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Generalized affine maps and generalized convex functions

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Abstract

We study some classes of generalized convex functions, using a generalized derivative approach. We establish some links between these classes and we devise some optimality conditions for constrained optimization problems. In particular, we get Lagrange-Kuhn-Tucker multipliers for mathematical programming problems.

Key words: colinconv, colinfine, generalized derivative, mathematical programming, optimality conditions, pseudoconvex function, pseudolinear function, quasiconvex function.

Mathematics Subject Classification: 26B25, 46G05, 49K27, 90C26, 90C32

1 Introduction

Prompted by the needs of mathematical economics, some mathematicians have scrutinized several notions of generalized convexity or concavity. While some of them, such as quasiconvexity and pseudoconvexity are rather general, some others are rather peculiar. This is the case for functions which are both pseudoconvex and pseudoconcave. Initially, these functions have been called pseudomonotone (Martos [45]); nowadays these functions are usually called pseudolinear, but we prefer to call them pseudoaffine as a function which is both convex and concave is affine; moreover, this class includes affine functions which are not linear. Albeit especial, this class of functions contains interesting, non trivial examples and enjoys striking properties, as shown in [1], [2], [5], [7], [8], [13], [16], [18], [27], [31], [33], [36], [51], [54], [59], [61] among others. In all references dealing with this subject, pseudoaffine functions are defined through the use of a directional derivative function or a Dini derivative. We also follow this line, taking a general bifunction h as a substitute for such a derivative. The use of subdifferentials for dealing with this class will be made elsewhere.

It is our purpose here to study related classes of functions, to give characterizations and to detect some relationships. In particular, we introduce and study properties which are related to some quasiconvexity, pseudoconvexity and invexity properties. We also present extensions to vector-valued maps and study some composition properties (section 4).

Some relations between solutions of variational inequalities and optimization problems were discussed in the papers quoted above. We also tackle this question in section 5; in particular, we study necessary and sufficient optimality conditions in the form of a Karush-Kuhn-Tucker condition when the data belong to the class of functions under study. We provide examples allowing a comparison with previous results in [38], [41], [43] and [44].

2 Pseudoconvex, quasiconvex and protoconvex functions

In the sequel, X, Y are normed vector spaces (n.v.s.), C is a nonempty subset of X and $f : C \rightarrow \mathbb{R}$; it is convenient to extend f by $+\infty$ outside C . We set $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{P} :=]0, +\infty[$. For $w, x \in X$, we set $[w, x] := \{(1-t)w + tx : t \in [0, 1]\}$, $]w, x[:= [w, x] \setminus \{x\}$ and $]w, x[:= [w, x] \setminus \{w\}$. We define the *visibility cone* $V(C, x)$ of C at $x \in C$ as the cone generated by $C - x$:

$$V(C, x) := \mathbb{P}(C - x) := \{r(c - x) : r \in \mathbb{P}, c \in C\}.$$

The *visibility bundle* of C is the set

$$VC := \{(x, u) \in C \times X : \exists r \in \mathbb{P}, w \in C, u = r(w - x)\} = \bigcup_{x \in C} \{x\} \times V(C, x).$$

It contains the *radial tangent bundle* of C which is the set

$$T^r C := \{(x, u) \in C \times X : \exists (r_n) \rightarrow 0_+, x + r_n u \in C \forall n\} = \bigcup_{x \in C} \{x\} \times T^r(C, x).$$

We also use the *tangent bundle* of C which is the set

$$TC := \{(x, u) \in C \times X : \exists (r_n) \in \mathbb{P}^{\mathbb{N}}, (u_n) \rightarrow u, x + r_n u_n \in C \forall n\} = \bigcup_{x \in C} \{x\} \times T(C, x).$$

Here $T^r(C, x) := \{u \in X : \exists (r_n) \rightarrow 0_+, x + r_n u \in C\}$ is the *radial tangent cone* to C at $x \in C$ and $T(C, x) := \{u \in X : \exists (r_n) \rightarrow 0_+, (u_n) \rightarrow u, x + r_n u_n \in C \forall n\}$ is the *tangent cone* to C at x . One has the obvious inclusions $T^r(C, x) \subset T(C, x)$, $T^r(C, x) \subset V(C, x)$. When C is *starshaped* at $x \in C$, in the sense that $[x, w] \subset C$ for every $w \in C$, one has $V(C, x) = T^r(C, x)$. When C is a convex subset of X , as it will be frequently the case in the sequel, $T(C, x)$ is the closure of $V(C, x) = T^r(C, x)$.

A bifunction $h : VC \rightarrow \overline{\mathbb{R}}$ will stand for a generalized derivative of f . All the usual generalized directional derivatives have the common feature that they are positively homogeneous functions of the direction u . Thus, we assume that for all $x \in C$, $h(x, \cdot)$ is positively homogeneous and $h(x, 0) = 0$. There are several possible choices for h . Among them are the Dini derivatives of f . The upper and the lower radial derivatives (or upper and lower Dini derivatives) of f at $x \in C$, in the direction $u \in T^r(C, x)$, are defined by

$$\begin{aligned} D^+ f(x, u) &= \limsup_{t \rightarrow 0_+, x+tu \in C} \frac{1}{t} [f(x + tu) - f(x)]; \\ D_+ f(x, u) &= \liminf_{t \rightarrow 0_+, x+tu \in C} \frac{1}{t} [f(x + tu) - f(x)]. \end{aligned}$$

If for some $(x, u) \in T^r C$ the relation $D^+ f(x, u) = D_+ f(x, u)$ holds, then the *radial derivative* of f at x in the direction u is the common value of the preceding limits:

$$f'(x, u) = D^+ f(x, u) = D_+ f(x, u).$$

This derivative is often called the *directional derivative* of f at x in the direction u , but we prefer to keep this term for the limit

$$f'(x, u) = \lim_{\substack{(t,v) \rightarrow (0_+, u) \\ x+tv \in C}} \frac{1}{t} [f(x + tv) - f(x)] \quad (x, u) \in TC,$$

when it exists. Of course, when this directional derivative (also called the Hadamard derivative) exists for some $(x, u) \in T^r C$, the radial derivative exists and has the same value. Note that the limit in the definition of $f'(x, u)$ is $+\infty$ if u does not belong to the tangent cone $T(C, x)$ to C at x which is the set of limits of sequences of the form $(t_n^{-1}(x_n - x))$ where $(t_n) \rightarrow 0_+$ and $x_n \in C$ for every $n \in \mathbb{N}$. Similarly, $D_+ f(x, u)$ is $+\infty$ if u does not belong to the $T^r(C, x)$.

Other generalized derivatives can be used. For instance, one can use the lower directional derivative (or contingent derivative or lower Hadamard derivative) df given by

$$df(x, u) := \liminf_{\substack{(t,v) \rightarrow (0_+, u) \\ x+tv \in C}} \frac{1}{t} [f(x+tv) - f(x)]$$

and its upper version, or some epiderivative. One can also use the Clarke-Rockafellar derivative f^\dagger of f ([10], [55]) or the moderate derivative f^\diamond of Michel-Penot ([46]).

Slightly modifying a notion introduced in [57], we say that a mapping $s : X \rightarrow \overline{\mathbb{R}}$ is *subodd* on $C \subset X$ if the following condition is satisfied

$$(S) \quad w, x \in C \implies s(w - x) \geq -s(x - w).$$

Clearly, if s is finite valued and positively homogenous, s satisfies (S) if s is sublinear. We say that $s : C - C \rightarrow \overline{\mathbb{R}}$ *reverses signs* on $C \subset X$ if the following condition is satisfied

$$(R) \quad w, x, y \in C, x \in]w, y[\implies s(w - x)s(y - x) \leq 0.$$

Conditions (R) and (S) are independent, as the following simple examples show.

Example 1. Let $s_1 : \mathbb{R} \rightarrow \mathbb{R}$ be given by $s_1(u) = |u|$; then s_1 is subodd, but $s_1(u)s_1(-u) > 0$ for all $u \in \mathbb{R} \setminus \{0\}$.

Example 2. Let $s_2 : \mathbb{R} \rightarrow \mathbb{R}$ be given by $s_2(u) = 0$ if $u \geq 0$ and $s_2(u) = u$ if $u < 0$; then $s_2(u)s_2(-u) \leq 0$ for all $u \in \mathbb{R}$ but $s_2(1) \not\geq -s_2(-1)$.

Both conditions (R) and (S) imply the following condition:

$$(T) \quad w, x, y \in C, x \in]w, y[, s(w - x) < 0 \implies s(y - x) \geq 0.$$

Note that s satisfies condition (R) if, and only if s and $-s$ satisfy condition (T). On the other hand, s and $-s$ satisfy condition (S) if, and only if s is odd on $C - C$.

A function $f : C \rightarrow \mathbb{R}$ is said to be *quasiconvex* at $x \in C$ if for all $w \in C$ and $x_t := tx + (1-t)w \in C$ for $t \in [0, 1]$ then $f(x_t) \leq \max\{f(x), f(w)\}$.

Let us recall the following definitions (see [24], [30], [32], [36] and [57]).

Definition 1 Given a bifunction $h : VC \rightarrow \overline{\mathbb{R}}$, a function $f : C \rightarrow \mathbb{R}$ is said to be

(a) *h-pseudoconvex* at $x \in C$, if

$$w \in C, f(w) < f(x) \implies h(x, w - x) < 0.$$

(b) *h-quasiconvex* at $x \in C$, if

$$w \in C, f(w) < f(x) \implies h(x, w - x) \leq 0.$$

(c) *h-protoconvex* at $x \in C$, if

$$w \in C, f(w) \leq f(x) \implies h(x, w - x) \leq 0.$$

- (d) h -pseudoconcave at $x \in C$, if $-f$ is $-h$ -pseudoconvex at $x \in C$.
- (e) h -quasiconcave at $x \in C$, if $-f$ is $-h$ -quasiconvex at $x \in C$.
- (f) h -protoconcave at $x \in C$, if $-f$ is $-h$ -protoconvex at $x \in C$.

We say that f is h -pseudoconvex on C or, in short, h -pseudoconvex (resp. h -quasiconvex, h -protoconvex, h -pseudoconcave, h -quasiconcave, h -protoconcave) if f is h -pseudoconvex (resp. h -quasiconvex, h -protoconvex, h -pseudoconcave, h -quasiconcave, h -protoconcave) at every $x \in C$.

Clearly, f is h -pseudoconcave (resp. h -quasiconcave, h -protoconcave) if, and only if, for any $w, x \in C$,

$$\begin{aligned} f(x) < f(w) &\Rightarrow h(x, w - x) > 0 \\ \text{(resp. } f(x) < f(w) &\Rightarrow h(x, w - x) \geq 0, \\ \text{(resp. } f(x) \leq f(w) &\Rightarrow h(x, w - x) \geq 0.) \end{aligned}$$

Clearly,

$$\begin{aligned} f \text{ is } h\text{-pseudoconvex} &\Rightarrow f \text{ is } h\text{-quasiconvex,} \\ f \text{ is } h\text{-protoconvex} &\Rightarrow f \text{ is } h\text{-quasiconvex.} \end{aligned}$$

Let us observe that h -pseudoconvexity at x and h -protoconvexity at x are independent properties. Both properties are consequences of the following condition:

$$w \in C, f(w) \leq f(x) \Rightarrow h(x, w - x) < 0.$$

This rather stringent condition will not be considered here.

These definitions are closely related. Let us note that here the terminology slightly differs from the one in [30], [37] and [57]. Such a change will allow a better concordance with the classical terminology for the case a subdifferential is used (see [2], [23], [39], [52]). Here, we do not need any assumption on the bifunction h , but in some cases, we will require some comparisons with lower and upper Dini derivatives.

Proposition 2 *If f has no local minimizer on $f^{-1}(f(x))$ and if $h(x, \cdot)$ is lower semicontinuous (l.s.c.) then h -quasiconvexity at x coincides with h -protoconvexity at x .*

Proof. Let $w \in C$ be such that $f(w) = f(x)$. Since f has no local minimizer on $f^{-1}(f(x))$, there exists $(w_n) \rightarrow w$ such that $f(w_n) < f(w) = f(x)$. Since f is h -quasiconvex at x , $h(x, w_n - x) \leq 0$ hence, $h(x, w - x) \leq 0$ by lower semicontinuity of $h(x, \cdot)$. Thus, f is h -protoconvex at x . \square

Links with usual quasiconvexity are given in the next proposition. It shows that, under mild assumptions, quasiconvexity, h -quasiconvexity and h -protoconvexity coincide. We will use the following complement to the Three Points Lemma of [33]; as there, we say that f satisfies the *nonconstancy property*, in short, the NC-property, if there is no line segment $[a, b]$ in C on which f is constant.

Lemma 3 *(Three Points Lemma) Let C be convex. Let $f : C \rightarrow \mathbb{R}$ be either radially continuous or radially lower semicontinuous and satisfying the NC-property, let $w, z \in C$ with $w \neq z$ and let $y = z + t(w - z)$, with $t \in]0, 1[$ be such that $f(y) > f(w) \geq f(z)$. Then there exist some $x, x' \in [w, z]$ such that $f(x') < f(x)$ and $D_+f(x, x' - x) > 0$.*

Proof. If f is radially l.s.c. and satisfies the NC-property, the result is given in [33, Lemma 10.1]. If f is radially continuous, by the intermediate value theorem, there exists some $p \in]0, t[$ such that $f(w) < f(u) < f(y)$ for $u := z + p(w - z)$. Taking p close enough to t , we may assume that $f(w) < f(v)$ for all $v \in [u, y]$.

Let $s := \sup\{r \in [p, t] : f(z + r(w - z)) \leq f(u)\}$ and let $v := z + s(w - z)$. Since f is radially l.s.c., one has $f(v) \leq f(u) < f(y)$. By the mean value theorem [15], there exists some $x \in [v, y[$ such that $D_+f(x, y - v) \geq f(y) - f(v) > 0$. Then $D_+f(x, w - x) > 0$ and $f(w) < f(x)$ since $f(z + t'(w - z)) > f(w)$ for all $t' \in [p, t]$. Setting $x' := w$, one has the conclusion. \square

Proposition 4 *Let C be convex.*

(a) *If f is quasiconvex and if $h \leq D^+f$, then f is h -protoconvex, hence h -quasiconvex.*

(b) *Conversely, if f is h -quasiconvex, if $h \geq D_+f$, and if f is either radially continuous or radially lower semicontinuous on C and satisfies the NC-property, then f is quasiconvex.*

(c) *If f is radially differentiable and if $h := f'$, then f is quasiconvex, if, and only if, it is h -quasiconvex, if, and only if, it is h -protoconvex.*

Proof. (a) Suppose f is quasiconvex and $f(w) \leq f(x)$ for some $w, x \in C$. Then for $t \in]0, 1[$ we have $f(x + t(w - x)) \leq f(x)$, hence $D^+f(x, w - x) \leq 0$. Thus $h(x, w - x) \leq 0$ if $h(x, \cdot) \leq D^+f(x, \cdot)$.

(b) Suppose that f is not quasiconvex. Then, there exist w, y, z such that $f(y) > \max\{f(z), f(w)\} =: f(w)$ where $y = z + t(w - z)$, $t \in]0, 1[$. By the preceding lemma, there exist $x, x' \in [w, z]$ such that $h(x, x' - x) \geq D_+f(x, x' - x) > 0$ and $f(x') < f(x)$: a contradiction with h -quasiconvexity of f .

(c) In view of the fact that a radially differentiable function is radially continuous, the assertion follows from (a) and (b). \square

A comparison between h -pseudoconvexity and quasiconvexity is given in the next proposition. Note that its assertion (b) does not involve Dini derivatives; it has been proved in [57, Theorem 5.1] when for all $x \in C$, if $h(x, \cdot)$ is subodd. Its last assertion is well known.

Proposition 5 *Suppose f is h -pseudoconvex on a convex set C . If one of the following two conditions is satisfied, then f is quasiconvex:*

(a) *f is radially lower semicontinuous on C and $h \geq D_+f$.*

(b) *for all $x \in C$, $h(x, \cdot)$ satisfies (T).*

In particular, if f is radially differentiable and f is h -pseudoconvex with $h := f'$ (in short, f is pseudoconvex), then f is quasiconvex.

Proof. (a) It is proved in [33, Theorem 10.5].

(b) Suppose $h(x, \cdot)$ satisfies (T) for all $x \in C$ and f is h -pseudoconvex. To prove that f is quasiconvex, it suffices to show that if there exist $w, x, y \in C$ with $x \in]w, y[$ such that $f(x) > f(w)$ and $f(x) > f(y)$, one is led to a contradiction. By h -pseudoconvexity of f , one gets $h(x, w - x) < 0$ and $h(x, y - x) < 0$. Since $h(x, \cdot)$ satisfies (T), one obtains the required contradiction. \square

When f is both h -pseudoconvex and h -protoconvex, we get a special form of invexity. Here, in the relation

$$\forall w, x \in C \quad f(w) \geq f(x) + h(x, v(w, x))$$

which characterizes h -invexity of f on C , the vector $v(w, x)$ is of the form $\lambda(w, x)(w - x)$, with $\lambda(w, x) \in \mathbb{P}$, i.e. $v(w, x)$ is positively colinear to the vector $w - x$. This fact explains the terminology chosen in the next statement. This terminology may also have an interest in the case one needs to avoid a possible confusion with the notion of semiconvexity used in [60], where semiconvexity is

equivalent with quasiconvexity and invexity or a confusion with the notion used for the study of Hamilton-Jacobi equations [9].

Proposition 6 *For any function $f : C \rightarrow \mathbb{R}$ and any bifunction $h : VC \rightarrow \overline{\mathbb{R}}$, the following assertions are equivalent:*

- (a) f is h -semiconvex (i.e. f is h -pseudoconvex and h -protoconvex).
- (b) f is h -colinver: there exists $\lambda : C \times C \rightarrow \mathbb{P}$ such that for all $w, x \in C$,

$$f(w) \geq f(x) + \lambda(w, x)h(x, w - x).$$

Proof. (b) \Rightarrow (a) is obvious.

(a) \Rightarrow (b) Let f be h -pseudoconvex and h -protoconvex. Let $x, w \in C$.

If $h(x, w - x) = 0$ then one has $f(w) \geq f(x)$ by h -pseudoconvexity of f and one can take $\lambda(w, x) = 1$ or any element in \mathbb{P} .

If $h(x, w - x) > 0$ then $f(w) > f(x)$ by h -protoconvexity of f . Then one can take

$$\lambda(w, x) = [f(w) - f(x)]/h(x, w - x).$$

If $h(x, w - x) < 0$ then one can take

$$\lambda(w, x) := \max\{[f(w) - f(x)]/h(x, w - x), 0\} + 1,$$

and then $f(w) \geq f(x) + \lambda(w, x)h(x, w - x)$. □

Remark 1. When f is convex and if h is the radial derivative of f , then (b) is satisfied with $\lambda(w, x) = 1$. As mentioned above, the specific choice of the function v given by $v(w, x) := \lambda(w, x)(w - x)$ as a vector colinear with $w - x$ pin points this class of functions among invex functions.

3 Pseudoaffine, protoaffine and colinfine functions

In the present section we study classes of functions which are still more restrictive. Their interests lie in their striking properties. In particular, they enjoy nice composition properties with the classes studied in the preceding section. Their behaviors will be presented in Proposition 10 and their mutual relationships will be examined in Proposition 13 below.

The terminology we adopt in the following definition is new. It stems from the fact that a function which is both convex and concave is affine.

Definition 7 *Given a bifunction $h : VC \rightarrow \overline{\mathbb{R}}$, a function $f : C \rightarrow \mathbb{R}$ is said to be*

- (a) h -pseudoaffine at $x \in C$ if f is h -pseudoconvex and h -pseudoconcave at $x \in C$.
- (b) h -protoaffine at $x \in C$ if f is h -protoconvex and h -protoconcave at $x \in C$.
- (c) h -semiaffine at $x \in C$ if f is h -semiconvex and h -semiconcave at $x \in C$.
- (d) h -colinfine at $x \in C$ if there exists some function $\lambda : C \times C \rightarrow \mathbb{P}$ such that

$$\forall w \in C \quad f(w) - f(x) = h(x, \lambda(w, x)(w - x)). \tag{1}$$

We say that f is h -pseudoaffine on C (in short, h -pseudoaffine) (resp. h -protoaffine, h -semiaffine, h -colinfine) if f is h -pseudoaffine (resp. h -protoaffine, h -semiaffine, h -colinfine) at every $x \in C$. In the case h is the radial derivative of f , we simply say that f is *pseudoaffine* (resp. *protoaffine*, *semiaffine*,

colinfine) if f is h -pseudoaffine (resp. h -protoaffine h -semiaffine, h -colinfine) on C . Observe that a radially differentiable function f is pseudoaffine (resp. protoaffine) if, and only if, f and $-f$ are pseudoconvex (resp. quasiconvex).

The first concept of the preceding definition has been introduced by Chew and Chow [13] in the differentiable case and has been used in [1], [7], [8], [31], [59], [36] [61].

It is easy to show that f is h -colinfine if, and only if, f is h -colinvex and $-f$ is $-h$ -colinvex.

The following observation is immediate. We will prove a related property in Proposition 13.

Proposition 8 *Given $h : VC \rightarrow \overline{\mathbb{R}}$, a function $f : C \rightarrow \mathbb{R}$ is h -colinfine if, and only if, it is h -semiaffine.*

Proof. Suppose that f is h -colinfine. Then there exists some function $\lambda : C \times C \rightarrow \mathbb{P}$ such that

$$\forall w, x \in C \quad f(w) - f(x) = \lambda(w, x)h(x, w - x).$$

Hence, f is h -conlinvex and $-f$ is $-h$ -colinvex. By Proposition 6, one has that f is h -semiconvex and $-f$ is $-h$ -semiconvex. Thus, f is h -semiaffine.

Conversely, suppose that f is h -semiaffine. Using either Proposition 6, or the definition of a semiaffine function, we see that $f(w) > f(x)$ if, and only if, $h(x, w - x) > 0$, $f(w) < f(x)$ iff $h(x, w - x) < 0$ and $f(w) = f(x)$ iff $h(x, w - x) = 0$. One can take any $\lambda(w, x) \in \mathbb{P}$ if $h(x, w - x) = 0$ and $\lambda(w, x) = [f(w) - f(x)]/h(x, w - x)$ otherwise. Then f is h -colinfine. \square

Now, let us give an answer to the question: when is a colinfine function a convex function?

Proposition 9 *Let f be h -colinfine on a convex set C and let $\lambda : C \times C \rightarrow \mathbb{P}$ be such that*

$$f(w) - f(x) = \lambda(w, x)h(x, w - x).$$

Assume that for all $w, x \in C$, $t \in [0, 1]$ one has $\lambda(x_t, x) \leq \lambda(w, x)$ where $x_t := x + t(w - x)$. Then f is convex on C .

Proof. Let $w, x \in C$, $x_t := x + t(w - x)$ for $t \in [0, 1]$. Since f is h -colinfine, one has

$$\begin{aligned} f(x_t) - f(x) &= \lambda(x_t, x)h(x, x_t - x) = t\lambda(x_t, x)h(x, w - x), \\ t[f(w) - f(x)] &= t\lambda(w, x)h(x, w - x). \end{aligned}$$

By assumption, we have $\lambda(x_t, x) \leq \lambda(w, x)$, hence $f(x_t) - f(x) \leq t(f(w) - f(x))$. Thus, f is convex. \square

Even for a smooth bifunction h , a h -colinfine function maybe nonsmooth, and not even continuous.

Example 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x + 1$ for $x > 0$, $f(x) = -x^2$ for $x \leq 0$, and let $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be given by $h(x, u) = u$ for all $x, u \in \mathbb{R}$. Then f is h -colinfine. Note that h is neither the radial derivative nor one of the Dini derivatives of f . The function f is not continuous at 0 and f is neither convex nor concave.

Example 4. If h is as in the preceding example, the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) := x^3$ is not pseudoaffine but f is h -colinfine.

The preceding example shows that a function may be h_1 -colinfine for some bifunction $h_1 : (x, u) \rightarrow u$ but not h_2 -colinfine (and not even h_2 -pseudoconvex) for some other bifunction $h_2 := f'$, even when f , h_1 and h_2 are smooth.

The next statement shows that h -colinfine functions have a very special behavior. Here, we do not need any continuity of f .

Proposition 10 *Let $f : C \rightarrow \mathbb{R}$. There exists $h : VC \rightarrow \overline{\mathbb{R}}$ such that f is h -colinfine on C if and only if for all $w, x \in C$, the function $t \mapsto f(x+t(w-x))$ is either increasing or decreasing or constant on the interval on which it is defined.*

Proof. Let f be h -colinfine on C . Let $w, x \in C$. If $f(w) = f(x)$ then $f(x+t(w-x)) = f(x)$ for all $t \in \mathbb{R}$ such that $x_t := x+t(w-x) \in C$ since we have $h(x, x_t - x) = th(x, w-x) = 0$.

If $f(w) > f(x)$ then $h(x, w-x) > 0$. For all $t > 0$ such that $x_t := x+t(w-x) \in C$, one has $h(x, x_t - x) > 0$ by homogeneity. Thus $f(x+t(w-x)) > f(x)$. Also, given $u := x+r(w-x) \in C$ and $v := x+s(w-x) \in C$ with $s > r$, one has $f(v) > f(u)$. Otherwise, one would have either $f(v) = f(u)$ and then $f(w) = f(x)$ by what precedes or $f(v) < f(u)$ and then $h(v, u-v) > 0$, $h(v, x-v) < 0$ (as $f(x) < f(v)$), a contradiction since $x-v = q(u-v)$ for some $q > 0$. A similar proof shows that for every $t > 0$ we have $f(w+t(x-w)) < f(w)$ and for $s > r$, $f(w+s(x-w)) < f(w+r(x-w))$ when the involved points are in C .

If $f(x) > f(w)$, then we also have the conclusion by interchanging the role of w and x in what precedes.

Conversely, a possible choice for h is as follow. Given $(x, u) \in VC$ with $\|u\| = 1$, we can find some $t \in \mathbb{P}$ such that $w := x+tu \in C$. If $f(w) = f(x)$, for $r \in \mathbb{P}$ we take $h(x, ru) = 0$ and $\lambda(x+ru, x) := 1$. If $f(w) \neq f(x)$, then for $r \in \mathbb{P}$, we set

$$h(x, ru) := rt^{-1}(f(w) - f(x)), \quad \lambda(x+ru, x) := \frac{t(f(x+ru) - f(x))}{r(f(x+tu) - f(x))}.$$

Then h is clearly positively homogeneous in its second variable, $\lambda(x+ru, x) \in \mathbb{P}$ by our assumption, and if $(x+ru, x) \in C \times C$ we have $f(x+ru) - f(x) = \lambda(x+ru, x)h(x, ru)$, as required. \square

From Proposition 10, the set of all minimizers and the set of all maximizers of a non constant h -colinfine function are contained in the boundary its domain.

Corollary 11 *Let f be h -colinfine and let $w, x \in C$. Then for any $x_t := x+t(w-x)$, $t \in \mathbb{R} \setminus \{0, 1\}$ such that $x_t \in C$, the followings statements are equivalent:*

- (a) $f(x) = f(w)$.
- (b) $h(x, w-x) = 0$.
- (c) $h(w, x-w) = 0$.
- (d) $f(x_t) = f(x)$.
- (e) $h(x_t, w-x) = 0$.
- (f) $f(x_t) = f(w)$.

Proof. One has (a) \Leftrightarrow (b) \Leftrightarrow (c) by the definition of a h -colinfine function and (a) \Leftrightarrow (d) \Leftrightarrow (f) by the preceding proposition. If $t > 0$ then $x_t - x = t(w-x)$, hence (d) \Leftrightarrow (e); Otherwise $t < 0$ then $w - x_t = (1-t)(w-x)$, hence (f) \Leftrightarrow (e). \square

Although the preceding definitions are quite restrictive, they are satisfied in some cases of significant interest.

Example 5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and let $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be positively homogeneous in its second variable and such that $h(x, 1) > 0$, $h(x, -1) < 0$. Then f is h -colinfine. Note that we do not require h is the directional derivative nor some Dini derivative of f .

Example 6. Let X be a n.v.s., let $a, b \in X^*$, $\alpha, \beta \in \mathbb{R}$. For $C := \{x \in X : bx + \beta > 0\}$, let $f : C \rightarrow \mathbb{R}$ be given by

$$\forall x \in C \quad f(x) = (ax + \alpha)/(bx + \beta).$$

Since

$$f'(x, w - x) = (awbx - axbw + (\beta a - \alpha b)(w - x))/(bx + \beta)^2,$$

one has

$$f(w) - f(x) = \frac{awbx - axbw + (\beta a - \alpha b)(w - x)}{(bw + \beta)(bx + \beta)} = \frac{bx + \beta}{bw + \beta} f'(x, w - x)$$

and f is colinfine on C with $\lambda(w, x) := (bx + \beta)(bw + \beta)^{-1}$. But f is not convex when $b \neq 0$.

Example 7. Let $g : X \rightarrow \mathbb{R}$ be colinfine. For $C := \{x \in X : g(x) > 0\}$, let $f(x) = \sqrt{g(x)}$. Since g is colinfine, we have $g(w) - g(x) = \lambda_g(w, x)g'(x, w - x)$ with $\lambda_g(w, x) > 0$. Since

$$f'(x, w - x) = \frac{1}{2\sqrt{g(x)}}g'(x, w - x),$$

one has

$$f(w) - f(x) = (g(w) - g(x))/(\sqrt{g(w)} + \sqrt{g(x)}) = \lambda_g(w, x)g'(x, w - x)/(\sqrt{g(w)} + \sqrt{g(x)}).$$

Thus f is colinfine with $\lambda_f(w, x) := 2\lambda_g(w, x)\sqrt{g(x)}/(\sqrt{g(w)} + \sqrt{g(x)})$.

Example 8. More generally, let X be a n.v.s., $g : X \rightarrow \mathbb{R}$ be colinfine on $C := \{x \in X : g(x) > 0\}$. Then, for any $p \in \mathbb{N}$, $f(\cdot) := g^p(\cdot) := (g(\cdot))^p$ is colinfine on C . In fact, since g is colinfine on C , for any $w, x \in C$ we have $g(w) - g(x) = \lambda_g(w, x)g'(x, w - x)$ with $\lambda_g(w, x) > 0$. Since $f'(x, w - x) = p(g(x))^{p-1}g'(x, w - x)$, we get

$$\begin{aligned} f(w) - f(x) &= g^p(w) - g^p(x) = (g(w) - g(x))\left(\sum_{k=0}^{p-1} g^k(w)g^{p-k-1}(x)\right) \\ &= \lambda_g(w, x)g'(x, w - x)\left(\sum_{k=0}^{p-1} g^k(w)g^{p-k-1}(x)\right). \end{aligned}$$

Thus f is colinfine with $\lambda_f(w, x) := \lambda_g(w, x)p^{-1}g^{1-p}(x)\left(\sum_{k=0}^{p-1} g^k(w)g^{p-k-1}(x)\right)$.

Composition properties which will be established in the next section will enlarge the field of colinfine functions. For the moment, we delineate some properties and characterizations.

In the sequel, we will consider the following hypothesis:

(H⁺) If $h(x, w - x) > 0$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) > f(x)$.

(H⁻) If $h(x, w - x) < 0$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) < f(x)$.

(H₀⁺) If $h(x, w - x) > 0$, $f(w) = f(x)$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) > f(x)$.

(H₀⁻) If $h(x, w - x) < 0$, $f(w) = f(x)$ for some $w, x \in C$, then there exists $z \in]w, x[$ such that $f(z) < f(x)$.

We say that f satisfies (H⁺) (resp. (H⁻), (H₀⁺), (H₀⁻)) if it satisfies (H⁺) (resp. (H⁻), (H₀⁺), (H₀⁻)). For the rest of this section, we assume that C is convex.

Remark 2. (a) Clearly (H⁺) implies (H₀⁺) and (H⁻) implies (H₀⁻).

(b) These assumptions are rather mild. Hypothesis (H⁺) (resp. (H⁻)) is weaker than h -protoconvexity (resp. h -quasiconcavity) of f . They are automatically satisfied if h is the radial derivative of f . More generally, we can make a precise comparison with the Dini derivatives of f .

Lemma 12 (a) If, for all $x \in C$, $h(x, \cdot) \leq D^+f(x, \cdot)$ (resp. $h(x, \cdot) \geq D_+f(x, \cdot)$) then assumption (H^+) (resp. (H^-)) is satisfied. In fact, if for all $w, x \in C$,

$$h(x, w - x) > 0 \implies D^+f(x, w - x) > 0 \quad (2)$$

$$\text{(resp. } h(x, w - x) < 0 \implies D_+f(x, w - x) < 0), \quad (3)$$

then f satisfies (H^+) (resp. (H^-)).

(b) Conversely, if condition (H^+) (resp. (H^-)) is satisfied, then for all $w, x \in C$ one has

$$h(x, w - x) > 0 \implies D^+f(x, w - x) \geq 0$$

$$\text{(resp. } h(x, w - x) < 0 \implies D_+f(x, w - x) \leq 0).$$

Proof. (a) Suppose (2) holds. Given $w, x \in C$ such that $h(x, w - x) > 0$, one has $D^+f(x, w - x) > 0$, so that there exists a sequence $(t_n) \rightarrow 0_+$ satisfying $f(x + t_n(w - x)) > f(x)$. Taking n large enough, we get some $z := x + t_n(w - x) \in (w, x)$ such that $f(z) > f(x)$. Similarly, if (3) holds, then f satisfies (H^-) .

(b) Conversely, when f satisfies (H^+) and $w, x \in C$ are such that $h(x, w - x) > 0$, for every $(\varepsilon_n) \rightarrow 0_+$ one also has $h(x, \varepsilon_n(w - x)) > 0$, so that there exists $t_n \in]0, \varepsilon_n[$ such that $f(x + t_n(w - x)) > f(x)$. Thus $D^+f(x, w - x) \geq 0$. The case of (H^-) is similar. \square

Let us deal with a characterization which has been obtained in [1], [7], [13], [31], [36], [37], [59], [61] under additional assumptions (lower semicontinuity of f or the assumption that h is equal to the radial derivative of f or to one of the Dini derivatives of f). In the following statement, given functions f, h , we say that f satisfies (R) if for every $w, x, y \in C$, $x \in]w, y[$ the function $s := h(x, \cdot)$ satisfies (R).

Proposition 13 (a) A function $f : C \rightarrow \mathbb{R}$ is h -pseudoaffine and satisfies (R), (H_0^+) , (H_0^-) if and only if f is h -colinfine.

(b) If f is h -pseudoaffine, if $D_+f \leq h \leq D^+f$ and if either f satisfies (R) or is radially continuous, then f is h -colinfine.

Proof. (a) Let f be h -pseudoaffine and let (R), (H_0^+) , (H_0^-) be satisfied. Given $w, x \in C$, let us find some $\lambda(w, x) \in \mathbb{P}$ such that $f(w) - f(x) = \lambda(w, x)h(x, w - x)$. If $f(w) < f(x)$ then, by h -pseudoconvexity we have $h(x, w - x) < 0$ and we can take

$$\lambda(w, x) = [f(w) - f(x)]/h(x, w - x).$$

If $f(w) > f(x)$ then, by h -pseudoconcavity, we have $h(x, w - x) > 0$ and we can take

$$\lambda(w, x) = [f(w) - f(x)]/h(x, w - x).$$

In the case $f(w) = f(x)$, we will prove $h(x, w - x) = 0$, so that we take $\lambda(w, x) = 1$. We may suppose $x \neq w$. If $h(x, w - x) > 0$, by hypothesis (H_0^+) , there exists $z \in]w, x[$ such that $f(z) > f(x) = f(w)$. Thus, $h(z, x - z) < 0$ and $h(z, w - z) < 0$, a contradiction with (R). Similarly, if $h(x, w - x) < 0$, hypothesis (H_0^-) yields a contradiction with (R).

Conversely, suppose f is h -colinfine. Then f is h -semiaffine by Proposition 8. Let us first check that for all $x \in C$ condition (R) is satisfied by f , or, equivalently, that, for all $x \in C$, $h(x, \cdot)$ and $-h(x, \cdot)$ satisfy condition (T). Let $w, y \in C$ be such that $x \in]w, y[$ and $h(x, w - x) < 0$; we have to

show that $h(x, y - x) \geq 0$. Suppose $h(x, y - x) < 0$. Since f is h -colinfine, we have $f(y) < f(x)$. By Proposition 10 we obtain $f(x) < f(w)$, hence $h(x, w - x) > 0$, a contradiction. Thus, (R) is satisfied. Now let us prove that (H^+) is satisfied. Let $w, x \in C$ be such that $h(x, w - x) > 0$. Then for all $z \in]x, w[$, $h(x, z - x) > 0$ by homogeneity, hence $f(z) > f(x)$ for all $z \in]x, w[$. Thus (H^+) is satisfied. Similarly, (H^-) is satisfied. Now (H_0^+) , (H_0^-) are satisfied too, by Remark 2(a).

(b) When f is h -pseudoaffine and satisfies (R) and when $D_+f \leq h \leq D^+f$, conditions (H_0^+) , (H_0^-) are fulfilled by Remark 2 and Lemma 12; then the conclusion follows from (a). In the case f is h -pseudoaffine, radially continuous and $D_+f \leq h \leq D^+f$, one has that f and $-f$ are quasiconvex by Proposition 5(a) and then f and $-f$ are h -protoconvex and $-h$ -protoconvex by Proposition 4(a). Thus f is h -protoaffine and then f is h -semiaffine. By Proposition 8, f is h -colinfine. \square

Let us consider the property:

$$(E_0) \quad \forall x, w \in C, \quad h(x, w - x) = 0 \Rightarrow f(x) = f(w).$$

Remark 3. (a) This property is satisfied if f is h -pseudoaffine or if $h(x, u) \neq 0$ for all $x \in C$, $u \in X \setminus \{0\}$.

(b) If f is h -protoaffine the reverse implication $f(x) = f(w) \Rightarrow h(x, w - x) = 0$ holds.

(c) Condition (E_0) implies that if there exist $w, x \in C$ such that $h(x, w - x) = 0$, then for all $t > 0$, $f(x + t(w - x)) = f(x)$.

A related property is presented in the following corollary.

Corollary 14 *Let f satisfy properties (R), (H^+) , (H^-) . If f is h -pseudoaffine on C then the following condition is satisfied:*

$$(E) \quad \forall x, w \in C, \quad h(x, w - x) = 0 \Leftrightarrow f(x) = f(w).$$

The converse is true if f is radially continuous on C . In fact, if f is radially continuous on C and satisfies conditions (E) and (H^+) (resp. (H^-)) then f is h -pseudoconvex (resp. h -pseudoconcave).

Proof. The first assertion follows from Proposition 13(a) by definition of a h -colinfine function.

To prove the converse assertion, changing f and h into $-f$ and $-h$ respectively, it suffices to show that if $f(w) < f(x)$, then one has $h(x, w - x) < 0$. Since f satisfies condition (E_0) , it is enough to prove that assuming $h(x, w - x) > 0$ leads to a contradiction. By hypothesis (H^+) , there exists $z \in]w, x[$ such that $f(z) > f(x)$. By continuity of f on $[w, z]$, since $f(x) \in]f(w), f(z)[$, there exists $u \in [w, z]$ such that $f(u) = f(x)$. Then $u \neq x$ and there exists $r \in \mathbb{P}$ such that $w - x = r(u - x)$. By condition (E), $0 = h(x, u - x) = h(x, w - x)$, a contradiction with our assumption $h(x, w - x) > 0$. \square

Corollary 15 *If $h \leq D^+f$ and if f is radially continuous on C and satisfies condition (E) then f is h -pseudoconvex.*

If $h \geq D_+f$ and if f is radially continuous on C and satisfies condition (E) then f is h -pseudoconcave.

The second assertion of the next corollary slightly extends assertion ii) of Theorem 4.13 in [8]; see also [13], [31], [34].

Corollary 16 *If D_+f and D^+f are finite, if $D_+f \leq h \leq D^+f$ and if f satisfies condition (E) then f is h -pseudoaffine. In particular, if f is radially differentiable on C and such that for every $x \in C$ the derivative $f'(x, \cdot)$ is sublinear, then f is pseudoaffine on C if, and only if f satisfies condition (E).*

The following illuminating Theorem has been given in [8, Theorems 4.13, 4.14], [31], [34] in the case the function f is differentiable and $h := f'$. Here f is nonsmooth. We start with a preparatory lemma dealing with the one-dimensional case.

Lemma 17 *Let C be an open interval of \mathbb{R} and let $\varphi : C \rightarrow \mathbb{R}$ be continuous, satisfy (H^+) and (H^-) with respect to h_φ , where $h_\varphi : C \times \mathbb{R} \rightarrow \mathbb{R}$ is given for some function $\ell_\varphi : \mathbb{R} \rightarrow \mathbb{R}$ by $h_\varphi(t, r) := \ell_\varphi(t)r$ for all $(t, r) \in C \times \mathbb{R}$. Then φ is h_φ -colinfine if, and only if, either (E_0) is satisfied and there exists some $t \in C$ such that $\ell_\varphi(t) = 0$ or the sign of ℓ_φ is constant on level sets of φ .*

Proof. Suppose that φ is h_φ -colinfine. Then there exists $\lambda : C \times C \rightarrow \mathbb{P}$ such that for any $s \neq t$ in C

$$\frac{\varphi(s) - \varphi(t)}{s - t} = \lambda(s, t)\ell_\varphi(t). \quad (4)$$

In addition, by Proposition 10, φ is constant or increasing or decreasing. By relation (4), if φ is constant then $\ell_\varphi = 0$ and (E_0) is satisfied; otherwise the sign of ℓ_φ is constant.

For the converse, let us first suppose (E_0) is satisfied and $\ell_\varphi(t) = 0$ for some $t \in C$. Then, for any $s \in C$, $\ell_\varphi(t)(s - t) = 0$, so that, by (E_0) , $\varphi(s) = \varphi(t)$. Hence, φ is constant and, in view of conditions (H^+) and (H^-) , we must have $\ell_\varphi(r) = 0$ for all $r \in C$: then φ is h_φ -colinfine.

Now suppose that the sign of ℓ_φ is constant on level sets of φ . Given $r \in C$ such that $\ell_\varphi(r) > 0$, we will prove that $\varphi(t) > \varphi(r)$ for all $t > r$. Let us first prove that we cannot have $\varphi(t) < \varphi(r)$. If $\varphi(t) < \varphi(r)$, let $s := \sup\{p \in [r, t] : \varphi(p) = \varphi(r)\}$. Since φ is continuous, we have $s < t$ and $\varphi(s) = \varphi(r)$. Then our assumption yields $\ell_\varphi(s) > 0$, so that, by condition (H^+) there exists some $p \in]s, t[$ with $\varphi(p) > \varphi(s)$ and the intermediate value theorem yields some $q \in [p, t]$ with $\varphi(q) = \varphi(s) = \varphi(r)$, a contradiction with the definition of s .

Now let us suppose that $\varphi(t) = \varphi(r)$. Then $\ell_\varphi(t) > 0$ by our assumption, so that $h_\varphi(t, r - t) = \ell_\varphi(t)(r - t) < 0$. Then condition (H^-) yields some $t' \in]r, t[$ such that $\varphi(t') < \varphi(t) = \varphi(r)$. Replacing t by t' , the first part of the proof yields the expected contradiction.

A similar proof shows that $\varphi(q) < \varphi(r)$ for all $q < r$. Changing φ and ℓ_φ into their opposites, we see that when $\ell_\varphi(r) < 0$, the function φ is decreasing on C . Thus, φ is h_φ -colinfine. \square

Theorem 18 *Let C be an open, convex subset in X , $f : C \rightarrow \mathbb{R}$ be radially continuous and $h : VC \rightarrow \mathbb{R}$ be linear and continuous in its second variable, with $\ell(x) := h(x, \cdot)$ for all $x \in C$.*

(a) *Let f be h -colinfine. Then, for all $w \in C$, the set $K_w := \{u \in X : h(w, u) = 0\}$ is a linear subspace of X . Moreover, for every $x \in C$ such that $f(x) = f(w)$ one has $K_x = K_w$.*

(b) *If f is h -colinfine on C then for any $w, x \in C$ such that $f(w) = f(x)$, the linear forms $\ell(w)$ and $\ell(x)$ are positively colinear:*

$$w, x \in C : f(w) = f(x) \Rightarrow \exists r \in \mathbb{P}, \quad \ell(w) = r\ell(x). \quad (5)$$

(c) *Conversely, suppose that f and h satisfies (E_0) , (H^+) and (H^-) , in particular, $D_+f \leq h \leq D_+f$ and $\ell(x) \neq 0$ for all $x \in C$. If $x \mapsto \frac{\ell(x)}{\|\ell(x)\|}$ is constant on the level sets of f , then f is h -colinfine.*

(d) *Assume that $C = X$, f satisfies (E_0) , (H^+) and (H^-) and $\ell(x) \neq 0$ for all $x \in C$. Then f is h -colinfine if, and only if, $x \mapsto \frac{\ell(x)}{\|\ell(x)\|}$ is constant on X .*

Proof. (a) Let $u, v \in K_w$. Since C is open, $T^r(C, w) = \{w\} \times X$ and for $t \in \mathbb{R}_+$ with t small enough one has $w + tu \in C$ and $f(w + tu) = f(w) + \lambda(w + tu, w)th(w, u) = f(w)$. Thus, by Proposition 10, $f(w + ru) = f(w)$ for all $r \in \mathbb{R}$ such that $w + ru \in C$. It follows that $h(w, -tu) = 0$ for $t > 0$ small enough and $-u \in K_w$. Now for $t > 0$ small enough, we also have $w + tv \in C$ and $w + t(u + v) \in C$. Since $f(w + tv) = f(w) = f(w + tu)$, by Proposition 10 again, we obtain

$$f\left(w + t\frac{u+v}{2}\right) = f\left(\frac{w+tu}{2} + \frac{w+tv}{2}\right) = f(w).$$

Thus $u + v \in K_w$. It follows that K_w is a linear subspace.

Now let $w, x \in C$ be such that $f(w) = f(x)$ and let $u \in K_w$. Since f is h -colinfine, we have $x - w \in K_w$, hence $x - w + tu \in K_w$ for all $t \in \mathbb{R}$ since K_w is a linear subspace. Thus, for $|t|$ small enough, we have $x + tu \in C$ and $f(x + tu) = f(w) = f(x)$. Therefore $h(x, u) = 0$ and $u \in K_x$. So, $K_w \subset K_x$. The symmetry of the roles of w and x yields $K_w = K_x$.

(b) Let $w, x \in C$ be such that $f(w) = f(x)$. Since $K_w = \ker l(w)$, $K_x = \ker l(x)$ and $K_w = K_x$ either $l(w)$ and $l(x)$ are both 0 and (5) is satisfied with $r = 1$ or both of them are non zero and there exists $r \in \mathbb{R} \setminus \{0\}$ such that $l(w) = rl(x)$. Let us prove that r is positive. Suppose that $r < 0$. Pick $u \in X$ such that $\langle l(x), u \rangle = 1$ and set $z_1 = w + tu$ and $z_2 = x + tu$, with $t > 0$ small enough to ensure that $z_1, z_2 \in C$. Then, since f is h -colinfine, one has $f(z_1) < f(w) = f(x) < f(z_2)$ and $\langle l(x), w - x \rangle = h(x, w - x) = 0$. Since f is radially continuous, there exists $z \in]z_1, z_2[$ such that $f(z) = f(x)$. Thus, there exists some $s \in]0, 1[$ such that $z := sz_1 + (1-s)z_2$, hence $z - x = sw - sx + tu$ and

$$0 = h(x, z - x) = l(x)(z - x) = l(x)(sw - sx + tu) = t > 0,$$

a contradiction. Hence, the case $r < 0$ is excluded and (b) is established.

(c) Let $w, x \in C$ and for $t \in C_{w,x} := \{t \in \mathbb{R} : x_t \in C\}$ with $x_t := (1-t)x + tw$, set $\varphi(t) := f(x_t)$. For $t \in C_{w,x}$, let $\ell_\varphi(t) := l(x_t)(w - x)$ and $h_\varphi(t, r) := \ell_\varphi(t)r$ for $r \in \mathbb{R}$. Let us show that φ satisfies (H^+) , (H^-) and (E_0) with respect to h_φ . Suppose that $h_\varphi(t, s - t) > 0$ for $t, s \in C_{w,x}$, with $s \neq t$. Then $h(x_t, x_s - x_t) = (s - t)\ell(x_t)(w - x) = h_\varphi(t, s - t) > 0$. Since f satisfies (H^+) , there exists $q \in (t, s)$ such that $\varphi(q) := f(x_q) > f(x_t) := \varphi(t)$. Thus, φ satisfies (H^+) . With a similar proof, we see that φ satisfies (H^-) . Suppose that $h_\varphi(t, s - t) = 0$ for $t, s \in C_{w,x}$, with $s \neq t$. Then $h(x_t, x_s - x_t) = h_\varphi(t, s - t) = 0$. Since f satisfies (E_0) , $\varphi(s) := f(x_s) = f(x_t) := \varphi(t)$, hence φ satisfies (E_0) .

Suppose that $\ell_\varphi(t) \neq 0$ for all $t \in C_{w,x}$. We shall prove that the sign of ℓ_φ is constant on the level sets of φ . Suppose that $r, s \in C_{w,x}$ are such that $\varphi(r) = \varphi(s)$ and $\ell_\varphi(r)\ell_\varphi(s) < 0$. We may assume that $r < s$, $\ell_\varphi(r) > 0$ and $\ell_\varphi(s) < 0$. Set $x_r := (1-r)x + rw$ and $x_s := (1-s)x + sw$. Since $f(x_r) = \varphi(r) = \varphi(s) = f(x_s)$, we have $h_\varphi(r, 1) = \ell(x_r)(w - x) > 0$ and $0 > h_\varphi(s, 1) = \ell(x_s)(w - x) = \frac{\|\ell(x_s)\|}{\|\ell(x_r)\|}\ell(x_r)(w - x) > 0$: a contradiction. Thus, there exists some $t \in C$ such that $\ell_\varphi(t) = 0$ or the sign of ℓ_φ is constant on level sets of φ and then φ is h_φ -colinfine, by the preceding lemma.

Since for any $w, x \in C$, φ is h_φ -colinfine and

$$f(w) - f(x) = \varphi(1) - \varphi(0) = \lambda_\varphi(1, 0)h_\varphi(0, 1 - 0) = \lambda(w, x)h(x, w - x)$$

where $\lambda(w, x) := \lambda_\varphi(1, 0) > 0$; thus f is a h -colinfine function.

(d) If $C = X$ and the mapping $x \mapsto \frac{\ell(x)}{\|\ell(x)\|}$ is constant on X , then f is h -colinfine by (c).

Conversely, suppose that f is h -colinfine and there exist $w, x \in C$ such that $\frac{\ell(w)}{\|\ell(w)\|} \neq \frac{\ell(x)}{\|\ell(x)\|}$.

From (b), we have $f(w) \neq f(x)$. Let K_w and K_x be as in (a). Since $\frac{\ell(w)}{\|\ell(w)\|} \neq \frac{\ell(x)}{\|\ell(x)\|}$, there exists $\bar{z} \in (w + K_w) \cap (x + K_x)$. One has $\bar{z} - w \in K_w$, $\bar{z} - x \in K_x$, hence $\ell(w)(\bar{z} - w) = 0$ and $\ell(x)(\bar{z} - x) = 0$. Since f is h -colinfine, we have $f(x) = f(\bar{z}) = f(w)$: a contradiction. \square

The next proposition is related to the preceding results; however, the assumptions are different.

Proposition 19 *Let C be open, let f be radially continuous and $\ell(x) := h(x, \cdot) \in X^* \setminus \{0\}$ for all $x \in C$.*

If f is h -quasiconvex then f is h -pseudoconvex.

If f is h -protoaffine then f is h -colinfine and condition (E) is satisfied.

Proof. Let $w, x \in C$ be such that $f(w) < f(x)$. Then $\ell(x)(w - x) \leq 0$ by h -quasiconvexity of f . Suppose that $\ell(x)(w - x) = 0$. Since $\ell(x) \neq 0$, we can find $u \in X$ such that $\ell(x)(u) = 1$. By the radial continuity of f , there exists $\varepsilon > 0$ such that $y := w + \varepsilon u \in C$ and $f(y) < f(x)$. By h -quasiconvexity of f , $0 \geq \ell(x)(y - x) = \varepsilon \ell(x)(u) = \varepsilon > 0$, a contradiction. Hence, $\ell(x)(w - x) < 0$ and then f is h -pseudoconvex.

Since a h -protoconvexity function is h -quasiconvex, if f is h -protoconvex and $\ell(x) \neq 0$, then f is h -pseudoconvex. Similarly, if f is h -protoaffine then f is h -pseudoaffine, hence h -semiaffine and h -colinfine by Proposition 8. The last assertion follows from Remark 3. \square

The following example illustrates the preceding theorem; note that since f is not differentiable, [8, Theorems 4.13, 4.14], [31], [34] cannot be applied.

Example 9. (a) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x_1, x_2) = x_2$ for $x_2 \leq 0$, $f(x_1, x_2) = x_2^2$ for $x_2 > 0$, and let $h : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by for all $u \in \mathbb{R}^2$, $h((x_1, x_2), (u_1, u_2)) = u_2$ for $x_2 \leq 0$ and $h((x_1, x_2), (u_1, u_2)) = x_2 u_2$ for $x_2 > 0$. Then, as easily seen, f satisfies (E₀), (H⁺) and (H⁻), and $x \mapsto \frac{\ell(x)}{\|\ell(x)\|}$ is constant with $\ell(x) := h(x, \cdot)$. Thus, f is h -colinfine.

We deduce from [42], [51] a characterization of continuous h -colinfine functions. We suppose that X is finite dimensional.

Proposition 20 *A continuous (resp. lower semicontinuous), nonconstant function $f : X \rightarrow \mathbb{R}$ is h -colinfine for some $h : X \times X \rightarrow \mathbb{R}$ if, and only if, there exist a continuous linear form ℓ on X , $\ell \neq 0$ and a continuous (resp. lower semicontinuous) increasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f = g \circ \ell$.*

Proof. The condition is clearly sufficient: for $w, x \in X$ one has $f(w) - f(x) = \lambda(w, x)h(x, w - x)$, with $h(x, u) := \ell(u)$ and

$$\lambda(w, x) := \frac{g(\ell(w)) - g(\ell(x))}{\ell(w) - \ell(x)} \text{ for } \ell(w) \neq \ell(x), \quad \lambda(w, x) := 1 \text{ for } \ell(w) = \ell(x)$$

Conversely, let $f : X \rightarrow \mathbb{R}$ be l.s.c. (resp. continuous) and h -colinfine, hence protoaffine. According to [42], [51], there exist a continuous linear form ℓ on X and a l.s.c. (resp. continuous) nondecreasing function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f = g \circ \ell$. Since f is nonconstant, $\ell \neq 0$. Let us prove that g is increasing. Suppose that there exist $r_1 < r_2$ in \mathbb{R} such that for all $r \in [r_1, r_2]$, $g(r) = g(r_1) = g(r_2)$. Since f is nonconstant, g is nonconstant and then for $t > 1$ large enough, $g(r_1 + t(r_2 - r_1)) > g(r_2)$ or $g(r_2 + t(r_1 - r_2)) < g(r_1)$. Without loss of generality, we can assume that $g(r_1 + t(r_2 - r_1)) > g(r_2)$ for t large enough. Since ℓ is linear and continuous with $\ell \neq 0$, there exist $w, x \in X$ such that $\ell(w) = r_1 < \ell(x) = r_2$. Taking t large enough $\ell(w + t(x - w)) > \ell(x) = r_2$ and then

$f(w + t(x - w)) = g(\ell(w + t(x - w))) = g(r_1 + t(r_2 - r_1)) > g(r_1) = f(w) = f(x)$. By Proposition 10, f is not h -colinfine. Therefore g is increasing. \square

Note that if f is constant, then one can take g constant and $\ell = 0$. Conversely, if g is constant or if $\ell = 0$ then $g \circ \ell$ is constant.

4 Vector-valued maps and preservation properties

Extensions of the preceding concepts to vector-valued maps can be given. The cases of quasiconvex and pseudoconvex maps have been studied in detail by several authors ([8], [40], [17]...). The cases of colinfine and semiaffine maps seem to be new. Our purpose here is limited. We just intend to use vector-valued concepts in order to get composition properties, and thus some more means for constructing examples. In the sequel, Y is another n.v.s., with dual space Y^* .

Definition 21 *Given a map $H : VC \rightarrow Y$ which is positively homogeneous in its second variable, a map $F : C \rightarrow Y$ is said to be H -colinfine at $x \in C$ if there exists some function $\lambda : C \times C \rightarrow \mathbb{P}$ such that for all $w \in C$*

$$F(w) - F(x) = H(x, \lambda(w, x)(w - x)).$$

It is H -colinfine on C if it is H -colinfine at every $x \in C$.

An immediate composition result can be stated.

Proposition 22 *Let X, Y, Z be n.v.s., let C (resp. D) be a convex subset of X (resp. Y) and let $H : VC \rightarrow Y$, $K : VD \rightarrow Z$. If $F : C \rightarrow Y$ and $G : D \rightarrow Z$ are H -colinfine on C and K -colinfine on D respectively and if $F(C) \subset D$, then $G \circ F$ is L -colinfine on C , for $L : VC \rightarrow Z$ given by $L(x, x') := K(F(x), H(x, x'))$ for $(x, x') \in VC$.*

To get an extension of the concepts of h -pseudoconvexity and h -colinvexity, one needs an order structure on Y . Thus we assume that Y is ordered by a closed convex cone Y_+ . We denote by Y_+^* the dual cone of Y_+ : $Y_+^* := \{y^* \in Y^* : \forall y \in Y_+ \langle y^*, y \rangle \geq 0\}$.

Definition 23 *Given a map $F : C \rightarrow Y$.*

(a) *F is said to be H -pseudoconvex at $x \in C$ if*

$$w \in C, H(x, w - x) \geq 0 \implies F(w) \geq F(x).$$

It is H -pseudoconvex on C if it is H -pseudoconvex at every $x \in C$.

(b) *F is said to be H -protoconvex at $x \in C$ if*

$$w \in C, F(w) \leq F(x) \implies H(x, w - x) \leq 0.$$

It is H -protoconvex on C if it is H -protoconvex at every $x \in C$.

(c) *F is said to be H -semiconvex at $x \in C$ if it is both H -pseudoconvex and H -protoconvex at $x \in C$. It is said to be H -protoaffine (resp. H -pseudoaffine) at x if it is H -protoconvex (resp. H -pseudoconvex) at x and if $-F$ is $-H$ -protoaffine (resp. $-H$ -pseudoaffine) at x .*

(d) *F is said to be H -colinvex on C if there exists some function $\lambda : C \times C \rightarrow \mathbb{P}$ such that for all $w, x \in C$*

$$F(w) - F(x) \geq H(x, \lambda(w, x)(w - x)).$$

If $Y_+ := \{0\}$, F is H -colinfine on C if, and only if, F is H -colinconv on C . If F is H -colinconv on C then F is clearly H -pseudoconvex on C .

Let us give another means to extend the previous notions to the vectorial case.

Definition 24 A map $F : C \rightarrow Y$ is said to be scalarly H -pseudoconvex (resp. scalarly H -protoconvex, scalarly H -semiconvex, scalarly H -pseudoaffine, scalarly H -semiaffine) on C if, for all $y^* \in Y_+^*$, the function $y^* \circ F$ is $y^* \circ H$ -pseudoconvex (resp. $y^* \circ H$ -protoconvex, $y^* \circ H$ -semiconvex, $y^* \circ H$ -pseudoaffine, $y^* \circ H$ -semiaffine).

Proposition 25 A map $F : C \rightarrow Y$ is H -pseudoconvex (resp. H -protoconvex, H -semiconvex, H -pseudoaffine, H -semiaffine) if, and only if, it is scalarly H -pseudoconvex (resp. scalarly H -protoconvex, scalarly H -semiconvex, scalarly H -pseudoaffine, scalarly H -semiaffine).

Proof. Let us prove that if F is scalarly H -pseudoconvex then it is H -pseudoconvex; the converse is obvious. Let $w, x \in C$ be such that $H(x, w-x) \geq 0$. Then, for all $y^* \in Y_+^*$ we have $(y^* \circ H)(x, w-x) \geq 0$, hence $(y^* \circ F)(w) \geq (y^* \circ F)(x)$. Then, the bipolar theorem ensures that $F(w) \geq F(x)$. The proofs of the other cases are similar. \square

Given some n.v.s. X, Y , a subset C of X , we say that a map $F : C \rightarrow Y$ is *directionally differentiable* (resp. radially differentiable) at $x \in C$ if for any $u \in T(C, x)$ (resp. $u \in T^r(C, x)$) the quotient $(1/t)(F(x+tv) - F(x))$ (resp. $(1/t)(F(x+tu) - F(x))$) has a limit as $(t, v) \rightarrow (0_+, u)$ with $x+tv \in C$ (resp. $t \rightarrow 0_+$). The preceding limit is then called the *directional derivative* (resp. the radial derivative) of F at x in the direction u and is denoted by $F'(x, u)$. Such a concept is a natural extension of the notion described above for real-valued functions.

If $F : C \rightarrow Y$ has a radial or a directional derivative, we say that F is colinfine if H is the radial derivative and F is H -colinfine. We use a similar convention for the other concepts introduced above.

Example 10. Given n.v.s. X, Y , $A \in L(X, Y)$, $y \in Y$, $b \in X^*$, $\beta \in \mathbb{R}$, $C := \{x \in X : b(x) + \beta > 0\}$, the map $F : C \rightarrow Y$ given by

$$F(x) := (A(x) + y)/(b(x) + \beta)$$

is a colinfine map on C , as a computation similar to the one of Example 6 shows.

Example 11. Given $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Y_+ := \mathbb{R}_+^2$, let $F : X \rightarrow Y$ be given by

$$F(x) := (x, x^3).$$

Then F is quasiconvex but it is not pseudoconvex as $y^* \circ F$ is not pseudoconvex when y^* is the second projection. Note that F is of class C^∞ and its derivative is never 0.

Remark 4. Here, H -colinfine maps and H -scalarly semiaffine maps are different from vector pseudo-linear in [61], where its components are colinfine. Also, our definition of a pseudoaffine map differs from the one in [8, Definition 4.13], even when one takes for H the derivative of F .

Let us give some preservation properties. The first one is immediate.

Proposition 26 Given $H : VC \rightarrow Y$ and H -colinfine (resp. H -scalarly semiaffine) maps $F : C \rightarrow Y$, $G : C \rightarrow Y$ and $r \in \mathbb{R}$, the map $F + G$ is H -colinfine (resp. H -scalarly semiaffine) and rF is rH -colinfine (resp. rH -scalarly semiaffine).

Now let us consider composition properties.

Proposition 27 Let X, Y, Z be n.v.s., Y and Z being ordered by closed convex cones Y_+ and Z_+ respectively. Let C (resp. D) be a convex subset of X (resp. Y) and let $F : C \rightarrow Y, G : D \rightarrow Z, H : VC \rightarrow Y, K : VD \rightarrow Z$. Assume that $F(C) + Y_+ \subset D$, F is H -colinconv on C and that for every $(y, v) \in VD, v' \in Y_+$ one has $K(y, v + v') \geq K(y, v)$. Let $L : VC \rightarrow Z$ be given by $L(x, x') := K(F(x), H(x, x'))$ for $(x, x') \in VC$.

- (a) If G is K -pseudoconvex on D , then $G \circ F$ is L -pseudoconvex on C .
- (b) If G is K -protoconvex on D , then $G \circ F$ is L -protoconvex on C .
- (c) If G is K -colinconv on D , then $G \circ F$ is L -colinconv on C .

Proof. (a) Since F is H -colinconv on C , given $w, x \in C$ one can find $\lambda(w, x) \in \mathbb{P}$ such that $F(w) - F(x) = \lambda(w, x)H(x, w - x) + v'$ with $v' \in Y_+$. Since G is K -pseudoconvex on D , for $u := F(w), v := F(x)$ one has

$$\begin{aligned} L(x, w - x) \geq 0 &\Rightarrow K(F(x), H(x, w - x)) \geq 0 \Rightarrow K(F(x), \lambda(w, x)H(x, w - x) + v') \geq 0 \\ &\Rightarrow K(F(x), F(w) - F(x)) \geq 0 \Rightarrow G(u) \geq G(v). \end{aligned}$$

so that $G \circ F$ is L -pseudoconvex on C .

(b) Given $w, x \in C$ such that $G(F(w)) \leq G(F(x))$ one has $K(F(x), F(w) - F(x)) \leq 0$ by K -quasiconvexity of G , hence $K(F(x), H(x, w - x)) \leq 0$ since F is H -colinconv and $K(F(x), \cdot)$ is nondecreasing.

(c) let $u := F(w), v := F(x)$. Then $G(u) - G(v) \geq \mu(u, v)K(v, u - v) \geq \mu(u, v)K(v, \lambda(w, x)H(x, w - x))$, for some $\mu(u, v) \in \mathbb{P}$ by K -colinconvity of G . \square

Remark 5. Note that assertion (c) yields Proposition 22 by taking $Y_+ := \{0\}, Z_+ := \{0\}$.

Taking $Z := \mathbb{R}$, the preceding proposition yields a means to generate more examples of generalized convex functions. Observing that affine mappings being colinconv, the same can be said for precompositions with affine maps.

Now let us give a chain rule in which the first map is not supposed to be colinconv.

Proposition 28 Let $f : C \rightarrow \mathbb{R}, g : \mathbb{R} \rightarrow \mathbb{R}$ and let $h : VC \rightarrow \mathbb{R}, k : \mathbb{R}^2 \rightarrow \mathbb{R}$ be positively homogeneous in their second variables and let $\ell : T^r C \rightarrow \mathbb{R}$ be given by $\ell(x, u) := k(f(x), h(x, u))$.

- (a) If f is h -pseudoconvex, if g is nondecreasing and if $k(r, -1) < 0$ for all $r \in f(C)$, then $g \circ f$ is ℓ -pseudoconvex.
- (b) If f is h -protoconvex, if g is increasing and if $D^+g \geq k$, then $g \circ f$ is ℓ -protoconvex;
- (c) If f is h -semiconvex, if g is increasing and if $k(y, -1) < 0$ for all $y \in f(C)$, then $g \circ f$ is ℓ -semiconvex.

Note that when f and g have radial derivatives and $h = f', k = g'$, then ℓ is the radial derivative of $g \circ f$ (observe that the radial derivative of g is also the directional derivative of g).

Proof. (a) Let $w, x \in C$ be such that $\ell(x, w - x) = k(f(x), h(x, w - x)) \geq 0$. Since $k(f(x), -1) < 0$, we have $h(x, w - x) \geq 0$. Since f is h -pseudoconvex we have $f(w) \geq f(x)$ and since g is nondecreasing, we get $(g \circ f)(w) \geq (g \circ f)(x)$. Thus $g \circ f$ is ℓ -pseudoconvex.

(b) Let $w, x \in C$ be such that $\ell(x, w - x) = k(f(x), h(x, w - x)) > 0$. Since g is nondecreasing and $k \leq D^+g$, we have $k(f(x), -1) \leq D^+g(f(x), -1) \leq 0$; thus, we must have $h(x, w - x) > 0$ (recall that $k(f(x), 0) = 0$). Since f is h -protoconvex, we have $f(w) > f(x)$ and since g is increasing, we get $(g \circ f)(w) > (g \circ f)(x)$.

(c) We already know that $g \circ f$ is ℓ -protoconvex. Let us prove that $g \circ f$ is ℓ -protoconvex by slightly modifying (b). Let $w, x \in C$ be such that $\ell(x, w - x) = k(f(x), h(x, w - x)) > 0$. Since $k(f(x), -1) < 0$, we have $h(x, w - x) > 0$; thus, as above, we get $(g \circ f)(w) > (g \circ f)(x)$. \square

5 Characterizations of solution sets

In the present section, we apply the previous concepts to the constrained minimization problem

$$(C) \quad \text{minimize } f(x) \text{ subject to } x \in C,$$

where C is a subset of X and $f : C \rightarrow \mathbb{R}$. We denote by S the solution of the constrained problem (C). Let $h : VC \rightarrow \mathbb{R}$ be a generalized directional derivative of f which is positively homogeneous in the second variable and $h(\cdot, 0) = 0$.

We say that f satisfies (H^-) at $a \in C$ (relatively to h) if $h(a, x - a) < 0$ for some $x \in C$, then there exists $z \in]a, x[$ such that $f(z) < f(a)$. Thus, f satisfies (H^-) if, and only if, f satisfies (H^-) at every $a \in C$.

Proposition 29 *Let f be h -pseudoconvex. If $a \in C$ is such that $h(a, x - a) \geq 0$ for all $x \in C$, then $a \in S$.*

Conversely, if $a \in S$, if C is starshaped at a , and if f satisfies (H^-) at a (or in particular, if $h(a, \cdot) \geq D_+ f(a, \cdot)$), then $h(a, x - a) \geq 0$ for all $x \in C$.

Proof. The first assertion follows from the definition of h -pseudoconvexity of f at a .

Conversely, let us suppose f satisfies (H^-) at $a \in S$. We must prove $h(a, x - a) \geq 0$, for all $x \in C$. If there exists some $x \in C$ such that $h(a, x - a) < 0$, then by (H^-) and starshapedness of C , there exists $z \in]a, x[\subset C$ such that $f(z) < f(a)$: this is a contradiction with the assumption that a is a solution to (C). \square

The following variant is a direct consequence of the definition of an h -semiaffine function (or of the fact that f is h -colinfine).

Proposition 30 *Suppose that f is h -semiaffine on C and C is starshaped at a . Then*

(a) *$a \in S$ if and only if $h(a, x - a) \geq 0$ for all $x \in C$ and if and only if $h(x, a - x) \leq 0$ for all $x \in C$;*

(b) *If a, b are in S , then $(a + \mathbb{R}(b - a)) \cap C$ is included in S ;*

(c) *If a is a solution to (C) then the set S of solutions to (C) is given by $S = S_h$, where $S_h := \{x \in C : h(a, x - a) = 0\}$;*

(d) *If the feasible set C is convex, then a local minimizer to (C) is a global minimizer to (C).*

Let us consider now the case in which the constraint set C is defined by a finite family of inequalities, so that problem (C) turns into the mathematical programming problem

$$(M) \quad \text{minimize } f(x) \text{ subject to } x \in C := \{x \in X : g_1(x) \leq 0, \dots, g_m(x) \leq 0\},$$

where $f : W \rightarrow \mathbb{R}$, $g_i : W \rightarrow \mathbb{R}$ and W is an open convex subset of X .

We assume that $h : W \times X \rightarrow \mathbb{R}$, $h_i : W \times X \rightarrow \mathbb{R}$ ($i \in \mathbb{N}_m := \{1, \dots, m\}$) are generalized directional derivatives of f and g_i , respectively, which are positively homogeneous in their second variables and such that $h(\cdot, 0) = h_i(\cdot, 0) = 0$. Given $a \in C$, let $I(a) := \{i \in \mathbb{N}_m : g_i(a) = 0\}$ and let $P(a)$ stand for a set of $i \in I(a)$ such that g_i is h_i -pseudoconcave at a , while $N(a) := I(a) \setminus P(a)$.

Each continuous linear functional x^* on X satisfying $\langle x^*, \cdot \rangle \leq h(x, \cdot)$ is said to be a subderivative of f with respect to h at x . The set $\partial^h f(x)$ of all subderivatives at x is called the h -subdifferential of f at x ; it is a weak* closed convex subset of X^* . Obviously, for $a \in X$,

$$h(a, u) \geq 0, \forall u \in X \Leftrightarrow 0 \in \partial^h f(a).$$

The subdifferential of a convex function $k : X \rightarrow \mathbb{R}_\infty$ at $x \in \text{dom } k := k^{-1}(\mathbb{R})$ is the set

$$\partial k(x) := \{x^* : k(w) - k(x) \geq \langle x^*, w - x \rangle, \forall w \in X\}.$$

The sufficient optimality criteria which follows is an easy consequence of the preceding proposition. If f and g_i ($i \in \mathbb{N}_m$) are differentiable, then the sufficient optimality condition is the same as the classical result in [41] and [8, Thm. 4.5].

Proposition 31 *Let f be h -pseudoconvex at $a \in W$ and let g_i be h_i -protoconvex at a for $i = 1, \dots, m$. If a is such that the following conditions are satisfied for some $y_i \in \mathbb{R}_+$, $i = 1, \dots, m$, then a is a solution to problem (\mathcal{M}) :*

$$h(a, x - a) + \sum_{i=1}^m y_i h_i(a, x - a) \geq 0, \quad \forall x \in C, \quad (6)$$

$$g_1(a) \leq 0, \dots, g_m(a) \leq 0, \quad y_1 g_1(a) = 0, \dots, y_m g_m(a) = 0. \quad (7)$$

Proof. Suppose on the contrary that there exists some $x \in C$ such that $f(x) < f(a)$. By h -pseudoconvexity of f at a , we have $h(a, x - a) < 0$.

Then, for $i \in I(a)$, by h_i -protoconvexity of g_i at a , we have $h_i(a, x - a) \leq 0$. Consequently, we have

$$\begin{aligned} h(a, x - a) &< 0, \\ h_i(a, x - a) &\leq 0, \quad \forall i \in I(a). \end{aligned}$$

Multiplying each side of the last inequalities by y_i and adding the obtained sides to the ones of the preceding relation, since $y_i = 0$ for $i \notin I(a)$, we get

$$0 \leq h(a, x - a) + \sum_{i=1}^m y_i h_i(a, x - a) < 0,$$

a contradiction. □

Corollary 32 *Let f be h -pseudoconvex at $a \in X$ and let g_i be h_i -protoconvex at a for $i = 1, \dots, m$. If a is such that the following conditions are satisfied for some $y_i \in \mathbb{R}_+$, $i = 1, \dots, m$, then a is a solution to problem (\mathcal{M}) :*

$$\begin{aligned} 0 &\in \partial^h f(a) + y_1 \partial^{h_1} g_1(a) + \dots + y_m \partial^{h_m} g_m(a), \\ g_1(a) &\leq 0, \dots, g_m(a) \leq 0, \quad y_1 g_1(a) = 0, \dots, y_m g_m(a) = 0. \end{aligned}$$

Proof. Let $a^* \in \partial^h f(a)$, $a_i^* \in \partial^{h_i} g_i(a)$ for $i = 1, \dots, m$ be such that

$$a^* + y_1 a_1^* + \dots + y_m a_m^* = 0.$$

Since for every $x \in C$ and $i \in \mathbb{N}_m$ we have

$$\begin{aligned}\langle a^*, x - a \rangle &\leq h(a, x - a), \\ \langle a_i^*, x - a \rangle &\leq h_i(a, x - a),\end{aligned}$$

multiplying each side of the last inequalities by y_i and adding the obtained sides to the ones of the preceding relation, since $y_i = 0$ for $i \notin I(a)$, we get

$$0 = \langle a^*, x - a \rangle + \sum_{i=1}^m y_i \langle a_i^*, x - a \rangle \leq h(a, x - a) + \sum_{i=1}^m y_i h_i(a, x - a)$$

and the preceding proposition applies. \square

Now let us give a necessary optimality condition in a limiting form. We assume now that X is a Banach space.

Proposition 33 *Let $a \in S$. Assume that h, h_i ($i \in I(a)$) are finite, sublinear and l.s.c. in their second variables. Assume that f satisfies condition (H^-) at a (relatively to h), g_i are u.s.c. at a for $i \notin I(a)$. Suppose that for $i \in N(a)$, g_i satisfies condition (H^-) at a (relatively to h_i) and g_i is quasiconvex at a . Then there exists $0 \neq (y_0, (y_n)) \in \mathbb{R}_+ \times \mathbb{R}_+^{N(a)}$ such that*

$$0 \in \text{cl}\{y_0 \partial^h f(a) + \sum_{n \in N(a)} y_n \partial^{h_n} g_n(a) + \sum_{p \in P(a)} y_p \partial^{h_p} g_p(a) : (y_p) \in \mathbb{R}_+^{P(a)}\}. \quad (8)$$

In particular, if $\partial^{h_i} g_i(a) \neq \emptyset$ for $i \in \mathbb{N}_m \setminus I(a)$ then

$$0 \in \text{cl}\{y_0 \partial^h f(a) + \sum_{i=1}^m y_i \partial^{h_i} g_i(a) : y_i \in \mathbb{R}_+, y_i g_i(a) = 0, y_0 + \sum_{n \in N(a)} y_n = 1\}.$$

Proof. Let $a \in S$. In the first step, we prove that the system

$$\begin{pmatrix} h(a, x - a) < 0 \\ h_i(a, x - a) < 0 \quad \forall i \in N(a) \\ h_i(a, x - a) \leq 0 \quad \forall i \in P(a) \end{pmatrix}$$

has no solution x in X .

Let us suppose, on the contrary, that x satisfy these inequalities. Then, since W is open, there exists $\delta > 0$ such that $x_t := a + t(x - a) \in W$ for $t \in [0, \delta]$. Thus, for $i \in P(a)$ one has $h_i(a, x_t - a) \leq 0$, so that $g_i(x_t) \leq g_i(a)$ for $t \in]0, \delta]$ by h_i -pseudoconcavity of g_i at a . For $i \in N(a)$, and $t \in]0, \delta]$, since $h_i(a, x_t - a) < 0$, by condition (H^-) there exists $z_t := x_s \in]x_t, a[\cap W$ such that $g_i(z_t) < g_i(a)$. In addition, as g_i is quasiconvex, one has $g_i(x_r) \leq g_i(a)$ for all $r \in [0, s]$. Moreover, since for $i \notin I(a)$, $g_i(a) < 0$ and g_i is u.s.c, we can take s small so that $g_i(x_r) < 0$ for $r \in [0, s]$. Since $h(a, x_s - a) < 0$, by condition (H^-) we get $f(x_r) < f(a)$ for some $r \in [0, s]$, a contradiction with $a \in S$. Our claim is proved.

In the second step, we prove that relation (8) is satisfied. According to the first step, the system

$$\begin{pmatrix} -h(a, x - a) > 0 \\ -h_n(a, x - a) > 0 \quad \forall n \in N(a) \\ -h_p(a, x - a) \geq 0 \quad \forall p \in P(a) \end{pmatrix}$$

has no solution x in X . Since $h(a, \cdot)$ and $h_i(a, \cdot)$ are sublinear and l.s.c., using the generalized Farkas lemma [21, Theorem 3], there exists $0 \neq (y_0, (y_n)) \in \mathbb{R}_+ \times \mathbb{R}_+^{N(a)}$ such that

$$0 \in \text{cl}\left\{\partial \left(y_0 h(a, \cdot) + \sum_{n \in N(a)} y_n h_n(a, \cdot) \right) (0) + \partial \left(\sum_{p \in P(a)} y_p h_p(a, \cdot) \right) (0) : (y_p) \in \mathbb{R}_+^{P(a)}\right\}.$$

Since the Attouch-Brézis qualification condition for a sum of convex functions is fulfilled, this condition can be rewritten as in (8), taking into account the notation $\partial^h f(a) = \partial h(a, \cdot)(0)$ and $\partial^{h_i} g_i(a) = \partial h_i(a, \cdot)(0)$ for $i \in I(a)$. The last assertion is obtained by taking $y_i = 0$ for $i \in \mathbb{N}_m \setminus I(a)$ and by using a normalization, replacing $(y_0, (y_i))$ by $(t^{-1}y_0, (t^{-1}y_i))$, where $t := y_0 + \sum_{n \in N(a)} y_n$. \square

In the case we can take for $P(a)$ the empty set, we get the following consequence.

Corollary 34 *Let $a \in S$. Assume that h, h_i ($i \in I(a)$) be finite, sublinear and l.s.c. in their second variables. Assume that f satisfies condition (H^-) at a (relatively to h), g_i are u.s.c. at a for $i \notin I(a)$. Suppose that for $i \in I(a)$, g_i satisfy condition (H^-) at a (relatively to h_i) and g_i is quasiconvex at a . Then there exists $(y_0, (y_i)) \in \mathbb{R}_+ \times \mathbb{R}_+^{I(a)}$ such that $(y_0, (y_i)) \neq 0$ and*

$$0 \in \text{cl}\{y_0 \partial^h f(a) + \sum_{i \in I(a)} y_i \partial^{h_i} g_i(a)\}.$$

Moreover, if the qualification condition

$$0 \notin \text{clco}\left(\bigcup_{i \in I(a)} \partial^{h_i} g_i(a)\right) \tag{9}$$

holds, then there exists $(y_i) \in \mathbb{R}_+^{I(a)}$ such that

$$0 \in \partial^h f(a) + \sum_{i \in I(a)} y_i \partial^{h_i} g_i(a).$$

Proof. The first assertion is a direct consequence of the preceding proposition, with $N(a) = I(a)$, $P(a) = \emptyset$. The second assertion stems from the fact that if $y_0 = 0$, replacing (y_i) by $(t^{-1}y_i)$ with $t := \sum_{i \in I(a)} y_i$ we get $0 \in \text{cl}(\sum_{i \in I(a)} y_i \partial^{h_i} g_i(a)) \subset \text{clco}(\bigcup_{i \in I(a)} \partial^{h_i} g_i(a))$, a contradiction with (9). \square

The case $P(a) = I(a)$, $N(a) = \emptyset$ is considered in the following statement; then we have $y_0 \neq 0$, or, equivalently, $y_0 = 1$ after normalization.

Corollary 35 *Let $a \in S$. Assume that h, h_i ($i \in I(a)$) are sublinear and l.s.c. in their second variables. Assume that f satisfies condition (H^-) at a (relatively to h), g_i are h_i -pseudoconcave at a for $i \in I(a)$ and g_i is u.s.c at a for $i \notin I(a)$. Then*

$$0 \in \text{cl}\{(\partial^h f(a) + y_1 \partial^{h_1} g_1(a) + \dots + y_m \partial^{h_m} g_m(a) : y_i \geq 0, y_i g_i(a) = 0)\}. \tag{10}$$

Let us gather a necessary condition deduced from the preceding statement with the sufficient condition of Corollary 32.

Corollary 36 *Let $a \in W$. Assume that f is h -pseudoconvex at a and g_i is h_i -colinfine at a for some bifunctions $h : W \times X \rightarrow \mathbb{R}$, $h_i : W \times X \rightarrow \mathbb{R}$ ($i \in \{1, \dots, m\}$) which are such that $h(a, \cdot)$ and $h_i(a, \cdot)$ are positively homogeneous and $h \geq D_+ f$. Assume that g_i is u.s.c for $i \notin I(a)$. Then $a \in C$ is a solution to (\mathcal{M}) if there exists $y := (y_1, \dots, y_m) \in \mathbb{R}_+^m$ such that $y_i g_i(a) = 0$ for $i = 1, \dots, m$ and*

$$0 \in \partial^h f(a) + y_1 \partial^{h_1} g_1(a) + \dots + y_m \partial^{h_m} g_m(a). \quad (11)$$

If $h(a, \cdot)$ and $h_i(a, \cdot)$ ($i \in I(a)$) are sublinear and continuous and if the following qualification condition holds, condition (11) is necessary for a to be in S :

$$0 \notin \text{co} \left(\bigcup_{i \in I(a)} \partial^{h_i} g_i(a) \right). \quad (12)$$

Proof. The sufficient condition is obvious from Corollary 32. For the necessary condition, since f is h -colinfine at a , it satisfies condition (H^-) by Remark 2 (b). Given a sequence (ε_n) in \mathbb{P}_+ with limit 0, relation (10) yields for each $i \in I(a)$ some sequences $(y_{i,n})$ in \mathbb{R}_+ , (y_n^*) , $(z_{i,n}^*)$ in X^* such that $y_n^* \in \partial^h f(a)$, $z_{i,n}^* \in \partial^{h_i} g_i(a)$ for all $n \in \mathbb{N}$ and

$$\left\| y_n^* + \sum_{i \in I(a)} y_{i,n} z_{i,n}^* \right\| \leq \varepsilon_n. \quad (13)$$

Let us show that the sequences $(y_{i,n})$ are bounded. If one of them is unbounded, taking a subsequence if necessary, we may assume that $(\|y_n\|) \rightarrow +\infty$, where $y_n = (y_{i,n})_{i \in I(a)} \in \mathbb{R}_+^{I(a)}$, the space $\mathbb{R}^{I(a)}$ being endowed with the norm given by $\|y\| := \sum_{i \in I(a)} |y_i|$. Since for $i \in I(a)$ the functions $h_i(a, \cdot)$ are sublinear and continuous, the sets $\partial^{h_i} g_i(a)$ are weak* compact; $\partial^h f(a)$ is also weak* compact. It follows that we can find a limit point (u, z^*) of the sequence (y_n, z_n^*) , where $u_n := y_n / \|y_n\| \in \mathbb{R}_+^{I(a)}$. Dividing by $\|y_n\|$ both sides of relation (13) and taking limits we get

$$\left\| \sum_{i \in I(a)} u_i z_i^* \right\| = 0.$$

Since $z_i^* \in \partial^{h_i} g_i(a)$ and $u := (u_i)_{i \in I(a)}$ is in the canonical simplex of $\mathbb{R}^{I(a)}$, we get a contradiction with (12).

Since the sequence (y_n) is bounded, and since (y_n^*) , $(z_{i,n}^*)$ are also bounded in X^* , we can take weak* limit points y , y^* , z_i^* and pass to the limit in (13). Thus, we get $y \in \mathbb{R}_+^{I(a)}$, $y^* \in \partial^h f(a)$, $z_i^* \in \partial^{h_i} g_i(a)$ such that

$$y^* + \sum_{i \in I(a)} y_i z_i^* = 0.$$

Hence condition (11) is satisfied. \square

The next example illustrates the corollary; note that since f is not differentiable at $a = 0$, [41, page 153] cannot be applied. Since f is not a Plastria function at a , [38, Theorem 10] cannot be applied either. Moreover, f is not locally Lipschitz at a ; thus [43, Prop. 6.1] and [44, Prop. 6.3] cannot be applied. Finally, let us note the usefulness of admitting that the bifunction h may differ from the Dini derivatives of f .

Examples 12. Let f be defined by $f(x) = x + 1$ for $x > 0$, $f(x) = x^2$ for $x \leq 0$ and let $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g_1(x) = -x$. Let $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be given by $h(0, u) = |u|$ for $u \in \mathbb{R}$ and let $h_1 := g'_1$. Then f, g_1 satisfy condition (H^-) at $a = 0$ and g_1 is quasiconvex. We can take $y_1 = 1$ as a multiplier since $\partial^h f(a) = [-1, 1]$, $\partial g_1(a) = \{-1\}$.

Examples 13. One may take for f a continuous function, for instance f given by $f(x) = \sqrt{x}$ for $x > 0$, $f(x) = x^2$ for $x \leq 0$, with g_1 and $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ as in the preceding example.

In convex mathematical programming, the multipliers are the same for all solutions. A related result has been given in [16] in the case the function f is pseudoaffine, differentiable, $h := f'$ and the functions g_i ($i \in \{1, \dots, m\}$) are linear. Here f and g_i are nonlinear and even nonsmooth.

Proposition 37 *Let $a \in S$ be such that there exists $y = (y_i) \in \mathbb{R}_+^m$ satisfying the optimality conditions (6) and (7).*

(a) *If f is h -colinvec at a and g_i is h_i -protoconvex at a , then $\sum_{i \in I(a)} y_i h_i(a, b - a) = 0$ for every $b \in S$.*

(b) *Let f be h -colinvec at a and $g = (g_i)$ be H -colinvec at a where $H = (h_i)_{i \in \mathbb{N}_m}$. Let L be the Lagrangian given by $L(x, z) := f(x) + zg(x)$ for $(x, z) \in C \times \mathbb{R}^m$ and let $b \in C$. Then, $b \in S$ if, and only if, $yg(b) = 0$ and $L(a, y) = L(b, y)$.*

Proof. (a) Let $b \in S$. Since f is h -colinvec at a , there exists $\lambda(b, a) \in \mathbb{P}$ such that $0 = f(b) - f(a) \geq \lambda(b, a)h(a, b - a)$. By condition (6), one has

$$\sum_{i \in I(a)} y_i h_i(a, b - a) \geq -h(a, b - a) \geq 0.$$

On the other hand, since g_i is h_i -protoconvex at a and for all $i \in I(a)$, $g_i(b) \leq g_i(a) = 0$, one has $h_i(a, b - a) \leq 0$ for all $i \in I(a)$. Hence $\sum_{i \in I(a)} y_i h_i(a, b - a) = 0$.

(b) Let $b \in S$. Since g is H -colinvec at a then there exists $\mu(b, a) \in \mathbb{P}$ such that for any $i \in I(a)$,

$$g_i(b) = g_i(b) - g_i(a) \geq \mu(b, a)h_i(a, b - a).$$

By (a), $y \in \mathbb{R}_+^m$ and $b \in C$, $0 \geq yg(b) \geq \mu(b, a) \sum_{i \in I(a)} y_i h_i(a, b - a) = 0$ and thus $L(a, y) := f(a) + yg(a) = f(a) = f(b) = f(b) + yg(b) =: L(b, y)$.

Conversely, let $b \in C$. If $yg(b) = 0$ and $L(a, y) = L(b, y)$ then $f(a) = L(a, y) = L(b, y) = f(b) + yg(b) = f(b)$, hence $b \in S$. \square

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