 0}^+(x_n)) \exp\left(-\frac{1}{2X(x_n)^2}\right).$$

In particular the sequence $\left(\exp\left(-\frac{1}{2}X(x_n)^{-2}\right)\right)_n$ converges towards some complex number $\lambda^+(\omega) \in \mathbb{C}$. Therefore

$$(2.6) \quad \varphi(0, y) = (0, \lambda^+(\omega) \psi^+(y\omega^{-1}))$$


 FIGURE 2.2. The function g .

and $\lambda^+(\omega) \neq 0$. Obviously the same construction can be carried out on \mathcal{V}^- , giving rise to a non-zero vanishing function $\omega \mapsto \lambda^-(\omega)$ such that for all y one has the same relation $\varphi(0, y) = (0, \lambda^-(\omega) \psi^-(y\omega^{-1}))$. Hence $\lambda^+(\omega) \psi^+(y\omega^{-1}) = \lambda^-(\omega) \psi^-(y\omega^{-1})$ for every $(0, y) \in \Delta$. Now we fix ω small enough in order that $|\omega|^{-1} \Delta$ contain the points $(0, y) \in \Delta$ and $(0, y+2)$ for at least one value of y . Using the 2-periodicity of $\psi^\pm - Id$ we derive

$$\lambda^+(\omega) \psi^+(y+2) = \lambda^-(\omega) \psi^-(y+2) + 2(\lambda^+(\omega) - \lambda^-(\omega)),$$

meaning $\lambda^+(\omega) = \lambda^-(\omega)$. Therefore the first two conditions of (2.4) yields $\psi^+(c+1+\alpha) = \psi^+(c+1-\alpha)$ for all $c \in \mathbb{C}$. Since ψ^+ is one-to-one the only possibility is $\alpha = 0$. \square

2.4. The transverse homeomorphisms. To go back to our purpose we first find admissible ψ^\pm .

Proposition 2.7. *Let $|\alpha| < \frac{1}{10}$ and $\eta := (1 - |\operatorname{Re}(\alpha)|)/3$. We define g as to be the simplest piece-wise real affine map g on $[0, 2]$ such that $g|_{[0, \eta]} := 0$, $g|_{[2\eta, 2-2\eta]} := 1$ and $g|_{[2-\eta, 2]} := 0$. We still denote by g its 2-periodic extension to \mathbb{R} . Then the following functions*

$$\begin{aligned} c \mapsto \psi^+(c) &:= c + \alpha g(\operatorname{Re}(c)) \\ c \mapsto \psi^-(c) &:= c - \alpha + \alpha g(1 + \operatorname{Re}(c - \alpha)) \end{aligned}$$

form a pair of homeomorphisms solution to (2.4). Moreover for all $c \in \mathbb{C}$:

$$\left| \frac{\psi^\pm(c)}{c} - 1 \right| < \min\left(\frac{1}{6}, \frac{1}{10|\operatorname{Re}(c)|}\right).$$

Proof. The fact that (ψ^+, ψ^-) is solution to the system (2.4) is clear enough. Besides $|\alpha| < \frac{1}{10}$ so

$$\sup_{t \in [-1, 1]} \left| \frac{g(t)}{t} \right| \leq \frac{1}{2\eta} < \frac{5}{3},$$

while for $|t| \geq 1$ one has $g(t) \leq 1$. Hence

$$\left| \frac{\psi^+(c)}{c} - 1 \right| < |\alpha| \min\left(\frac{5}{3}, \frac{1}{|\operatorname{Re}(c)|}\right).$$

The same kind of estimate arises for the other map, where if $|Re(c)| \in [0, 1]$:

$$\begin{aligned} \left| \frac{\psi^-(c)}{c} - 1 \right| &= \left| \frac{\alpha}{c} (g(1 - Re(\alpha) + Re(c)) - 1) \right| \\ &< \frac{5}{3} |\alpha| < \frac{1}{6} \end{aligned}$$

since $1 + Re(\alpha) - \eta > \frac{3}{5}$. To end the proof we only need to notice that $\psi^\pm - Id$ is $\frac{1}{6}$ -Lipschitz, thus one-to-one and onto \mathbb{C} . \square

2.5. The homeomorphism φ . We look now for φ on $\overline{\mathbb{D}} \times \mathbb{C}$ as we'll extend it to the entire projective plane in the next section. As was noticed in Proposition 2.6 we cannot preserve globally the x -variable in the four directions $\{\cos \arg x^2 = 0\}$. Hence we build a new (sectorial) variable $X^\pm(x, y)$ which will mostly be the identity except in the neighbourhood of those forbidden directions. We define

$$X^\pm(x, y) := x(1 - 2x^2 \log f^\pm(x, c^\pm(x, y)))^{-1/2}$$

where f^\pm is a functional parameter which will be adjusted in the sequel to suit our needs, and where c^\pm is the function holomorphic on \mathcal{V}^\pm defined by the relation

$$y_{\alpha, c^\pm(x, y)}(x) = y.$$

For any fixed x the partial function $y \mapsto c^\pm(x, y)$ is a diffeomorphism of \mathbb{C} .

In order to send a leave of \mathcal{F}_α^\pm into a leave of \mathcal{F}_0^\pm while changing the transverse structure we take the new y -variable as being the following

$$Y^\pm(x, y) := y_{0,0}^\pm(X^\pm(x, y)) + f^\pm(x, c^\pm(x, y)) \psi^\pm(c^\pm(x, y)) \exp\left(-\frac{1}{2x^2}\right)$$

and set

$$(2.7) \quad \varphi^\pm(x, y) := (X^\pm(x, y), Y^\pm(x, y)).$$

If we want that φ^+ and φ^- glue on each connected component of $\mathcal{V}^+ \cap \mathcal{V}^-$ we must require that for all $c \in \mathbb{C}$:

$$\begin{cases} f^+(x, c) = f^-(x, c + 1 + \alpha) & , \forall Re(x) < 0 \\ f^-(x, c) = f^+(x, c + 1 - \alpha) & , \forall Re(x) > 0 \end{cases}.$$

If we moreover wish that φ^\pm extend to Id on $\{x = 0\}$ the parameters f^\pm must satisfy the additional condition that for all y_0 :

$$\lim_{(x, y) \rightarrow (0, y_0)} f^\pm(x, c^\pm(x, y)) \frac{\psi^\pm(c^\pm(x, y))}{c^\pm(x, y)} = 1.$$

The following lemma is straightforward to prove :

Lemma 2.8. *Let (χ_1, χ_2) be the simplest non-negative affine partition of unity of the circle $\mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ such that $\chi_1\left(\frac{\pi}{4} + k\frac{\pi}{2}\right) = 1$ for any $k \in \mathbb{Z}/4\mathbb{Z}$, $\chi_2(\theta) = 1$ whenever $|\cos(2\theta)| > \delta$ for some small, fixed $\delta > 0$ and $\chi_1 + \chi_2 = 1$. Define the functions*

$$f^\pm(x, c) := \chi_1(\arg x) \frac{c}{\psi^\pm(c)} + \chi_2(\arg x).$$

These functions satisfy the following properties :

- (1) $|f^\pm(x, c) - 1| < \frac{1}{5}$ and $|2x^2 \log f^\pm(x, c)| < \frac{3}{5} |x|^2$ whenever $|x| \leq 1$,
- (2) f^\pm is continuous on $\overline{\mathbb{D}} \times \mathbb{C}$,

- (3) f^\pm is constant to 1 on $\mathcal{V}^+ \cap \mathcal{V}^-$,
- (4) $\lim_{(x,y) \rightarrow (0,y_0)} f^\pm(x, c^\pm(x,y)) \frac{\psi^\pm(c^\pm(x,y))}{c^\pm(x,y)} = 1$ for all $y_0 \in \mathbb{C}$,
- (5) for all fixed x the maps $c \mapsto f^\pm(x, c) \psi^\pm(c)$ are homeomorphisms of the complex line.

As a consequence the map φ thus defined is a continuous map from $\overline{\mathbb{D}} \times \mathbb{C}$ conjugating the foliations \mathcal{F}_α and \mathcal{F}_0 on this domain.

Proposition 2.9. *The map φ is one-to-one and thus defines a homeomorphism from $\overline{\mathbb{D}} \times \mathbb{C}$ onto its image $W \times \mathbb{C}$ which, up to rescaling φ in the first coordinate for both source and target spaces, contains $\overline{\mathbb{D}} \times \mathbb{C}$.*

Proof. Firstly we shall prove the latter claim. Let us write $\varphi = (X, Y)$. Because of the first statement of the previous lemma we have

$$\left| \frac{X(x, y)}{x} - 1 \right| \leq A|x|^2$$

for some $A > 0$ and all $|x| \leq 1$. This implies that for $|x| < \delta$ small enough $\varphi(\mathcal{V}^\pm \cap \delta\mathbb{D} \times \mathbb{C})$ contains a sector $W^\pm := \{x : |x| < r, |\arg x \mp \frac{\pi}{2}| < \frac{3\pi}{4} - \theta\}$, where θ can be chosen as small as we wish by decreasing δ . Up to rescaling the x - and X -coordinates we can then assume that $\overline{\mathbb{D}} \subset X(\overline{\mathbb{D}} \times \mathbb{C})$. Because of (5) we also derive that $\varphi(\mathcal{V}^\pm \cap \delta\mathbb{D} \times \mathbb{C})$ contains a sector of lesser aperture $W^\pm \times \mathbb{C}$, so that $\overline{\mathbb{D}} \times \mathbb{C} \subset \varphi(\overline{\mathbb{D}} \times \mathbb{C})$ as required.

To prove that φ is one-to-one we first notice that since φ preserves the sector decomposition of $\overline{\mathbb{D}} \times \mathbb{C}$ and since each leaf of the sectorial foliations \mathcal{F}_α^\pm is the graph of a function holomorphic on W^\pm , we only need to prove that the restriction of φ^\pm to some transversal $\{x = x_0\}$ is one-to-one. Let us choose $x_0 := \pm 1$, so that $f^\pm(x_0, c) = 1$. We thus have that

$$\varphi^\pm(x_0, y) = \left(x_0, y_{0,0}(x_0) + \psi^\pm(c^\pm(x_0, y)) \exp\left(-\frac{1}{2x_0^2}\right) \right),$$

which completes the proof. □

3. EXTENDING THE HOMEOMORPHISM

The foliations under consideration have exactly three singularities, located in homogeneous coordinates at $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. We will use the following three affine charts of $\mathbb{C}P^2$:

$$\begin{aligned} \mathbb{C}^2 &= \{(x, y)\} = \{[x : y : 1]\} \\ \mathbb{C}^2 &= \{(s, t)\} = \{[1 : t : s]\} \\ \mathbb{C}^2 &= \{(u, v)\} = \{[u : 1 : v]\} \end{aligned}$$

with transition maps

$$y = tx, 1 = sx, x = uy, 1 = vy, v = us, 1 = ut \quad .$$

The singular point $[0 : 0 : 1]$ has been extensively studied in the previous sections and this study gave rise to a topological conjugacy between the foliations \mathcal{F}_0 and \mathcal{F}_α from $\overline{\mathbb{D}} \times \mathbb{C}$ onto its image.

Lemma 3.1. *One can extend φ to a homeomorphism of $\mathbb{C} \times \mathbb{C}$ such that the extension, still noted φ , preserves each fiber of Π outside $\overline{\mathbb{D}} \times \mathbb{C}$ and still conjugates \mathcal{F}_0 and \mathcal{F}_α .*

Proof. This result is straightforward. Choose $0 < r < 1$ and consider the simplest affine non-negative partition of unity (ξ_1, ξ_2) of $\mathbb{R}_{\geq 0}$ where $\xi_1 = 1$ on $[0, r]$, $\xi_2 = 1$ outside $[0, 1]$ and $\xi_1 + \xi_2 = 1$. By setting

$$\begin{aligned} (\forall (x, c) \in \overline{\mathbb{D}} \times \mathbb{C}) \quad \hat{f}^\pm(x, c) &:= f^\pm(x, c) \xi_1(|x|) + \xi_2(|x|) \\ (\forall (x, c) \notin \overline{\mathbb{D}} \times \mathbb{C}) \quad \hat{f}^\pm(x, c) &:= 1 \end{aligned}$$

and defining φ by (2.7) the reader can easily show that our claim is true, as in Proposition 2.8. \square

Once this is stated we have to check that φ extends to a homeomorphism of $\mathbb{C}P^2$. It is only a matter of writing things in appropriate charts since as expected φ extends to Id along the line at infinity $\{s = 0\} \cup \{v = 0\}$.

Proposition 3.2. *φ extends to a global homeomorphism of $\mathbb{C}P^2$, which implies that the main theorem is true.*

Proof. Let us write φ in the chart (s, t) near $\{0\} \times \mathbb{C}$:

$$\tilde{\varphi}(s, t) := \left(s, s \left(y_{0,0}^\pm \left(\frac{1}{s} \right) + \psi^\pm(\tilde{c}^\pm(s, t)) \exp\left(-\frac{s^2}{2}\right) \right) \right)$$

where as before \tilde{c}^\pm is uniquely defined on \mathcal{V}^\pm by

$$y_{\alpha, \tilde{c}^\pm(s, t)}^\pm \left(\frac{1}{s} \right) = \frac{t}{s},$$

for all $(s, t) \in \mathbb{C}_{\neq 0} \times \mathbb{C}$. Following (2.3) we have

$$y_{\alpha, 0}^\pm \left(\frac{1}{s} \right) = -\exp\left(-\frac{s^2}{2}\right) \int_s^{\pm\infty i} \left(1 + \frac{\alpha}{z}\right) \exp\left(\frac{z^2}{2}\right) \frac{dz}{z}$$

so that

$$\lim_{s \rightarrow 0} s y_{\alpha, 0}^\pm \left(\frac{1}{s} \right) = 0.$$

On the other hand if $(s, t) \in \mathcal{V}^\pm$ then $t = s \left(y_{0,0}^\pm \left(\frac{1}{s} \right) + \tilde{c}^\pm(s, t) \exp(-s^2/2) \right)$ so that setting

$$\tilde{\varphi}(0, t) := (0, t)$$

defines a continuous extension of $\tilde{\varphi}$ to $\mathbb{C} \times \mathbb{C}$, because $\psi^\pm(\tilde{c}^\pm(s, t)) - \tilde{c}^\pm(s, t)$ remains bounded as $(s, t) \rightarrow (0, t_0)$.

Finally we shall check that φ admits a limit at $[0 : 1 : 0]$. We write it in the chart (u, v) as

$$\hat{\varphi}(u, v) := \left(\hat{U}^\pm(u, v), \hat{V}^\pm(u, v) \right)$$

where

$$\begin{aligned} \hat{X}^\pm(u, v) &= \frac{u}{v \sqrt{1 - 2 \frac{u^2}{v^2} \log f^\pm\left(\frac{u}{v}, \hat{c}^\pm(u, v)\right)}} \\ (3.1) \quad \hat{\mathbb{I}}^\pm(u, v) &= \frac{1}{y_{0,0}^\pm\left(\hat{X}^\pm(u, v)\right) + f^\pm\left(\frac{u}{v}, \hat{c}^\pm(u, v)\right) \psi^\pm\left(\hat{c}^\pm(u, v)\right) \exp\left(-\frac{v^2}{2u^2}\right)} \\ \hat{U}^\pm(u, v) &= \hat{V}^\pm(u, v) \hat{X}^\pm(u, v) \end{aligned}$$

and $\hat{c}^\pm(u, v)$ is uniquely defined by $v^{-1} = y_{\alpha,0}(u/v) + \hat{c}^\pm(u, v) \exp(-v^2/2u^2)$ for $(\frac{u}{v}, \frac{1}{v}) \in \mathcal{V}^\pm$. Let us split $\mathbb{C}_{\neq 0}^2$ into the sets

$$\begin{aligned} \mathcal{C}^> &:= \left\{ (u, v) : \left| \frac{u}{v} \right| > 1 \right\} \\ \mathcal{C}^\leq &:= \left\{ (u, v) : \left| \frac{u}{v} \right| \leq 1 \right\}. \end{aligned}$$

We recall that $f^\pm(\frac{u}{v}, \hat{c}^\pm(u, v)) = 1$ whenever $(u, v) \in \mathcal{C}^>$. On the other hand $\hat{X}^\pm(\mathcal{C}^\leq)$ is bounded. Hence it suffices to show that $\hat{V}^\pm(u, v) = O(v)$ as $(u, v) \rightarrow 0$ in order to prove that $\hat{\varphi}$ extends continuously to $(0, 0)$ by $\hat{\varphi}(0, 0) := (0, 0)$.

Because $x \mapsto y_{0,0}^\pm(x)$ is smooth as a real map there exists a constant $A > 0$ such that for all $(u, v) \in \mathcal{C}^<$ one has

$$\left| y_{0,0}^\pm(\hat{X}^\pm(u, v)) - y_{0,0}^\pm\left(\frac{u}{v}\right) \right| \leq A \left| \frac{u^3}{v^3} \right|$$

whereas this estimate is true with $A := 0$ when $(u, v) \in \mathcal{C}^>$. We then derive :

$$\begin{aligned} \left| \frac{1}{\hat{V}^\pm(u, v)} - \frac{1}{v} \right| &\leq A \left| \frac{u^3}{v^3} \right| + \left| \hat{c}^\pm(u, v) - f^\pm\left(\frac{u}{v}, \hat{c}^\pm(u, v)\right) \right| \psi^\pm(\hat{c}^\pm(u, v)) \left| \exp(-v^2/2u^2) \right| \\ \left| \frac{v}{\hat{V}^\pm(u, v)} - 1 \right| &\leq A |u| \left| \frac{u}{v} \right|^2 + B |v| \left| \exp(-v^2/2u^2) \right| \end{aligned}$$

for some $B > 0$. Hence there exists $B^> > 0$ such that for all $(u, v) \in \mathcal{C}^>$:

$$\left| \frac{v}{\hat{V}^\pm(u, v)} - 1 \right| < B^> |v|.$$

Consider now $(u, v) \in \mathcal{C}^\leq$. Clearly there exists $B_+^\leq > 0$ such that if $\operatorname{Re}\left(\frac{u^2}{v^2}\right) \geq 0$ then

$$\left| \frac{v}{\hat{V}^\pm(u, v)} - 1 \right| < B_+^\leq (|u| + |v|)$$

while according to (3.1) there exists $B_-^\leq > 0$ such that if $\operatorname{Re}\left(\frac{u^2}{v^2}\right) < 0$ then

$$\left| \hat{V}^\pm(u, v) \right| < B_-^\leq |v|.$$

□

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