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François Ledrappier

LINEAR DRIFT AND ENTROPY FOR REGULAR COVERS

FRANÇOIS LEDRAPPIER

ABSTRACT. We consider a regular Riemannian cover $\widetilde{M}$ of a compact Riemannian manifold. The linear drift ℓ and the Kaimanovich entropy h are geometric invariants defined by asymptotic properties of the Brownian motion on $\widetilde{M}$. We show that $\ell^2 \leq h$.

[1] Let $\pi : \widetilde{M} \to M$ be a regular Riemannian cover of a compact manifold: $\widetilde{M}$ is a Riemannian manifold and there is a discrete group G of isometries of $\widetilde{M}$ acting freely and such that the quotient $M = G \setminus \widetilde{M}$ is a compact manifold. The quotient metric makes M a compact Riemannian manifold.

We consider the Laplacian Δ on $\widetilde{M}$, the corresponding heat kernel $\widetilde{p}(t, \widetilde{x}, \widetilde{y})$ and the associated Brownian motion $\widetilde{X}_t, t \geq 0$. The following quantities were introduced by Guivarc'h [**Gu**] and Kaimanovich [**K1**], respectively, as almost everywhere limits on the space of trajectories of the Brownian motion $\widetilde{X}$:

- the linear drift $\ell := \lim_{t\to\infty} \frac{1}{t} d_{\widetilde{M}}(\widetilde{X}_0, \widetilde{X}_t)$.
- the entropy $h := \lim_{t\to\infty} -\frac{1}{t} \ln \widetilde{p}(t, \widetilde{X}_0, \widetilde{X}_t)$.

In this note we prove the following

Theorem A. *Let $\pi : \widetilde{M} \to M$ be a regular Riemannian cover of a compact manifold. With the above notations, we have:*

$$(1) \qquad\qquad \ell^2 \leq h.$$

In the case when $\widetilde{M}$ is the universal covering of a compact manifold with negative curvature, inequality (1) is due to V. Kaimanovich ([**K1**]). Moreover in that case, there is equality in (1) if, and only if, the manifold $\widetilde{M}$ is a symmetric space of negative curvature (se the discussion below in section 1). For a general cover, it follows from (1) that, whenever $h = 0$ (which is equivalent to the Liouville property of $\widetilde{M}$ [**D**], [**KV**]), then $\ell = 0$. This was shown in [**KL1**] by using discretization of the Brownian motion and a qualitative result for random walks. Indeed, similar quantities can be defined for a symmetric random walk on a finitely generated group, where the distance on the group is the word distance. A precise result similar to (1) is not known for discrete random walks. There are estimates for symmetric random walks with finite support ([**Va**]) or finite second moment ([**EK**]).

Let v be the volume entropy of $\widetilde{M}$

$$v \;=\; \lim_{R\to\infty} \frac{\ln \operatorname{vol}(B_{\widetilde{M}}(x_0, R))}{\ln R},$$

where $B_{\widetilde{M}}(x_0, R)$ is the ball of radius R in $\widetilde{M}$ about a given point x_0 and vol is the Riemannian volume. It holds: $h \leq \ell v$ ([**Gu**]).

Corollary 0.1. *Let $\pi : \widetilde{M} \to M$ be a regular Riemannian cover of a compact manifold. With the above notations, $\ell \leq v$ and $h \leq v^2$. Either equality $\ell = v, h = v^2$ implies equality in (1).*

Let λ be the bottom of the spectrum of the Laplacian on $\widetilde{M}$:

$$\lambda \;:=\; \inf_{f\in C^2_K(\widetilde{M})} \frac{\int_{\widetilde{M}} \|\nabla f\|^2}{\int_{\widetilde{M}} f^2}.$$

Clearly (by considering C^2_K approximations to the functions $e^{-sd(x_o,.)}$ for $s > v/2$), we have $4\lambda \leq v^2$. It can be shown that $4\lambda \leq h$ ([**L1**], Proposition 3). Therefore,

Corollary 0.2. *Let $\pi : \widetilde{M} \to M$ be a regular Riemannian cover of a compact manifold. With the above notations, equality $4\lambda = v^2$ implies equality in (1).*

Our proof of (1) is based on the construction of a compact bundle space X_M over M which is laminated by spaces modeled on $\widetilde{M}$ and of a laminated Laplacian. In the case when M has negative curvature and $\widetilde{M}$ is the universal cover of M, the bundle space contains the unit tangent bundle $T^1 M$ and the lamination on $T^1 M$ is the weak stable foliation of the geodesic flow. The foliated Laplacian and the associated harmonic measure are useful tools for the geometry and the dynamics of the geodesic flow (see [**Ga**], [**K1**], [**L2**], [**Y**], [**H**]). In Section 1, we construct the lamination in the general case and state the properties of the harmonic measures which lead to Theorem A. The laminated Laplacian defines a laminated Brownian motion, a diffusion on X_M with the property that the trajectories remain in the same leaf for all time. Section 2 describes this diffusion. The rest of the paper is devoted to proving propositions 1.1 and 1.2.

1. The Busemann Lamination

We consider the *Busemann compactification* of the metric space $\widetilde{M}$: since the space $\widetilde{M}$ is a complete manifold, it is a *proper* metric space (closed bounded subsets are compact). Fix a point $x_0 \in \widetilde{M}$ and define, for $x \in \widetilde{M}$ the function $\xi_x(z)$ on $\widetilde{M}$ by:

$$\xi_x(z) \;=\; d(x,z) - d(x,x_0).$$

The assignment $x \mapsto \xi_x$ is continuous, one-to-one and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of $\widetilde{M}$. The Busemann compactification $\widehat{M}$ of $\widetilde{M}$ is the closure of $\widetilde{M}$ for that topology. The space $\widehat{M}$ is a compact separable space. The *Busemann boundary* $\partial\widetilde{M} := \widehat{M} \setminus \widetilde{M}$ is made of Lipschitz continuous functions ξ on $\widetilde{M}$ such that $\xi(x_0) = 0$. Elements of $\partial\widetilde{M}$ are called *horofunctions*. Observe that we may extend by continuity the action of G from $\widetilde{M}$ to $\widehat{M}$, in such a way that for ξ in $\widehat{M}$ and g in G,

$$g.\xi(z) \;=\; \xi(g^{-1}z) - \xi(g^{-1}(x_0)).$$

We define now the *horospheric suspension* X_M of M as the quotient of the space $\widetilde{M} \times \widehat{M}$ by the diagonal action of G. The projection onto the first component in $\widetilde{M} \times \widehat{M}$ factors into a projection from X_M to M so that the fibers are isometric to $\widehat{M}$. It is clear that the space X_M is metric compact. If $M_0 \subset \widetilde{M}$ is a fundamental domain for M, one can represent X_M as $M_0 \times \widehat{M}$ in a natural way.

To each point $\xi \in \widehat{M}$ is associated the projection W_ξ of $\widetilde{M} \times \{\xi\}$. As a subgroup of G, the stabilizer G_ξ of the point ξ acts discretely on $\widetilde{M}$ and the space W_ξ is homeomorphic to the quotient of $\widetilde{M}$ by G_ξ. We put on each W_ξ the smooth structure and the metric inherited from $\widetilde{M}$. The manifold W_ξ and its metric vary continuously on X_M. The collection of all $W_\xi, \xi \in \widehat{M}$ form a continuous lamination $\mathcal{W}_M$ with leaves which are manifolds locally modeled on $\widetilde{M}$. In particular, it makes sense to differentiate along the leaves of the lamination and we denote $\Delta^{\mathcal{W}}$ the laminated Laplace operator acting on functions which are smooth along the leaves of the lamination. A Borel measure on X_M is called *harmonic* if it satisfies, for all f for which it makes sense,

$$\int \Delta^{\mathcal{W}} f \, dm \;=\; 0.$$

By [**Ga**], there exist harmonic measures and the set of harmonic probability measures is a weak* compact set of measures on X_M. Moreover, if m is a harmonic measure and $\widetilde{m}$ is the G-invariant measure which extends m on $\widetilde{M} \times \widehat{M}$, then ([**Ga**]), there is a finite measure ν on $\widehat{M}$ and, for ν-almost every ξ, a positive harmonic function $k_\xi(x)$ with $k_\xi(x_0) = 1$ such that the measure m can be written as;

$$\widetilde{m} \;=\; k_\xi(x)(dx \times \nu(d\xi)).$$

The harmonic probability measure m is called ergodic if it is extremal among harmonic probability measures. In that case, for ν-almost every ξ, the following limits exist along almost every trajectory of the Brownian motion (see [**K2**] and section 3 below):

- the linear drift of m $\ell(m) := \lim_{t\to\infty} \frac{1}{t}\xi(\widetilde{X}_t)$.
- the transverse entropy $k(m) := \lim_{t\to\infty} -\frac{1}{t}\ln k_\xi(\widetilde{X}_t)$.

The proof of Theorem A reduces to the three following results;

Proposition 1.1. *With the above notations, there exists an ergodic harmonic measure such that $\ell(m) = \ell$.*

Proposition 1.2. *For all harmonic measure m, we have $\ell^2(m) \leq k(m)$ with equality only if the harmonic functions k_ξ are such that $\nabla^{\mathcal{W}} \ln k_\xi = -\ell(m)\nabla^{\mathcal{W}}\xi$ m-almost everywhere.*

Proposition 1.3. *For all ergodic harmonic measure m, we have $k(m) \leq h$.*

The proof of Proposition 1.1 is an extension of the proof of the Furstenberg formula in [**KL2**] and is given in section 4. Kaimanovich ([**K1**]) proved Proposition 1.2 under the hypothesis that the horofunctions are of class C^2 by applying Itô's formula to the function ξ. In the general case, horofunctions are only uniformly 1-Lipschitz, but the integrated formulas of [**K1**] are still valid (see Section 3). See [**K2**] and Section 3 for Proposition 1.3.

Assume that $\widetilde{M}$ is the universal cover of a negatively curved compact manifold M. Then, $\widetilde{M}$ is homeomorphic to an open ball and the Busemann compactification

is homeomorphic to the closure of the ball. In particular, for all $x \in \widetilde{M}$, the Busemann boundary is homeomorphic to the unit sphere in the tangent space $T_x\widetilde{M}$: a unit vector v defines a unique geodesic $\sigma_v(t)$ such that $\sigma_v(0) = x, \dot{\sigma}_v(0) = v$. As $t \to \infty$, $\xi_{\sigma_v(t)}$ converges in $\widehat{M}$ towards the Busemann function $\xi_{\sigma_v(+\infty)}$ and $v \mapsto \xi_{\sigma_v(+\infty)}$ defines the homeomorphism between $T_x^1\widetilde{M}$ and $\partial\widetilde{M}$. In particular, $\partial\widetilde{M}$ is a closed G-invariant subset of $\widehat{M}$. We can identify $\widetilde{M} \times \partial\widetilde{M}$ with the unit tangent bundle $T^1\widetilde{M}$. The induced action of G is the natural differential action on $T^1\widetilde{M}$. The quotient T^1M is therefore identified with a closed subset of X_M. For $\xi \in \partial\widetilde{M}$, unit vectors v such that $\sigma_v(+\infty) = \xi$ form a stable manifold for the geodesic flow. The lamination $\mathcal{W}$ in T^1M is the usual stable lamination of the geodesic flow. In this case, there is a unique harmonic probability measure m (see [**Ga**], [**L2**], [**Y**]; the proof shows that any harmonic measure on X_M has to be carried by $(\widetilde{M} \times \partial\widetilde{M})/G$ and the support of the harmonic measure is the whole $(\widetilde{M} \times \partial\widetilde{M})/G$ (see [**A**], [**S**]; by compactness, the curvature is pinched betwen two negative constants). Proposition 1.1 (i.e. $\ell(m) = \ell$) and 1.2 are due to Kaimanovich ([**K1**]). By Proposition 1.2, if we have equality in (1), then the Busemann functions are such that $\Delta\xi$ is a constant, the manifold $\widetilde{M}$ is asymptotically harmonic. It follows then from the combined works of Y. Benoist, G.Besson, G. Courtois, P. Foulon, S. Gallot and F. Labourie ([**FL**], [**BFL**], [**BCG**]) that the manifold $\widetilde{M}$ is a symmetric space.

2. Laminated Brownian motion

The operator $\Delta^{\mathcal{W}}$ is Markovian ($\Delta^{\mathcal{W}}1 = 0$) and in this section, we construct the corresponding diffusion on X_M. As we detail now, this diffusion is derived from the Brownian motion on $\widetilde{M}$.

We define subspaces of trajectories in $C(\mathbb{R}_+, M), C(\mathbb{R}_+, \widetilde{M}), C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ and $C(\mathbb{R}_+, \mathcal{W}_M)$ and natural identifications: $C(\mathbb{R}_+, M), C(\mathbb{R}_+, \widetilde{M})$ are the spaces of continuous functions from $\mathbb{R}_+$ into respectively M and $\widetilde{M}$ with the natural projection from $C(\mathbb{R}_+, \widetilde{M})$ to $C(\mathbb{R}_+, M)$; the space $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ is the space of continuous functions from $\mathbb{R}_+$ into $\widetilde{M} \times \widehat{M}$ which are *constant* on the second component, with the forgetful projection from $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ to $C(\mathbb{R}_+, \widetilde{M})$; the group G acts on $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ by postcomposition; the quotient space of G-orbits in $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ is the space $C(\mathbb{R}_+, \mathcal{W}_M)$, with the natural projection from $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ to $C(\mathbb{R}_+, \mathcal{W}_M)$. Elements of $C(\mathbb{R}_+, \mathcal{W}_M)$ can be seen as trajectories on X_M which are included in a single leaf of the lamination $\mathcal{W}$. Translations over $\mathbb{R}_+$ act by precomposition on all our spaces of trajectories and the translation by t will be denoted σ_t on each of them.

The operator Δ is uniformly elliptic on $\widetilde{M}$. The fundamental solution of the equation $\frac{\partial u}{\partial t} = \Delta u$ is the heat kernel $\widetilde{p}(t, \widetilde{x}, \widetilde{y})$. There is a unique family of probabilities

$\widetilde{P}_x, x \in \widetilde{M}$, on $C(\mathbb{R}_+, \widetilde{M})$ such that $\{\omega(t), t \in \mathbb{R}_+\}$ is a Markov process with generator Δ. This means for example that we have for $f_j \in C_c(\widetilde{M}), j = 0, 1, 2, 0 < s_1 < s_2$,

$$\int f_0(\omega(0))f_1(\omega(s_1))f_2(\omega(s_2))d\widetilde{P}_x$$
$$= \int f_0(x)f_1(y_1)f_2(y_2)\widetilde{p}(s_1, x, y_1)\widetilde{p}(s_2 - s_1, y_1, y_2)dy_1 dy_2.$$

The family of measures $\widetilde{P}_x, x \in \widetilde{M}$ defines the Brownian motion on $\widetilde{M}$. See e.g. [**P**] Chapter 4.8 for the following:

Proposition 2.1. *Let $\widetilde{m}$ a locally finite positive measure on $\widetilde{M}$. The following properties are equivalent:*
the measure $\widetilde{m}$ satisfies, for all $f \in C_c^2(\widetilde{M})$, $\int \Delta f d\widetilde{m} = 0$,
the measure $\widetilde{m}$ is of the form $k(y)dy$ where k is a positive harmonic function,
the measure $\widetilde{m}$ is $\widetilde{p}$ invariant, i.e. for all $t > 0$, all $f \in C_c^2(\widetilde{M})$,

$$\int_{\widetilde{M}} \left(\int_{\widetilde{M}} f(y)\widetilde{p}(t, x, y)dy \right) d\widetilde{m}(x) = \int_{\widetilde{M}} f(x)d\widetilde{m}(x),$$

the measure $\widetilde{P}_{\widetilde{m}} := \int_{\widetilde{M}} \widetilde{P}_x d\widetilde{m}(x)$ on $C(\mathbb{R}_+, \widetilde{M})$ is σ-invariant.

By uniqueness, the family of measures $\widetilde{P}_x$ is G-equivariant and projects as a family of measures $P_x, x \in M$ on $C(\mathbb{R}_+, M)$ which defines the Brownian motion on M, with the same properties as above. In particular, the heat kernel $p(u, x, y)$ on M is given by

$$p(u, x, y) = \sum_{g \in G} \widetilde{p}(u, \widetilde{x}, g\widetilde{y}),$$

where $\widetilde{x}, \widetilde{y}$ are lifts in $\widetilde{M}$ of the points x, y in M. The Lebesgue probability measure Leb $:= \frac{1}{\text{vol}M}\text{vol}$ on M satisfies for all $f \in C^2(M)$, $\int \Delta f d\text{Leb} = 0$ and for all $t > 0$, $\int_M \left(\int_M f(y)p(t, x, y)dy \right) d\text{Leb}(x) = \int_M f(x)d\text{Leb}(x)$. Moreover, the probability measure $P = \int_M P_x d\text{Leb}(x)$ is invariant under the time shift σ. The probability Leb is the only one with any of those properties. Indeed, by Proposition 2.1, the G-invariant lift of such a measure m to $\widetilde{M}$ has to be of the form $k(y)d\text{vol}(y)$ where k is a G-invariant positive harmonic function, and G-invariant positive harmonic functions on $\widetilde{M}$ are lifts of positive harmonic functions on the compact manifold M and are therefore constant.

Let ξ_0 be a point in $\widehat{M}$. There is a one-to-one correspondence between the set of trajectories in $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ satisfying $\xi(0) = \xi_0$ (and therefore $\xi(t) = \xi_0$ for all t) and $C(\mathbb{R}_+, \widetilde{M})$. For all $x \in \widetilde{M}$, the measure $\widetilde{P}_x$ defines a measure $\widetilde{Q}_{x,\xi_0}$ on the set of trajectories in $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$ satisfying $(\omega(0), \xi(0)) = (x, \xi_0)$. The family $\widetilde{Q}_{x,\xi}$ describes the Brownian motion along the leaves of the trivial fibration of $\widetilde{M} \times \widehat{M}$ into $\widetilde{M} \times \{\xi\}$'s. In particular $\widetilde{q}(u, (x, \xi), (y, \eta)) = \widetilde{p}(u, x, y)\delta_\xi(\eta)$ is the Markov kernel of the diffusion with law $\widetilde{Q}_{x,\xi}$ and we may write for $f_j \in C_c(\widetilde{M} \times \widehat{M}), j = 0, 1, 2$

and $0 < s_1 < s_2$,

$$\int f_0(\omega(0),\xi(0))f_1(\omega(s_1),\xi(s_1))f_2(\omega(s_2),\xi(s_2))d\widetilde{Q}_{x,\xi}$$

$$= \int f_0(x,\xi)f_1(y_1,\xi)f_2(y_2,\xi)\widetilde{p}(s_1,x,y_1)\widetilde{p}(s_2-s_1,y_1,y_2)dy_1dy_2.$$

Proposition 2.2. *Let $\widetilde{m}$ a locally finite positive measure on $\widetilde{M}\times\widehat{M}$. The following properties are equivalent:*

the measure $\widetilde{m}$ satisfies, for all $f \in C_c^2(\widetilde{M}\times\widehat{M})$, $\int \Delta_x f d\widetilde{m} = 0$,

the measure $\widetilde{m}$ is of the form $k_\xi(y)dy \otimes d\nu(\xi)$ where ν is a finite measure on $\widehat{M}$, $(x,\xi) \mapsto k_\xi(x)$ is measurable and for ν almost all ξ, $k_\xi(y)$ is a positive harmonic function on $\widetilde{M}$,

the measure $\widetilde{m}$ is $\widetilde{q}$ invariant, i.e. for all $t > 0$, all $f \in C_c^2(\widetilde{M}\times\widehat{M})$,

$$\int_{\widetilde{M}\times\widehat{M}}\left(\int_{\widetilde{M}}f(y,\xi)\widetilde{p}(t,x,y)dy\right)d\widetilde{m}(x,\xi) = \int_{\widetilde{M}\times\widehat{M}}f(x,\xi)d\widetilde{m}(x,\xi),$$

the measure $\widetilde{Q}_{\widetilde{m}} := \int_{\widetilde{M}\times\widehat{M}}\widetilde{Q}_{x,\xi}d\widetilde{m}(x,\xi)$ on $C_{\widehat{M}}(\mathbb{R}_+,\widetilde{M}\times\widehat{M})$ is σ-invariant.

Proof. . It is clear that a measure of the form $k_\xi(y)dy \otimes d\nu(\xi)$ satisfies the other properties. Conversely, since $\widetilde{m}$ is locally finite, we can find a positive continuous function b on $\widetilde{M}\times\widehat{M}$ such that $b\widetilde{m}$ is a finite measure. Write $b\widetilde{m}$ as $\int_{\widehat{M}}\widetilde{m}_\xi d\nu(\xi)$ for a finite measure ν on $\widehat{M}$ and a measurable family $\xi \mapsto \widetilde{m}_\xi$ of probabilities on $\widetilde{M}$. Then, for ν almost every ξ, Proposition 2.1 applies to the measure $b^{-1}\widetilde{m}_\xi$. □

In this paper, we normalize ν and the k_ξs by choosing $k_\xi(x_0) = 1$.

Finally, the family of measures $\widetilde{Q}_{x,\xi}$ is G equivariant, and defines a family $Q_w, w \in X_M$ of measures on $C(\mathbb{R}_+, \mathcal{W}_M)$. The family Q_w describes the laminated Brownian motion. By construction, all trajectories of the laminated Brownian motion remain on the leaf of the initial point $w(0)$. In the identification of X_M with $M_0 \times \widehat{M}$, the Markov transition probabilities $q(t,(x,\xi),d(y,\eta))$ of the diffusion with law $Q_{x,\xi}$ are given by:

$$q(t,(x,\xi),d(y,\eta)) = \sum_{g\in G}\widetilde{q}(t,(\widetilde{x},\xi),g_\star d(\widetilde{y},\eta)) = \sum_{g\in G}\widetilde{p}(t,\widetilde{x},g\widetilde{y})d\widetilde{y}\delta_{g^{-1}\xi}(\eta),$$

where $\widetilde{x}, \widetilde{y}$ are lifts in $\widetilde{M}$ of the points x, y in M.

Proposition 2.3. *There is a one-to-one correspondence between:*

(1) *harmonic probability measures m on X_M,*
(2) *G-invariant measures $\widetilde{m}$ which satisfy the equivalent conditions of Proposition 2.2 and such that $\widetilde{m}(M_0 \times \widehat{M}) = 1$,*
(3) *probability measures m on X_M which projects on M onto the Lebesgue probability measure and which can be written in local $\Phi(\mathbb{D}^d \times T)$ charts $m = \int_T \left(\int k_t(x)dx\right)d\nu(t)$, where the function $(x,t) \mapsto k_t(x)$ is measurable and, for ν almost all $t \in T$, $k_t(x)$ is a positive harmonic function,*
(4) *probability measures m on X_M which can be written in a $M_0 \times \widehat{M}$ representation*

$$m = \int_{M_0}\left(\int_{\widehat{M}}d\mu_x(\xi)\right)dLeb(x),$$

where $x \mapsto \mu_x$ is a measurable family of measures on $\widehat{M}$ with the same negligible sets and such that for almost every $\xi \in \widehat{M}$, $k_\xi(x) := \frac{d\mu_x}{d\mu_{x_0}}(\xi)$ is obtained as the restriction to a fundamental domain of a positive harmonic function on $\widetilde{M}$,

(5) probability measures m on X_M which are invariant under q: for all $f \in C^2(X_M)$, all $t > 0$, we have

$$\int_{X_M} \left(\int_{X_M} f(y, \eta) q(t, (x, \xi), d(y, \eta)) \right) dm((x, \xi)) = \int_{X_M} f \, dm.$$

(6) probability measures m on X_M such that $Q_m := \int_{X_M} Q_w dm(w)$ is a σ-invariant measure on $C(\mathbb{R}_+, \mathcal{W}_M)$.

Proof. Let m be a harmonic probability measure on X_M. By writing the harmonic equation for functions which are constant on the fibers, we see that the projection of m onto M is a harmonic probability measure and thus is Leb. Write $\widetilde{m}$ for the unique G-invariant measure on $\widetilde{M} \times \widehat{M}$ such that the restriction to any fundamental domain projects to m. The measure $\widetilde{m}$ satisfies for all $f \in C^2_c(\widetilde{M} \times \widehat{M})$, $\int \Delta_x f \, d\widetilde{m} = 0$. Conversely, the restriction of such a measure to $(M_0 \times \widehat{M})$ is finite and harmonic. This shows the equivalence of properties (1) and (2).

Moreover, by proposition 2.2, the measure $\widetilde{m}$ is of the form $k_\xi(y) dy \otimes d\nu(\xi)$ where ν is a finite measure on $\widehat{M}$, $(x, \xi) \mapsto k_\xi(x)$ is measurable and for ν almost all ξ, $k_\xi(y)$ is a positive harmonic function on $\widetilde{M}$. When we restrict to the image of a $\mathbb{D}^d \times T$ chart, this gives the description of property (3). Conversely, assume that m satisfies property (3). Using if necessary a partition of unity we may take the function f in $C^2(X_M)$ with support inside the image of a $\Phi(\mathbb{D}^d \times T)$ chart. Then:

$$\int \Delta^{\mathcal{W}} f \, dm = \int_T \left(\int \Delta_x f(x, t) k_t(x) dx \right) d\nu(t)$$

and the inner integral vanishes for all $t \in T$ such that $x \mapsto k_t(x)$ is a harmonic function, that is, for almost every t.

Assume m satisfies property (3). Putting together the $\mathbb{D}^d \times T$ charts into a measurable $M_0 \times \widehat{M}$ representation, we have a measure which projects on a measure ν on $\widehat{M}$ and such that the conditional on M are proportional to $k_\xi(x) d\mathrm{vol}(x)$. In other words, the measure m writes as $m = \dfrac{k_\xi(x)}{\int_{M_0} k_\xi(x) dx} dx \otimes d\nu$. Since we assume that the projection onto M_0 is Leb, we can write $m = \int_{M_0} \left(\int_{\widehat{M}} d\mu_x(\xi) \right) d\mathrm{Leb}(x)$, where

$$\mu_x(d\xi) = \frac{k_\xi(x) \mathrm{vol} M}{\int_{M_0} k_\xi(x) dx} d\nu(\xi).$$

We indeed have $\frac{d\mu_x}{d\mu_y}(\xi) = \frac{k_\xi(x)}{k_\xi(y)}$. This shows that property (3) implies property (4). The converse is proven analogously, by setting $\nu = \mu_{x_0}$.

Properties (5) and (6) are equivalent to (1) by general theory of diffusions with a finite invariant measure. The point to check is that a σ-invariant measurable set B in $C(\mathbb{R}_+, \mathcal{W}_M)$ is of the form $x \in B_0$, where B_0 is a Q-invariant subset of X_M. It follows from the Markov property that the set B has Q_w measure 0 or 1 for m-almost every $w \in X_M$. Take $B_0 = \{w : Q_w(B) = 1\}$. $\qquad\square$

Proposition 2.3 is due to Garnett ([**Ga**]). We included a proof in the suspension case for notational purposes. A harmonic measure is called ergodic harmonic if it cannot be decomposed into a convex combination of other harmonic measures. By proposition 2.3, an ergodic harmonic measure is also extremal for properties (5) and (6) and therefore the time shift σ_t is ergodic on $(C(\mathbb{R}_+, \mathcal{W}_M), Q_m)$. By proposition 2.3 a harmonic measure can be written, in a $M_0 \times \widehat{M}$ representation as

$$\int_M d\mu_x(\xi) d\mathrm{Leb}(x) \quad \text{where } \frac{d\mu_x}{d\mu_y}(\xi) \; = \; \frac{k_\xi(x)}{k_\xi(y)}$$

and $k_\xi(x)$ is a positive harmonic function for $\nu(= \mu_{x_0})$-almost every ξ. In particular, for $f \in C^2(X_M)$ with support in the interior of M_0, we may write:

$$\begin{aligned}
\int f \, dm &= \int_{M_0} \int_{\widehat{M}} f(x, \xi) d\mu_x(\xi) d\mathrm{Leb}(x) \\
&= \int_{M_0} \left(\int_{\widehat{M}} f(x, \xi) k_\xi(x) d\mathrm{Leb}(x) \right) d\nu(\xi)
\end{aligned}$$

Integrating by parts the inner integral, the following formulas follow, for all $f, g \in C^2(X_M)$:

$$\begin{aligned}
\int \Delta^{\mathcal{W}} f \, dm &= -\int \langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm = 0 \\
\int g \Delta^{\mathcal{W}} f \, dm &= -\int \langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} g \rangle dm - \int g \langle \nabla^{\mathcal{W}} f, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm,
\end{aligned}$$

where $\nabla^{\mathcal{W}} g$ denotes the gradient of the function g along the leaves of the lamination $\mathcal{W}$ and $\langle, \rangle$ the leafwise scalar product. The second formula extends by approximation to vector fields Y which are C^1 along the leaves and such that Y and $\mathrm{div}^{\mathcal{W}} Y$ are continuous:

$$(2) \qquad \int \mathrm{div}^{\mathcal{W}} Y \, dm \; = \; -\int \langle Y, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm.$$

3. ASYMPTOTICS OF HARMONIC MEASURES

In this section, we state two formulas as Proposition 3.1 and 3.2. We deduce from them Proposition 1.2 and, using Propositions 1.1 and 1.3, Theorem A.

Let m be an ergodic harmonic measure on X_M. Recall that m can be written as $\int_{M_0} k_\xi(x) d\nu(\xi) d\mathrm{Leb}(x)$ for some positive harmonic function $k_\xi(x)$ defined for ν-almost every ξ. The probability measure Q_m is invariant and ergodic under the shift on the space of trajectories $C(\mathbb{R}_+, \mathcal{W}_M)$. There are two natural additive functional on $C(\mathbb{R}_+, \mathcal{W}_M)$ which are defined as G-invariant functionals on $C_{\widehat{M}}(\mathbb{R}_+, \widetilde{M} \times \widehat{M})$: the horospherical displacement

$$L(t, \omega, \xi) := \xi(\omega(t)) - \xi(\omega(0))$$

and the harmonic kernel

$$K(t, \omega, \xi) := \ln \frac{k_\xi(\omega(0))}{\kappa_\xi(\omega(t))}.$$

The functional $K(t, \omega, \xi)$ is defined for Q_m-almost every (ω, ξ), but for all $t \geq 0$. We have $L(t + s, \omega, \xi) = L(t, \omega, \xi) + L(s, \sigma_t(\omega), \xi)$ and, for Q_m-almost every (ω, ξ), $K(t + s, \omega, \xi) = K(t, \omega, \xi) + K(s, \sigma_t(\omega), \xi)$.

By the ergodic theorem, the two following limits exist Q_m-almost everywhere and are constant Q_m-almost everywhere:

$$\ell(m) \; := \; \lim_{T \to \infty} \frac{1}{T} L(T, \omega, \xi) \; \text{ and } \; k(m) \; := \; \lim_{T \to \infty} \frac{1}{T} K(T, \omega, \xi).$$

By our description of the measure Q_m in Section 2, for ν-almost every ξ, the numbers $\ell(m)$ and $k(m)$ can also be seen as the limits along almost every trajectory of the Brownian motion of respectively $\frac{1}{t}\xi(\widetilde{X}_t)$ and $-\frac{1}{t}\ln k_\xi(\widetilde{X}_t)$. This is the way they were introduced in Section 1. In particular, since the functions ξ are Lipschitz, for all ergodic harmonic measure m,

$$(3) \qquad\qquad\qquad \ell(m) \; \leq \; \ell.$$

The analogous result $k(m) \leq h$ is Proposition 1.3. Kaimanovich introduced in [**K2**] the *reverse entropy* of an ergodic harmonic measure as the number $h'(m)$ such that, for Q_m-almost every trajectory in $C(\mathbb{R}_+, \widetilde{M})$,

$$h'(m) \; = \; \lim_{t \to \infty} -\frac{1}{t} \ln \left(\widetilde{p}(t, \omega(0), \omega(t)) \frac{k_\xi(\omega(0))}{k_\xi(\omega(t))} \right).$$

Clearly, $h'(m) = h - k(m)$. Proposition 1.3 follows from the observation that the number $h'(m)$ is nonnegative, since it can be seen as the entropy of a conditional process (see [**K1**], section 4)).

Observe that, by σ-invariance and ergodicity, for all $\tau > 0$, we have:

$$\ell(m) \; = \; \frac{1}{\tau} \int \big(L(\tau, \omega, \xi) \big) dQ_m \qquad k(m) \; = \; \frac{1}{\tau} \int \big(K(\tau, \omega, \xi) \big) dQ_m.$$

For a non-ergodic harmonic measure, we *define* $\ell(m)$ and $k(m)$ by these formulas. We have:

Proposition 3.1. *Let m be a harmonic measure. Then:*

$$k(m) \; = \; \int_{X_M} \|\nabla^{\mathcal{W}} \ln k_\xi\|^2 dm.$$

Proposition 3.2. *Let m be a harmonic measure. Then:*

$$\ell(m) \; = \; -\int_{X_M} \langle Z_\xi, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm,$$

where the vector field Z_ξ is defined m-almost everywhere by $Z_\xi := \nabla^{\mathcal{W}} \xi$.

Recall that ξ is defined as the uniform limit of difference of distances. It follows that ξ is 1-Lipschitz and by Rademacher Theorem, $\nabla \xi$ is defined Lebesgue-almost everywhere on $\widetilde{M}$. Since m is harmonic, its conditional on the leaves of $\mathcal{W}$ are absolutely continuous, and $Z_\xi := \nabla^{\mathcal{W}} \xi$ is defined m-almost everywhere. Moreover, $\|Z_\xi\| \leq 1$ m-almost everywhere. Schwarz inequality, Propositions 3.1 and 3.2 yield that, for any harmonic measure m,

$$\ell^2(m) = \left| \int_{X_M} \langle Z_\xi, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm \right|^2 \quad \leq \quad \int_{X_M} |\langle Z_\xi, \nabla^{\mathcal{W}} \ln k_\xi \rangle|^2 dm$$

$$\leq \quad \int_{X_M} \|\nabla^{\mathcal{W}} \ln k_\xi\|^2 dm \; = \; k(m),$$

with equality only if $\nabla^{\mathcal{W}} \ln k_\xi \; = \; -\ell(m) Z_\xi$ m-almost everywhere. This proves Proposition 1.2. We also have:

Corollary 3.3. *Let $\widetilde{M}$ be a regular Riemannian cover of a compact manifold; then,*

$$\ell^2 \;\leq\; h.$$

If there is equality $\ell^2 = h$, then there is an ergodic harmonic measure m on X_M such that $\ln k_\xi(x) = -\ell\xi(x)$ m-almost everywhere in the case $\ell > 0$, $k_\xi(x) = 1$ m-almost everywhere in the case $\ell = 0$.

Proof. By Proposition 1.2, for any harmonic measure m, $\ell^2(m) \leq k(m)$ with equality only if the harmonic functions k_ξ are such that, $\nabla^{\mathcal{W}} \ln k_\xi(x) = -\ell(m)\nabla^{\mathcal{W}}\xi(x)$ for m-almost every (x,ξ). If $\ell(m) = 0$, k_ξ is constant for ν-almost every ξ. If $\ell(m) > 0$, since both functions $\ln k_\xi$ and ξ vanish at x_0, $\ln k_\xi = -\ell(m)\xi$ for ν-almost every ξ. All this applies to the measure m_0 given by proposition 1.1 so that:

$$\ell^2 \;=\; \ell^2(m_0) \;\leq\; k(m_0) \;\leq\; h,$$

with $\ell^2 = h$ only if $\ell^2(m_0) = k(m_0) = h$ and therefore $\ln k_\xi(x) = -\ell\xi(x)$ m_0-almost everywhere in the case $\ell > 0$, $k_\xi(x) = 1$ m_0-almost everywhere in the case $\ell = 0$.
$\square$

Observe that in the case $\ell^2 = h > 0$, the harmonic measure given by Corollary 3.3 gives full measure to $\widetilde{M} \times \partial\widetilde{M}/G$ because $e^{-\ell d(x,z)}$ cannot be a harmonic function in z. In general the support of m is smaller than $\widetilde{M} \times \partial\widetilde{M}/G$: consider for instance $\widetilde{M} = \mathbb{H}^2 \times \mathbb{H}^2$. We have $\ell^2 = h = 2$. The space $\partial\widetilde{M}$ can be parametrized by (ξ_1, ξ_2, θ), where $\xi_j \in \partial\mathbb{H}^2$ for $j = 1, 2$ and θ is an angle in $[0, \pi/4]$; the horofunction $\xi_{\xi_1,\xi_2,\theta}$ is given by:

$$\xi_{\xi_1,\xi_2,\theta}(z_1, z_2) \;=\; \cos\theta \, \xi_1(z_1) + \sin\theta \, \xi_2(z_2).$$

The function $e^{-\sqrt{2}\xi}$ satisfies

$$\Delta e^{-\sqrt{2}\xi_{\xi_1,\xi_2,\theta}} \;=\; \big(2 - \sqrt{2}(\cos\theta + \sin\theta)\big)e^{-\sqrt{2}\xi_{\xi_1,\xi_2,\theta}}.$$

This is a harmonic function only if $\theta = \pi/4$. The support of the measure m given by Corollary 3.3 is included in $\widetilde{M}_{\pi/4}/G$, where $\widetilde{M}_{\pi/4} := \{(x,\xi); x \in \widetilde{M}, \xi = (\xi_1, \xi_2, \theta) \in \partial\widetilde{M}$ and $\theta = \pi/4\}$. The discussion is similar for any symmetric space of non-positive curvature which is not of negative curvature.

Theorem A is the first part of Corollary 3.3. When $h = 0$, Corollary 3.3 adds that there is an ergodic measure m with $k_\xi(x) = 1$ m-almost everywhere: in terms of Proposition 2.3 (3) the measure m is, in local charts, the product of the Lebesgue measure on the leaves and some transverse holonomy-invariant measure ν. When $h > 0$ and equality $\ell^2 = h$ holds, one can conclude from Corollary 3.3 that $(\widehat{M}, \nu)$ represents all bounded harmonic functions on $\widetilde{M}$ (cf. [**K1**]).

It remains to prove Propositions 1.1, 3.1 and 3.2. Proposition 1.1 is proven in Section 4. Proposition 3.1 is due to Kaimanovich. We give a proof in section 5, because it follows the same computation as in the proof of Proposition 3.2 in Section 6.

4. Proof of Proposition 1.1

Let X_M be a horospheric suspension as above. We construct the measure m by a limiting procedure (compare [**KL2**], proof of Theorem 7). To define a measure on X_M, we usually describe it as a G-invariant measure on $\widetilde{M} \times \widehat{M}$. It will project

onto M as Leb, and in particular, it will be a probability measure, as soon as it projects onto $\widetilde{M}$ as $\widetilde{\text{Leb}} := \frac{dx}{\text{vol}(M)}$. Set:

$$\nu_t := \int_{\widetilde{M}} \left(\xi_*(\widetilde{p}(t,x,y)dy)\right) \frac{dx}{\text{vol}(M)},$$

where, for a measure μ on $\widetilde{M}$, $\xi_*(\mu)$ is the pushed-forward of μ by the mapping $\xi : \widetilde{M} \to \widehat{M}$. The measure ν_t is a G-invariant measure on $\widetilde{M} \times \widehat{M}$ which projects on $\widehat{\text{Leb}}$ and we can write, for $f \in C_c(\widetilde{M} \times \widehat{M})$,

$$\int f(x,\xi)d\nu_t(x,\xi) = \int f(x,\xi_y)\widetilde{p}(t,x,y)\frac{dxdy}{\text{vol}(M)}.$$

We then form the measure $\int \widetilde{q}(s,.,.)d\nu_t = \int \widetilde{q}(s,(x,\xi),d(y,\eta))d\nu_t(x,\xi)$. The measure $\int \widetilde{q}(s,.,.)d\nu_t$ is a G-invariant measure on $\widetilde{M} \times \widehat{M}$ which projects on $\widetilde{\text{Leb}}$. Observe that $\int \widetilde{q}(s,.,.)d\nu_t = \nu_{t+s}$. Indeed, we may write, for $f \in C_c(\widetilde{M} \times \widehat{M})$,

$$\int fd\left(\int \widetilde{q}(s,.,.)d\nu_t\right) = \int f(y,\eta)\widetilde{q}(s,(x,\xi_z),d(y,\eta))\widetilde{p}(t,x,z)\frac{dzdx}{\text{vol}(M)}$$
$$= \int f(y,\xi_z)\widetilde{p}(s,x,y)\widetilde{p}(t,x,z)\frac{dydzdx}{\text{vol}(M)}.$$

By the symmetry and the semigroup property of $\widetilde{p}$, $\int \widetilde{p}(s,x,y)\widetilde{p}(t,x,z)dx = \widetilde{p}(t+s,y,z)$ and we find, as claimed,

$$\int fd\left(\int \widetilde{q}(s,.,.)d\nu_t\right) = \int f(y,\xi_z)\widetilde{p}(t+s,y,z)\frac{dydz}{\text{vol}(M)} = \int fd\nu_{t+s}.$$

The set of measures on X_M which project on Leb on M is a convex weak* compact set of probability measures on X_M. Any limit point of $\frac{1}{T}\int_0^T \nu_t dt$ is a harmonic measure. Indeed, by the above observation

$$\frac{1}{T}\int_0^T \nu_t dt = \frac{1}{T}\int_0^{T-1}\int \widetilde{q}(s,.,.)d\nu_1 ds + O(1/T),$$

so that, if $m_0 = \lim_k \frac{1}{T_k}\int_0^{T_k} \nu_t dt$, we have

$$\int \widetilde{q}(1,.,.)dm_0 = \lim_k \left(\frac{1}{T_k}\int_0^{T_k-1}\int \widetilde{q}(s+1,.,.)d\nu_1 ds + O(1/T_k)\right) = m_0.$$

Take m_0 such a limit. We choose a fundamental domain M_0 for M and we compute $\ell(m_0)$:

$$\tau\ell(m_0) =$$

$$= \int \big(\xi(\omega(\tau)) - \xi(\omega(0))\big)dQ_m$$

$$= \int_{M_0 \times \widehat{M}} \left(\int \big(\xi(y) - \xi(x)\big)\widetilde{p}(\tau, x, y)dy\right) dm_0(x, \xi)$$

$$= \lim_k \frac{1}{T_k} \int_0^{T_k} \int_{M_0 \times \widehat{M}} \left(\int \big(\xi(y) - \xi(x)\big)\widetilde{p}(\tau, x, y)dy\right) d\nu_t(x, \xi)dt$$

$$= \lim_k \frac{1}{T_k} \int_0^{T_k} \int_{M_0 \times \widetilde{M}} \left(\int \big(\xi_z(y) - \xi_z(x)\big)\widetilde{p}(\tau, x, y)dy\right) \widetilde{p}(t, x, z)\frac{dxdz}{\mathrm{vol}(M)}dt$$

$$= \lim_k \frac{1}{T_k} \int_0^{T_k} \int_{M_0 \times \widetilde{M} \times \widetilde{M}} \big(d(z, y) - d(z, x)\big)\widetilde{p}(\tau, x, y)\widetilde{p}(t, x, z)\frac{dxdydz}{\mathrm{vol}(M)}dt$$

$$= \lim_k \frac{1}{T_k} \int_0^{T_k} \Big(\int_{M_0 \times \widetilde{M} \times \widetilde{M}} d(z, y)\widetilde{p}(\tau, x, y)\widetilde{p}(t, x, z)\frac{dxdydz}{\mathrm{vol}(M)}$$
$$- \int_{M_0 \times \widetilde{M} \times \widetilde{M}} d(z, x)\widetilde{p}(\tau, x, y)\widetilde{p}(t, x, z)\frac{dxdydz}{\mathrm{vol}(M)}\Big) \ dt$$

$$= \lim_k \frac{1}{T_k} \int_0^{T_k} \Big(\int_{M_0 \times \widetilde{M}} d(z, y)\widetilde{p}(t + \tau, z, y)\frac{dydz}{\mathrm{vol}(M)}$$
$$- \int_{M_0 \times \widetilde{M}} d(z, x)\widetilde{p}(t, x, z)\frac{dxdz}{\mathrm{vol}(M)}\Big) \ dt$$

$$= \lim_k \frac{1}{T_k} \int_0^{\tau} \left(\int_{M_0 \times \widetilde{M}} d(x, z)\widetilde{p}(T_k + t, x, z)\frac{dxdz}{\mathrm{vol}(M)} \right) dt$$
$$- \lim_k \frac{1}{T_k} \int_0^{\tau} \left(\int_{M_0 \times \widetilde{M}} d(x, z)\widetilde{p}(t, x, z)\frac{dxdz}{\mathrm{vol}(M)} \right) dt.$$

The last term goes to 0 as $T_k \to \infty$. Recall that ℓ is defined by the subadditive ergodic theorem so that $\ell = \lim_{T \to \infty} \frac{1}{T} \int_{M_0 \times \widetilde{M}} d(x, z)\widetilde{p}(T, x, z)\frac{dxdz}{\mathrm{vol}(M)}$. We have indeed $\tau\ell(m_0) = \tau\ell$. The above measure m_0 is not necessarily ergodic, but since $\ell(m) \le \ell$ for all harmonic measures and $m \mapsto \ell(m)$ is linear, there are ergodic measures m in the extremal decomposition of m_0 which satisfy $\ell(m) = \ell$.

5. Proof of Proposition 3.1

Let m be a harmonic measure on X_M. We have to show that $k(m) = \int_{X_M} \|\nabla^{\mathcal{W}} \ln k_\xi\|^2 dm$. We compute:

$$\tau k(m) = \int \ln \frac{k_\xi(\omega(0))}{\kappa_\xi(\omega(\tau))}dQ_m$$

$$= \int_{M_0 \times \widehat{M}} \left(\int_{\widetilde{M}} \widetilde{p}(\tau, x, y)(\ln k_\xi(x) - \ln k_\xi(y))dy\right) dm(x, \xi)$$

The inner integral is

$$\int_{\widetilde{M}} \widetilde{p}(\tau, x, y)(\ln k_\xi(x) - \ln k_\xi(y))dy \ =$$

$$= \int_{\widetilde{M}} \int_0^\tau \frac{\partial}{\partial s} \widetilde{p}(s, x, y)(\ln k_\xi(x) - \ln k_\xi(y))dsdy$$

$$= \int_{\widetilde{M}} \int_0^\tau \Delta_y \widetilde{p}(s, x, y)(\ln k_\xi(x) - \ln k_\xi(y))dsdy$$

$$= -\int_0^\tau \int_{\widetilde{M}} \widetilde{p}(s, x, y)\Delta_y \ln k_\xi(y)dsdy$$

$$= \int_0^\tau \int_{\widetilde{M}} \widetilde{p}(s, x, y)\|\nabla_y \ln k_\xi(y)\|^2 dsdy.$$

Observe that the function

$$\varphi(y, \xi) = \|\nabla_y \ln k_\xi(y)\|^2$$

is G-invariant on $\widetilde{M} \times \widehat{M}$. Therefore the integral

$$\int_{M_0 \times \widehat{M}} \left(\int_{\widetilde{M}} \widetilde{p}(s, x, y)\varphi(y, \xi)dy \right) dm(x, \xi)$$

is $\int \varphi(\omega(s), \xi(s))dQ_m$. By invariance, we have, for all $s > 0$,

$$\int_{M_0 \times \widehat{M}} \left(\int_{\widetilde{M}} \widetilde{p}(s, x, y)\varphi(y, \xi)dy \right) dm(x, \xi) \ = \ \int \|\nabla_y \ln k_\xi(y)\|^2 dm$$

and the formula follows.

6. Proof of Proposition 3.2

Recall that the horofunctions are Lipschitz, so that $Z_\xi = \nabla^{\mathcal{W}} \xi$ exists almost everywhere along the leaves and satisfies $\|Z_\xi\|^2 \leq 1$ m-almost everywhere. In particular the expression $\int_{X_M} \langle Z_\xi, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm$ makes sense as soon as m has absolutely continuous conditional measures along the leaves. In this section, we prove that, if m is a harmonic measure, $\ell(m) = -\int_{X_M} \langle Z_\xi, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm$.

We follow the same computation as in Section 5, except that, for technical reasons we choose $\varepsilon > 0$ and write:

$$\tau\ell(m) \ = \ \int \left(L(\varepsilon + \tau, \omega, \xi) - L(\varepsilon, \omega, \xi) \right) dQ_m$$

$$= \ \int_{M_0 \times \widehat{M}} \left(\int_{\widetilde{M}} \widetilde{p}(\varepsilon + \tau, x, y)(\xi(y) - \xi(x))dy \right) dm(x, \xi)$$

$$- \int_{M_0 \times \widehat{M}} \left(\int_{\widetilde{M}} \widetilde{p}(\varepsilon, x, y)(\xi(y) - \xi(x))dy \right) dm(x, \xi)$$

$$= \ \int_{M_0 \times \widehat{M}} \left(\int_{\widetilde{M}} \widetilde{p}(\tau, x, z)(\varphi_\varepsilon(z, \xi) - \varphi_\varepsilon(x, \xi))dz \right) dm(x, \xi),$$

where

$$\varphi_\varepsilon(x, \xi) := \int_{\widetilde{M}} \widetilde{p}(\varepsilon, x, y)\xi(y)dy.$$

Observe that, since the manifold $\widetilde{M}$ has bounded Ricci curvature, for any $s > 0$ there is a constant $C(s)$ such that, for all $x, y \in \widetilde{M}$, $\widetilde{p}(s, x, y) \leq Ce^{-(d(x,y)/C)^2}$([**CLY**]). This shows that the function φ_ε is well defined and that we can separate in the above computation the integrals of $\xi(y)$ and $\xi(x)$.

For all $\varepsilon > 0$, the function φ_ε is smooth and satisfies:

$$\varphi_\varepsilon(gx, g\xi) = \varphi_\varepsilon(x, \xi) - \xi(g^{-1}x_0).$$

The inner integral is

$$\int_{\widetilde{M}} \widetilde{p}(\tau, x, z)(\varphi_\varepsilon(z, \xi) - \varphi_\varepsilon(x, \xi))dz =$$

$$= \int_{\widetilde{M}} \int_0^\tau \frac{\partial}{\partial s}\widetilde{p}(s, x, z)(\varphi_\varepsilon(z, \xi) - \varphi_\varepsilon(x, \xi))ds dz$$

$$= \int_{\widetilde{M}} \int_0^\tau \Delta_z\widetilde{p}(s, x, z)(\varphi_\varepsilon(z, \xi) - \varphi_\varepsilon(x, \xi))ds dz$$

$$= \int_0^\tau \int_{\widetilde{M}} \widetilde{p}(s, x, z)\Delta_z\varphi_\varepsilon(z, \xi)ds dz.$$

The function $\Delta_z\varphi_\varepsilon(z, \xi) = \operatorname{div}_z\nabla_z\varphi_\varepsilon(z, \xi)$ is G-invariant and as before, we have, for all $s > 0$,

$$\int_{M_0 \times \widehat{M}} \left(\int_{\widetilde{M}} \widetilde{p}(s, x, z)\Delta_z\varphi_\varepsilon(z, \xi)dz \right) dm(x, \xi) = \int \operatorname{div}^{\mathcal{W}}\nabla^W\varphi_\varepsilon(y, \xi)dm(y, \xi).$$

Using equation (2), the latter integral is $-\int \langle \nabla^{\mathcal{W}}\varphi_\varepsilon, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm$, so that

$$\ell(m) = -\int \langle \nabla^{\mathcal{W}}\varphi_\varepsilon, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm.$$

Fix ξ. As $\varepsilon \to 0$, the functions $\varphi_\varepsilon(x, \xi)$ are uniformly Lipschitz and converge towards ξ uniformly on compact sets. Their gradients, seen as their weak gradients, converge in L_{loc}^∞ towards the gradient Z_ξ of the limit ([**EG**], Theorem 4.2.3). This proves the formula in Proposition 3.2, namely:

$$\ell(m) = -\int_{X_M} \langle Z_\xi, \nabla^{\mathcal{W}} \ln k_\xi \rangle dm.$$

Proposition 3.2 was proven by Kaimanovich ([**K1**]) with the additional hypothesis that the horofunctions are of class C^2. In the above proof, we can, in that case, take directly $\varepsilon = 0$. Recall that the horofunctions are of class C^2 when M has nonpositive sectional curvature and $\widetilde{M}$ is the universal cover of M ([**HI**]).

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LPMA, Boîte Courrier 188, 4, Place Jussieu, 75252 PARIS cedex 05, France
E-mail address: `francois.ledrappier@upmc.fr`