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► **To cite this version:**

Bruno Machet. Mixing at 1-loop for quasi-degenerate fermions in a SU(2)_L gauge theory of weak interactions. 2009. hal-00422646v2

HAL Id: hal-00422646

<https://hal.science/hal-00422646v2>

Preprint submitted on 8 Oct 2009 (v2), last revised 15 Jun 2010 (v3)

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**MIXING AT 1-LOOP FOR QUASI-DEGENERATE FERMIONS
IN A $SU(2)_L$ GAUGE THEORY OF WEAK INTERACTIONS**

B. Machet ^{1 2}

Abstract: 1-loop transitions induce diagonal and non-diagonal kinetic-like, momentum (p^2) dependent interactions between fermionic bare mass eigenstates. We correspondingly re-examine the Cabibbo procedure, which requires in particular the simultaneous re-diagonalization of kinetic and mass terms. When two fermions get close to degeneracy, a resonance exists, at which the “Cabibbo” angle becomes close to maximal and is simply connected with the masses of all four fermions inside the concerned two generations and to that of the W gauge boson. It proves also, then, the closest to its classical value. Mass splittings are furthermore shown to make slightly non-unitary the connection between bare flavor (or mass) states and 1-loop mass eigenstates. Still, the 1-loop Cabibbo (PMNS) matrix $\mathcal{C}(p^2)$, the expression of which is dictated by gauge invariance, stays unitary.

PACS: 12.15.Ff 12.15.Lk 14.60.Pq **Keywords:** mixing, radiative corrections, mass-splitting

1 Introduction

The origin of large mixing angles observed in leptonic charged currents is still largely unknown [1]. A widespread belief is that it is linked to a quasi-degeneracy of neutrinos, but this connection was never firmly established. And it cannot be on simple grounds. Indeed, the mixing angles that are “observed” in neutrino oscillations are the ones occurring in charged currents, which combine the individual mixing matrices of fermions with different electric charges ¹; the path that goes from the quasi-degeneracy of one of the two doublets to large mixing in the PMNS matrix [2], one example of which is proposed below, cannot thus be completely straightforward. Furthermore, homographic transformations on a (mass) matrix, while changing its eigenvalues, do not change its eigenvectors, neither, accordingly, mixing angles; an infinity of different mass spectra can thus be associated with a given mixing angle.

We will focus here on two pairs of fermions, making up two generations. For the sake of convenience, we shall call them generically (d, s) and (u, c) . The first will be supposed to be close to degeneracy and the

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¹The electronic (ν_e), muonic (ν_μ), and tau (ν_τ) neutrinos are defined as the neutrinos that couple, inside charged currents, to the mass eigenstates of charged leptons. They are accordingly related to the neutrino mass eigenstates ($\nu_{em}, \nu_{\mu m}, \nu_{\tau m}$) by

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = K_\ell^\dagger K_\nu \begin{pmatrix} \nu_{em} \\ \nu_{\mu m} \\ \nu_{\tau m} \end{pmatrix}, \quad (1)$$

where K_ℓ and K_ν are the mixing matrices respectively of charged leptons and neutrinos. This connection is seen to involve the hermitian conjugate $K_\ell^\dagger K_\nu$ of the PMNS matrix.

second largely split. Results can easily be translated to the leptonic sector, which we rather have in mind. The Cabibbo angle θ_c [3] is then replaced by the corresponding entry θ_{PMNS} of the PMNS matrix.

This study implements the necessary (re-)diagonalization of both kinetic and mass terms of a quadratic Lagrangian to suitably determine an orthogonal set of mass eigenstates. While this requirement is always and simply taken care of at the classical level by a bi-unitary diagonalization of the mass matrix, it is generally overlooked as soon as radiative corrections are concerned [4] [5] [6] [7] [8]. In particular, only considering self-mass contributions to determine the renormalized mass states from the renormalized mass matrix exposes to the problem that they are not orthogonal since there still exist kinetic-like transitions between them. We show that the re-diagonalization of kinetic terms can have important effects. The first is that, for non-degenerate systems, bare mass (or flavor) states are non-unitarily related to 1-loop mass eigenstates. The second is that a resonance appears, at which large mixing in charged currents (PMNS, Cabibbo) gets indeed associated with (d, s) quasi-degeneracy.

The procedure goes as follows (for the sake of convenience, calculations are done in the bare mass basis). Though, at 1-loop, diagonal and non-diagonal corrections to kinetic terms only occur at $\mathcal{O}(g^2)$, casting them back to their canonical form $\bar{\Psi} \not{p} \mathbb{I} \Psi$ (\mathbb{I} stands for the unit matrix) involves an angle ω , which depends on the classical Cabibbo angle θ_c , on the masses of all four fermions (m_d, m_s, m_u, m_c), and on m_W . It does not depend on the coupling constant g and gets close to $(-\theta_c)$ when the two fermions are much closer to degeneracy than the second pair in the same two generations. The corresponding transformation, \mathcal{V} , which is slightly non-unitary as soon as the fermions are non-degenerate, modifies the mass matrix which stays no longer diagonal, but keeps nevertheless symmetric. Its re-diagonalization requires accordingly a simple rotation $\mathcal{R}(\xi)$ which, being in particular unitary, does not change anymore the kinetic terms. So, after the two transformations \mathcal{V} and $\mathcal{R}(\xi)$ have been performed, the quadratic Lagrangian is fully diagonal, which determines the 1-loop mass eigenstates.

Like ω , the angle ξ depends on θ_c and on the fermion masses. In the same situation as above (two fermions quasi-degenerate and the other pair far from degeneracy) $\tan 2\xi$ exhibits a pole. The pole condition is a simple relation between θ_c , the masses of all four fermions and the W mass. When it is fulfilled, ξ becomes maximal and very close to the 1-loop value of the Cabibbo angle. We also demonstrate that the (d, s) mixing angle is then also the closest to its classical value, and the (u, c) mixing angle extremely close to it. This makes the Cabibbo angle at resonance both close to maximal and the closest to its classical value.

That mass splittings trigger non-unitary relations between bare flavor states and 1-loop mass states is not a new result [9][10][11][12]. It turns out however, that, unlike individual mixing matrices, the 1-loop Cabibbo matrix $\mathcal{C}(p^2)$ occurring in charged currents stays unitary. It is a consequence of gauge invariance, which in particular connects, through the covariant derivative of fermion fields, kinetic terms to gauge currents, both at the classical level and including radiative corrections. The expression of the 1-loop Cabibbo matrix $\mathcal{C}(p^2)$ is thus directly dictated by that of the 1-loop kinetic terms, which is one more reason to pay a special attention to them.

This study also exhibits the instability of quasi-degenerate systems with respect to small (perturbative) variations of the Lagrangian. This is definitely not a new result, but it may call for techniques that go beyond perturbation theory.

In the last section, we briefly show that 1-loop corrections have a negligible influence on the propagation of neutrinos in matter and, more specifically, on the MSW [13][14] resonance.

For the sake of simplicity, we shall work in a pure $SU(2)_L$ theory of weak interactions instead of the standard $SU(2)_L \times U(1)$ electroweak model. Since the theory is renormalizable, we shall use the unitary gauge, devoid of the intricacies due to scalar fields and which, consistently working at order g^2 , yields finite results for the quantities of concern to us. While we cannot, accordingly, verify the gauge independence of the results (independence on the ξ parameter in an R_ξ gauge), gauge invariance, as already mentioned, is of primordial importance in this work.

2 1-loop transitions between non-degenerate fermions. Re-diagonalizing the quadratic Lagrangian

2.1 1-loop transitions

Like in the Standard Model of electroweak interactions [15], the diagonalization of the classical mass matrix by a bi-unitary transformation leads to the classical mass eigenstates, for example s_m^0 and d_m^0 , with classical masses m_s and m_d . They are orthogonal with respect to the classical Lagrangian (which is akin to the property that no transition between them occurs at the classical level). However, at 1-loop, gauge interactions induce diagonal and non-diagonal transitions between them. For example, Fig. 1 describes non-diagonal $s_m^0 \rightarrow d_m^0$ transitions, mediated by the W^\pm gauge bosons. Diagonal transitions are mediated either by W_μ^\pm or by W_μ^3 .

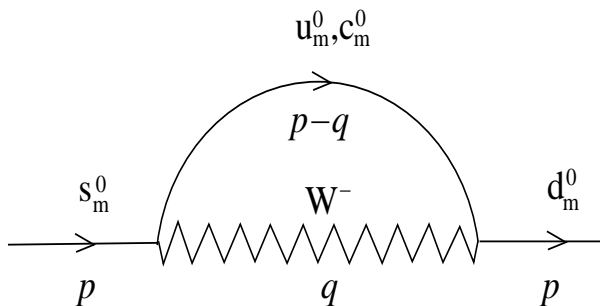


Fig. 1: $s_m^0 \rightarrow d_m^0$ transition at 1-loop

We investigate in this work how the Cabibbo procedure implements in the presence of these transitions [16]. The one depicted in Fig. 1 contributes as a left-handed, kinetic-like, p^2 -dependent interaction of the type

$$\sin \theta_c \cos \theta_c (h(p^2, m_u, m_W) - h(p^2, m_c, m_W)) \bar{d}_m^0 \not{p} (1 - \gamma_5) s_m^0, \quad (2)$$

that we abbreviate, with transparent notations, into

$$s_c c_c (h_u - h_c) \bar{d}_m^0 \not{p} (1 - \gamma_5) s_m^0. \quad (3)$$

It depends on the classical Cabibbo angle $\theta_c = \theta_d - \theta_u$. The function h is dimensionless.

It is simple matter to realize that all (diagonal and non-diagonal) 1-loop transitions between s_m^0 and d_m^0 mediated by W^\pm gauge bosons transform their kinetic terms into

$$\begin{aligned} & \begin{pmatrix} \bar{d}_m^0 & \bar{s}_m^0 \end{pmatrix} \left[\mathbb{I} \not{p} + \begin{pmatrix} c_c^2 h_u + s_c^2 h_c & s_c c_c (h_u - h_c) \\ s_c c_c (h_u - h_c) & s_c^2 h_u + c_c^2 h_c \end{pmatrix} \not{p} (1 - \gamma_5) \right] \begin{pmatrix} d_m^0 \\ s_m^0 \end{pmatrix} \\ = & \begin{pmatrix} \bar{d}_m^0 & \bar{s}_m^0 \end{pmatrix} \left[\mathbb{I} \not{p} + \left(\frac{h_u + h_c}{2} + (h_u - h_c) \mathcal{T}_x(2\theta_c) \right) \not{p} (1 - \gamma_5) \right] \begin{pmatrix} d_m^0 \\ s_m^0 \end{pmatrix}, \end{aligned} \quad (4)$$

where we noted

$$\mathcal{T}_x(\theta) = \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}. \quad (5)$$

To the contributions (4) we must add the diagonal transitions mediated by the W_μ^3 gauge boson. The kinetic terms for left-handed d_m^0 and s_m^0 quarks then become (omitting the fermionic fields and the dependence on p^2, \dots)²

²From now onwards, to lighten the notations, we shall frequently omit the dependence on p^2 and on the masses.

$$\begin{aligned}
K_d &= \mathbb{I} + H_d ; \\
H_d &= \frac{h_u + h_c}{2} + (h_u - h_c) \mathcal{T}_x(2\theta_c) + \begin{pmatrix} h_d & \\ & h_s \end{pmatrix}, \tag{6}
\end{aligned}$$

where $h_d = h(p^2, m_d, m_W)$ and $h_s = h(p^2, m_s, m_W)$. Likewise, in the (u, c) sector, one has

$$\begin{aligned}
K_u &= \mathbb{I} + H_u ; \\
H_u &= \frac{h_d + h_s}{2} + (h_d - h_s) \mathcal{T}_x(2\theta_c) + \begin{pmatrix} h_u & \\ & h_c \end{pmatrix}. \tag{7}
\end{aligned}$$

In the unitary gauge, for $m_i^2, m_j^2, p^2 \ll m_W^2$, one has

$$h_i - h_j \approx \frac{g^2}{4\pi^2} \frac{m_j^2 - m_i^2}{p^2} \left[1 + \frac{m_W^2}{p^2} \ln \frac{m_W^2 - p^2}{m_W^2} \right] \approx \frac{g^2}{8\pi^2} \frac{m_i^2 - m_j^2}{m_W^2} \left[1 + \frac{2}{3} \left(\frac{p^2}{m_W^2} \right)^2 + \dots \right]. \tag{8}$$

We shall now diagonalize the quadratic part of the effective 1-loop Lagrangian, which means putting the pure kinetic terms back to the unit matrix and, at the same time, re-diagonalizing the mass matrix. This is accordingly a two-steps procedure.

Since the kinetic terms of right-handed fermions are not modified, we shall only be concerned with the left-handed ones.

2.2 First step: re-diagonalizing kinetic terms back to the unit matrix

The pure kinetic terms K_d for (d_m^0, s_m^0) written in (6) can be cast back to their canonical form by a p^2 -dependent non-unitary transformations $\mathcal{V}_d(p^2, \dots)$ according to

$$\mathcal{V}_d^\dagger K_d \mathcal{V}_d = \mathbb{I}. \tag{9}$$

The procedure to find \mathcal{V}_d is the following. Let $(1 + t_+)$ and $(1 + t_-)$, $t_+, t_- = \mathcal{O}(g^2)$, be the eigenvalues of the symmetric matrix K_d . One has explicitly

$$t_\pm = \frac{h_u + h_c + h_d + h_s}{2} \pm \frac{1}{2} \sqrt{(h_u - h_c)^2 + (h_d - h_s)^2 + 2(h_u - h_c)(h_d - h_s) \cos 2\theta_c}. \tag{10}$$

K_d can be diagonalized by a rotation $\mathcal{R}(\omega_d) \equiv \begin{pmatrix} \cos \omega_d & \sin \omega_d \\ -\sin \omega_d & \cos \omega_d \end{pmatrix}$ according to

$$\mathcal{R}(\omega_d)^\dagger K_d \mathcal{R}(\omega_d) = \begin{pmatrix} 1 + t_+ & \\ & 1 + t_- \end{pmatrix}, \tag{11}$$

with

$$\tan 2\omega_d = \frac{-(h_u - h_c) \sin 2\theta_c}{(h_u - h_c) \cos 2\theta_c + h_d - h_s}, \tag{12}$$

or, equivalently,

$$\cos 2\omega_d = \frac{(h_u - h_c) \cos 2\theta_c + h_d - h_s}{t_+ - t_-}, \quad \sin 2\omega_d = -\frac{(h_u - h_c) \sin 2\theta_c}{t_+ - t_-}, \tag{13}$$

in which $(t_+ - t_-)$ can be immediately obtained from (10)³. (12) defines ω_d in particular as a function of θ_c , $\omega_d = \omega_d(\theta_c, \dots)$. Since both numerator and denominator of (12) are $\mathcal{O}(g^2)$, ω_d does not depend on the coupling constant g . Nor does it depend on p^2 , for the same reason.

The diagonal matrix obtained in (11) is not yet the required unit matrix, but one simply gets to it by renormalizing the columns of $\mathcal{R}(\omega_d)$ respectively by $\frac{1}{\sqrt{1+t_+}}$ and $\frac{1}{\sqrt{1+t_-}}$. The looked-for non-unitary matrix \mathcal{V}_d writes finally

$$\mathcal{V}_d = \begin{pmatrix} \frac{c_{\omega_d}}{\sqrt{1+t_+}} & \frac{s_{\omega_d}}{\sqrt{1+t_-}} \\ -\frac{s_{\omega_d}}{\sqrt{1+t_+}} & \frac{c_{\omega_d}}{\sqrt{1+t_-}} \end{pmatrix}. \quad (14)$$

It differs from the rotation $\mathcal{R}(\omega_d)$ only at $\mathcal{O}(g^2)$ and satisfies

$$\mathcal{V}_d \mathcal{V}_d^\dagger = \frac{1}{(1+t_+)(1+t_-)} \left(\mathbb{I} + \frac{t_+ + t_-}{2} - (t_+ - t_-) \mathcal{T}_x(-2\omega_d) \right), \quad \mathcal{V}_d^\dagger \mathcal{V}_d = \begin{pmatrix} \frac{1}{1+t_+} & \\ & \frac{1}{1+t_-} \end{pmatrix}. \quad (15)$$

Unlike ω_d , it slightly depends on p^2 .

For $|h_d - h_s| \ll |h_u - h_c|$, which is the situation that we shall always keep in mind, $(t_+ - t_-) \approx (h_u - h_c)$ and the expression for $\sin 2\omega_d$ in (13) shows that $\omega_d(\theta_c) \approx -\theta_c$ (see also footnote 3). So, when the pair (d, s) is close to degeneracy and (u, c) far from it, \mathcal{V}_d becomes close to a rotation $\mathcal{R}(-\theta_c)$.

The non-degeneracy of (d, s) or (u, c) is enough to trigger a non-trivial non-unitarity of \mathcal{V}_d . Instead, for $m_d = m_s$ and $m_u = m_c$, $t_+ = t_- = t$, which entails $\mathcal{V}_d \mathcal{V}_d^\dagger \approx \mathcal{V}_d^\dagger \mathcal{V}_d = \frac{\mathbb{I}}{1+t}$: in this twice degenerate case, a simple rotation to which is added a global diagonal wave function renormalization is enough to diagonalize the (d, s) kinetic terms at 1-loop.

2.3 Second step: re-diagonalization of the mass matrix

By (9), the kinetic terms⁴ $(\overline{d_{mL}^0}, \overline{s_{mL}^0}) K_d \not{p} \begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix}$ at 1-loop for left-handed d and s in the bare mass basis rewrite $(\overline{d_{mL}^0}, \overline{s_{mL}^0}) (\mathcal{V}_d^{-1})^\dagger \mathcal{V}_d^{-1} \not{p} \begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix}$, which leads to defining d_{mL}^1 and s_{mL}^1 such that $\begin{pmatrix} d_{mL}^1 \\ s_{mL}^1 \end{pmatrix} = \mathcal{V}_d^{-1} \begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix}$. The mass terms $(\overline{d_{mL}^0}, \overline{s_{mL}^0}) M_d \begin{pmatrix} d_{mR}^0 \\ s_{mR}^0 \end{pmatrix} + h.c.$, with $M_d = \text{diag}(m_d, m_s)$, rewrite accordingly $(\overline{d_{mL}^1}, \overline{s_{mL}^1}) \mathcal{V}_d^\dagger M_d \begin{pmatrix} d_{mR}^0 \\ s_{mR}^0 \end{pmatrix} + h.c.$. Hence, the mass matrix that needs to be re-diagonalized is $\mathcal{V}_d^\dagger M_d$. It is done through two unitary transformations $\mathcal{R}(\xi_d)$ and $S(\xi_d)$ such that $\mathcal{R}(\xi_d)^\dagger (\mathcal{V}_d^\dagger M_d) S(\xi_d) = \text{diag}(\mu_d, \mu_s)$. Since $\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d$ is a real symmetric matrix

$$\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d = \mathcal{V}_d^\dagger \begin{pmatrix} m_d^2 & \\ & m_s^2 \end{pmatrix} \mathcal{V}_d = \begin{pmatrix} \frac{m_d^2 c_{\omega_d}^2 + m_s^2 s_{\omega_d}^2}{1+t_+} & -\frac{s_{\omega_d} c_{\omega_d} (m_s^2 - m_d^2)}{\sqrt{(1+t_+)(1+t_-)}} \\ -\frac{s_{\omega_d} c_{\omega_d} (m_s^2 - m_d^2)}{\sqrt{(1+t_+)(1+t_-)}} & \frac{m_d^2 s_{\omega_d}^2 + m_s^2 c_{\omega_d}^2}{1+t_-} \end{pmatrix}, \quad (16)$$

³(12) also rewrites $\frac{\sin 2(\omega_d + \theta_c)}{\sin 2\omega_d} = -\frac{h_d - h_s}{h_u - h_c}$, which shows that $\omega_d \rightarrow -\theta_c$ when $|m_s - m_d| \ll |m_u - m_c|$.

⁴The subscript L refers to left-handed fermions and R to right-handed ones.

$\mathcal{R}(\xi_d)$ can be taken as a rotation, according to

$$\mathcal{R}(\xi_d)^\dagger \left(\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d \right) \mathcal{R}(\xi_d) = \begin{pmatrix} \mu_d^2(p^2, \dots) & \\ & \mu_s^2(p^2, \dots) \end{pmatrix}. \quad (17)$$

Since $\mathcal{R}(\xi_d)$ is unitary, it preserves the canonical form of the kinetic terms that had been rebuilt in subsection 2.2. It satisfies

$$\tan 2\xi_d = \frac{-(m_d^2 - m_s^2) \sqrt{(1+t_+)(1+t_-)} \sin 2\omega_d}{(m_d^2 - m_s^2) \left(1 + \frac{t_+ + t_-}{2} \right) \cos 2\omega_d - (m_d^2 + m_s^2) \frac{t_+ - t_-}{2}}. \quad (18)$$

Through $\omega_d(\theta_c, \dots)$, (18) defines ξ_d in particular as a function of θ_c , $\xi_d = \xi_d(\theta_c, \dots)$.

We shall see in subsection 4.3 that, for quasi-degenerate (d, s) and largely split (u, c) , $\xi_d(\theta_c)$ ultimately becomes the 1-loop Cabibbo angle, which is accordingly implicitly expressed by (18) as a function of the masses of fermions and gauge fields.

Since the mass terms rewrite $(\overline{d_{mL}^1}, \overline{s_{mL}^1}) \mathcal{R}(\xi_d) \text{diag}(\mu_d, \mu_s) S(\xi_d)^\dagger \begin{pmatrix} d_{mR}^0 \\ s_{mR}^0 \end{pmatrix} + h.c.$, the 1-loop left-handed mass eigenstates $(d_{mL}(p^2, \dots), s_{mL}(p^2, \dots))$ are defined by $(\overline{d_{mL}}, \overline{s_{mL}}) = (\overline{d_{mL}^1}, \overline{s_{mL}^1}) \mathcal{R}(\xi_d)$, which leads to

$$\begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix} = \mathcal{V}_d \mathcal{R}(\xi_d) \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}. \quad (19)$$

By construction, at this order and at any given p^2 , there exists no transition between $d_{mL}(p^2, \dots)$ and $s_{mL}(p^2, \dots)$, which are thus, by definition, orthogonal.

2.3.1 The resonance (pole of $\tan 2\xi_d$)

One should keep careful about approximations. For example, we have seen in subsection 2.2 that $|m_d - m_s| \ll |m_u - m_c|$ yields $\omega_d \approx -\theta_c$; however (18) should not be approximated by $\tan 2\xi_d \approx \tan 2\theta_c$, nor the conclusion drawn that the combined actions of \mathcal{V} and $\mathcal{R}(\xi_d)$ sum up to zero or close to it. Indeed, though the $\mathcal{O}(g^2)$ quantity $(t_+ - t_-)$ in the denominator of (18) is indeed small, it cannot be neglected with respect to the other contribution when d and s are quasi-degenerate and when $\cos 2\omega_d$ is small enough, that is, for θ_c close to maximal.

(18) is indeed characterized by the existence of a pole. It occurs (neglecting terms of $\mathcal{O}(g^{>2})$) for

$$2 \frac{m_d^2 - m_s^2}{m_d^2 + m_s^2} \cos 2\omega_d(\theta_c) \approx t_+ - t_-. \quad (20)$$

That $|\cos 2\omega_d|$ must be smaller than 1 requires $\left| \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2} \right| \geq \frac{t_+ - t_-}{2}$. In practice, for known fermions of the (d, s) type, it is always satisfied, ensuring the existence of this pole. This is however not the case for fermions of the (u, c) type when they are largely split while (d, s) is close to degeneracy. Indeed, no value of θ_c can then satisfy the equivalent of (20) and (12) for $(d, s) \leftrightarrow (u, c)$ (in particular because $\omega_u \rightarrow 0$, see also subsection 4.3), such that, unlike $\tan 2\xi_d$, $\tan 2\xi_u$ cannot have a pole.

Using the expression of $\cos 2\omega_d$ given in (13), (20) becomes

$$2 \frac{m_d^2 - m_s^2}{m_d^2 + m_s^2} \left((h_u - h_c) \cos 2\theta_c + (h_d - h_s) \right) = (t_+ - t_-)^2 \stackrel{(10)}{=} (h_u - h_c)^2 + (h_d - h_s)^2 + 2(h_u - h_c)(h_d - h_s) \cos 2\theta_c, \quad (21)$$

which, for (d, s) close to degeneracy, $|m_s - m_d| \ll |m_c - m_u|$, can be approximated by

$$\frac{m_d^2 - m_s^2}{m_d^2 + m_s^2} \approx \frac{1}{2} \frac{(h_u - h_c)^2}{(h_u - h_c) \cos 2\theta_c + (h_d - h_s)}. \quad (22)$$

Then, the pole of $\tan 2\xi_d$ occurs for

$$\cos 2\theta_c|_{pole} = \frac{1}{2} \left[(h_u - h_c) \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} - \frac{h_d - h_s}{h_u - h_c} \right], \quad (23)$$

in which we recall that d, s, u, c, θ_c are generic notations standing either for quarks or for leptons. The corresponding value of θ_c is always, in practice, close to maximal. Indeed, as far as the modulus of the second contribution to the r.h.s. of (23) is concerned: in the leptonic sector it is smaller than 10^{-16} (the reasonable upper bound of neutrino $(mass)^2 = \mathcal{O}(1 eV^2)$ divided by the smallest difference between the $(mass)^2$ of charged leptons, which is very close to m_μ^2); in the quark sector, it is smaller than $\frac{1}{2} \frac{m_s^2}{m_c^2 - m_u^2} \approx 5 \cdot 10^{-3}$. Concerning now the first contribution: for leptons, the ratio $\left| \frac{m_{\nu_i}^2 + m_{\nu_j}^2}{m_{\nu_i}^2 - m_{\nu_j}^2} \right|$ is smaller than 10^5 while $\frac{1}{2} |h_{\ell_1} - h_{\ell_2}|$ is smaller than $\frac{g^2}{16\pi^2} \frac{m_\tau^2}{m_W^2} \approx 2 \cdot 10^{-7}$; for quarks, the ratio $\left| \frac{m_{q_i}^2 + m_{q_j}^2}{m_{q_i}^2 - m_{q_j}^2} \right|$ for $q_i, q_j = (d, s, b)$ is always $\mathcal{O}(1)$ and $\frac{1}{2} |h_u - h_c| \approx 10^{-7}$ (we will not consider the top quark here, for which (8) is no longer valid).

In this same configuration, the resonance condition (20) can be written (remember that $\omega_d \approx -\theta_c$, see subsection 2.2),

$$(m_s^2 - m_d^2) \cos 2\omega_d \stackrel{|m_d - m_s| \ll |m_u - m_c|}{\approx} \frac{G_F}{\sqrt{2}\pi^2} (m_c^2 - m_u^2) \frac{m_d^2 + m_s^2}{2}. \quad (24)$$

For the (ν_e, ν_μ, e, μ) system, the r.h.s. of (24), which is independent of the energy of the neutrinos, becomes $\frac{G_F}{\sqrt{2}\pi^2} m_\mu^2 \frac{m_{\nu_e}^2 + m_{\nu_\mu}^2}{2} \approx 3 \cdot 10^{-8} \frac{m_{\nu_e}^2 + m_{\nu_\mu}^2}{2} (eV^2)$; for the $(\nu_\mu, \nu_\tau, \mu, \tau)$ system, it becomes $\frac{G_F}{\sqrt{2}\pi^2} m_\tau^2 \frac{m_{\nu_\mu}^2 + m_{\nu_\tau}^2}{2} \approx 2.12 \cdot 10^{-6} \frac{m_{\nu_\mu}^2 + m_{\nu_\tau}^2}{2} (eV^2)$. By comparison, the resonance condition for $\nu_e \leftrightarrow \nu_\mu$ oscillations in matter writes [17][18] $\Delta m^2 \cos 2\theta = 2\sqrt{2}EGFN_e$, where E is the energy of the neutrinos and N_e the electron density. In the core of the sun, where $N_e \approx 100N_A cm^{-3}$ (N_A is the Avogadro number) and for neutrinos of energy $\approx 10 MeV$, one gets $2\sqrt{2}EGFN_e \approx 10^{-4} (eV^2)$ [18], while in the core of the earth, where $N_e \approx 6 N_A cm^{-3}$ and for $1 MeV$ neutrinos, $\Delta m^2 \cos 2\theta \approx 10^{-6} (eV^2)$ [18].

2.3.2 1-loop masses

The re-diagonalization of kinetic terms indirectly contributes to a renormalization of the masses: $m_d \rightarrow \mu_d(p^2, \dots), m_s \rightarrow \mu_s(p^2, \dots)$. For $\frac{t_+ - t_-}{2} \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2} \cos 2\omega_d \ll 1$ and $\frac{t_+ - t_-}{2} \frac{m_s^2 + m_d^2}{m_s^2 - m_d^2} \cos 2\omega_d \ll 1$ ⁵, one gets from (16)

$$\begin{aligned} \mu_s^2(p^2, \dots) &\approx m_s^2 \left(1 - \frac{t_+ + t_-}{2} \right) - m_d^2 \frac{t_+ - t_-}{2} \cos 2\omega_d, \\ \mu_d^2(p^2, \dots) &\approx m_d^2 \left(1 - \frac{t_+ + t_-}{2} \right) + m_s^2 \frac{t_+ - t_-}{2} \cos 2\omega_d; \end{aligned} \quad (25)$$

(25) is not valid for $m_d = m_s$.

⁵The first condition is immediately seen to be always satisfied. The second too, unless (d, s) are extremely close to degeneracy or degenerate, which does not occur for any known fermions.

This yields in particular, still when the two conditions mentioned at the beginning of this subsection are satisfied,

$$\frac{\mu_s^2 - \mu_d^2}{\mu_s^2 + \mu_d^2} \approx \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2} - (t_+ - t_-) \frac{m_s^4 + m_d^4}{(m_s^2 + m_d^2)^2} \cos 2\omega_d, \quad (26)$$

which becomes, for $m_s \approx m_d$ ($m_s \neq m_d$)

$$\begin{aligned} \frac{\mu_s^2 - \mu_d^2}{\mu_s^2 + \mu_d^2} &\stackrel{m_s \approx m_d}{\approx} \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2} - \frac{t_+ - t_-}{2} \cos 2\omega_d \\ &\stackrel{(13)}{\approx} \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2} - \frac{1}{2} (h_u - h_c) \cos 2\theta_c = \frac{m_s^2 - m_d^2}{m_s^2 + m_d^2} + \frac{g^2}{16\pi^2} \frac{m_c^2 - m_u^2}{m_W^2} \cos 2\theta_c. \end{aligned} \quad (27)$$

Supposing $\cos 2\theta_c > 0$ and $m_c > m_u$, $\frac{\mu_s^2 - \mu_d^2}{\mu_s^2 + \mu_d^2}$ goes to a minimum, identical to its classical value, when θ_c becomes maximal. This is the case, as we saw in subsection 2.3.1, at the ξ_d resonance. Note that a similar property is satisfied in the case of the MSW resonance (see for example [18]).

The classically degenerate case $m_d = m_s$ is most easily studied directly from (16). Degeneracy gets lifted at 1-loop since the renormalized masses become, then $\mu_d^2 = \frac{m_{d,s}^2}{1+t_+}$, $\mu_s^2 = \frac{m_{d,s}^2}{1+t_-}$, such that $\frac{\mu_s^2 - \mu_d^2}{\mu_s^2 + \mu_d^2} \approx \frac{h_c - h_u}{2} \approx \frac{g^2}{16\pi^2} \frac{m_c^2 - m_u^2}{m_W^2}$. It turns out to be the limit of (27) for $m_d = m_s$ and θ_c maximal.

3 Individual mixing matrix and 1-loop mixing angle

3.1 1-loop and classical mass eigenstates are non-unitarily related

According to (19), the left-handed 1-loop mass eigenstates (d_{mL}, s_{mL}) are related to the bare ones via the product of a p^2 -dependent non-unitary transformation \mathcal{V}_d by a p^2 -dependent unitary one $\mathcal{R}(\xi_d)$. We have seen in subsection 2.2 that mass splittings are at the origin of the non-unitarity of \mathcal{V}_d . The two bases are accordingly non-unitarily related [9] [10] [11] [12].

On mass-shell (respectively at $p^2 = \mu_d^2(p^2)$ and $p^2 = \mu_s^2(p^2)$), one deduces from (19) the expressions of the physical mass eigenstates

$$\begin{aligned} d_{mL}^{phys} &\equiv d_{mL}(p^2 = \mu_d^2(p^2)) = [(\mathcal{V}_d \mathcal{R}(\xi_d))^{-1}]_{11}(p^2 = \mu_d^2(p^2)) d_{mL}^0 + [(\mathcal{V}_d \mathcal{R}(\xi_d))^{-1}]_{12}(p^2 = \mu_d^2(p^2)) s_{mL}^0, \\ s_{mL}^{phys} &\equiv s_{mL}(p^2 = \mu_s^2(p^2)) = [(\mathcal{V}_d \mathcal{R}(\xi_d))^{-1}]_{21}(p^2 = \mu_s^2(p^2)) d_{mL}^0 + [(\mathcal{V}_d \mathcal{R}(\xi_d))^{-1}]_{22}(p^2 = \mu_s^2(p^2)) s_{mL}^0, \end{aligned} \quad (28)$$

such that a second source of non-unitarity, due to $\mu_d \neq \mu_s$, adds to the one due to \mathcal{V}_d (as we mentioned at the end of subsection 2.2, both $m_d = m_s$ and $m_u = m_c$ are necessary to dim the non-unitarity of \mathcal{V}). Since bare mass states are unitarily related to bare flavor states, the physical mass eigenstates are also non-unitarily related to the latter.

3.2 Individual mixing matrix and 1-loop mixing angle

Classical flavor eigenstates and 1-loop mass eigenstates are related to each other according to

$$\begin{pmatrix} d_{fL}^0 \\ s_{fL}^0 \end{pmatrix} = C_{d0} \begin{pmatrix} d_{mL}^0 \\ s_{mL}^0 \end{pmatrix} \stackrel{(19)}{=} C_{d0} \mathcal{V}_d \mathcal{R}(\xi_d) \begin{pmatrix} d_{mL} \\ s_{mL} \end{pmatrix}, \quad (29)$$

where $C_{d0} \equiv \mathcal{R}(\theta_d)$ is the classical mixing matrix of the (d, s) pair. The individual mixing matrix at 1-loop is thus given by

$$C_d(p^2, \dots) = C_{d0} \mathcal{V}_d \mathcal{R}(\xi_d) = \mathcal{R}(\theta_d) \mathcal{V}_d \mathcal{R}(\xi_d), \quad (30)$$

which slightly deviates from unitarity. Since $\mathcal{V}_d \approx \mathcal{R}(\omega_d) + \mathcal{O}(g^2)$ (see (14)), \mathcal{C}_d stays nevertheless close to a rotation

$$\mathcal{C}_d(p^2, \dots) \approx \mathcal{R}(\theta_d + \omega_d + \xi_d) + \mathcal{O}(g^2). \quad (31)$$

The quantity $(\omega_d + \xi_d)$ is seen to renormalize the classical mixing angle θ_d ; it satisfies, from (18), the relation (neglecting, as in (20), the terms proportional to $\frac{t_+ + t_-}{2}$ which are of order $g^{>2}$)

$$\tan 2(\omega_d + \xi_d) \approx \frac{-\tan 2\omega_d \left[\frac{t_+ - t_-}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} \frac{1}{\cos 2\omega_d} \right]}{1 + \tan^2 2\omega_d - \left[\frac{t_+ - t_-}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} \frac{1}{\cos 2\omega_d} \right]}. \quad (32)$$

In practice, $\tan 2(\omega_d + \xi_d)$ stays small, and so does, accordingly, $(\omega_d + \xi_d)$. Indeed, renormalization effects could become large only close to the pole of (32). The latter, which occurs for

$$\frac{1}{\cos 2\omega_d} = \frac{t_+ - t_-}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}, \quad (33)$$

that is, for $\frac{1}{\cos 2\omega_d} = \mathcal{O}(g^2) \times \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}$, is usually unphysical because it corresponds to $|\cos 2\omega_d| > 1$. $|\cos 2\omega_d|$ could become smaller than 1 only if, generically, $\left| \frac{m_d^2 - m_s^2}{m_d^2 + m_s^2} \right| < \frac{t_+ - t_-}{2} \approx \frac{g^2}{16\pi^2} \frac{m_c^2 - m_u^2}{m_W^2}$, which is never satisfied for known fermions, quarks or leptons ⁶.

From (32), (12) and (13) one also gets $\tan 2(\omega_d + \xi_d)$ as a function of θ_c and the classical masses

$$\tan 2(\omega_d + \xi_d) \approx \frac{\frac{1}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} (h_u - h_c) \sin 2\theta_c}{1 - \frac{1}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} ((h_u - h_c) \cos 2\theta_c + h_d - h_s)}. \quad (34)$$

3.3 1-loop mixing angles at the resonance

3.3.1 The (d, s) mixing angle

The interest of introducing the quantity $f(\omega_d) \equiv \left[\frac{t_+ - t_-}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} \frac{1}{\cos 2\omega_d} \right]$ in (32) is that, according to (20), it is equal to 1 at the pole of $\tan 2\xi_d$. So, when ξ_d becomes maximal, $\tan 2(\omega_d + \xi_d) = -\frac{1}{\tan 2\omega_d}$, which is very small. Indeed, for $|m_d - m_s| \ll |m_u - m_c|$, $|h_d - h_s| \ll |h_u - h_c|$ and (20) yields $\cos 2\omega_d \approx \frac{h_u - h_c}{2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2} = \frac{g^2}{16\pi^2} \frac{m_u^2 - m_c^2}{m_W^2} \frac{m_d^2 + m_s^2}{m_d^2 - m_s^2}$ which is very small for known fermions. So, at the pole, ω_d is very close to maximal ⁷.

Furthermore, from the expression (32) of $\tan 2(\omega_d + \xi_d)$, one deduces

$$\frac{d \tan 2(\omega_d + \xi_d)}{d\omega_d} = \frac{2f(\omega_d)(f(\omega_d) - 1)}{1 - f(\omega_d) \cos^2 2\omega_d}, \quad (35)$$

which provides a characterization of the resonance: $|\omega_d + \xi_d|$ is then minimal, such that the 1-loop (d, s) mixing angle is the closest to its classical value θ_d .

Note: From the expression (12) for $\tan 2\omega_d$ and (21) for $\cos 2\omega_d$, one gets $\frac{d\omega_d}{d\theta_c} \equiv \frac{1}{2} \cos^2 2\omega_d \frac{d \tan 2\omega_d}{d\theta_c} = -\frac{(h_u - h_c) [(h_u - h_c) + (h_d - h_s) \cos 2\theta_c]}{(t_+ - t_-)^2} \frac{|m_d - m_s| \ll |m_u - m_c|}{\approx} -1$, such that $\frac{d \tan 2(\omega_d + \xi_d)}{d\theta_c} \approx -\frac{d \tan 2(\omega_d + \xi_d)}{d\omega_d}$. So, at the pole, the dependence of $(\omega_d + \xi_d)$ on θ_c vanishes.

⁶For example, in the $(\nu_\mu, \nu_\tau, \nu, \tau)$ sector, the condition writes $\left| \frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{m_{\nu_\tau}^2 + m_{\nu_\mu}^2} \right| < \frac{g^2}{16\pi^2} \frac{m_\tau^2 - m_\mu^2}{m_W^2}$, the r.h.s. of which $\approx 1.9 \cdot 10^{-7}$, while the l.h.s. is experimentally known to be $\mathcal{O}(10^{-3})$ if one considers that the neutrino mass scale is $\mathcal{O}(eV)$. The mismatch is similar in the $(\nu_e, \nu_\tau, e, \tau)$ sector and worse in the (ν_e, ν_μ, e, μ) sector.

⁷And so is accordingly θ_c (see the end of subsection 2.2).

3.3.2 The (u, c) mixing angle

In the same configuration $|m_d - m_s| \ll |m_u - m_c|$, from the expression equivalent to (12) in the (u, c) sector, $\tan 2\omega_u = \frac{-(h_d - h_s) \sin 2\theta_c}{(h_d - h_s) \cos 2\theta_c + h_u - h_c}$, one deduces that, since $|h_u - h_c| \gg |h_d - h_s|$, $\omega_u \rightarrow 0$. Then, from the equivalent of (32), $\tan 2(\omega_u + \xi_u)$ gets suppressed with respect to $\tan 2\omega_u$ by an additional factor $\mathcal{O}(g^2)$.

One also gets $\frac{d \tan 2(\omega_u + \xi_u)}{d\omega_u} \approx (h_u - h_c)$ and $\frac{d \tan 2(\omega_u + \xi_u)}{d\theta_c} \approx \frac{(h_d - h_s) \sin 2\theta_c}{\tan 2\omega_d}$. This is extremely small at the resonance of $\tan 2\xi_d$ where ω_d is close to maximal, and it is furthermore damped by the small factor $(h_d - h_s)$. Accordingly, for largely split (u, c) , not only, at the pole of $\tan 2\xi_d$, ω_u and $(\omega_u + \xi_u)$ are small (such that the 1-loop (u, c) mixing angle stays, like the (d, s) mixing angle, very close to its classical value), but, also, $(\omega_u + \xi_u)$ depends extremely weakly on θ_c (though, unlike for $(\omega_d + \xi_d)$, we cannot speak of a minimum here).

4 The 1-loop Cabibbo matrix

4.1 The effective Lagrangian at 1-loop (in the bare mass basis)

$SU(2)_L$ gauge invariance demands the replacement, in the Lagrangian, of the partial derivative ∂ by the covariant derivative D . This is how, at the classical level and in the bare mass basis, calling $\Psi_m^0 T = (u_{mL}^0, c_{mL}^0, d_{mL}^0, s_{mL}^0)$, the kinetic + gauge terms write in their standard form $i \overline{\Psi}_m^0 \overleftrightarrow{D}_\mu \gamma^\mu \Psi_m^0 \equiv \frac{i}{2} (\overline{\Psi}_m^0 \gamma^\mu (D_\mu \Psi_m^0) - \overline{(D_\mu \Psi_m^0)} \gamma^\mu \Psi_m^0)$, such that

$$\mathcal{L}_{class} = \overline{\Psi}_m^0 (\mathbb{I} (i\partial_\mu) + g\vec{T} \cdot \vec{W}_\mu) \gamma^\mu \Psi_m^0 + \dots \quad (36)$$

The T 's are the (Cabibbo rotated) $SU(2)$ generators

$$T^3 = \frac{1}{2} \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & -1 \end{pmatrix}, T^+ = \begin{pmatrix} & & & \mathcal{C}_0 \\ & & & \\ & & & \\ & & & \end{pmatrix}, T^- = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \mathcal{C}_0^\dagger \end{pmatrix}, \quad (37)$$

where \mathcal{C}_0 is the classical Cabibbo matrix

$$\mathcal{C}_0 = \mathcal{R}(\theta_c) = \begin{pmatrix} \cos \theta_c & \sin \theta_c \\ -\sin \theta_c & \cos \theta_c \end{pmatrix} = \mathcal{C}_{u0}^\dagger \mathcal{C}_{d0} = \mathcal{R}(\theta_u)^\dagger \mathcal{R}(\theta_d). \quad (38)$$

Gauge currents and their $SU(2)_L$ algebra are thus directly related to kinetic terms by gauge invariance and the resulting Lagrangian is both gauge invariant and hermitian.

We shall use the same procedure to determine the Lagrangian after 1-loop transitions have been accounted for. Still in the bare mass basis Ψ_m^0 , we have seen in subsection 2.1 that the kinetic terms, which are classically proportional, in momentum space, to $\mathbb{I} \not{p}$ get renormalized at 1-loop into $A(p^2, m_i, m_W) \not{p}$, with

$$A(p^2, \dots) = \left(\frac{K_u(p^2, \dots)}{K_d(p^2, \dots)} \right) = \mathbb{I} + \left(\frac{H_u(p^2, \dots)}{H_d(p^2, \dots)} \right); \quad (39)$$

p_μ stands, there, for the common momentum of the ingoing and outgoing fermions, as depicted in Fig. 1.

The 1-loop kinetic + gauge Lagrangian that we will hereafter consider is accordingly $i \overline{\Psi}_m^0 \overleftrightarrow{A} D_\mu \gamma^\mu \Psi_m^0 \equiv \frac{i}{2} (\overline{\Psi}_m^0 \gamma^\mu (A D_\mu \Psi_m^0) - \overline{(A D_\mu \Psi_m^0)} \gamma^\mu \Psi_m^0)$, which yields

$$\mathcal{L}_{1-loop} = \overline{\Psi}_m^0 \left(A (i\partial_\mu) + \frac{g}{2} (A \vec{T} + \vec{T} A) \cdot \vec{W}_\mu \right) \gamma^\mu \Psi_m^0 + \dots \quad (40)$$

It has the required properties of gauge invariance and, thanks to the presence of the symmetric expression $A\vec{T} + \vec{T}A$, of hermiticity⁸. Gauge invariance has in particular dictated the 1-loop expression of the gauge currents, from which we shall now deduce that of the 1-loop Cabibbo matrix.

4.2 The Cabibbo matrix stays unitary at 1-loop

The 1-loop Cabibbo matrix in the bare mass basis can be read directly from the expression $\frac{g}{2}\overline{\Psi}_m^0(A\vec{T} + \vec{T}A)\gamma^\mu\Psi_m^0$ of the gauge currents that results from (40). This yields

$$\mathcal{C}^{bm}(p^2, \dots) = \frac{1}{2} \left[\underbrace{(\mathbb{I} + H_u)}_{K_u(p^2, \dots)} \mathcal{C}_0 + \mathcal{C}_0 \underbrace{(\mathbb{I} + H_d)}_{K_d(p^2, \dots)} \right]. \quad (41)$$

A naive calculation⁹ could erroneously lead to the conclusion that \mathcal{C}^{bm} is non-unitary, but it relies on the implicit assumption that the bare mass basis is orthogonal, which is untrue at 1-loop. The pure kinetic terms in (40) are, in particular, not normalized to \mathbb{I} but to the non-diagonal matrix A . So, it is suitable to go to the orthogonal basis of 1-loop mass eigenstates by using the relation (19). Because of the unitarity of the $\mathcal{R}(\xi)$ rotations, one has $]\mathcal{V}_{u,d}\mathcal{R}(\xi_{u,d})]^\dagger K_{u,d}[\mathcal{V}_{u,d}\mathcal{R}(\xi_{u,d})] \equiv \mathcal{R}(\xi_{u,d}^\dagger)[\mathcal{V}_{u,d}^\dagger K_{u,d}\mathcal{V}_{u,d}]\mathcal{R}(\xi_{u,d}) \stackrel{(9)}{=} \mathcal{R}(\xi_{u,d})^\dagger\mathcal{R}(\xi_{u,d}) = \mathbb{I}$, such that the pure kinetic terms get now normalized to \mathbb{I} . And, as we show next, the 1-loop Cabibbo matrix $\mathfrak{C}(p^2, \dots)$ rewrites, then, as a rotation. It becomes indeed in this basis¹⁰

$$\mathfrak{C}(p^2, \dots) = [\mathcal{V}_u \mathcal{R}(\xi_u)]^\dagger \mathcal{C}^{bm}(p^2, \dots) [\mathcal{V}_d \mathcal{R}(\xi_d)]. \quad (43)$$

Transforming the general expressions (43) and (41) with the help of (9) which entails $K_d = (\mathcal{V}_d^{-1})^\dagger \mathcal{V}_d^{-1}$ ($K_u = (\mathcal{V}_u^{-1})^\dagger \mathcal{V}_u^{-1}$), yields

$$\mathfrak{C} = \frac{1}{2} \mathcal{R}(\xi_u)^\dagger \left[\mathcal{V}_u^{-1} \mathcal{C}_0 \mathcal{V}_d + \mathcal{V}_u^\dagger \mathcal{C}_0 (\mathcal{V}_d^{-1})^\dagger \right] \mathcal{R}(\xi_d) = \frac{1}{2} \mathcal{R}(\xi_u)^\dagger \left[\mathcal{V}_u^{-1} \mathcal{C}_0 \mathcal{V}_d + ((\mathcal{V}_u^{-1} \mathcal{C}_0 \mathcal{V}_d)^{-1})^\dagger \right] \mathcal{R}(\xi_d). \quad (44)$$

⁸Hermiticity is, instead, not achieved if one considers a kinetic Lagrangian of the form $i\overline{\Psi}_m^0 \overrightarrow{AD}_\mu \gamma^\mu \Psi_m^0$ (with \rightarrow instead of \leftrightarrow above AD_μ).

⁹Using $\mathcal{C}_0 = \mathcal{R}(\theta_d - \theta_u)$ and the expressions (6) (7) for K_d and K_u , one finds $\mathcal{C}^{bm}(\mathcal{C}^{bm})^\dagger \neq \mathbb{I}$. However, this does not mean that the Cabibbo matrix is non-unitarity because these expressions are written in a basis which is non-orthogonal at 1-loop. Consider indeed, for example, the relation $\mathcal{C}_{11}^* \mathcal{C}_{12} + \mathcal{C}_{21}^* \mathcal{C}_{22} \neq 0$. It traduces the non-orthogonality of the two vectors $\mathcal{C} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{C}_{12} \\ \mathcal{C}_{22} \end{pmatrix}$ and $\mathcal{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \mathcal{C}_{11} \\ \mathcal{C}_{21} \end{pmatrix}$ when their scalar product is evaluated with the metric (1, 1). However,

this metric is the correct one only at the classical level, at which $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which represent fermions in bare mass space, are orthogonal since no transition occurs between the two of them; but it is no longer so at 1-loop (see Fig. 1). And, indeed, when one goes, as we do next in the text, to the orthogonal basis of 1-loop mass eigenstates, this apparent non-unitarity gets exactly canceled. Likewise, for any matrix U , the relation $UU^\dagger = 1$ traduces unitarity only if U is expressed in an orthogonal basis of states (*i.e.* no transition exists between them at the order that is considered).

¹⁰One can easily demonstrate that $\mathfrak{C}(p^2, \dots) = \mathcal{C}_u^\dagger \mathcal{C}_d + \mathcal{O}(g^2)$, reminiscent of the classical relation $\mathcal{C}_0 = \mathcal{C}_{u0}^\dagger \mathcal{C}_{d0}$, as follows. Since H_u and H_d in (41) are $\mathcal{O}(g^2)$, the terms proportional to them in (43) can be calculated with the expressions of $\mathcal{R}(\xi_d)$ and \mathcal{V}_d at $\mathcal{O}(g^0)$, that is, for $t_+ = 0 = t_-$; one can accordingly take in there $\mathcal{R}(\xi_d) \stackrel{(18)}{\rightarrow} \mathcal{R}(-\omega_d)$ and $\mathcal{V}_d \stackrel{(14)}{\rightarrow} \mathcal{R}(\omega_d)$, such that $\mathcal{V}_d \mathcal{R}(\xi_d) \rightarrow \mathbb{I}$. The same approximation can be done in the (u, c) sector. The resulting expression for \mathfrak{C} is

$$\mathfrak{C}(p^2, \dots) \stackrel{\mathcal{O}(g^2)}{\approx} \mathcal{R}(\xi_u)^\dagger \mathcal{V}_u^\dagger \mathcal{C}_0 \mathcal{V}_d \mathcal{R}(\xi_d) + \frac{1}{2} \underbrace{(\mathcal{H}_u \mathcal{C}_0 + \mathcal{C}_0 \mathcal{H}_d)}_{\mathcal{O}(g^2)}, \quad (42)$$

which leads to the announced formula after using (38), and (30) and its equivalent for \mathcal{C}_u . That $\mathfrak{C}(p^2)$ is unitary means that the non-unitarity of $\mathcal{C}_u^\dagger \mathcal{C}_d$ gets compensated by that of $\frac{1}{2}(\mathcal{H}_u \mathcal{C}_0 + \mathcal{C}_0 \mathcal{H}_d)$.

Noticing that t_+ and t_- given in (10) are the same for (d, s) and (u, c) and using the expression (14) for the \mathcal{V} 's, one gets $\mathcal{V}_u^{-1}\mathcal{C}_0\mathcal{V}_d = \begin{pmatrix} \cos(\omega_u - \omega_d - \theta_c) & -\sqrt{\frac{1+t_+}{1+t_-}} \sin(\omega_u - \omega_d - \theta_c) \\ \sqrt{\frac{1+t_-}{1+t_+}} \sin(\omega_u - \omega_d - \theta_c) & \cos(\omega_u - \omega_d - \theta_c) \end{pmatrix}$, which leads finally to

$$\mathfrak{C}(p^2, \dots) = \mathcal{R}\left((\theta_d + \omega_d(p^2, \dots) + \xi_d(p^2, \dots)) - (\theta_u + \omega_u(p^2, \dots) + \xi_u(p^2, \dots))\right) + \mathcal{O}(g^{(>2)}). \text{ q.e.d.} \quad (45)$$

$\mathfrak{C}(p^2)$ stays thus unitary for any common value of p^2 at which its entries are evaluated ¹¹.

In the basis of 1-loop mass eigenstates, the Lagrangian \mathcal{L} rewrites

$$\mathcal{L} = \begin{pmatrix} \overline{u_{mL}} & \overline{c_{mL}} & \overline{d_{mL}} & \overline{s_{mL}} \end{pmatrix} (p^2, \dots) \left(\not{p} + g \vec{\mathfrak{X}}(p^2, \dots) \cdot \vec{W}_\mu \gamma^\mu + \dots \right) \begin{pmatrix} u_{mL} \\ c_{mL} \\ d_{mL} \\ s_{mL} \end{pmatrix} (p^2, \dots) + \dots, \quad (46)$$

with “1-loop” $SU(2)_L$ generators $\vec{\mathfrak{X}}(p^2, \dots)$ depending now on p^2 and on the masses

$$\mathfrak{X}^3(p^2, \dots) = \frac{1}{2} \begin{pmatrix} 1 & & \\ & & \\ & & -1 \end{pmatrix}, \mathfrak{X}^+(p^2, \dots) = \begin{pmatrix} & & \mathfrak{C}(p^2, \dots) \\ & & \\ & & \end{pmatrix}, \mathfrak{X}^-(p^2, \dots) = \begin{pmatrix} & & \\ & & \mathfrak{C}^\dagger(p^2, \dots) \\ & & \end{pmatrix}. \quad (47)$$

Our procedure has accordingly preserved the $SU(2)_L$ structure of gauge currents at 1-loop, which guarantees in particular that the corresponding Ward identities are satisfied.

4.3 The 1-loop Cabibbo angle and its large value at the resonance

(45) shows that the Cabibbo angle $\theta_c = \theta_d - \theta_u$ gets renormalized by $(\omega_d + \xi_d) - (\omega_u + \xi_u)$.

At the pole of $\tan 2\xi_d$, we have seen that $(\omega_d + \xi_d)$ (see subsection 3.3.1), and $(\omega_u + \xi_u)$ (see subsection 3.3.2) become extremely small. So behaves accordingly $(\omega_d + \xi_d) - (\omega_u + \xi_u)$, such that the 1-loop Cabibbo angle stays, then, very close to its classical value, which is, according to (23), close to maximal.

Another property is that, when d and s are close to degeneracy, while u and c are far from it,

$$\mathfrak{C}(p^2, \dots) \approx \mathcal{R}(\xi_d(\theta_c)). \quad (48)$$

This results from $\omega_d(\theta_c) \approx -\theta_c$ demonstrated at the end of subsection 2.2; ω_d cancels, then, $(\theta_d - \theta_u)$ in (45). $(\omega_u + \xi_u)$, that we just showed to be very small, does not alter this property. The 1-loop value of the Cabibbo angle finally becomes close to $\xi_d(\theta_c)$ as given by (18); it is accordingly determined by the rotation that casts back, at 1-loop, the mass matrix of the quasi-degenerate pair to diagonal. The role of the (u, c) fermion pair far from degeneracy is blurred; indeed, the only trace left is the dependence, at the resonance, of ξ_d and θ_c on the masses of all four fermions (m_d, m_s, m_u, m_c) . But the situation is nevertheless different from a naive “alignment” of (u, c) mass and flavour eigenstates. This would mean a vanishing (u, c) mixing angle, while what we get is the quasi-vanishing of the sole quantum correction $(\omega_u + \xi_u)$; we have no control on the classical θ_u .

¹¹This may not be in contradiction with the non-unitarity claimed in [10] and [12] when the two external fermions legs are on different mass-shell, since, then, two different p^2 are involved. See also the end of section 7.

5 Instability close to degeneracy

Quasi-degenerate systems are known to be unstable with respect to small perturbations. This property is easily verified here, through the amount by which classical mixing angles are renormalized when 1-loop transitions are accounted for. It undergoes indeed large variations when the classical masses span a very small interval in the neighborhood of degeneracy: we first consider the case of exact classical degeneracy ($m_d = m_s$), secondly the pole of (32), which corresponds to a situation where d and s are extremely close to degeneracy (see subsection 3.2), and, last, the pole of $\tan 2\xi$, which also corresponds to quasi-degenerate fermions, but not as close as previously. The two first cases are very likely purely academic (see [20] and subsection 3.2), but this is no relevance for what we want to show here. The third case can be physically relevant, as we noticed in subsection 2.3.1.

- For exact classical degeneracy $h_d = h_s$ such that, by the expression of $\sin 2\omega_d$ in (13), $\omega_d = -\theta_c$. (16) shows then that $\mathcal{V}_d^\dagger M_d M_d^\dagger \mathcal{V}_d$ stays diagonal, and, so, $\xi_d = 0$ ¹². The classical (d, s) mixing angle θ_d is renormalized (see (31)) by $(\omega_d + \xi_d) = -\theta_c$ and becomes $\theta_d - \theta_c = \theta_u$, the classical mixing angle of the (u, c) pair.

According to (45), the Cabibbo mixing angle gets renormalized from its classical value θ_c to $\theta_c + (\omega_d + \xi_d) - (\omega_u + \xi_u) = -(\omega_u + \xi_u)$ which is very close to vanishing when the pair (u, c) is largely split. To such a system is accordingly associated a near-vanishing 1-loop Cabibbo angle. Renormalization effects can thus be large.

- At the pole of (32), by definition, the renormalization of θ_d becomes maximal ($\pm \frac{\pi}{4}$).
- At the pole of $\tan 2\xi_d$, it becomes instead minimally small (see subsection 3.3.1).

So, in a close neighborhood of degeneracy, the renormalization $(\omega_d + \xi_d)$ of θ_d undergoes large variations. So does the one of the Cabibbo angle.

6 Propagation in matter

Coherent forward scattering of neutrinos in matter has been shown [19] [18] to modify $M_\nu M_\nu^\dagger$ (M_ν is the classical mass matrix for neutrinos). In the space of electronic and muonic neutrinos (ν_e, ν_μ) that couple to charged leptons mass eigenstates (see footnote 1), $\nu_e = \cos(\theta_{PMNS})\nu_{em} - \sin(\theta_{PMNS})\nu_{\mu m}$, $\nu_\mu = \sin(\theta_{PMNS})\nu_{em} + \cos(\theta_{PMNS})\nu_{\mu m}$, it becomes

$$\begin{aligned} (M_\nu M_\nu^\dagger)^{(\nu_e, \nu_\mu)} &= \frac{1}{2}(m_{\nu_e}^2 + m_{\nu_\mu}^2 + A_{CC}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} A_{CC} - (m_{\nu_\mu}^2 - m_{\nu_e}^2) \cos(2\theta_{PMNS}) & (m_{\nu_\mu}^2 - m_{\nu_e}^2) \sin(2\theta_{PMNS}) \\ (m_{\nu_\mu}^2 - m_{\nu_e}^2) \sin(2\theta_{PMNS}) & -A_{CC} + (m_{\nu_\mu}^2 - m_{\nu_e}^2) \cos(2\theta_{PMNS}) \end{pmatrix}, \end{aligned} \quad (49)$$

with $A_{CC} = 2\sqrt{2}EG_F N_e$, E being the energy of the (ν_e) ultra-relativistic neutrinos and N_e the electronic density of the medium, which can vary in space. Equivalently, in the bare mass space in which we have been working, $M_\nu M_\nu^\dagger$ becomes

$$\begin{aligned} (M_\nu M_\nu^\dagger)^{bm} &= \mathcal{R}(\theta_{PMNS})^\dagger (M_\nu M_\nu^\dagger)^{(\nu_e, \nu_\mu)} \mathcal{R}(\theta_{PMNS}) \\ &= \begin{pmatrix} m_{\nu_e}^2 & \\ & m_{\nu_\mu}^2 \end{pmatrix} + \frac{A_{CC}}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + A_{CC} \mathcal{T}_x(2\theta_{PMNS}). \end{aligned} \quad (50)$$

¹²This is in agreement with (18) and (20) which show that $\tan 2\xi_d$ has no pole when $m_d = m_s$.

After kinetic terms have been cast back to their canonical form, one has to re-diagonalize, according to the procedure explained in subsection 2.3, the expression (50) sandwiched between \mathcal{V}_ν^\dagger and \mathcal{V}_ν as given by the equivalent of (14) for the neutrino sector. Since $A_{CC} = \mathcal{O}(g^2)$, when transforming the parts proportional to it, we can neglect in \mathcal{V}_u all terms $\propto t_\pm$, which amounts to taking $\mathcal{V}_\nu \approx \mathcal{R}(\omega_\nu)$. So doing, to the expression for $\mathcal{V}_\nu^\dagger M_\nu M_\nu^\dagger \mathcal{V}_\nu$ equivalent to (16) for the neutrino sector, one finds that one must add

$$\mathcal{R}(\omega_u)^\dagger \left[\frac{A_{CC}}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + A_{CC} \mathcal{T}_x(2\theta_{PMNS}) \right] \mathcal{R}(\omega_u) \equiv A_{CC} \left[\frac{1}{2} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \mathcal{T}_x(2(\theta_{PMNS} + \omega_u)) \right].$$

The pole of $\tan 2\xi_d$ occurs now for $\frac{m_{\nu_e}^2 c_{\omega_\nu}^2 + m_{\nu_\mu}^2 s_{\omega_\nu}^2}{1+t_+} - \frac{m_{\nu_e}^2 s_{\omega_\nu}^2 + m_{\nu_\mu}^2 c_{\omega_\nu}^2}{1+t_-} + A_{CC} \cos 2(\theta_{PMNS} + \omega_\nu) = 0$, which can be approximated to $(m_{\nu_e}^2 - m_{\nu_\mu}^2) \cos 2\omega_\nu - (m_{\nu_e}^2 + m_{\nu_\mu}^2) \frac{t_+ - t_-}{2} + A_{CC} \cos 2(\theta_{PMNS} + \omega_\nu) = 0$. For $|m_{\nu_e} - m_{\nu_\mu}| \ll |m_e - m_\mu|$, $\omega_\nu \approx -\theta_{PMNS}$ (see subsection 2.2) and the pole equation becomes

$$(m_{\nu_e}^2 - m_{\nu_\mu}^2) \cos 2\omega_\nu - (m_{\nu_e}^2 + m_{\nu_\mu}^2) \frac{t_+ - t_-}{2} + A_{CC} = 0. \quad (51)$$

It differs from the one in vacuum (20) by the term A_{CC} , and we also recall that the relation at the classical MSW resonance writes [19][18] $(m_{\nu_e}^2 - m_{\nu_\mu}^2) \cos(2\theta_{PMNS}) + A_{CC} = 0$. Looking at the magnitude of the different contributions shows that 1-loop effects are negligible in matter and do not modify the MSW resonance. Indeed, the orders of magnitudes roughly match, between $(m_{\nu_e}^2 - m_{\nu_\mu}^2)$ which is known to be $\mathcal{O}(10^{-5} eV^2)$ and A_{CC} , which, as we mentioned at the end of subsection 2.3.1, can vary between 10^{-4} to $10^{-6} eV^2$, depending whether one considers the sun or the earth. This is, roughly speaking, why the MSW resonance can happen even for “small values of the mixing angle” $\cos 2\omega_\nu \cos 2\theta_{PMNS} \sim 1$. By contrast, the term in the middle of (51), coming from 1-loop corrections, is much smaller since, considering that $(m_{\nu_e}^2 + m_{\nu_\mu}^2) \approx 1 eV^2$, $\frac{t_+ - t_-}{2} \approx \frac{g^2}{16\pi^2} \frac{m_\mu, m_\tau^2}{m_W^2} \approx 2 \cdot 10^{-9}$.

7 Summary and prospects

We have paid in this study special attention to 1-loop transitions between fermions. They spoil the diagonality of kinetic terms which must be first cast back into their canonical form before the mass matrix is suitably re-diagonalized and orthogonal 1-loop mass eigenstates suitably determined.

A first property that we encountered is that, for non-degenerate systems, bare mass states and 1-loop mass states are non-unitarily related.

A second property is that the 1-loop mixing matrix $\mathfrak{C}(p^2)$ occurring in charged currents (Cabibbo, PMNS ...) stays unitary at $\mathcal{O}(g^2)$, whatever be the common scale p^2 at which all its entries are evaluated.

The third point concerns the 1-loop value of the “Cabibbo” angle (or its equivalent for leptons). We have shown that it exhibits a pole when, inside the two generations of fermions that are considered, one pair is close to degeneracy (we have been concerned with the case when the other pair of fermions is far from degeneracy). The pole condition stands as a relation between the classical θ_c , the masses of the four fermions and that of the W . It appears furthermore as a stability condition since, then, the (d, s) mixing angle at 1-loop is the closest to its classical value. As far as the (u, c) mixing angle is concerned, it also stays extremely close to its classical value. At resonance, the renormalization of θ_c appears thus to be not only very small but “minimal”, and the resonance condition, *de facto*, a condition between the 1-loop Cabibbo angle, which is close to maximal, and the set of masses. We have also shown that, then, the 1-loop masses get the closest to each other. A new characterization of maximal mixing thus appears, a nice aspect of which is that it connects large mixing *in charged currents*, as observed, to the quasi-degeneracy of one of the two doublets (the widespread belief postulates instead the alignment of mass and flavor states for the largely split fermions to establish the same connection). The final statement stays however weak: “For two fermions close to degeneracy and the second pair far from it, there exists a resonance at which both the 1-loop “Cabibbo angle” and $\Delta m^2/m^2$ get the closest to classical; the former is then close to maximal ($\pm\pi/4$) and the second minimal”. This seems suited to the $(\nu_\mu, \nu_\tau, \mu, \tau)$ system, but

we have, at the moment, no other criterion why a specific quartet is at resonance and not the others. We have indeed seen that, for all values of known fermion masses, the pole condition (23) corresponds to a large mixing angle. So, if θ_{23} is large, one has also to explain why, for example, the Cabibbo angle is not ¹³. The resonant phenomenon that we presented in this work is accordingly, at the moment, just one among a series of mechanisms that can participate to the building of large mixing angles (see for example [13] [14] [8]). They are eventually supplemented with symmetry considerations [8] [21], which is not the case here (with the sole exception of gauge invariance).

Another standard property that we encountered is that the mixing among quasi-degenerate fermions is highly unstable. We witnessed this at 1-loop by evolving a pair of fermions from exact degeneracy, to approximate degeneracy: the renormalization ($\omega + \xi$) of the corresponding mixing angle undergoes large variations and so does the one of the Cabibbo angle. So, even if, as we noticed, real systems are in a regime where renormalization turns out finally to be small (and even minimal), one should not forget that this comes from a cancellation between the two angles ξ and ω which are “non-perturbatively” close to θ_c and eventually large. So, it may happen that the results that we obtained inaccurately describe the physics of quasi-degenerate fermions and that an appropriate treatment requires techniques that go beyond perturbation theory. That the resonance at 1-loop seems to correspond to some “quasi-classical” configuration could be a sign in this direction.

A last issue should be evoked before concluding this work. The 1-loop effective Lagrangian was made diagonal in the basis $d_{mL}(p^2, \dots), s_{mL}(p^2, \dots)$, in which p_μ stands for the common 4-momentum of d and s (see Fig. 1). This means that there exist no more non-diagonal transitions between them and that $d_{mL}(p^2, \dots)$ and $s_{mL}(p^2, \dots)$ are, by definition, orthogonal at 1-loop. However, as soon as a mass splitting exists between d and s , both cannot be simultaneously on mass-shell and the physical fermions (28), which belong to two different sets of orthogonal states, are themselves expected to be non-orthogonal. So, unless subtle cancellations take place, non-diagonal transitions are likely to occur between them at this order, which is akin to saying that the 1-loop Lagrangian is itself no longer diagonal when re-expressed in terms of them. The existence and relevance of these transitions should be scrutinized, together with the opportunity to introduce counterterms [22][16][11] to cancel them (see Appendix A). At the same time, unlike $\mathcal{C}(p^2)$ in (41), which is defined for an overall global p^2 , the “on mass-shell” Cabibbo matrix is expected to exhibit some slight non-unitarity. This contrasts with the work [7] in which it is conjectured to be unitary ¹⁴ and where, seemingly, the basis of orthogonal states has been chosen to be that of physical d and s . Ours and the approach in [7] are thus *a priori* slightly nonequivalent and making a closer connection between the two requires rewriting our results in terms of physical fermions together with performing a complete treatment and re-diagonalization of the kinetic terms in [7]. One has in particular to face the issues of the hermiticity of the 1-loop Lagrangian (which was assumed in [7] for the mass terms and that we achieved in (40) for the whole Lagrangian), and that of the $SU(2)_L$ structure of its gauge currents (which we confirmed in (47)). This matter is currently under investigation.

Acknowledgments: *It is a pleasure to thank M.I. Vysotsky for comments and advice.*

¹³A solution has been proposed in [8] in which, in the quark sector, (d, s) and (u, c) mixing angles largely cancel each other while, in the lepton sector, the opposite occurs.

¹⁴see also footnote 9: unitarity is conjectured in [7] but in a basis that is likely to be non-orthogonal at 1-loop.

A Canceling transitions between non-degenerate physical states

The 1-loop quadratic effective Lagrangian (kinetic and mass terms), that we made diagonal (see section 2) in the basis $(d_{mL}(p^2, \dots), s_{mL}(p^2, \dots))$ of 1-loop mass eigenstates, can be generically rewritten, in the basis of physical eigenstates

$$\begin{aligned} \mathcal{L}^{1-loop} = & \left(\overline{d_{mL}^{phys}} \quad \overline{s_{mL}^{phys}} \right) \not{p} \begin{pmatrix} g_1(p^2) & g_2(p^2) \\ g_3(p^2) & g_4(p^2) \end{pmatrix} \begin{pmatrix} d_{mL}^{phys} \\ s_{mL}^{phys} \end{pmatrix} + \left(\overline{d_{mR}^{phys}} \quad \overline{s_{mR}^{phys}} \right) \not{p} \mathbb{I} \begin{pmatrix} d_{mR}^{phys} \\ s_{mR}^{phys} \end{pmatrix} \\ & - \left(\overline{d_{mL}^{phys}} \quad \overline{s_{mL}^{phys}} \right) \begin{pmatrix} \rho_1(p^2) & \rho_2(p^2) \\ \rho_3(p^2) & \rho_4(p^2) \end{pmatrix} \begin{pmatrix} d_{mR}^{phys} \\ s_{mR}^{phys} \end{pmatrix} \\ & - \left(\overline{d_{mR}^{phys}} \quad \overline{s_{mR}^{phys}} \right) \begin{pmatrix} \sigma_1(p^2) & \sigma_2(p^2) \\ \sigma_3(p^2) & \sigma_4(p^2) \end{pmatrix} \begin{pmatrix} d_{mL}^{phys} \\ s_{mL}^{phys} \end{pmatrix} + \dots \end{aligned} \quad (52)$$

Indeed, combined with (19) which relates bare mass states to 1-loop mass eigenstates, (28) entails that the coefficients of the linear relation between the latter and physical states are functions of (p^2, \dots) . Hermiticity requires the (supposedly real and presumably $\mathcal{O}(g^2)$) quantities $g_2, g_3, \sigma_2, \sigma_3, \rho_2, \rho_3$ to satisfy the relations $g_3 = g_2, \rho_2 = \sigma_3, \rho_3 = \sigma_2$. Furthermore, since right-handed fermions are not concerned by 1-loop transitions, $(1 + \gamma^5)d_m^{phys} = (1 + \gamma^5)d_m^0$ and $(1 + \gamma^5)s_m^{phys} = (1 + \gamma^5)s_m^0$.

As seen on (52), non-diagonal, p^2 -dependent transitions spring again, now between physical states, which can be traced back to their non-degeneracy. Though more detailed investigations are needed, the resulting ‘‘oscillations’’, like $e^- \leftrightarrow \mu^-$, look embarrassing. They can however be canceled by introducing specific counterterms, as we now show. The procedure is similar to the one used in [22]. Canceling, for example, (on mass-shell s) \rightarrow (on mass-shell d) transitions can be done by adding to (52) four kinetic and mass-like counterterms, concerning both chiralities of fermions:

$$-A_d \overline{d_m^{phys}} \not{p} (1 - \gamma^5) s_m^{phys} - B_d \overline{d_m^{phys}} (1 - \gamma^5) s_m^{phys} - E_d \overline{d_m^{phys}} \not{p} (1 + \gamma^5) s_m^{phys} - D_d \overline{d_m^{phys}} (1 + \gamma^5) s_m^{phys}. \quad (53)$$

Since s_m^{phys} is on mass-shell, one gets the condition (we call respectively μ_s and μ_d the 1-loop physical masses of s and d , that is, the square roots of the values of p^2 solutions of $p^2 = \mu_s^2(p^2)$ and $p^2 = \mu_d^2(p^2)$) (see subsection 2.3.2))

$$\begin{aligned} & g_2(\mu_s^2) \overline{d_m^{phys}} (1 + \gamma^5) \mu_s s_m^{phys} - \rho_2(\mu_s^2) \overline{d_m^{phys}} (1 + \gamma^5) s_m^{phys} - \sigma_2(\mu_s^2) \overline{d_m^{phys}} (1 - \gamma^5) s_m^{phys} \\ = & A_d \overline{d_m^{phys}} (1 + \gamma^5) \mu_s s_m^{phys} + B_d \overline{d_m^{phys}} (1 - \gamma^5) s_m^{phys} + E_d \overline{d_m^{phys}} (1 - \gamma^5) \mu_s s_m^{phys} + D_d \overline{d_m^{phys}} (1 + \gamma^5) s_m^{phys}, \end{aligned} \quad (54)$$

and since d_m^{phys} is also on mass-shell,

$$\begin{aligned} & g_2(\mu_d^2) \overline{d_m^{phys}} (1 - \gamma^5) \mu_d s_m^{phys} - \rho_2(\mu_d^2) \overline{d_m^{phys}} (1 + \gamma^5) s_m^{phys} - \sigma_2(\mu_d^2) \overline{d_m^{phys}} (1 - \gamma^5) s_m^{phys} \\ = & A_d \overline{d_m^{phys}} (1 - \gamma^5) \mu_d s_m^{phys} + B_d \overline{d_m^{phys}} (1 - \gamma^5) s_m^{phys} + E_d \overline{d_m^{phys}} (1 + \gamma^5) \mu_d s_m^{phys} + D_d \overline{d_m^{phys}} (1 + \gamma^5) s_m^{phys}. \end{aligned} \quad (55)$$

Equating the terms with identical chiralities in (54) and (55) yields the four equations

$$\begin{aligned} \mu_s g_2(\mu_s^2) - \rho_2(\mu_s^2) &= \mu_s A_d + D_d, \\ -\sigma_2(\mu_s^2) &= \mu_s E_d + B_d, \\ \mu_d g_2(\mu_d^2) - \sigma_2(\mu_d^2) &= \mu_d A_d + B_d, \\ -\rho_2(\mu_d^2) &= \mu_d E_d + D_d, \end{aligned} \quad (56)$$

which have the $\mathcal{O}(g^2)$ solutions

$$\begin{aligned}
A_d &= \frac{\mu_s^2 g_2(\mu_s^2) - \mu_d^2 g_2(\mu_d^2) + \mu_s (\rho_2(\mu_d^2) - \rho_2(\mu_s^2)) - \mu_d (\sigma_2(\mu_s^2) - \sigma_2(\mu_d^2))}{\mu_s^2 - \mu_d^2}, \\
E_d &= \frac{\mu_d \mu_s (g_2(\mu_s^2) - g_2(\mu_d^2)) + \mu_d (\rho_2(\mu_d^2) - \rho_2(\mu_s^2)) - \mu_s (\sigma_2(\mu_s^2) - \sigma_2(\mu_d^2))}{\mu_s^2 - \mu_d^2}, \\
B_d &= -\sigma_2(\mu_s^2) - \mu_s E_d, \\
D_d &= -\rho_2(\mu_d^2) - \mu_d E_d.
\end{aligned} \tag{57}$$

Likewise, four counterterms $\tilde{A}_d, \tilde{E}_d, \tilde{B}_d, \tilde{D}_d$ can get rid of the on mass-shell $d_m^{phys} \rightarrow s_m^{phys}$ transitions. Hermiticity (see above) constrains them to satisfy $\tilde{A}_d = A_d, \tilde{E}_d = E_d, \tilde{B}_d = D_d, \tilde{D}_d = B_d$. Similar additions can be done in the (u, c) sector.

The bare Lagrangian to which these $\mathcal{O}(g^2)$ counterterms have been added is suitable for a perturbative expansion. It can, as before, be performed in the bare mass basis, in which the counterterms can be re-expressed with the help of (28); it has the advantage that the $SU(2)_L$ generators are simply expressed by (37). The $\mathcal{O}(g^2)$ effective Lagrangian that is, then, obtained after adding 1-loop transitions like we did in subsection 2.1 includes now:

- * $\mathcal{O}(g^2)$ kinetic-like diagonal and non-diagonal terms for left-handed fermions which are either p^2 -dependent (the same as in subsection 2.1), or p^2 -independent (counterterms);
- * $\mathcal{O}(g^2)$ kinetic-like non-diagonal and diagonal¹⁵ counterterms for right-handed fermions which do not depend on p^2 ;
- * $\mathcal{O}(g^2)$ non-diagonal and diagonal mass counterterms which do not depend on p^2 , and which spoil the diagonality of the mass matrix at this order.

Kinetic (left and right) and mass terms can be diagonalized. A bi-unitary transformation is now in general necessary for the latter, which leads to $diag(\tilde{\mu}_d(p^2), \tilde{\mu}_s(p^2))$. This procedure defines left $(\tilde{d}_{mL}(p^2), \tilde{s}_{mL}(p^2))$ and right $(\tilde{d}_{mR}(p^2), \tilde{s}_{mR}(p^2))$ mass eigenstates, which, since they diagonalize the whole 1-loop quadratic Lagrangian, are by definition orthogonal at this order. They are expected to differ from the ones obtained in section 2. So do the effective masses; one expects $\tilde{\mu}_d(p^2) \neq \mu_d(p^2), \tilde{\mu}_s(p^2) \neq \mu_s(p^2)$, which leads to distinguish mass-shells with and without counterterms. Since the latter have been shaped to cancel non-diagonal transitions between $d_{mL}(p^2 = \mu_d^2(p^2))$ and $s_{mL}(p^2 = \mu_s^2(p^2))$ at 1-loop, this Lagrangian is also expected to describe orthogonal two such states. Thus, these should be unitarily related with any set $(\tilde{d}_{mL}(p^2), \tilde{s}_{mL}(p^2))$ (which is orthogonal by construction).

The appropriate $SU(2)_L$ structure of gauge currents is generated by replacing each derivative acting on a left-handed fermion by the corresponding $SU(2)_L$ covariant derivative. The difference with subsection 4.1 is again that the kinetic-like left-handed counterterms must now be included in the procedure. Left-handed charged gauge currents then exhibit a Cabibbo matrix at 1-loop which slightly differs from that given in subsection 4.2. It is expected to be unitary when the Lagrangian is expressed in the orthogonal basis $d_{mL}(p^2 = \mu_d^2(p^2))$ and $s_{mL}(p^2 = \mu_s^2(p^2))$.

By modifying the theory in particular on mass-shell, the introduction of counterterms could only ensure orthogonality of states which are no longer physical in their presence. And, because of this discrepancy between physical states with and without the counterterms, non-diagonal transitions are still likely to exist between $\tilde{d}_{mL}(p^2 = \tilde{\mu}_d^2(p^2))$ and $\tilde{s}_{mL}(p^2 = \tilde{\mu}_s^2(p^2))$ (on mass-shell (with counterterms) states). The hope is that they are of order $g^{>2}$, but this should be carefully checked.

¹⁵In the basis of physical states, the kinetic-like right-handed counterterms are only non-diagonal but diagonal ones can be generated when going to the bare mass basis.

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