Natural closures, natural compositions and natural sums of monotone operators

Jean-Paul Penot

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Abstract

We introduce new methods for defining generalized sums of monotone operators and generalized compositions of monotone operators with linear maps. Under asymptotic conditions we show these operations coincide with the usual ones. When the monotone operators are subdifferentials of convex functions, a similar conclusion holds. We compare these generalized operations with previous constructions by Attouch-Baillon-Théra, Revalski-Théra and Pennanen-Revalski-Théra. The constructions we present are motivated by fuzzy calculus rules in nonsmooth analysis. We also introduce a convergence and a closure operation for operators which may be of independent interest.
Key words: asymptotic analysis, composition, fuzzy sum, monotone operator, natural topology, natural sum, variational sum.

Mathematics Subject Classification: 47H05

Fermeture naturelle, composition et somme d'opérateurs maximaux monotones

Résumé:

Par des méthodes nouvelles, nous introduisons des sommes généralisées d'opérateurs monotones et des compositions généralisées d'un opérateur monotone et d'une application linéaire. Sous des conditions asymptotiques, nous montrons que ces opérations coïncident avec les opérations usuelles. Quand les opérateurs monotones sont les sous-différentiels de fonctions convexes, une conclusion semblable a lieu. Nous comparons ces opérations généralisées avec des constructions dues à Attouch-Baillon-Théra, Revalski-Théra et Pennanen-Revalski-Théra. Les constructions que nous présentons sont motivées par les règles de calcul flou de l'analyse non-lisse. Nous introduisons aussi une convergence et une fermeture pour les opérateurs qui peuvent avoir un intérêt propre.

Mots clés : analyse asymptotique, composition naturelle, opérateur monotone, somme floue, somme naturelle, topologie naturelle.

Dedicated to Haim Brézis on the occasion of his sixthieth birthday

1 Introduction

It is well-known that the sum of two maximal monotone operators and the composition of a maximal monotone operator with a continuous linear map are not always maximal monotone operators. Several studies have been devoted to sufficient conditions ensuring preservation of maximal monotonicity (see [3], [6], [13], [17], [57] for instance). Another direction has been

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openned by Attouch-Baillon-Théra in [5]. In that paper the authors introduce a generalized sum called the variational sum which coincides with the ordinary sum when the latter is maximal monotone. In [53], [54] and [41] this construction is extended from the setting of Hilbert spaces to the case of reflexive Banach spaces. In both cases the construction relies on the Yosida approximations of the operators. One may wonder whether the use of this specific approximation is crucial or not.

In order to tackle such a question, we introduce here other generalized operations which stem from quite different an idea. It arises from an insight in nonsmooth analysis in which the role of fuzziness is crucial. In general, the subdifferential of a sum of two lower semicontinuous functions cannot be given as the sum of the subdifferentials of the functions. Instead, in nice enough Banach spaces (say Asplund spaces for the Fréchet subdifferential) it can be approximated by sums of elements of the subdifferentials at neighboring points ([10], [21], [22], [24], [25], [38], [39]...). For closed convex functions, a similar phenomenon holds in any Banach space ([45], [61], [62]). Taking into account the close relationships between convexity and monotonicity (see for instance [47], [48], [59] and their references), we extend to monotone operators the fuzziness which occurs with subdifferentials (sections 3 and 4). In such a way, we get a notion which is more flexible than the one obtained by using the Yosida regularization; in particular, it can be defined for any type of operators. Since several types of generalized monotone operators are known to be of great importance for the solutions of nonlinear equations and variational inequalities (see [12], [34] and [2] for instance) such a feature is of interest. Also, it is not clear whether the variational sum of [5] (resp. the variational composition of [41]) depends on the specific type of regularization. We give a partial answer for the case of the generalized Yosida regularization studied in [49]. Moreover, our generalized sum and composition contain the ordinary sum and ordinary composition respectively, a property which may not be satisfied with the variational concepts. On the other hand, we have to check in each specific situation whether the obtained operator is not too large and whether it is still monotone when the operators are monotone; we present some results in this direction. We also present some conditions ensuring that the extensions we study coincide with the ordinary composition or sum. These conditions rely on some concepts of asymptotic analysis. In section 6 we compare the notions obtained with such a process with the notions introduced previously with approximations, showing coincidence under the assumption that the "natural" closure of the composition is maximal monotone; a similar result holds for sums.

In section 2 we introduce a topology on the product $X \times X^*$ of a Banach space with its dual space which has the pleasant property that the closure of a monotone operator in that topology (called hereafter the natural closure) is large enough and is still monotone; in particular, any maximal monotone operator is closed in that topology. The simple construction we give arises from the observation made in [42] (and probably elsewhere) that the duality coupling is not continuous for the product of the strong topology with the weak* topology unless X is finite dimensional. It is also related to the fact that a Banach space X such that the graph of any maximal monotone operator $M : X \rightrightarrows X^*$ is closed for the product of the norm topology with the bounded weak* topology is finite dimensional ([8, Thm 3]). Such a fact shows the necessity to adopt another topology when dealing with monotone operators in infinite dimensional spaces. This closure process may have an independent interest, but it is clearly related with the constructions we adopt for generalized sums and compositions.

Let us observe that both approximations and fuzziness appear in calculus rules for subdifferentials of nonsmooth, nonconvex functions: a decoupling procedure is usually applied to get them via a penalization method which has some similarity with the Moreau-Yosida approximation. Therefore, it is natural that these two means can be used for calculus rules of monotone operators. It is our hope that concrete applications will make use of both processes; in a subsequent work we intend to deal with the case of elliptic operators and Nemyckii operators in the classical sense or in the sense of [1].

2 The natural closure of an operator

In the sequel X is a Banach space with dual space X^*; we are mainly interested in the case X is reflexive. The domain of a multimapping (or operator) M is denoted by $\mathrm{D}(M)$; we identify M with its graph $\mathrm{G}(M)$ whenever there is no risk of confusion. The following topology on the product of X with its dual X^* has been used in [49]; for related observations see [42]. The symmetry of the notion could be increased by considering the case of two spaces in separating duality.

Definition 1 *The coarsest topology on $X \times X^*$ which makes continuous the mappings*

$$\begin{aligned} &(x, x^*) \mapsto \langle w^*, x\rangle && (w^* \in X^*) \\ &(x, x^*) \mapsto \langle x^*, w\rangle && (w \in X) \\ &(x, x^*) \mapsto \langle x^*, x\rangle && \end{aligned}$$

will be called the natural topology ν on $X \times X^*$.

It follows that for every $(w, w^*) \in X \times X^*$the functions $(x, x^*) \mapsto \langle w^*, x\rangle + \langle x^*, w\rangle$ are continuous for the topology ν; thus ν is stronger than the weak topology on $X \times X^*$. When X is finite dimensional, the natural topology ν is just the product topology. The convergence associated with the topology ν is given by:

$((x_i, x_i^*))_{i\in I} \to (x, x^*)$ for ν iff $(x_i)_{i\in I} \xrightarrow{\sigma} x$, $(x_i^*)_{i\in I} \xrightarrow{\sigma^*} x^*$, $(\langle x_i^*, x_i\rangle)_{i\in I} \to \langle x^*, x\rangle$,
where σ (resp. σ^*) is the weak (resp. weak*) topology on X (resp. X^*).

Thus the convergence associated with ν is weaker than the product convergence of the strong convergence on X with the weak-star convergence of bounded nets on X^*. In particular, a sequence $((x_n, x_n^*))_n$ converges to (x, x^*) for ν if $(x_n) \to x$ and $(x_n^*) \xrightarrow{\sigma^*} x^*$. The natural convergence is also weaker than the topology γ considered by Gossez in [26] which is defined as the product of the strong topology on X with the weakest topology on X^* for which the functions $x^* \mapsto \|x^*\|$ and $x^* \mapsto \langle x^*, w\rangle$ for $w \in X$ are continuous. In fact, if a net $((x_i, x_i^*))_{i\in I}$ converges to (x, x^*) for the topology γ, there exist $k \in I$ and $r > 0$ such that $\|x_i^*\| \le r$ for $i \in I$, $i > k$, so that

$$\begin{aligned} |\langle x_i^*, x_i\rangle - \langle x^*, x\rangle| &\le |\langle x_i^*, x_i - x\rangle| + |\langle x_i^* - x^*, x\rangle| \\ &\le \|x_i^*\| . \|x_i - x\| + |\langle x_i^* - x^*, x\rangle| \to 0 \end{aligned}$$

and $((x_i, x_i^*))_{i\in I}$ converges to (x, x^*) for the natural topology ν.

Although ν is not compatible with the linear structure of $X \times X^*$, it enjoys separate compatibility in the following sense: if two nets $((x_i, x_i^*))_{i\in I}$ $((x_i, y_i^*))_{i\in I}$ converge to (x, x^*) and (x, y^*) respectively for the topology ν, then, for any $r, s \in \mathbb{R}$, the net $((x_i, rx_i^* + sy_i^*))_{i\in I}$ converges to $(x, rx^* + sy^*)$ for ν; a similar assertion holds when interchanging the roles of the variables. In particular, if a multimapping $M : X \rightrightarrows X^*$ has convex images, its closure $\overline{M}$ for the natural topology also has convex images. We also note that if $P := M \times N$, where $M : X \rightrightarrows X^*$, $N : Y \rightrightarrows Y^*$, then $\overline{P} = \overline{M} \times \overline{N}$, as the natural topology on $X \times Y \times X^* \times Y^*$ is the product of the natural topologies on $X \times X^*$ and $Y \times Y^*$, as easily checked.

Although ν is rather weak, it is adapted to the study of monotone operators as it satisfies the following properties.

Proposition 2 *Any maximal monotone operator (identified with its graph) is closed in the natural topology.*

This property is an immediate consequence of the fact that the closure of (the graph of) a monotone operator in the natural topology is a monotone operator.

In turn, this property follows from a result of independent interest contained in the next lemma. Here we make use of the notion of *monotone polar* T^0 of a subset T of $X \times X^*$ explicitely introduced to us by Martínez-Legaz, (but implicitely used in [13], [52] for example), given by

$$T^0 := \{(x, x^*) \in X \times X^* : \forall (w, w^*) \in T, \ \langle x^* - w^*, x - w \rangle \geq 0\}.$$

Clearly, one has the equivalence

$$S \subset T^0 \Leftrightarrow T \subset S^0,$$

hence, taking $S := T^0$, $T \subset T^{00}$. Moreover, the definitions show the following equivalences

$$T \text{ is monotone } \Leftrightarrow T \subset T^0, \tag{1}$$

$$T \text{ is maximal monotone } \Leftrightarrow T = T^0. \tag{2}$$

It is shown in [52, Thm 2.5] that if M is a linear monotone operator, then M is maximal monotone iff its domain $\mathrm{D}(M)$ is dense in X and $\mathrm{D}(M^0) = \mathrm{D}(M)$.

Lemma 3 *Let S and T be subsets of $X \times X^*$ such that $S \subset T^0$. Then, the natural closures $\overline{S}$ and $\overline{T}$ of S and T respectively satisfy $\overline{S} \subset \overline{T}^0$.*

In particular, for any subset T of $X \times X^*$, one has $\overline{T}^0 = T^0$ and T^0 is closed in the natural topology.

Proof. Let (x, x^*) be in the closure $\overline{S}$ of S and let $(w, w^*) \in T$. Then there exists a net $((x_i, x_i^*))_{i \in I}$ in S with limit (x, x^*) in the natural topology, so that one has $\langle w^*, x \rangle = \lim_i \langle w^*, x_i \rangle$, $\langle x^*, w \rangle = \lim_i \langle x_i^*, w \rangle$, $\langle x^*, x \rangle = \lim_i \langle x_i^*, x_i \rangle$ and one gets

$$\langle x^* - w^*, x - w \rangle = \langle x^*, x \rangle - \langle x^*, w \rangle - \langle w^*, x \rangle + \langle w^*, w \rangle \tag{3}$$

$$= \lim_i (\langle x_i^*, x_i \rangle - \langle x_i^*, w \rangle - \langle w^*, x_i \rangle + \langle w^*, w \rangle) \tag{4}$$

$$= \lim_i \langle x_i^* - w^*, x_i - w \rangle \geq 0.$$

Thus $\overline{S} \subset T^0$. It follows that $T \subset \overline{S}^0$, hence, by what precedes, $\overline{T} \subset \overline{S}^0$ or $\overline{S} \subset \overline{T}^0$.

Taking $S = T^0$ and observing that $\overline{T}^0 \subset T^0$, we get $\overline{T^0} \subset \overline{T}^0 \subset T^0$, so that T^0 is closed and coincides with $\overline{T}^0$. □

The preceding closedness result can be reformulated as follows. Here we say that an operator S is *co-monotone* if there exists $T \subset X \times X^*$ such that $S = T^0$.

Corollary 4 *Any co-monotone operator S is closed in the natural topology.*

In particular, any maximal monotone operator being co-monotone, we get the closedness result of Proposition 2.

Taking $S = T$ in the first assertion of Lemma 3, we get the announced preservation of monotonicity:

Corollary 5 *For any monotone operator T, the closure $\overline{T}$ of T in the natural topology is monotone.*

Remark. This result is also a consequence of Proposition 2 since by Zorn lemma T is contained in a maximal monotone operator M, so that $\overline{T}$ is contained in $\overline{M} = M$, hence is monotone. We owe to C. Zalinescu the observation that one can avoid the use of the Zorn lemma by using (1) and by applying Lemma 3 with $S = T$. □

The weakness of the topology ν is an advantage when looking for a maximal monotone extension of a monotone operator M: since the natural closure $\overline{M}$ of M is large and still monotone, it is more likely maximal monotone than the closure for the norm topology or for the Gossez topology. In the same line of thought, Proposition 2 shows that there is no hope of finding a maximal monotone operator (whose graph) is not closed in the natural topology.

A comparison with enlargements is made in the following corollary; for more on the topic of enlargements, see [14], [15], [16], [36], [54], [60] for instance.

Corollary 6 *For any monotone operator M, the closure $\overline{M}$ of M in the natural topology satisfies $\overline{M} \subset M^0 = \bigcap_{\varepsilon>0} M^\varepsilon$, where*

$$M^\varepsilon := \{(x, x^*) \in X \times X^* : \ \forall (w, w^*) \in M, \ \langle x^* - w^*, x - w\rangle \geq -\varepsilon\}.$$

Proof. This follows from Lemma 3 by setting $S := M$, $T := \bigcap_{\varepsilon>0} M^\varepsilon$, observing that, for every $\varepsilon > 0$, M^ε is closed in the natural topology and that $S \subset S^0 \subset M^\varepsilon$ □

The following result shows that a classical argument about limits of operators (see [4, Prop. 3.59]) has a natural place in the present framework.

Proposition 7 *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of monotone operators and let A be a maximal monotone operator such that $A \subset \liminf_{n\in\mathbb{N}} A_n$ (for the strong topology on $X \times X^*$). Then, the sequential $\limsup_{n\in\mathbb{N}} A_n$ in the natural topology is contained in A, so that $A = \lim_{n\in\mathbb{N}} A_n$ in the natural topology and in the strong topology.*

Proof. Let (x, x^*) be the natural limit of a sequence $((x_n, x_n^*))_{n\in\mathbb{N}}$ such that $(x_n, x_n^*) \in A_{p(n)}$ for an increasing map $n \mapsto p(n)$ of $\mathbb{N}$ into $\mathbb{N}$. Given $(w, w^*) \in A$ we can find a sequence $((w_n, w_n^*))_{n\in\mathbb{N}} \to (w, w^*)$ strongly. Then, since a weakly converging sequence is bounded, the definition of the natural convergence shows that

$$\langle x^* - w^*, x - w\rangle = \lim_{n\in\mathbb{N}}(\langle x_n^*, x_n\rangle - \langle x_n^*, w_{p(n)}\rangle - \langle w_{p(n)}^*, x_n\rangle + \langle w_{p(n)}^*, w_{p(n)}\rangle) \geq 0.$$

Since A is maximal monotone, we get $(x, x^*) \in A$. □

3 Passages between sums and composition

Before extending these usual operations, let us note that they are closely related.

Let us first note that the sum $S := M + N$ of two operators $M, N : X \rightrightarrows X^*$ can be obtained as a composition $S = A^T \circ (M \times N) \circ A$, where $A : X \to X \times X$ is the diagonal mapping given by $A(x) := (x, x)$ and $(M \times N)(x', x'') := M(x') \times N(x'')$. This follows from the computation of A^T :

$$A^T(x^*, y^*) = x^* + y^*.$$

We note that $M \times N$ is monotone whenever M and N are monotone (and even maximal monotone if M and N are maximal monotone and X is reflexive).

Now let us show that if $A : X \to Y$ is a densely defined linear mapping and if $M : Y \rightrightarrows Y^*$ is an operator, the composition $A^T \circ M \circ A$ can be obtained with the help of the sum operation. For that purpose, we associate to A and M the operators $B, C : X \times Y \rightrightarrows X^* \times Y^*$ given by

$$\begin{aligned} B(x, y) &:= \{(A^T y^*, -y^*) : y^* \in Y^*\} \text{ if } y = A(x), \ \ \emptyset \text{ otherwise}, \\ C(x, y) &:= \{0\} \times M(y) \ \text{ for any } (x, y) \in X \times Y. \end{aligned}$$

Then we have

$$(B + C)(x, y) = \{(A^T y^*, -y^* + z^*) : y^* \in Y^*, \ z^* \in M(Ax)\} \text{ if } y = A(x), \ \ \emptyset \text{ otherwise.}$$

Therefore

$$\left(A^T \circ M \circ A\right)(x) = \{x^* \in X^* : (x^*, 0) \in (B + C)(x, Ax)\},$$

or equivalently, $(x, x^*) \in A^T \circ M \circ A$ iff $(x, Ax, x^*, 0) \in \mathrm{G}(B+C)$, the graph of $B+C$. We note that B is maximal monotone, that C is monotone when M is monotone and C is maximal monotone when C is maximal monotone (we are indebted to J. Revalski for this last observation). .

We choose to treat first composition, not only because composition is crucial from the point of view of the theory of categories, but also because it is notationally simpler.

4 Natural composition and fuzzy composition

Let X and Y be Banach spaces and let $A : X \to Y$ be a linear operator with closed graph and dense domain $\mathrm{D}(A)$. Its transpose A^T is the linear operator with closed graph defined by $x^* = A^T y^*$ if $x \mapsto \langle y^*, Ax \rangle$ is the restriction to $\mathrm{D}(A)$ of the continuous linear map x^*. In the sequel M is an operator from Y to Y^*.

The abundance of conditions we impose in the following definition arises from our wish to get as a natural composition an operator which is as close as possible to the ordinary composition. Also, such conditions are inspired by what occurs for the calculus of Fréchet or limiting subdifferentials in Asplund spaces ([9], [10], [21], [24], [25], [38], [39]) for which one is eager to get the most precise information about the subdifferential of a composite function, in spite of the fuzziness of the rule.

Definition 8 *The natural composition of M with A is the set $\left(A^T M A\right)_{nat}$ of pairs $(x, x^*) \in X \times X^*$ which are limits of nets $((x_i, x_i^*))_{i \in I}$ of $\mathrm{D}(A) \times X^*$ in the natural topology such that there exists a net $((y_i, y_i^*))_{i \in I}$ in M with $x_i^* = A^T y_i^*$ for each $i \in I$ and*
$$(\|y_i - Ax_i\|)_{i \in I} \to 0, \quad (\langle y_i^*, Ax_i - y_i \rangle)_{i \in I} \to 0.$$

Let us note that, under the assumption that $(x_i, x_i^*)_{i\in I} \to (x, x^*)$ in the natural topology, the condition $(\langle y_i^*, Ax_i - y_i\rangle)_{i\in I} \to 0$ is equivalent to the condition $(\langle y_i^*, y_i\rangle)_{i\in I} \to \langle x^*, x\rangle$: setting $\varepsilon_i := \langle y_i^*, Ax_i - y_i\rangle$, we have

$$\langle x_i^*, x_i\rangle = \langle A^T y_i^*, x_i\rangle = \langle y_i^*, Ax_i\rangle = \langle y_i^*, y_i\rangle + \varepsilon_i$$

When $x \in \mathrm{D}(A)$, since $\langle y_i^*, Ax_i - Ax\rangle = \langle x_i^*, x_i - x\rangle \to 0$, we have $(\langle y_i^*, y_i - Ax\rangle)_{i\in I} \to 0$. Moreover, when $\mathrm{D}(A) = X$ and A is continuous, we also have $(Ax_i)_{i\in I} \xrightarrow{\sigma} Ax$, $(y_i)_{i\in I} \xrightarrow{\sigma} Ax$.

Let us also introduce a variant whose definition is also motivated by the calculus of subdifferentials of lower semicontinuous convex functions in reflexive Banach spaces ([45], [61], [62]).

Definition 9 *(a) The fuzzy composition of M with A is the set $\left(A^T MA\right)_{fuz}$ of $(x, x^*) \in X \times X^*$ such that there exists sequences $((x_n, x_n^*))_{n\in\mathbb{N}}$ in $X \times X^*$, $((y_n, y_n^*))_{n\in\mathbb{N}}$ in M with $x_n^* = A^T(y_n^*)$ for each $n \in \mathbb{N}$, $((x_n, x_n^*))_{n\in\mathbb{N}} \to (x, x^*)$ strongly and $(\|y_n - Ax_n\|)_{n\in\mathbb{N}} \to 0$, $(\|y_n^*\| . \|y_n - Ax_n\|)_{n\in\mathbb{N}} \to 0$.*

(b) The weak fuzzy composition of M with A is the set $\left(A^T MA\right)_{wf}$ of $(x, x^*) \in X \times X^*$ such that there exists nets $((x_i, x_i^*))_{i\in I}$ in $X \times X^*$, $((y_i, y_i^*))_{i\in I}$ in M with $x_i^* = A^T(y_i^*)$ for each $i \in I$, $(x_i^*)_{i\in I}$ is bounded, $(x_i^*)_{i\in I} \to (x^*)$ weakly and $(\|x_i - x\|)_{i\in I} \to 0$, $(\|y_i - Ax_i\|)_{i\in I} \to 0$, $(\|y_i^*\| . \|y_i - Ax_i\|)_{i\in I} \to 0$.*

(c) The sequential weak fuzzy composition of M with A is the set $\left(A^T MA\right)_{swf}$ obtained by replacing nets by sequences in (b).

The following inclusions are obvious:

$$\left(A^T MA\right)_{fuz} \subset \left(A^T MA\right)_{swf} \subset \left(A^T MA\right)_{wf} \subset \left(A^T MA\right)_{nat}.$$

The following result is an easy consequence of the definitions. Let us note that the fuzzy composition (hence the natural composition $\left(A^T MA\right)_{nat}$) is always larger than the ordinary composition $A^T MA$; the similar inclusion for the variational composition of [41] requires particular assumptions.

Proposition 10 *The natural composition $\left(A^T MA\right)_{nat}$ of M with A contains the closure of $A^T MA := A^T \circ M \circ A$ in the natural topology: $\overline{A^T MA} \subset \left(A^T MA\right)_{nat}$. The strong closure of $A^T MA$ is contained in $\left(A^T MA\right)_{fuz}$. The sequential closure of $A^T MA$ in the product of the strong convergence with the weak* convergence is contained in $\left(A^T MA\right)_{swf}$.*

Proof. Given $(x, x^*) \in \overline{A^T MA}$, there exists a net $(x_i, x_i^*)_{i\in I}$ in $A^T MA$ with limit (x, x^*) in the natural topology. Then, setting $y_i := Ax_i$, and picking $y_i^* \in My_i$ such that $x_i^* = A^T y_i^*$ we see that the conditions of the definition of $\left(A^T MA\right)_{nat}$ are fulfilled: $(x, x^*) \in \left(A^T MA\right)_{nat}$. The inclusion of the strong closure of $A^T MA$ in $\left(A^T MA\right)_{fuz}$ is obtained similarly as is the last inclusion. □

Proposition 11 *For any operators $M, N \subset Y \times Y^*$ such that $M \subset N^0$ one has*

$$\left(A^T MA\right)_{fuz} \subset \left(A^T MA\right)_{swf} \subset \left(A^T MA\right)_{wf} \subset \left(A^T MA\right)_{nat} \subset (A^T NA)^0.$$

In particular $\left(A^T MA\right)_{nat} \subset \left(A^T M^0 A\right)^0$.

Proof. Let us show that for any $(x, x^*) \in \left(A^T M A\right)_{nat}$ and $(w, w^*) \in A^T N A$ we have

$$\langle x^* - w^*, x - w \rangle \geq 0. \tag{5}$$

For this purpose, let us pick nets $((x_i, x_i^*))_{i \in I}$ in $X \times X^*$, $((y_i, y_i^*))_{i \in I}$ in M as in Definition 8 and let $z^* \in NAw$ be such that $w^* = A^T z^*$. Then, using the observation following that definition and the fact that $(x_i, x_i^*)_{i \in I}$ converges to (x, x^*) in the natural topology, we have

$$\langle x^*, x \rangle = \lim_{i \in I} \langle y_i^*, y_i \rangle, \quad \langle x^*, w \rangle = \lim_{i \in I} \langle y_i^*, Aw \rangle, \quad \langle w^*, x \rangle = \lim_{i \in I} \langle z^*, Ax_i \rangle = \lim_{i \in I} \langle z^*, y_i \rangle,$$

so that relation (5) is a consequence of the relation $\langle y_i^* - z^*, y_i - Aw \rangle \geq 0$ for each $i \in I$. Taking $N := M^0$ we obtain $\left(A^T M A\right)_{nat} \subset \left(A^T M^0 A\right)^0$. □

From the characterizations of monotone and maximal monotone operators given in (2) we deduce the following consequences. They show that $\left(A^T M A\right)_{nat}$ is not necessarily large, although it contains the natural closure $\overline{A^T M A}$ of $A^T M A$.

Corollary 12 *If M is a monotone operator, then $\left(A^T M A\right)_{nat} \subset \left(A^T M A\right)^0$.*

Proof. Since M is monotone, one has $M \subset M^0$, hence $\left(A^T M A\right)_{nat} \subset \left(A^T M A\right)^0$. □

Thus, if M is a monotone operator such that $A^T M A$ is maximal monotone, then $\left(A^T M A\right)_{nat} = A^T M A$. More generally, we have a similar result with the natural closure $\overline{A^T M A}$ of $A^T M A$.

Corollary 13 *If M is a monotone operator such that the natural closure $\overline{A^T M A}$ of $A^T M A$ is maximal monotone, then $\left(A^T M A\right)_{nat} = \overline{A^T M A}$. If M is a monotone operator and if the strong closure of $A^T M A$ is maximal monotone, then it coincides with $\left(A^T M A\right)_{fuz}$.*

Proof. In view of Proposition 10 it suffices to show that $\left(A^T M A\right)_{nat} \subset \overline{A^T M A}$. Since, by the preceding corollary and Lemma 3, $\left(A^T M A\right)_{nat} \subset \left(A^T M A\right)^0 = \left(\overline{A^T M A}\right)^0$, this inclusion is a consequence of the relation $N^0 = N$ when N is maximal monotone. If the strong closure $\mathrm{cl}(A^T M A)$ of $A^T M A$ is maximal monotone, then one has $\mathrm{cl}(A^T M A) \subset \left(A^T M A\right)_{fuz} \subset \left(A^T M A\right)^0 = \left(\mathrm{cl}(A^T M A)\right)^0 = \mathrm{cl}(A^T M A)$ and equality holds. □

Also, the following result partly justifies our construction.

Theorem 14 *Let $A : X \to Y$ be linear and continuous, let $g : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let $f := g \circ A$. Then $\left(A^T \partial g A\right)_{nat} = \partial f$. If X is reflexive, one also has $\left(A^T \partial g A\right)_{fuz} = \partial f$.*

Proof. Let $(x, x^*) \in \partial f$ and let $y := Ax$. Using [45] Theorem 2.2 or [61], [62], we can find a net $(y_i, y_i^*)_{i \in I}$ in ∂g such that $(\|y_i - y\|)_{i \in I} \to 0$, $(x_i^*)_{i \in I} := (A^T y_i^*)_{i \in I}$ converges weakly* to x^*, $(\langle y_i^*, y_i - y \rangle)_{i \in I} \to 0$ and $(g(y_i))_{i \in I} \to g(y)$. Then, taking $x_i := x$, we see that $((x_i, x_i^*))_{i \in I}$ converges to (x, x^*) in the natural topology and $(\|y_i - Ax_i\|)_{i \in I} \to 0$, $(\langle y_i^*, Ax_i - y_i \rangle)_{i \in I} \to 0$. When X is reflexive, by [45] Theorem 2.2, one can take a sequence instead of a net and assume that (x_i^*) converges to x^*, so that we get $\partial f \subset \left(A^T \partial g A\right)_{fuz}$.

Now let $(x, x^*) \in \left(A^T \partial g A\right)_{nat}$. Let $(x_i, x_i^*)_{i \in I}$ converge to (x, x^*) in the natural topology and be such that there exists $(y_i, y_i^*) \in \partial g$ with $(\|y_i - Ax_i\|)_{i \in I} \to 0$, $(\langle y_i^*, Ax_i - y_i \rangle)_{i \in I} \to 0$,

$x_i^* = A^T(y_i^*)$ for each $i \in I$. Then, since $(Ax_i)_{i\in I}$ weakly converges to Ax, we obtain that $(y_i)_{i\in I}$ weakly converges to Ax and $\liminf_{i\in I} g(y_i) \geq g(Ax) = f(x)$ since g is convex and lower semicontinuous. Moreover, as observed above, $(\langle y_i^*, y_i\rangle)_{i\in I} \to \langle x^*, x\rangle$, and, for every $w \in X$, $(\langle y_i^*, Aw\rangle)_{i\in I} = (\langle x_i^*, w\rangle)_{i\in I} \to \langle x^*, w\rangle$, so that

$$f(w) = g(Aw) \geq \liminf_{i\in I}(g(y_i) + \langle y_i^*, Aw - y_i\rangle) \geq f(x) + \langle x^*, w - x\rangle.$$

Thus $x^* \in \partial f(x)$. □

Example. Let $M : Y \to Y^*$ be a monotone linear operator which is symmetric (i.e. $\langle My, z\rangle = \langle Mz, y\rangle$ for any $y, z \in Y$) and let $A : X \to Y$ be a continuous linear operator. Then $\left(A^T MA\right)_{nat}$ is larger than $A^T MA$:

$$\left(A^T MA\right)_{nat} = A^T MA + \{0\} \times \left(\mathrm{D}(A^T MA)\right)^{\perp}.$$

In fact, by [52, Thm 5.1], $M = \partial g$, where g is the lower semicontinuous function given by $g(y) = \frac{1}{2}\langle My, y\rangle$, so that $\left(A^T MA\right)_{nat} = \partial f$, where $f := g \circ A$; now, for $Q := A^T MA$, one has $f(x) = \frac{1}{2}\langle Qx, x\rangle$ and for $x \in \mathrm{D}(Q)$, $w^* \in (\mathrm{D}(Q))^{\perp}$ one has, for each $u \in \mathrm{D}(Q)$,

$$\frac{1}{2}\langle Q(x+u), x+u\rangle - \frac{1}{2}\langle Qx, x\rangle = \langle Q(x), u\rangle + \frac{1}{2}\langle Qu, u\rangle \geq \langle Qx + w^*, u\rangle,$$

so that $Qx + w^* \in \partial f(x)$; conversely, if $x^* \in \partial f(x)$, for each $u \in \mathrm{D}(Q)$ and each $t \geq 0$ one has

$$\langle Q(x), tu\rangle + \frac{1}{2}\langle tQu, tu\rangle = f(x + tu) - f(x) \geq \langle x^*, tu\rangle,$$

hence $\langle Q(x), u\rangle \geq \langle x^*, u\rangle$ and $x^* - Q(x) \in (\mathrm{D}(Q))^{\perp}$. □

Now let us give a criterion in order that the fuzzy composition coincides with the usual composition. For this purpose, we require the following notion introduced in [7] (in fact, it corresponds to the notion of [7] through a double passage to the inverse operator); see also [35], [50], [51], [58].

Definition 15 *The (sequential) asymptotic multimapping associated to a multimapping $M : Y \rightrightarrows Y^*$ is the multimapping whose value $M^\infty(y)$ at $y \in Y$ is the set of $z^* \in Y^*$ for which there exist sequences $(t_n)_{n\in\mathbb{N}} \to +\infty$, $(y_n)_{n\in\mathbb{N}} \to y$, $(y_n^*)_{n\in\mathbb{N}}$ in Y^* such that $(t_n^{-1}y_n^*)_{n\in\mathbb{N}} \to z^*$ and $y_n^* \in M(y_n)$ for every $n \in \mathbb{N}$.*

If $M = L^{-1}$, where $L : Y^* \to Y$ is linear with a closed graph and closed range, then $M^\infty(y) = \mathrm{Ker}L$ for $y \in \mathrm{D}(M)$, $M^\infty(y) = \emptyset$ for $y \in Y\backslash\mathrm{D}(M)$. In fact, given $y \in \mathrm{D}(M) = L(Y^*)$, $y^* \in \mathrm{Ker}L$, for every $w^* \in L^{-1}(y)$, $n \in \mathbb{N}$ we have $ny^* + w^* \in M(y)$ and $(n^{-1}(ny^* + w^*))_{n\in\mathbb{N}} \to y^*$; conversely, if $(y_n)_{n\in\mathbb{N}} \to y$, $y_n^* \in M(y_n)$ and $(t_n^{-1}y_n^*)_{n\in\mathbb{N}} \to z^*$ for some sequence $(t_n)_{n\in\mathbb{N}} \to +\infty$, we have $y_n \in L(Y^*)$, hence $y \in L(Y^*) = \mathrm{D}(M)$ and $(z^*, 0) = (\lim_n t_n^{-1}y_n^*, \lim_n t_n^{-1}y_n) = \lim_n(t_n^{-1}y_n^*, L(t_n^{-1}y_n^*)) \in \mathrm{G}(L)$, hence $L(z^*) = 0$.

We need another concept from [35] which is related to a general notion of compact net as in [43] which has been used in a similar way in [44] and elsewhere (for instance in [39] where it has been adopted).

Definition 16 *A multimapping $M : W \rightrightarrows Z$ between two normed vector spaces is (sequentially) asymptotically compact at $w \in W$ if for any sequence $((w_n, z_n))_{n\in\mathbb{N}}$ in M with $(w_n)_{n\in\mathbb{N}} \to w$ and $(t_n)_{n\in\mathbb{N}} := (\|z_n\|)_{n\in\mathbb{N}} \to \infty$ the sequence $(t_n^{-1}z_n)_{n\in\mathbb{N}}$ has a converging subsequence.*

If $W = X$ is a Banach space, $Z = X^*$, M is monotone, and if y is in the interior of the domain $\mathrm{D}(M)$ of M, then this condition is (vacuously) satisfied since M is bounded on a neighborhood of y. A less restrictive condition will be given in the next section, along with a notion of weak asymptotic multimapping. With such notions, a variant of the following criterion could be given as in the next section.

Proposition 17 *Suppose A is linear and continuous, $M : Y \rightrightarrows Y^*$ is maximal monotone, M is asymptotically compact at Ax and $M^\infty(Ax) \cap \mathrm{Ker}A^T = \{0\}$. Then one has $(A^T MA)_{fuz}(x) = (A^T MA)_{swf}(x) = A^T MA(x)$.*

Proof. Let $x^* \in (A^T MA)_{fuz}(x)$. Let $(x_n, x_n^*)_{n\in\mathbb{N}} \in X \times X^*$, $(y_n, y_n^*) \in M$ be sequences as in Definition 9. Suppose first that $(t_n)_{n\in\mathbb{N}} := (\|y_n^*\|)_{n\in\mathbb{N}}$ converges to $+\infty$. By the assumption of asymptotic compactness, we can find a subsequence $(t_{n(k)}^{-1} y_{n(k)}^*)_{k\in\mathbb{N}}$ of $(t_n^{-1} y_n^*)_{n\in\mathbb{N}}$ which converges to some z^*. Since $(y_n) \to y := Ax$, we have $z^* \in M^\infty(y)$. Moreover $(t_{n(k)}^{-1} x_{n(k)}^*)_{k\in\mathbb{N}}$ converges to 0 and, since A^T has a closed graph, we get $(z^*, 0) \in \mathrm{G}(A^T)$. Thus $z^* \in M^\infty(y) \cap \mathrm{Ker}A^T = \{0\}$, a contradiction with the fact that z^* is a unit vector. Thus $(y_n^*)_{n\in\mathbb{N}}$ has a bounded subsequence $(y_j^*)_{j\in J}$ which has a weak* cluster point y^*. Since M is maximal monotone, we have $y^* \in M(y)$. Then $(x_j^*)_{j\in J}$ has $A^T(y^*)$ as a weak* cluster point. Thus $x^* = A^T y^* \in (A^T MA)(x)$. The same argument being valid if one just has $(x_n^*)_{n\in\mathbb{N}} \to x^*$ in the weak* topology, we also get $x^* = A^T y^* \in (A^T MA)(x)$ when $x^* \in (A^T MA)_{swf}(x)$. □

Slight changes in the preceding proof using non sequential versions of Definitions 29, 28 below yield the equality $(A^T MA)_{wf}(x) = A^T MA(x)$. We rather give a similar result with the natural composition; we also need to introduce variants of the preceding asymptotic concepts.

Definition 18 *A multimapping $M : Y \rightrightarrows Y^*$ is naturally asymptotically compact at $y \in Y$ if, for any $z^* \in Y^*$ and any net $((y_i, y_i^*))_{i\in I}$ in M such that $(t_i)_{i\in I} := (\|y_i^*\|)_{i\in I} \to \infty$ and $((y_i, t_i^{-1} y_i^*))_{i\in I} \to (y, z^*)$ in the natural topology, one has $z^* \neq 0$.*

If Y is finite dimensional, any multimapping $M : Y \rightrightarrows Y^*$ is naturally asymptotically compact at every point.

Definition 19 *The natural asymptotic multimapping associated to a multimapping $M : Y \rightrightarrows Y^*$ is the multimapping whose value $M_{nat}^\infty(y)$ at $y \in Y$ is the set of $z^* \in Y^*$ for which there exist nets $(t_i)_{i\in I} \to +\infty$, $((y_i, y_i^*))_{i\in I}$ in M such that $(y_i)_{i\in I} \to y$, $((y_i, t_i^{-1} y_i^*))_{i\in I} \to (y, z^*)$ in the natural topology.*

Proposition 20 *Suppose A is linear and continuous, $M : Y \rightrightarrows Y^*$ is maximal monotone, M is naturally asymptotically compact at Ax and $M_{nat}^\infty(Ax) \cap \mathrm{Ker}A^T = \{0\}$. Then one has $(A^T MA)_{nat}(x) = A^T MA(x)$.*

Proof. Let $x^* \in (A^T MA)_{nat}(x)$. Let $(x_i, x_i^*)_{i\in I} \in X \times X^*$, $(y_i, y_i^*)_{i\in I} \in M$ be nets as in Definition 8. Then, as observed after Definition 8, $(\langle y_i^*, y_i \rangle)_{i\in I} \to \langle x^*, x \rangle$. Moreover, since A is weakly continuous, $(y_i)_{i\in I} \to y := Ax$ weakly. Suppose first that $(t_i)_{i\in I} := (\|y_i^*\|)_{i\in I}$ converges to $+\infty$. We can find a subnet $(t_j^{-1} y_j^*)_{j\in J}$ of $(t_i^{-1} y_i^*)_{i\in I}$ which weak* converges to some $z^* \in Y^*$. Then $(t_j^{-1} x_j^*)_{j\in J}$ weak* converges to 0 and, since A^T has a closed graph, we get $(z^*, 0) \in \mathrm{G}(A^T)$. Moreover, since $(\langle y_i^*, y_i \rangle)_{i\in I} \to \langle x^*, x \rangle$, we get $\left(\langle t_j^{-1} y_j^*, y_j \rangle\right)_{j\in J} \to 0 = \langle A^T z^*, x \rangle = \langle z^*, Ax \rangle =$

$\langle z^*, y\rangle$. Thus, $(y, z^*) \in M^\infty_{nat}$ and $z^* \in M^\infty_{nat}(y) \cap \mathrm{Ker} A^T = \{0\}$, a contradiction with $z^* \neq 0$. Thus $(y^*_i)_{i\in I}$ has a bounded subnet $(y^*_j)_{j\in J}$. Taking a further subnet if necessary, we may assume that $(y^*_j)_{j\in J}$ weak* converges to some point $y^* \in Y^*$. Then $(\langle y^*_j, y_j\rangle)_{j\in J} \to \langle x^*, x\rangle = \langle A^T y^*, x\rangle = \langle y^*, y\rangle$, so that $((y_j, y^*_j))_{j\in J} \to (y, y^*)$ in the natural topology. Since M is maximal monotone, hence closed in the natural topology, we have $y^* \in M(y)$. Thus $x^* = A^T y^* \in (A^T M A)(x)$. □

5 Fuzzy sum and natural sum

The connections we established between sum and composition enable us to transpose to sums the constructions we have made for composition. They lead to the following definitions.

Definition 21 *Given two operators $M, N : X \rightrightarrows X^*$ between a Banach space and its dual, their natural sum is the set $S := (M + N)_{nat}$ of $(x, x^*) \in X \times X^*$ such that there exist nets $(x_i, x^*_i)_{i\in I} \to (x, x^*)$ in the natural topology, $(u_i, u^*_i) \in M$, $(v_i, v^*_i) \in N$, $x^*_i = u^*_i + v^*_i$ for all $i \in I$,*
$(\|u_i - x_i\|)_{i\in I} \to 0$, $(\|v_i - x_i\|)_{i\in I} \to 0$, $(\langle u^*_i, u_i - x_i\rangle + \langle v^*_i, v_i - x_i\rangle)_{i\in I} \to 0$.

Definition 22 *Given two operators $M, N : X \rightrightarrows X^*$ between a Banach space and its dual, their fuzzy sum is the set $(M + N)_{fuz}$ of $(x, x^*) \in X \times X^*$ for which there exist sequences $(x_n)_{n\in\mathbb{N}}$ in X, $(u_n, u^*_n)_{n\in\mathbb{N}}$ in M, $(v_n, v^*_n)_{n\in\mathbb{N}}$ in N such that $(\|x_n - x\|)_{n\in\mathbb{N}} \to 0$, $(\|u_n - x\|)_{n\in\mathbb{N}} \to 0$, $(\|v_n - x\|)_{n\in\mathbb{N}} \to 0$,*
$(\|u^*_n + v^*_n - x^*\|)_{n\in\mathbb{N}} \to 0$ $(\|u^*_n\| \|u_n - x_n\| + \|v^*_n\| \|v_n - x_n\|)_{n\in\mathbb{N}} \to 0$.

Note that the last convergence implies that $(\langle u^*_n, u_n - x\rangle + \langle v^*_n, v_n - x\rangle)_{n\in\mathbb{N}} \to 0$.

The sequential weak fuzzy sum $(M + N)_{swf}$ of M and N is obtained by replacing the strong convergence of $(u^*_n + v^*_n)$ to x^* in the preceding definition by weak* convergence. One can also give a notion of non sequential weak fuzzy sum.

Although the correspondences of section 3 can serve as a guideline for the following results, we prefer to give direct (somewhat abridged) proofs.

Proposition 23 *The natural sum $(M+N)_{nat}$ of M and N contains $(M + N)_{fuz}$ and the closure $\overline{M + N}$ of $M+N$ in the natural topology. The fuzzy sum $(M + N)_{fuz}$ contains the strong closure of $M+N$. The (sequential) weak fuzzy sum $(M+N)_{swf}$ contains the sequential closure of $M+N$ in the product of the strong convergence with the weak* convergence.*

Proof. The inclusion $(M + N)_{fuz} \subset (M + N)_{nat}$ is obvious; the inclusion $\overline{M + N} \subset (M + N)_{nat}$ follows from the choice $u_i = x_i = v_i$ in Definition 21. The proof of the last inclusions is similar. □

The following results show that $(M + N)_{nat}$ is not necessarily large.

Proposition 24 *For any operators $M, N, S, T \subset X \times X^*$ with $M \subset S^0$, $N \subset T^0$ one has $(M + N)_{nat} \subset (S + T)^0$.*

Proof. Let us show that for any $(x, x^*) \in (M + N)_{nat}$, for any $w \in X$, $y^* \in S(w)$, $z^* \in T(w)$, $w^* = y^* + z^*$, we have

$$\langle x^* - w^*, x - w\rangle \geq 0. \tag{6}$$

For this purpose, let us pick nets $(x_i, x_i^*)_{i\in I} \to (x, x^*)$ in the natural topology, $(u_i, u_i^*)_{i\in I}$ in M, $(v_i, v_i^*)_{i\in I}$ in N, such that $x_i^* = u_i^* + v_i^*$, $(\|u_i - x_i\|)_{i\in I} \to 0$, $(\|v_i - x_i\|)_{i\in I} \to 0$, $(\langle u_i^*, x_i - u_i\rangle + \langle v_i^*, x_i - v_i$ 0 as in Definition 21. Then, we have

$$\begin{aligned}
\langle x^*, x\rangle &= \lim_{i\in I} \langle x_i^*, x_i\rangle = \lim_{i\in I} \left(\langle u_i^*, u_i\rangle + \langle v_i^*, v_i\rangle\right), \\
\langle x^*, w\rangle &= \lim_{i\in I} \left(\langle u_i^*, w\rangle + \langle v_i^*, w\rangle\right), \\
\langle w^*, x\rangle &= \lim_{i\in I} \left(\langle y^*, u_i\rangle + \langle z^*, v_i\rangle\right),
\end{aligned}$$

so that relation (6) is a consequence of the relations $\langle u_i^* - y^*, u_i - w\rangle \geq 0$, $\langle v_i^* - z^*, v_i - w\rangle \geq 0$ for all $i \in I$. □

Taking $S = M$, $T = N$, we get the following corollary.

Corollary 25 *If M and N are monotone operators, then $(M+N)_{nat} \subset (M+N)^0$. If $M+N$ is maximal monotone, then $(M+N)_{nat} = M+N$.*

Taking into account Proposition 23 and the relations $S^0 = \overline{S}^0$, $T^0 = T$ for T maximal monotone, we get the following consequence.

Corollary 26 *If M and N are monotone operators such that the natural closure $\overline{M+N}$ of $M+N$ is maximal monotone, then $(M+N)_{nat} = \overline{M+N}$. If moreover $\overline{M+N}$ coincides with the sequential closure of $M+N$ in the product of the strong topology with the weak* topology, then $(M+N)_{fuz} = (M+N)_{nat} = \overline{M+N}$. If M and N are monotone operators such that the strong closure $\mathrm{cl}(M+N)$ of $M+N$ is maximal monotone, then $(M+N)_{fuz} = \mathrm{cl}(M+N)$.*

The proof of following result relies on [45] Theorem 2.3 in a way which is similar to the proof of Theorem 14. Thus, we skip it.

Theorem 27 *Let $f, g : Y \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Then $(\partial f + \partial g)_{nat} = \partial(f+g)$. If X is reflexive, one also has $(\partial f + \partial g)_{fuz} = \partial(f+g)$.*

Now let us give a criterion in order that the sequential weak fuzzy sum $(M+N)_{swf}$ of M and N coincides with the usual sum $M+N$. For this purpose, we introduce variants of Definitions 15, 16, although the previous concepts could be used here too. Also, non sequential versions could be given.

Definition 28 *The sequential weakly* asymptotic multimapping to a multimapping $M : X \rightrightarrows X^*$ is the multimapping whose value $M_{sw}^{\infty}(x)$ at x is the set of weak* limits of bounded sequences $(t_n^{-1} x_n^*)_{n\in\mathbb{N}}$ such that $(t_n)_{n\in\mathbb{N}} \to +\infty$ and there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $(x_n)_{n\in\mathbb{N}} \to x$ and $x_n^* \in M(x_n)$ for each $n \in \mathbb{N}$.*

While the preceding definition yields a map which is larger than the sequential asymptotic map, the following definition is less restrictive than Definition 16.

Definition 29 *A multimapping $M : X \rightrightarrows X^*$ is sequentially weakly* asymptotically compact at $x \in X$ if for any sequence $((x_n, x_n^*))_{n\in\mathbb{N}}$ in M with $(x_n)_{n\in\mathbb{N}} \to x$ and $(t_n)_{n\in\mathbb{N}} := (\|x_n^*\|)_{n\in\mathbb{N}} \to \infty$ the sequence $(t_n^{-1} x_n^*)_{n\in\mathbb{N}}$ has a weak* converging subsequence whose limit is non null.*

We are ready to give our criterion. Its assumption about the closed unit ball of X^* could be dropped provided one takes weak* cluster points of sequences $(t_n^{-1}x_n^*)_{n\in\mathbb{N}}$ in the preceding definitions.

Proposition 30 *Suppose X is reflexive, or, more generally, that the closed unit ball of X^* is sequentially weak* compact. Suppose M, $N : X \rightrightarrows X^*$ are maximal monotone, M is sequentially weakly* asymptotically compact at x and $M^\infty_{sw}(x) \cap (-N^\infty_{sw}(x)) = \{0\}$. Then one has $(M+N)_{swf}(x) = M(x) + N(x)$.*

Proof. Let $x^* \in (M+N)_{swf}(x)$. Let $(u_n, u_n^*)_{n\in\mathbb{N}} \in M$, $(v_n, v_n^*)_{n\in\mathbb{N}} \in N$ be sequences such that $(u_n)_{n\in\mathbb{N}} \to x$, $(v_n)_{n\in\mathbb{N}} \to x$, $(\langle u_n^*, u_n - x\rangle + \langle v_n^*, v_n - x\rangle)_{n\in\mathbb{N}} \to 0$, $(x_n^*)_{n\in\mathbb{N}} := (u_n^* + v_n^*)_{n\in\mathbb{N}}$ weak* converges to x^*. Taking a subsequence if necessary, we may assume that $(t_n)_{n\in\mathbb{N}} := (\|u_n^*\|)_{n\in\mathbb{N}}$ converges to some $t \in \mathbb{R}_+ \cup \{+\infty\}$. Let us show that assuming $t = +\infty$ leads to a contradiction. Since the closed unit ball of X^* is weak* sequentially compact, taking a subsequence if necessary we may assume that $(t_n^{-1}u_n^*)_{n\in\mathbb{N}}$ weak* converges to some z^*. Since M is weakly* asymptotically compact at x we may suppose $z^* \neq 0$. Then, $(t_n^{-1}v_n^*)_{n\in\mathbb{N}}$ weak* converges to $-z^*$ and we get $z^* \in M^\infty_{sw}(x) \cap (-N^\infty_{sw}(x))$, $z^* \neq 0$, a contradiction with our asssumption. Thus $t \in \mathbb{R}_+$, and $(u_n^*)_{n\in\mathbb{N}}$ has a subsequence $(u_{n(j)}^*)_{j\in\mathbb{N}}$ which weak* converges to some u^*. Then $(v_{n(j)}^*)_{j\in\mathbb{N}}$ weak* converges to $v^* := x^* - u^*$. Since M and N are sequentially closed in the product of the strong topology with the weak* topology, we have $u^* \in M(x)$, $v^* \in N(x)$ and we get $x^* = u^* + v^* \in (M+N)(x)$. □

6 Comparisons with other notions

Let us introduce an extended composition and compare it with the natural composition. A comparison for sums would be similar.

Definition 31 *The natural extended composition of an operator $M : Y \rightrightarrows Y^*$ with a linear operator $A : X \to Y$ is the operator $(A^TMA)_{ext} := \bigcap_{\varepsilon>0} \overline{A^TM_\varepsilon A}$, where $\overline{A^TM_\varepsilon A}$ denotes the natural closure of $A^TM_\varepsilon A$.*

Proposition 32 *For any monotone operator $M : Y \rightrightarrows Y^*$ and any linear operator $A : X \to Y$ one has $(A^TMA)_{nat} \subset (A^TMA)_{ext}$.*

Proof. Given $(x, x^*) \in (A^TMA)_{nat}$, let us pick nets $(x_i, x_i^*)_{i\in I}$ in $X \times X^*$, $(y_i, y_i^*)_{i\in I}$ in M as in Definition 8. Given $\varepsilon > 0$ and $(z, z^*) \in M$, using the fact that $(\langle y_i^*, Ax_i - y_i\rangle)_{i\in I} \to 0$ and $(\|y_i - Ax_i\|)_{i\in I} \to 0$, for i large enough we have

$$\langle y_i^* - z^*, Ax_i - z\rangle \geq \langle y_i^* - z^*, y_i - z\rangle - \varepsilon \geq -\varepsilon,$$

hence $(Ax_i, y_i^*) \in M_\varepsilon$. It follows that $(x_i, x_i^*) \in A^TM_\varepsilon A$. Therefore $(x, x^*) \in \overline{A^TM_\varepsilon A}$ for any $\varepsilon > 0$. □

Now, let us compare the natural composition with a notion of variational composition introduced recently in [41]. For this purpose we suppose that Y is reflexive. In fact, in recalling the construction of [41], we slightly extend the definition of the variational composition.

Given a continuous increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(0) = 0$, $h(t) \to +\infty$, $h^{-1}(t) \to +\infty$ as $t \to +\infty$, we define the duality mapping J associated with h by

$$J(y) = \{y^* \in Y^* : \langle y^*, y\rangle = \|y^*\| \, \|y\| \, , \ \|y^*\| = h(\|y\|)\}.$$

This multimapping is just the subdifferential of the convex function $j(y) := H(\|y\|)$, where $H(t) := \int_0^t h(s)ds$. Since Y is reflexive, we endow it with a norm which is Fréchet differentiable off 0 and is locally uniformly rotund and whose dual norm has the same properties. Then j is Fréchet differentiable and J is the derivative of j. While the usual case corresponds to the choice $h(t) = t$, other choices are convenient; in particular, for L_p spaces, with $p > 1$, taking $h(t) = (1/p)t^{p-1}$ is advantageous (see [34, p. 173-179] for instance).

Definition 33 *The generalized Moreau-Yosida regularization (associated with the weight h) of the multimapping M from Y into Y^* is given for $t > 0$ by*

$$M_t := \left(M^{-1} + tJ^{-1}\right)^{-1}.$$

When M is maximal monotone, for each $t > 0$ the mapping M_t is single-valued and everywhere defined (see for instance [49] Proposition 4.4 and 4.5) and maximal monotone ([11] Proposition 2.4). Introducing the resolvant or proximal mapping P_t^M (often denoted by J_λ^M) associated with M as

$$P_t^M(y) = \left\{z \in Y : \ 0 \in J\left(\frac{z-y}{t}\right) + M(y)\right\} \qquad y \in Y, \tag{7}$$

[49] Proposition 3.5 yields

$$M_t(y) = J(t^{-1}y - t^{-1}P_t^M y) \in M(P_t^M y).$$

The first part of the following definition has been introduced in [41]; the second part is new.

Definition 34 *The variational composition $\left(A^T MA\right)_v$ of M with A is the set of $(x, x^*) \in X \times X^*$ such that there exists a parametrized family $(x_t)_{t>0}$ of X satisfying $(x, x^*) = \lim_{t \to 0_+}(x_t, A^T M_t A x_t)$. If this convergence is for the natural topology, we write $(x, x^*) \in \left(A^T MA\right)_{nv}$ and we say that (x, x^*) belongs to the natural variational composition.*

A first comparison is as follows.

Proposition 35 *Let M be a maximal monotone operator and let $\overline{A^T MA}$ be the natural closure of $A^T MA$. Then one has*

$$\left(A^T MA\right)_v \subset \left(A^T MA\right)_{nv} \subset \left(\overline{A^T MA}\right)^0 = \left(A^T MA\right)^0.$$

If $\overline{A^T MA}$ is maximal monotone then one has $\left(A^T MA\right)_{nv} \subset \overline{A^T MA}$.

If $\left(A^T MA\right)_{nv}$ (resp. $\left(A^T MA\right)_v$) is maximal monotone, then $\overline{A^T MA} \subset \left(A^T MA\right)_{nv}$ (resp. $\overline{A^T MA} \subset \left(A^T MA\right)_v$).

If both $\overline{A^T MA}$ and $\left(A^T MA\right)_{nv}$ (resp. $\left(A^T MA\right)_v$) are maximal monotone, then they coincide.

Proof. Our proof relies on a device in [41, Lemma 5] adapted to the generalized duality mapping we use, and in a crucial way, on the notion of monotone polar. Let $(w,w^*) \in A^T MA$, $z := Aw$, and let $z^* \in M(z)$ with $w^* := A^T z^*$. Let $(x_t, A^T M_t A x_t)_{t>0}$ be a parametrized family of $X \times X^*$ with limit (x, x^*) for the natural topology. Let us set $v_t := Ax_t$, $v_t^* := M_t v_t$, $x_t^* := A^T v_t^*$ and let us use the fact that there exists some $y_t \in M^{-1}(v_t^*)$ such that $v_t = y_t + tJ^{-1}(v_t^*)$. By the monotonicity of M^{-1} we have

$$\begin{aligned}\langle v_t^* - z^*, v_t - z\rangle &= \langle v_t^* - z^*, y_t + tJ^{-1}(v_t^*) - z\rangle \geq t\langle v_t^* - z^*, J^{-1}(v_t^*)\rangle \\ &\geq t\,\|v_t^*\|\, h^{-1}(\|v_t^*\|) - t\,\|z^*\|\, h^{-1}(\|v_t^*\|) \\ &\geq tk(\|z^*\|),\end{aligned}$$

where, for $s \in \mathbb{R}_+$, $k(s)$ denotes the infimum over $r \in \mathbb{R}_+$ of $(r-s)h^{-1}(r)$, which is finite since $h^{-1}(r) \to +\infty$ as $r \to +\infty$. Since

$$\langle x_t^* - w^*, x_t - w\rangle = \langle A^T v_t^* - A^T z^*, x_t - w\rangle = \langle v_t^* - z^*, v_t - z\rangle \geq tk(\|z^*\|),$$

passing to the natural limit, we get $\langle x^* - w^*, x - w\rangle \geq 0$. Since (w,w^*) is arbitrary in $A^T MA$, this means that $\left(A^T MA\right)_{nv} \subset \left(A^T MA\right)^0$. Now $\left(A^T MA\right)^0 = \left(\overline{A^T MA}\right)^0$.

When the natural closure $\overline{A^T MA}$ of $A^T MA$ is maximal monotone, we have $\left(\overline{A^T MA}\right)^0 = \overline{A^T MA}$ hence $\left(A^T MA\right)_{nv} \subset \overline{A^T MA}$. When $S := \left(A^T MA\right)_{nv}$ (resp. $S := \left(A^T MA\right)_v$) is maximal monotone, then from the inclusion $S \subset \left(\overline{A^T MA}\right)^0$ we deduce that $\overline{A^T MA} \subset S^0 = S$. The last assertion ensues from the two preceding ones. □

Now, let us turn to a comparison with the natural composition. We obtain it by combining Corollary 13 with some changes in the proof of [41] Theorem 7 which deals with the norm closure of $A^T MA$ and not its natural closure.

Theorem 36 *If M is a maximal monotone operator such that the natural closure $\overline{A^T MA}$ of $A^T MA$ is maximal monotone, then $\left(A^T MA\right)_{nv} = \left(A^T MA\right)_v = \left(A^T MA\right)_{nat} = \overline{A^T MA}$.*

Proof. When $\overline{A^T MA}$ is maximal monotone, in order to show that $(A^T M_t A)_{t>0}$ converges to $\overline{A^T MA}$ as $t \to 0$, by [4] Theorem 3.62, it suffices to prove that for any $u^* \in X^*$ the solution x_t of

$$u^* \in J(x_t) + \left(A^T M_t A\right)(x_t)$$

converges to the solution x of

$$u^* \in J(x) + (\overline{A^T MA})(x).$$

Then we shall get $\overline{A^T MA} = \left(A^T MA\right)_v \subset \left(A^T MA\right)_{nv} \subset \overline{A^T MA}$ and the equalities of the theorem since we have seen that $\left(A^T MA\right)_{nat} = \overline{A^T MA}$ when $\overline{A^T MA}$ is maximal monotone.

Taking again an arbitrary element (w,w^*) in $A^T MA$, $z^* \in MAw$ such that $w^* = A^T z^*$ and setting $v_t := Ax_t$, $v_t^* := M_t v_t$, $r_t := \|v_t^*\|$, we have, as in the preceding proof,

$$\langle A^T M_t A x_t - w^*, x_t - w\rangle \geq tk(\|z^*\|)$$

hence, by the definition of x_t,

$$\langle u^* - J(x_t) - w^*, x_t - w\rangle \geq tk(\|z^*\|). \tag{8}$$

With the definition of J, this inequality implies that

$$h(\|x_t\|)\,(\|w\| - \|x_t\|) + \|u^* - w^*\|\,\|x_t\| \geq \langle u^* - w^*, w\rangle + tk(\|z^*\|).$$

It follows that (x_t) is bounded. Let $\bar{x}$ be a weak limit point of (x_t) as $t \to 0$. Using relation (8) and the monotonicity of J under the form $\langle -J(x_t), x_t - w\rangle \leq \langle -J(w), x_t - w\rangle$, we get

$$\langle u^* - J(w) - w^*, x_t - w\rangle \geq tk(\|z^*\|). \tag{9}$$

The definition of the natural topology enables us to extend this relation to any $(w, w^*) \in \overline{A^T MA}$ and to get

$$\langle u^* - J(w) - w^*, \bar{x} - w\rangle \geq 0. \tag{10}$$

Since $\overline{A^T MA}$ is maximal monotone, $J + \overline{A^T MA}$ is maximal monotone too, and we get $u^* \in \left(J + \overline{A^T MA}\right)(\bar{x})$. By uniqueness of the solution of this inclusion, we have $\bar{x} = x$ and the whole family (x_t) weakly converges to x.

Returning to relation (8) which is valid for any (w, w^*) in $A^T MA$, hence for any (w, w^*) in $\overline{A^T MA}$ and using the inequality $\langle J(x_t), w\rangle \leq h(\|x_t\|)\,\|w\|$, we get

$$h(\|x_t\|)\,\|w\| - h(\|x_t\|)\,\|x_t\| + \langle u^* - w^*, x_t - w\rangle \geq tk(\|z^*\|).$$

Setting $r := \limsup_{t\to 0} \|x_t\|$ and passing to the limit in this inequality, we get, for any $(w, w^*) \in \overline{A^T MA}$,

$$h(r)\,\|w\| - h(r)r + \langle u^* - w^*, x - w\rangle \geq 0.$$

Taking $(w, w^*) = (x, u^* - J(x))$ which belongs to $\overline{A^T MA}$ by definition of x, we obtain $h(r)(\|x\| - r) \geq 0$. Thus $r \leq \|x\| = \|\bar{x}\| \leq \liminf_{t\to 0} \|x_t\|$, so that $\|x_t\| \to \|\bar{x}\|$ as $t \to 0$ and the Kadec-Klee property implies that (x_t) converges to $\bar{x} = x$. $\square$

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