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Towards an understanding of tradeoffs between regional wealth, tightness of a common environmental constraint and the sharing rules

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Abstract

Consider a country with two regions that have developed differently so that their current levels of energy efficiency differ. Each region's production involves the emission of pollutants, on which a regulator might impose restrictions. The restrictions can be related to pollution standards that the regulator perceives as binding the whole country (*e.g.*, imposed by international agreements like the Kyoto Protocol). We observe that the pollution standards define a common constraint upon the *joint* strategy space of the regions. We propose a game theoretic model with a coupled constraints equilibrium as a solution to the regulator's problem of avoiding excessive pollution. The regulator can direct the regions to implement the solution by using a political pressure, or compel them to employ it by using the coupled constraints' Lagrange multipliers as taxation coefficients. We specify a stylised model of the Belgian regions of Flanders and Wallonia that face a joint constraint, for which the regulator wants to develop a *sharing rule*. We analytically and numerically analyse the equilibrium regional production levels as a function of the pollution standards and of the sharing rules. We thus provide the regulator with an array of equilibria that he (or she) can select for implementation. For the computational results, we use NIRA, which is a piece of software designed to min-maximise the associated Nikaido-Isoda function.

Keywords: Coupled constraints, generalised Nash equilibrium, Nikaido-Isoda function; regional economics, environmental regulations.
JEL: C6, C7, D7

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1 Introduction

The aim of this paper[1] is to examine the impact of imposition of emission constraints on regional revenues and the national revenue of a two-region economy. The regions that are somewhat controllable by a sole regulator will be composed of industries of diverging specialisations. We will study the revenues as functions of the *sharing rules*, adoptable by a regulator, for spreading the burden of the constraints' satisfaction among the regions.

The need for regulation might result from the regulator's wish to comply with a national emissions' quota assigned to the country through an international agreement (like the Kyoto Protocol). The split of the quota among the regions is always a contentious issue in the context of dissimilar profiles of the regional industries. Moreover, the split will be a highly controversial matter if the regions are ethnically different.

An example of such a problem is a disagreement between Wallonia and Flanders (two Belgian regions) regarding sharing the pollution cleaning burden, recently studied in [3] and [9]. In those papers, an impact of "grandfathering" emission permits on regional revenues in a small open multi-sector (multi-regional) economic model (Heckscher-Ohlin type) is considered. Other countries with different regional industrial profiles (most European countries, Canada, *etc.*.) might be facing similar problems. Therefore the problem of pollution burden sharing across regions is of utmost theoretical and practical importance. Our paper provides a new insight into the relevant elements of the problem within a game-theoretic framework, not exploited yet in the related literature (*e.g.*, [3] and [9]).

In this paper, we treat the industries, or regions, as competitive agents and analyse the resulting equilibrium policies as well as the corresponding outputs and payoffs, as a consequence of adoption of a *sharing rule*, for apportioning a pollution quota to each region. What makes our paper essentially different from the above cited publications is that we allow for an emission constraint upon the agents' *joint* strategy space. Assuming the presence of an industry-independent regulator, we then vary the levels of the agents' responsibility for the coupled constraint's satisfaction and suggest which *sharing rules* might be preferred by the regulator.

The problem's setup in this paper is conceptually similar to that of [10], [11], [16], [4], [13], [19], [8] and also [5]. The common feature is that all those papers deal with coupled constraints games, in which competitive agents maximise their utility functions subject to constraints upon their *joint* strategy space. However, in this paper, we make explicit the relationship between a solution to the problem and the *weights*, which the regulator may use to distribute the responsibility for satisfaction of a joint constraint, among the agents. In that, we follow the seminal work [21] and use a *coupled constraints equilibrium* as a solution concept for the discussed problem. Under this solution concept the regulator can compute (for sufficiently concave games) the agents' strategies that are both unilaterally non-improvable (Nash) and such that the constraints imposed on the joint strategy space are satisfied.

[1] An earlier version of this paper was presented at *14th International Conference on Computing in Economics and Finance*, June 26-28, 2008, Sorbonne, Paris.

If the regulator can modify the agents' utilities and impose penalties for violation of the joint constraints then the game will become "decoupled" and the agents will implement the *coupled constraints equilibrium* in its "own interest", to avoid fines associated with excessive pollution. These penalties, which prevent excessive pollution, can be computed using the coupled constraints Lagrange multipliers. However, for this modification of the players' utilities to induce the required behaviour, a coupled constraints equilibrium needs to exist and be unique for a given distribution of the responsibilities for the joint constraints satisfaction, among the agents. We will prove that our model possesses the property of diagonal strict concavity (DSC), which will be sufficient for uniqueness of a coupled constraints equilibrium. Obviously, the game has to possess the same properties should the sharing rules be implemented through a political process rather by threatening the regions with penalties.

Reports on that the energy more-intensive sector's revenue is proportionally more affected by the environmental policy than that of its less-intensive counterpart are provided by [3] and [9]. Our model's results suggest that the decision on apportioning a higher or lower energy share to a region should depend on an analysis of externalities, which the regions exert on each other. We also report on the various degrees of market "distortion" as a consequence of the imposition of pollution quotas and of the alteration of the rules for sharing the burden of the joint constraints' satisfaction. We expect our model can help the regulator discover which rules imply an acceptable degree of market distortion.

For the results we use NIRA, which is a piece of software designed to min-maximise the Nikaido-Isoda function and thus compute a coupled constraints equilibrium (see [2, 17]). We also notice that a coupled constraints equilibrium could be obtained[2] as a solution to a quasi-variational inequality (see [12], [19]) or as a result of gradient *pseudo-norm* minimisation (see [21], [10], [11]).

What follows is a brief outline of what this paper contains. In Section 2 a stylised model of a two-region country is presented. Section 3 briefly explains the idea of a coupled constraints equilibrium and the algorithm that will be used to compute it. We present the calibrated model for a two-region environmental game in Section 5 and report on the equilibrium solutions. The concluding remarks summarise our findings, which include the economic interpretation of the results.

2 A two-region country model

2.1 A game with constraints upon the agents' *joint* strategy space

We understand a country as an entity, on whose territory Gross National Product (GNP) is generated. In some countries that have historically developed into *regions*, notwithstanding the inter-regional spill-over effects, the product created by one region's industry can be deemed somehow independent of the product created by the other region's industry. However, due to trade and, recently, environmental concerns, the supply of some production factors might be jointly constrained.

[2]We refer to [14] for a review on numerical solutions to coupled-constraint equilibria.

As a plausible application area for our analysis, we have mentioned in the Introduction the Belgian regions Flanders and Wallonia whose industries have developed distinctly over the last two centuries. In result, their contributions to the Belgian GNP and the overall "Belgian" pollution are significantly different.[3]

There is some evidence (see Table[4] I) on that Flanders produces about 60% of GNP using about 60% of energy consumed by Belgium. On the other hand, Wallonia contributes to the gross product in about 24% but "burns" 33% of the country's energy supply. The remainders of GNP and energy consumption are the contributions by Brussels. However, we will not consider Brussels an active player in the game we define below. We believe that, firstly, Wallonia and Flanders see each other as direct competitors and neither of them "cares" much about what the regulator decides about the Brussels' quota. Secondly, Brussels' energy consumption appears small and, perhaps, no quota will make a big difference to its contribution to GNP.

Table I: Estimations of energy consumption and gross product generation in Belgium, about 2000

	Energy Use		Gross Value Added		GVA/En.Use
	10^6 `boe`	%	10^6 Euros	%	Euro/`boe`
Belgium	270.065	100	223 812.0	100	828.73
Brussels	14.427	5.4	42 562.5	19	2950.20
Wallonia	89.832	33.3	52 819.1	24	587.98
Flanders	165.804	61.4	128 146.6	57	772.88

In Table I, we observe that Wallonia's usage of energy appears less effective than Flanders' is.

Intuitively, it seems possible to keep the overall pollution constant but vary its regional contributions and achieve an improvement of the whole country's performance. For example, using different *sharing rules* of apportioning the energy use between the two regions can force them use different equilibrium strategies, which may be more efficient from the social planner's point of view. It is the aim of the analysis conducted in this paper to help the "planners" to improve the latter.

Unless Wallonia's product is "badly" needed by Flanders, it may appear that encouraging Wallonia to use less energy (especially, if constrained) and allowing Flanders to use more of it, might be beneficial for the global revenue.

[3]We shall omit the international dimension of the pollution burden sharing problem and focus on the regional level. We treat the allowable emission quota allocated to the country as given and abstract from the world market of pollution permits.

[4]The statistical data cited in this paper come from *Belgostat* [1] and [7]. The energy consumption is expressed in the table in `boe`. A *barrel of oil equivalent* (`boe`) is a unit of energy based on the approximate energy released by burning one barrel of crude oil. One `boe` contains approximately 0.143 `toe` (a *ton of oil equivalent*) or $6.1178632 \cdot 10^9$ J or about 1.70 MWh. Using `boe` rather than any other energy unit has the advantage that the *price* of 1 `boe` is the price OPEC charges their clients.

Table II: Estimations of energy consumption and gross product generation *per unit of labour* in Belgium, about 2000

	Employment	Energy Use /Employment	Gross Value Added /Employment	GVA/Empl. /En. Use
		`boe`	Euros	Euro/`boe`
Belgium	4 085 677	66.10	54 780	0.0002
Brussels	640 992	22.51	66 401	0.0046
Wallonia	1 093 076	82.18	48 322	0.0005
Flanders	2 351 609	70.51	54 493	0.0003

Table II presents a slightly different picture. Here, the regional performance is expressed in relation to the size of labour, employed in each region. We see that Wallonia's usage of energy per worker is the highest and the revenue generated by a unit of labour is the lowest.[5] In the rest of this paper we will propose and analyse a few models to help politicians decide about the energy *sharing rules* that could lead to a socially preferred *equilibrium*.[6]

2.2 Model specification

In this paper we consider a country of two regional industries $i = w, f$ (we will use for $w =$Wallonia and for $f =$Flanders) that generate output according to a two-factor production function. Let e_i denote *energy* input per unit of labour in region i where[7] $e_i \in \mathbb{R}_+$; $e \equiv [e_i, e_{-i}] \in \mathbb{R}_+ \times \mathbb{R}_+$. One factor will be function $G_i(e_i)$ *concave* and *smooth*, dependent on energy used in sector i; the other factor $F_i(e_i, e_{-i})$, also a concave and smooth function, dependent on energy used in the whole country, will represent the effects of *learning-by-doing*, knowledge spill-overs, externalities' impact, interregional flows, *etc.*. Energy will be purchased by each sector at international price p.

If the regions are separated geographically (or constitute different political or ethnic "units") their decisions about the use of energy, feeding into the factors $G_i(\cdot), F_i(\cdot, \cdot)$, can be regarded as independent of one another. If, for various reasons, each region strives to maximise its Gross Regional *Revenue* (GRR) we should look for each region's input and output levels as results of an equilibrium solution to a non-cooperative two-agent (or, two-*region*) game. If, in addition, the amount of energy to be used by the entire country is restricted[8], a constraint needs to be added to the agents' joint strategy space. In result, the input and output will be

[5]The last column of Table II provides a rather complex and non standard measure of "efficiency", which is the revenue (in Euros) obtained out of one `boe` by one unit of labour. Here, Wallonia *may* appear more "economic" than the rest of the country in such per unit of labour terms. However, given the existing employment levels, it is impossible to claim on this basis that allowing Wallonia to use more energy would increase the whole country's revenue.

[6]As said in the Introduction, we are skeptical about the social planner's ability to enforce a Pareto efficient solution.

[7]Notation $_{-i}$ signifies the *other* player.

[8]*E.g.*, implied by a trade balance or the Kyoto Protocol.

determined as a *coupled constraints* equilibrium, see [21][9] .

We assume there is one identical good generated by each region and this good's price is normalised to 1. Below we propose a model for *regional revenue* Π_i, or the value added by region i that directly depends on this good's production. The two values sum up to the national revenue.

Each regional revenue (or net value-added) Π_i $i = w, f$ will be expressed in monetary units and modelled as the difference between the "gross" good's value, modelled as product $F_i(\cdot,\cdot)G_i(\cdot)$ (multiplied by price=1), and the cost of *energy* input e_i as follows:

$$\Pi_i(e) = \underbrace{F_i(e_i, e_{-i})}_{\text{spill-overs}, \, etc.} G_i(e_i) - p\, e_i \ . \tag{1}$$

One could consider several realisations of $G_i(\cdot)$ and $F_i(\cdot,\cdot)$. For example,

a. $F_i(e_i, e_{-i}) \equiv$ *constant* (no spill-overs, no externalities). This model would correspond to autarkic development of the regions; however it could also serve as a benchmark case, to isolate the elementary economic mechanisms of regional competition.

b. $F_i(e_i, e_{-i}) = (e_{-i})^{\delta_i}$ (strategic complementarities). This would be a simple formulation of strategic complementarities: region $-i$ exerts an externality on the other region. If $\delta_i > 0$ then the externality is positive.

c. $F(e_i, e_{-i}) = (e_i + e_{-i})^{\delta_i}$ (production spill-overs[10]). In essence the production of one region depends also of the total energy used by the country.

d. $G(e_i) = \alpha_i e_i^{\beta_i}$. Coefficient α_i is total factor productivity. If $F_i(\cdot)$ is like in (b.), $0 < \beta_i$, $0 < \delta_i$ and $\beta_i + \delta_i < 1$ then the first term ("output") of $\Pi_i(e)$ is a Cobb-Douglas production function with diminishing returns to scale.

e. $G(e_i) = \alpha_i \ln(e_i)$. To avoid $\ln(e_i) < 0$ we shall scale the model so that e_i will always check $e_i > 1$, see Section 4. In conjunction with $F_i(e_i, e_{-i}) =$ *constant* (see (a.)), the corresponding $\Pi_i(e)$ and $\Pi_{-i}(e)$ constitute (arguably) the simplest pair of the revenue functions that retain the production-function (strict) concavity feature.

Notice that (1) and the factor realisations (a.)-(e.) capture several basic facts (albeit with a different degree of accuracy) about regional economics that a game model should encapsulate. In particular, expression (1) says it is costly to use energy and all choices (a.)-(e.) reflect the fact that using energy increases output. On the other hand, in each realisation we have abstracted from labour. We assume that the variables are expressed in *per-unit-of-labour* terms.

We can summarise some features of the above choices as follows:

[9] Or see [10], [11], [16], [4], [13], [5].

[10] In this realisation, total factor productivity $F(e_i, e_{-i})$ is the same in both regions. This is so because it captures total non-appropriable knowledge derived from *all* productive activities in the economy, here proxied by the associated energy inputs. This specification is in the spirit of [20].

- if $\alpha_i > \alpha_{-i}$ then region i's total factor productivity is higher than region $-i$'s; this might suggest technology in region i is more energy efficient than that of region $-i$;
- if $\beta_i > \beta_{-i}$ then region i's output is more elastic to energy changes than region $-i$'s;
- if $\delta_i > \delta_{-i}$ than the externality produced by region $-i$ is more important for production of region i than the other way around.

To highlight the main features of energy usage rationalisation we will use a benchmark model that combines (a.) and (e.)

$$\Pi_i(e) = \alpha_i \, \ln(e_i) - p \, e_i \ . \tag{2}$$

This simple model will also help us to motivate the need for a numerical analysis of a more complicated model. It will be the combination of (c.) and (d.) as follows

$$\Pi_i(e) = \alpha_i \, (e_{-i})^{\delta_i} (e_i)^{\beta_i} - p \, e_i \ . \tag{3}$$

As said before, this choice captures the likely fact that region $-i$ (the "other" region) produces positive externality that feeds into the production of region i.

Let E_i denote the amount of energy used in region $i, \ i = w, f$. As e_i the amount of energy used by a unit of labour then

$$E_i = \eta_i e_i \tag{4}$$

where $\eta_i > 0$ is the quantity[11] of labour in region i.

Believing the sectors are "burning" (predominantly) oil to obtain energy, the emissions M_i can be assumed a linear function of energy E_i as follows

$$M_i = \kappa_i \, E_i \, , \quad \kappa_i > 0 \tag{5}$$

where κ_i characterises the "burning" technology of region i.

In case the whole country is striving to curb its emissions below $M > 0$ (where M could result from the Kyoto protocol) the maximisation of (1) (or (2) or (3)) needs to allow for the following constraint

$$M_i + M_{-i} \le M \quad \Rightarrow \quad \kappa_i \, E_i + \kappa_{-i} \, E_{-i} \le M \, . \tag{6}$$

If the emissions M_i generated by a unit of E_i were identical in each industry then $\kappa_i = \kappa_{-i} = \kappa$ and the constraint (6) could be rewritten as

$$\eta_i e_i + \eta_{-i} e_{-i} \le E \tag{7}$$

which is imposed upon the joint strategy space $\mathbb{R}_+ \times \mathbb{R}_+ \ni [e_i, e_{-i}]$ and where $E = \dfrac{M}{\kappa}$ is the energy available to the whole country.

[11] Variable e_i can be expressed in per-worker units, per hour, per worker·hour, *etc.*. The *quantity* η_i will correspond to the measure of e_i.

Given our assumption that the regions are "playing" a non-cooperative game, the optimal energy usage levels and the regional products can be obtained as a solution to a *coupled constraint* game defined as follows:

$$\left.\begin{array}{rll} \Pi_i(e^*) = & \max & \Pi_i(e_i, e^*_{-i}) \\ & e_i \quad \texttt{s.t.} & \eta_i e_i + \eta_{-i} e^*_{-i} \le E \qquad i = w, f \end{array}\right\} \tag{8}$$

where $e^* = [e^*_i,\, e^*_{-i}]$ and we wrote $\Pi_i(e_i, e^*_{-i})$ to stress that player i needs to allow for the optimal action of player $-i$, in deciding about her own optimal level e^*_i.

A solution to (8) is such that no region can improve its own payoff by a unilateral action without breaching (7). Hence, this solution is a "generalised" Nash-Cournot equilibrium as it is called in *e.g.*, [19], or a coupled constraint equilibrium as we call it. If it exists it depends on a vector of *weights* r_i, $(i = w, f)$ that can be viewed as a political instrument, which the central government can use to distribute the burden of satisfaction of the coupled constraint (7), among the players (regions or industries). We explain this concept in Section 3.

3 Constrained equilibria

3.1 Basic definitions and properties

An equilibrium defined by (8) is a *coupled constraint equilibrium*.

A coupled constraints equilibrium (CCE) is an extension of a standard Nash equilibrium in which players' strategy sets are allowed to depend upon other players' strategies. Coupled constraints equilibria are also known as generalised Nash equilibria. The competition between the regions subject to the energy constraint described as is an example of such a problem. Analytical solutions to CCE problems are not normally possible so Appendix A describes a numerical method for solving some such problems.

Coupled constraints equilibria are particularly useful in a class of problems where competing agents are subjected to regulation. Many electricity market and environmental problems belong to this class see *e.g.*, [10], [11], [4] where this concept has been applied in microeconomic contexts; see [8] for an international economics application. In general, CCE allows modelling of a situation in which the actions of one player condition how 'big' the actions of other players can be. Constraints in which the actions of one player do not affect the action space of another (as in Nash equilibrium problems) are called uncoupled.

In these games the constraints are assumed to be such that the resulting collective action set X is a closed convex subset of $\mathbb{R}^m$. If X_f is player-i's action set $(f = 1, 2, \dots F)$, $X \subseteq X_1 \times \cdots \times X_F$ is the collective action set (where $X = X_1 \times \cdots \times X_F$ represents the special case in which the constraints are uncoupled).

Allowing for the above, a solution to (8) can be explained as follows. Let the collective action $\mathbf{x}^*$ be the game solution and the players' payoff functions, Π_i, be continuous in all players' actions and concave in their own action. The Nash equilibrium can be written as

$$\Pi_i(\mathbf{x}^*) = \max_{\mathbf{x} \in X} \Pi_i(y_i | \mathbf{x}^*) \tag{9}$$

where $y_f|\mathbf{x}^* \in X$ denotes a collection of actions where the fth agent "tries" y_i while the remaining agents continue to play the collective action $\mathbf{x}^*$. Note that $\mathbf{x}^*$ is a column vector with elements x_g, $g = 1, 2, \ldots, f-1, f+1, \ldots, F$. At $\mathbf{x}^*$ no player can improve his own payoff through a unilateral change in his strategy so $\mathbf{x}^*$ is a Nash equilibrium point. If X is a closed and strictly convex set defined through coupled constraints (like (7)) then $\mathbf{x}^*$ is a CCE.

Games with coupled constraints rarely allow for an analytical solution and so numerical methods must be employed. In this paper, we will solve game (8), with the revenue functions defined by (3), using a method based on the Nikaido-Isoda function and a relaxation algorithm (hence the name: NIRA), as explained in Section 5.3.1.

Nonetheless, it is one thing to know that a numerical method to solve games with constraints exists, the other is to establish that the game has an equilibrium at all. Furthermore, since the NIRA algorithm will be shown to converge to a single equilibrium point it would be good to show that equilibrium is unique. We know from [21]that an equilibrium exists and is unique if the game is *diagonally strictly concave*. Below, we summarise the main ingredients needed for the confirmation that the constrained game (8) has a unique equilibrium; we also formulate existence and uniqueness theorem 3.1 for a two-player case, which we consider in this paper[12].

Denote $e \equiv [e_w, e_f]$, $r \equiv [r_w, r_f] \in \mathbb{R}^2_+$; let $\rho(e, r)$, the "combined payoff" (or *joint payoff function*) be

$$\rho(e, r) = r_w \Pi_w(e) + r_f \Pi_f(e)\,. \tag{10}$$

The pseudo-gradient $g(e, r)$ of $\rho(e, r)$ is

$$g(e, r) = \begin{bmatrix} r_w \dfrac{\partial \Pi_w(e)}{\partial e_w} \\ \\ r_f \dfrac{\partial \Pi_f(e)}{\partial x_f} \end{bmatrix}. \tag{11}$$

Definition 3.1. *The function $\rho(e, r)$ will be called* diagonally strictly concave *in e, if for every $e = (e_w, e_f)^T$ and $e' = (e'_w, e'_f)^T$, $e_w + e_f \leq E$ and fixed $r \in \mathbb{R}^2_+$, we have*

$$(e - e')^T g(e', r) + (e' - e)^T g(e, r) > 0, \tag{12}$$

where ^{T} means transposition.

We often call a game *diagonally strictly concave* whose joint payoff function is diagonally strictly concave.

Lemma 3.1. *If $\Pi_i(e), i = w, f$ are "enough" differentiable, a sufficient condition that $\rho(e, r)$ is diagonally strictly concave in e for fixed $r > 0$ is that the "pseudo-Hessian" symmetric matrix*

$$\mathcal{H}(e, r) = G(e, r) + G^T(e, r) \tag{13}$$

is negative definite for $e \in X$. Here the matrix $G(e, r)$ is the Jacobian with respect to e of gradient $g(e, r)$.

[12] The original Rosen theorem in [21] is valid for n players.

Theorem 3.1. *If $\rho(e, r)$ is diagonally strictly concave for some $r \in I\!R^2_+ \backslash\{0\}$, then the Nash equilibrium point of game (8) exists and is unique.*

We will apply this theorem to games considered in Sections 4 and 5.

3.2 Enforcement through taxation

Once a CCE, x^*, has been computed, it is possible to create an unconstrained ("decoupled") game that has x^* as its solution. This can be achieved through a simple modification to the players' payoff functions. If a regulator has computed x^* that is the CCE of a game, in which the agents' behaviour is *satisfactory*, then the regulator can implement a taxation scheme to induce the players to arrive at this solution. This can be accomplished by the use of penalty functions that punish the players for breaching the coupled constraints.

Penalty functions are weighted by the Lagrange multipliers obtained from the constrained game. For each constraint, players are taxed according to the function

$$T_{\ell,i}(\lambda, r_i, \mathbf{x}) = \frac{\lambda_\ell}{r_i} \max(0, Q_\ell(\mathbf{x}) - \overline{Q}_\ell) \tag{14}$$

where λ_ℓ is the Lagrange multiplier associated with the ℓth constraint and $Q_\ell(\mathbf{x})$ can be the amount of energy as described by the left hand side of (7). Symbol $\overline{Q}_\ell$, $\ell = 1, 2, \ldots L$ denotes the corresponding limits (L is the total number of constraints[13]); $\mathbf{x}$ is the vector of players' actions, r_i is player f's weight that defines their responsibility for the constraints' satisfaction.

If the weights r were identical $[1\,, 1\,, \ldots 1]$ then the penalty term for constraint ℓ is the same for each player f

$$T_{\ell,i}(\lambda, 1, \mathbf{x}) = \lambda_\ell \max(0, Q_\ell(\mathbf{x}) - \overline{Q}_\ell)\,. \tag{15}$$

Hence, if the weight for player f is for example $r_i > 1$ and the weights for the other players were 1, 1,... 1, then the responsibility of player f for the constraints' satisfaction is lessened.

The players' payoff functions, so modified, will be

$$\underline{\Pi}_i(\mathbf{x}) = \Pi_i(\mathbf{x}) - \sum_\ell T_{\ell i}(\lambda, r, \mathbf{x})\,. \tag{16}$$

Notice that under this taxation scheme the penalties remain "nominal" (*i.e.*, zero) if all constraints are satisfied.

The Nash equilibrium of the new unconstrained ("decoupled") game with payoff functions $\overline{\Pi}$ is implicitly defined by the equation

$$\underline{\Pi}(\mathbf{x}^{**}) = \max_{y_i \in I\!R^+} \underline{\Pi}(y_i|\mathbf{x}^{**}) \qquad \forall\, i, \tag{17}$$

(compare with equation (9)). For the setup of the problem considered in this paper $x^* = x^{**}$. That is, the CCE is equal to the unconstrained equilibrium with penalty functions for breaches of the constraints, weighted by the Lagrange multipliers (see [16], [13] and [14] for a more detailed discussion).

[13]Here, we have $\ell = L = 1$.

4 The benchmark problem

4.1 Existence and uniqueness of equilibrium

We will formulate and analyse a simple problem of energy apportioning between two regions with no externalities and no spill-over effects. We will also see that studying the solutions to this "simplest" game of requires some computational analysis.

Consider two "autarkic" regions $i = w, f$ with the revenue functions based on (2) (see page 7) *i.e.*,

$$\begin{aligned} \Pi_w(e) &= \ln(e_w) - p\, e_w && (18) \\ \Pi_f(e) &= \alpha_f \ln(e_f) - p\, e_f && (19) \end{aligned}$$

where, for simplicity, we assumed $\alpha_w = 1$. As said in Section 2.2, we will scale the model appropriately (see footnote 16) to avoid $e_i \leq 1$, $i = w, f$.

The regions face a joint constraint (compare (7))

$$e_w + \eta_f e_f \leq E \tag{20}$$

where, for simplicity, we assumed $\eta_w = 1$. Naturally $\eta_w \geq 0,\ \eta_f \geq 0$.

To claim equilibrium existence and uniqueness (see Theorem 3.1) we need to prove Lemma 3.1. Matrix $\mathcal{H}$ (see (13)) for game (8) with the revenue functions (18), (19) is

$$\mathcal{H}_1 = \begin{bmatrix} -\dfrac{r_w}{{e_w}^2} & 0 \\ 0 & -\dfrac{r_f \alpha_f}{{e_f}^2} \end{bmatrix}.$$

Clearly $\mathcal{H}_1$ is strictly negative definite. The constraint set determined by (20) and $e_w \geq 0,\ e_f \geq 0$ is convex. Hence the game defined by payoffs (18), (19) is *diagonally strictly convex.* Consequently, if we fix the weights $(r_w, r_f) \in \mathbb{R}^2_+$ and compute an equilibrium then the equilibrium is unique. The other theorems' results, proved in [21] for diagonally strictly convex games, which will be referred to below, are applicable to this game.

By definition, a coupled constraint equilibrium $(e^*_w, e^*_f, \lambda^*)$ is to be determined as the triple that satisfies:

$$\left.\begin{array}{rcrcrcl} \ln(e^*_w) - p\, e^*_w & \geq & \ln(e_w) & - & p\, e_w & + & \lambda_w\,(-e_w - \eta_f e^*_f + E) \\ \alpha_f \ln(e^*_f) - p\, e^*_f & \geq & \alpha_f \ln(e_f) & - & p\, e_f & + & \lambda_f\,(-e^*_w - \eta_f e_f + E) \end{array}\right\} \tag{21}$$

where $\lambda_w \geq 0,\ \lambda_f \geq 0$ and $\lambda_w(-e^*_w - \eta_f e^*_f + E),\ \lambda_f(-e^*_w - \eta_f e^*_f + E)$. From [21][14] we know that that for every concave game there exists a (Rosen-)*normalised* equilibrium point (e^*_w, e^*_f) with $\lambda^*_w = \dfrac{\lambda^*}{r_w}$, $\lambda^*_f = \dfrac{\lambda^*}{r_f}$ where λ^* is a *joint* Lagrange multiplier ("shadow" price) that corresponds to constraint (20).

[14]See Theorem 3 in [21].

If so and by the necessity of the Karush-Kuhn-Tucker conditions, in equilibrium $(e_w^*, e_f^*, \lambda^*)$ satisfy

$$\left.\begin{array}{rcl} -e_w^* - \eta_f e_f^* + E & \geq & 0 \\ \lambda^* & \geq & 0 \\ \lambda^*(-e_w^* - \eta_f e_f^* + E) & = & 0 \\ \dfrac{1}{e_w^*} - p - \lambda^* & = & 0 \\ \dfrac{\alpha_f}{e_f^*} - p - \eta_f \dfrac{\lambda^*}{r_f} & = & 0 \end{array}\right\} \tag{22}$$

where for simplicity we assumed $r_w = 1$.[15]

4.2 Properties of equilibrium

Given r_f, the coupled constraint equilibrium for the interesting case of $\lambda^* > 0$ is available albeit as a function of λ^*. The equilibrium energy usage[16] per worker is

$$e_w^* = \frac{1}{p + \lambda^*}, \qquad e_f^* = \frac{\alpha_f r_f}{r_f p + \eta_f \lambda^*} = \frac{\alpha_f}{p + \frac{\eta_f \lambda^*}{r_f}}\,. \tag{23}$$

However, these relations are functions of the equilibrium shadow price λ^*. (We notice that the marginal cost of violating the constraint is diminished r_f-times for the player whose $r_i \neq 1$.) To explicit the equilibrium shadow-price dependence on the problem parameters (including $r_f \neq 1$) one has to solve the following equation

$$\frac{1}{p + \lambda^*} + \frac{\alpha_f \eta_f}{p + \frac{\eta_f \lambda^*}{r_f}} = E \tag{24}$$

which is a quadratic equation in λ^*

$$\eta_f E (\lambda^*)^2 - (\eta_f + \eta_f r_f \alpha_f - Ep(\eta_f + r_f))\,\lambda^* - p r_f (1 + \eta_f \alpha_f - Ep) = 0\,.$$

[15]Rosen [21] analyses equilibria for weights $r_i \in \mathbb{R}_+$, $i = 1, 2, \ldots N$. We prove in Appendix B, for $N = 2$, equivalence between equilibria obtained for $[r_w, r_f] \in \mathbb{R}_+^2$, and when $r_w = 1$ and $r_f \in (0, \infty)$.

[16]For $e_i^* > 1$ so that we avoid negative output, $p + \lambda^* < 1$. To assure this result we can scale the model as follows: choose the energy unit so that price $p < 1$; then scale total factor productivity α_f (and/or the labour units) with respect to E so that the ratios appearing in (23) are larger than 1.

The roots of this equation are

$$\lambda_1^* = \frac{\eta_f(1+r_f\alpha_f) - Ep\,(\eta_f + r_f)}{2E\eta_f} \tag{25}$$

$$+ \frac{\sqrt{(\eta_f + \eta_f r_f \alpha_f)^2 - 2\,Ep\eta_f\left(\eta_f - r_f - \eta_f r_f \alpha_f + \alpha_f r_f^2\right) + E^2p^2\,(\eta_f - r_f)^2}}{2E\eta_f}$$

$$\lambda_2^* = \frac{\eta_f(1+r_f\alpha_f) - Ep\,(\eta_f + r_f)}{2E\eta_f} \tag{26}$$

$$- \frac{\sqrt{(\eta_f + \eta_f r_f \alpha_f)^2 - 2\,Ep\eta_f\left(\eta_f - r_f - \eta_f r_f \alpha_f + \alpha_f r_f^2\right) + E^2p^2\,(\eta_f - r_f)^2}}{2E\eta_f}$$

Any *exact* conclusions about the relationship between r_f and the positive root λ^* are parameter specific and would require simulation. We will solve numerically a more realistic game in Section 5 to analyse this relationship. Here, however, we can formulate Proposition 4.1 to demonstrate a few simple general properties, which the shadow price λ^* satisfies in coupled constraint equilibrium (described by (23) and (24)).

Proposition 4.1. *In the unique coupled constraint equilibrium of the benchmark bi-regional game (18)-(20), the constraint's shadow price λ^* possesses the following properties:*

a. if the amount of available energy E decreases, the shadow price λ^* increases;

b. if the price of energy p increases, then the shadow price λ^* decreases;

c. if r_f increases, which means a diminution of the responsibility of region f, λ^* increases.

d. if labour supply η_f or total factor productivity α_f increase, λ^* increases.

The proofs are elementary and based on the analysis of (24) re-written as

$$\frac{\alpha_f \eta_f}{p + \frac{\eta_f \lambda^*}{r_f}} = E - \frac{1}{p + \lambda^*}\,. \tag{27}$$

For example, (a.) follows from the observation that if λ^* decreased, rather than *increased* as is claimed in the proposition, then the right hand side of (27) would be negative. Similar reasoning proves the other items of the proposition.

Remark 4.1. *Notice that properties (a.)-(d.) are insufficient to claim that the privileged player's (here: region f whose $r_f > 1$) marginal cost of violating the constraint (see (14)) will always decrease for $r_f > 1$. For the claim to be true, $\frac{d\lambda^*}{dr_f} < 1$. We will examine a similar relationship in the game with externalities, solved numerically in the next section.*

Equilibrium revenue per worker (as a function of λ^*) is

$$\Pi_w^* = \ln\left(\frac{1}{p+\lambda^*}\right), \qquad \Pi_f^* = \alpha_f \ln\left(\frac{\eta_f \alpha_f}{p + \frac{\eta_f \lambda^*}{r_f}}\right) \tag{28}$$

and the country's (total) revenue

$$\Pi_w^* + \Pi_f^* = \ln\left(\frac{1}{p+\lambda^*}\right) + \alpha_f \ln\left(\frac{\eta_f \alpha_f}{p + \frac{\eta_f \lambda^*}{r_f}}\right) = \ln\left(\frac{1}{p+\lambda^*}\right)\left(\frac{\eta_f \alpha_f}{p + \frac{\eta_f \lambda^*}{r_f}}\right)^{\alpha_f}. \tag{29}$$

The latter *may* grow in r_f but only *if* the growth of λ^* is slower than the rise of r_f, see Remark 4.1. We will analyse the issue of improvements of $\Pi_w^* + \Pi_f^*$ due to the changes in r_w and r_f numerically in Section 5.

We also notice that, for this benchmark game *without* externalities or spill-over effects, the symmetric Pareto-optimal ("efficient") solution $(\bar{e}_w, \bar{e}_f, \bar{\lambda}) \geq 0$, which satisfies

$$\ln(\bar{e}_w) - p\,\bar{e}_w + \ln(\bar{e}_f) - p\,\bar{e}_f \geq \ln(e_w) - p\,e_w + \ln(e_f) - p\,e_f + \bar{\lambda}\,(-e_w - \eta_f e_f + E) \tag{30}$$

coincides[17] with e_w^*, e_f^*, λ^* given in (23) and (24).

5 A game with externalities

5.1 Uniqueness and equilibrium conditions

Here we consider two competitive regions that face a joint constraint (compare (7))

$$\eta_w\, e_w + \eta_f\, e_f \leq E \tag{31}$$

and whose outputs are enhanced by positive externalities feeding into the opponents' revenue functions as in (3) (see page 7) *i.e.*,

$$\Pi_i(e) = \alpha_i\,(e_{-i})^{\delta_i}(e_i)^{\beta_i} - p\,e_i\,, \quad i = w, f\;. \tag{32}$$

The pseudo-Hessian for game (8) with the revenue functions (32) is

[17]The symmetric Pareto-optimal solution is when the weights for each player's payoff are identical. Hence, we mean the coincidence with the game when $r_f = 1$.

$$\mathcal{H}_2 =$$

$$\begin{bmatrix} r_w\,\alpha_w\,{e_f}^{\delta_w}{e_w}^{\beta_w-2}\beta_w\,(\beta_w-1) & \dfrac{r_f\,\alpha_f\,{e_w}^{\delta_f}\delta_f\,{e_f}^{\beta_f}\beta_f + r_w\,\alpha_w\,{e_f}^{\delta_w}\delta_w\,{e_w}^{\beta_w}\beta_w}{2\,e_w\,e_f} \\ \dfrac{r_f\,\alpha_f\,{e_w}^{\delta_f}\delta_f\,{e_f}^{\beta_f}\beta_f + r_w\,\alpha_w\,{e_f}^{\delta_w}\delta_w\,{e_w}^{\beta_w}\beta_w}{2e_w\,e_f} & r_f\,\alpha_f\,{e_w}^{\delta_f}{e_f}^{\beta_f-2}\beta_f\,(\beta_f-1) \end{bmatrix}$$

We can clearly see that $\mathcal{H}_2^{1,1} < 0$. However, to determine the diagonal strict concavity of the game we also need $\det \mathcal{H}_2 > 0$. This result appears parameter dependent but will be satisfied at least for "large" e_w and e_f. Notice that the product $\mathcal{H}^{1,2}\mathcal{H}^{2,1}$, to be subtracted from $\mathcal{H}^{1,1}\mathcal{H}^{2,2} > 0$ (to compute the determinant) is vanishing for large e_w and e_f; this is so because the powers of e_w and e_f in the denominator are greater than in the numerator.

We will check later in Section 5.3 (*i.e.*, after calibration) that $\mathcal{H}_2$ is negative definite. Hence, if we fix the weights r_w, r_f and compute an equilibrium for game (8) this equilibrium is unique (certainly, for some "large" levels of energy consumed).

As in Section 4, a coupled constraint equilibrium is the triple $(e_w^*, e_f^*, \lambda^*)$ that satisfies:

$$\left.\begin{array}{rcllcl} \alpha_w\,(e_f^*)^{\delta_w}(e_w^*)^{\beta_i} - p\,e_w^* & \geq & \alpha_w\,(e_f^*)^{\delta_w}(e_w)^{\beta_i} - & p\,e_w & + & \lambda_w\,(-\eta_w e_w - \eta_f e_f^* + E) \\ \alpha_f\,(e_w^*)^{\delta_f}(e_f^*)^{\beta_i} - p\,e_f^* & \geq & \alpha_f\,(e_w^*)^{\delta_w}(e_f)^{\beta_i} - & p\,e_f & + & \lambda_f\,(-\eta_w e_w^* - \eta_f e_f + E) \end{array}\right\} \tag{33}$$

where $\lambda_w \geq 0,\ \lambda_f \geq 0$ and $\lambda_w(-e_w^* - \eta_f e_f^* + E), \lambda_f(-e_w^* - \eta_f e_f^* + E)$. As in Section 4, we invoke the results obtained in [21]. In particular we look for an equilibrium point (e_w^*, e_f^*) with $\lambda_w^* = \dfrac{\lambda^*}{r_w},\ \lambda_f^* = \dfrac{\lambda^*}{r_f}$ where λ^* is a *joint* Lagrange multiplier ("shadow" price) that corresponds to constraint (31).

If so and by the necessity of the Karush-Kuhn-Tucker conditions, the triple $(e_w^*, e_f^*, \lambda^*)$ has to satisfy in equilibrium

$$\left.\begin{array}{rcl} -\eta_w e_w^* - \eta_f e_f^* + E & \geq & 0 \\ \lambda^* & \geq & 0 \\ \lambda^*(-\eta_w e_w^* - \eta_f e_f^* + E) & = & 0 \\ r_w\,\alpha_w\,{e_f}^{\delta_w}{e_w}^{\beta_w-1}\beta_w - r_w\,p - \eta_w\lambda^* & = & 0 \\ r_f\,\alpha_f\,{e_w}^{\delta_f}{e_f}^{\beta_f-1}\beta_f - r_f\,p - \eta_f\lambda^* & = & 0 \end{array}\right\}. \tag{34}$$

We can express the equilibrium strategies (e_w^*, e_f^*) as functions of λ^* for the interesting case of $\lambda^* > 0$. However, after the substitution of the strategies in the energy balance condition ($\eta_w e_w^* + \eta_f e_f^* = E$) the resulting equation is substantially "more" nonlinear than (24). It appears that its analytical solution is unavailable. We will solve the coupled constraint equilibrium problem (34) numerically using NIRA in Section 5.3.

5.2 A calibrated model

Given the data in Table II, we would like to establish plausible values for the 6 parameters ($\alpha_i, \beta_i, \delta_i$ for the two regions $i = w, f$) that characterise the regional revenue functions (32) (or (3)). The two functions (32) constitute two conditions that the parameters have to satisfy for a given value of price p.

We assume that the joint constraint (31) was not binding in 2000 *i.e.*, in the year for which the data were collected in Table II. Since we claim that the regions are "game players", we have the following two first-order *non*-coupled-constraint-equilibrium conditions

$$e_w^* = \left(\frac{\beta_w\,\alpha_w}{p}\right)^{\frac{1}{1-\beta_w}} (e_f^*)^{\frac{\delta_w}{1-\beta_w}} \tag{35}$$

$$e_f^* = \left(\frac{\beta_f\,\alpha_f}{p}\right)^{\frac{1}{1-\beta_f}} (e_w^*)^{\frac{\delta_f}{1-\beta_f}}\,, \tag{36}$$

which the coefficients also need to satisfy. Consequently, we have 4 equations in 6 variables.

Additionally, we have constraints on non-negativity of all parameters, [0, 1] membership of the exponents and that $\beta_i > \delta_i$. This means that we have *not-so-much* freedom in choosing the parameters. A Matlab constrained minimisation function[18] was used to minimise the sum of deviations (squared, weighted) between the computed and historical revenues and between the postulated equilibrium energy consumptions and the historical consumptions. A solution (without *any* claim of uniqueness) is presented in Table III.

Table III: Parameters of regional revenue functions

	Wallonia	Flanders
α	40592	43085
δ	0.002359	0.025848
β	0.048545	0.037365

We conjecture that the parameter values in Table III, *can* represent a stylised regional competition problem. As said, we make no claim on any sort of uniqueness of theses parameters. (In particular a change of unites could diminish the values of α_i but we will stick to the "natural" units: Euro and `boe`.)

The *calibrated* game with the revenue functions (32) and constraint (31) is particular in several respects and so will be the numerical solutions to the game, presented below. For example, $\delta_f > \delta_w$ suggests that Flanders relies on the positive externality that Wallonia produces more than the other way around[19]; the inequality $\beta_f < \delta_w$ jointly with $\alpha_f > \alpha_w$ may reflect better efficiency of Flanders' use of energy, albeit for a finite range. In brief, we believe that the calibrated model used

[18] `fmincon`.

[19] Traditionally, Wallonia was a coal and steel producer. Perhaps $\delta_f > \delta_w$ captures Flanders' reliance on those products.

in the rest of this paper enables us to analyse coupled constraint equilibria for a *case study*, which the model represents.

Notice that if $p = 30$ EURO/boe the revenue functions (32) reproduce the statistical data displayed in Table II:

$$40592(70.51)^{0.002359} \cdot (82.18)^{0.048545} - 30 \cdot 82.18 = 48\,332 \quad (37)$$
$$43085(82.18)^{0.025848} \cdot (70.51)^{0.037365} - 30 \cdot 70.51 = 54\,493 \quad (38)$$

5.3 Numerical solutions

We now come the numerical set-up. As announced in Section 3, we rely on the NIRA technique to solve for the coupled constrained equilibria. Needless to say, we have checked the existence-uniqueness condition (i.e. the determinant of $\mathcal{H}_2 > 0$ for the adopted parameter values) in all our computations.

5.3.1 The algorithm

The algorithm relies on the Nikaido-Isoda function which transforms the complex process of solving a (constrained) game into a far simpler (constrained) optimisation problem.

Definition 5.1. *Let Π_i be the payoff function for player i, X a collective strategy set as before and $r_i > 0$ be a given weighting*[20] *of player i. The* Nikaido-Isoda *function $\Psi : X \times X \to I\!R$ is defined as*

$$\Psi(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{F} r_i[\Pi_i(y_i|\mathbf{x}) - \Pi_i(\mathbf{x})] \quad (39)$$

Notice that (see [22]).

$$\Psi(\mathbf{x}, \mathbf{x}) \equiv 0 \qquad \mathbf{x} \in X. \quad (40)$$

Each summand from the Nikaido-Isoda function can be thought of as the improvement in payoff a player will receive by changing his action from x_i to y_i while all other players continue to play according to $\mathbf{x}$. Therefore, the function represents the sum of these improvements in payoff. Note that the *maximum* value this function can take, for a given $\mathbf{x}$, is always nonnegative. The function is everywhere non-positive when either $\mathbf{x}$ or $\mathbf{y}$ is a Nash equilibrium point, since in an equilibrium situation no player can make any improvement to their payoff. Consequently, each summand in this case can be at most zero at the Nash equilibrium point (see [16]).

When the Nikaido-Isoda function cannot be made (significantly) positive for a given $\mathbf{y}$, we have (approximately) reached the Nash equilibrium point. This observation is used to construct a termination condition for the relaxation algorithm,

[20] The weights were introduced in the context of formula (8), see the discussion after this formula on page 8. As said, they can be viewed as a political instrument the regulator might use to apportion the responsibility to the generators for the joint constraints' satisfaction (which might be distributed over periods, if the game was dynamic, see [6]).

which is used to min-maximise Ψ. An ε is chosen such that, when

$$\max_{\mathbf{y}\in\mathbb{R}^m} \Psi(\mathbf{x}^s, \mathbf{y}) < \varepsilon, \tag{41}$$

(where $\mathbf{x}^s$ is the s-th iteration approximation of x^*) the Nash equilibrium would be achieved to a sufficient degree of precision [16].

The *relaxation algorithm* can be forumlated as follows. Let the vector $Z(\mathbf{x})$ gives the 'best move' of each player when faced with the collective action $\mathbf{x}$.

Definition 5.2. *The* optimum response function *at point* $\mathbf{x}$ *is*

$$Z(\mathbf{x}) \in \arg\max_{\mathbf{y}\in X} \Psi(\mathbf{x}, \mathbf{y}). \tag{42}$$

The relaxation algorithm iterates the function Ψ to find the Nash equilibrium of a game. It starts with an initial estimate of the Nash equilibrium and iterates from that point towards $Z(\mathbf{x})$ until no more improvement is possible. At such a point every player is playing their optimum response to every other player's action and the Nash equilibrium is reached. The relaxation algorithm, when $Z(\mathbf{x})$ is single-valued, is

$$\mathbf{x}^{s+1} = (1-\alpha_s)\mathbf{x}^s + \alpha_s Z(\mathbf{x}^s) \qquad 0 < \alpha_s \leq 1, \quad s = 0, 1, 2, \ldots \tag{43}$$

From the initial estimate, an iterate step $s+1$ is constructed by a weighted average of the players' improvement point $Z(\mathbf{x}^s)$ and the current action point $\mathbf{x}^s$. Given concavity assumptions (see [22], [16]), this averaging ensures convergence to the Nash equilibrium by the algorithm. By taking a sufficient number of iterations of the algorithm, the Nash equilibrium $\mathbf{x}^*$ can be determined with a specified precision.

We observe that the "sum of improvements" in Ψ (39) depends on the weighting vector $r = (r_i)_{i\in F}$. Consequently, a manifold of equilibria indexed by r is expected to exist. However, for a *given* r and diagonal strict concavity of $\sum_{i\in F} r_i\Pi_i(x_i)$, uniqueness of equilibrium $\mathbf{x}^*$ is guaranteed, see [21] and [11]. In particular, Theorem A.1 in the appendix shows that NIRA converges to the unique equilibrium, for the value of r that was used in the definition of Ψ.

5.3.2 Sharing the constraint's burden in solidarity or the "status quo" solutions

We are interested to know how the regions respond to the imposition of the energy constraint (31). We will assume that the revenue functions' coefficients proposed in Table III do not depend on the energy use and solve the coupled constraint game (34) numerically for several values of E. We will keep the price p constant for these experiments.[21]

[21] In real life, a higher price of energy would cause the value added to increase. Our regional revenue model (32) does not allow for this effect so, the assumption of a constant price might correspond to the value added expressed in constant prices. Also, should the price increase substantially the constraint would not be binding.

In this section we examine the regions' reaction to the imposition of the energy constraint under the assumption that they share the responsibility for the constraint's satisfaction in solidarity *i.e.*, the weights are $r_w = r_f = 1$. To see the scope for the regulator's interventions we will analyse this *competitive* solution against the symmetric Pareto-*optimal* ("efficient") solution.

In the figures that follow the solid lines correspond to the former while the dotted lines represent the latter. We will use an upper bar $^-$ to denote the Pareto optimal solutions; asterisk * will be used for coupled constraint equilibrium solutions.

Figure 1, scaled in Euro *per-unit-of-labour*, and Figure 2, in Euro, show how the regional and national revenues change when energy constraints are introduced. The horizontal axis represents the energy availability so, tightening of the constraint corresponds to "moving" from right to left.

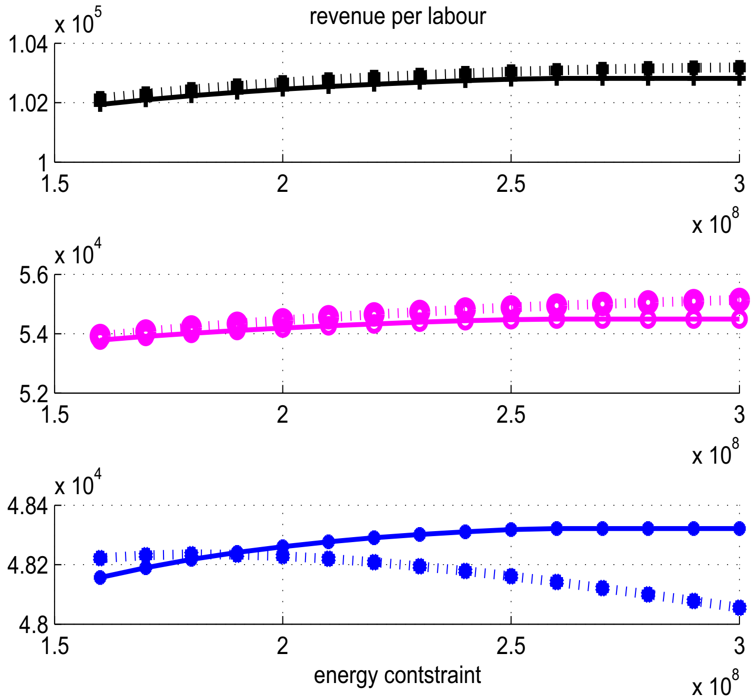

Figure 1: Revenue per unit of labour as a function of the energy constraint.

The bottom panel in each figure corresponds to the value added of Wallonia, the middle one represents the Flanders' value and the top graph is the sum of both and is meant to describe Belgium's (total) national revenue[22].

[22]As said in Section 2.1 we neglect the contributions of Brussels.

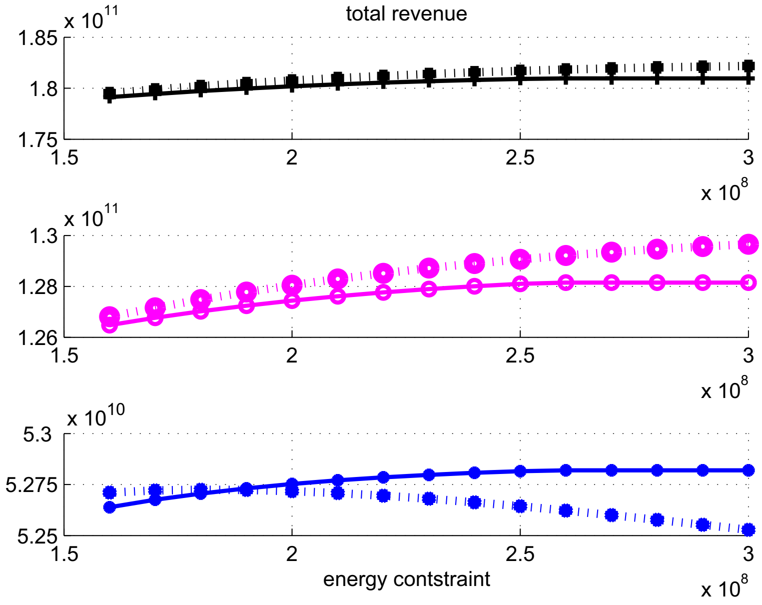

Figure 2: Total revenue as a function of the energy constraint.

As expected, all revenues diminish if the available energy decreases. However, under the efficient solution, Wallonia is supposed to increase its contributions to the national revenue. This would be a result of assigning Wallonia a higher share of the energy consumption, as shown in Figure 3. (We notice that the constraint becomes active at $E = 2.5 \cdot 10^8$ for the competitive solutions. The efficient solutions rely always on the whole energy quota, see Figure 5.)

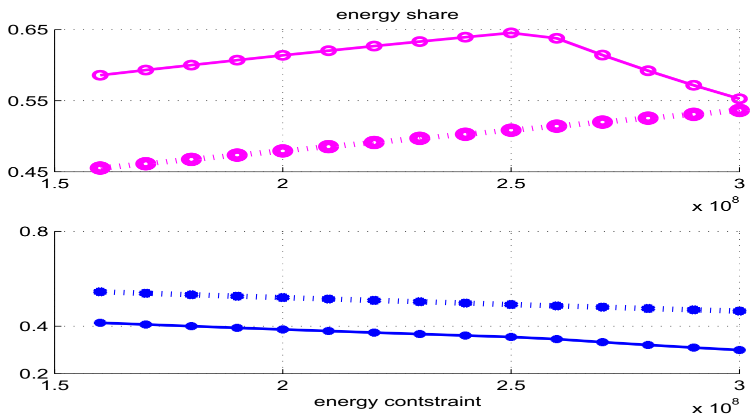

Figure 3: Wallonia's and Flanders' energy usage shares as functions of the energy constraints.

A higher energy share apportioned to Wallonia might be "required" for the revenue maximisation because Wallonia's positive externality is "needed" for Flanders' output. Also, notice that under the Pareto optimal solution, Flanders' revenue per-unit-of-labour *per-energy-input*, see Figure 4, is growing faster than Wallonia's. This appears to compensate a possible decrease of Flander's output due to a more favourable treatment of Wallonia.

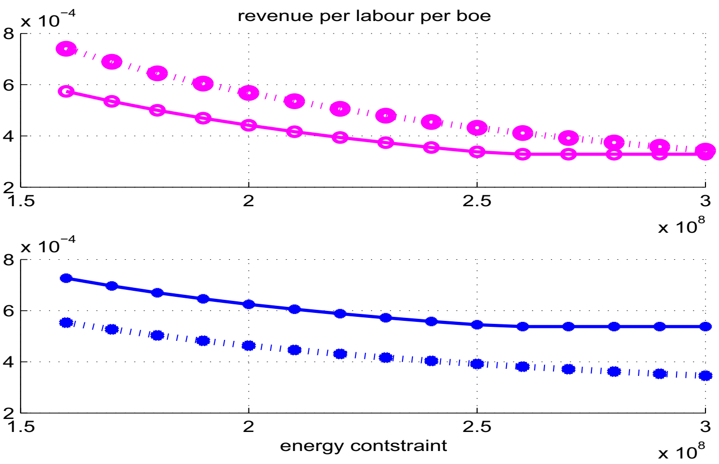

Figure 4: Wallonia's and Flanders' revenue generation efficiency as functions of the energy constraints.

It is interesting to notice that the competitive solutions are not only inefficient regarding the revenues they can generate but also in terms of the constraint's saturation. Figure 5 shows the constraint's slacks as the constraints are tightened. It is clear that the constraint is not binding for $E > 250\,000\,000$ boe for the competitive solutions. Conversely, the regions always work to their "full capacity" if a Pareto optimal solution is implemented, see the dotted line at the level of zero.

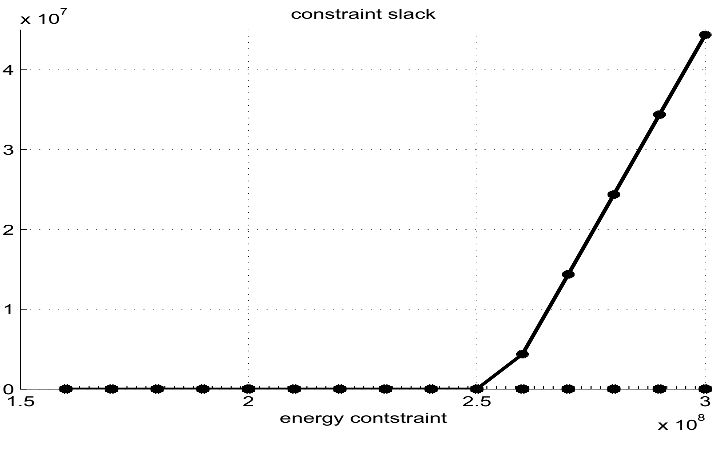

Figure 5: The constraint's slack as a function of tightening the constraints.

Finally, it is interesting to compare the shadow prices for the competitive and efficient solutions. Figure 6 documents that higher prices are needed to support the Pareto symmetric optimal solution, than for a game equilibrium.

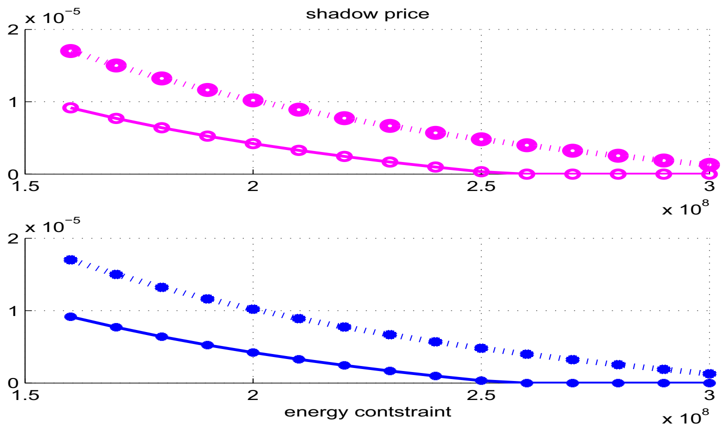

Figure 6: The shadow prices as functions of the energy constraints.

The observation about high shadow prices required for more socially acceptable outcomes will be exploited in the following section. By increasing one weight (r_w or r_f) the responsibility for the constraint's satisfaction will be lessen for one region. This will encourage this region to consume more energy. If this is the "right" region, the total revenue might increase in a new equilibrium.

We will see that an unequal weighting scheme $r_w = 1, r_f \neq 1, (r_f > 0)$ can produce a higher common-constraint shadow price and help the equilibrium become closer to the optimal solution.

5.3.3 Asymmetrical sharing rules

Here, we compute the regions' reactions to imposition of an energy constraint when the responsibility for the constraint's satisfaction is distributed unevenly. In particular, we will construct constrained equilibria when the marginal cost of violating the energy constraint for Flanders will be weighted by a series of $\frac{1}{r_f}$ as follows

$$\frac{1}{r_f} := 3,\ 2,\ 1.5,\ 1,\ \frac{3}{4},\ \frac{1}{2},\ \frac{1}{3}\,.$$

Obviously, these weights correspond to

$$r_f \ := \frac{1}{3},\ \frac{1}{2},\ \frac{3}{4},\ 1,\ 1.5,\ 2,\ 3\,.$$

The other region's weight[23] will be kept $r_w = 1$.

[23] See footnote 15 and Appendix B.

Most of the results in this section will be presented in three-dimensional spaces where the first dimension is $r_f \in \left\{\frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1, 1.5, 2, 3\right\}$ and the second dimension is the available energy $E \in [1.5,\ 3] \times 10^8$`boe`. The variable of interest will be presented in the third dimension.

Inspired by the observation that the shadow price for the symmetric Pareto optimal solution dominates the equilibrium shadow price (see Figure 6) we will examine whether varying r_f can indeed generate λ^* that would resemble $\bar{\lambda}$.

To help interpret the following 3D graphs we first show how the Pareto solution shadow price known from 6 can be represented in 3D, see Figure 7. In essence, this is the top *line* in Figure 6 (either panel) shown as a *surface* where all iso-lines are parallel to r_f (first dimension).

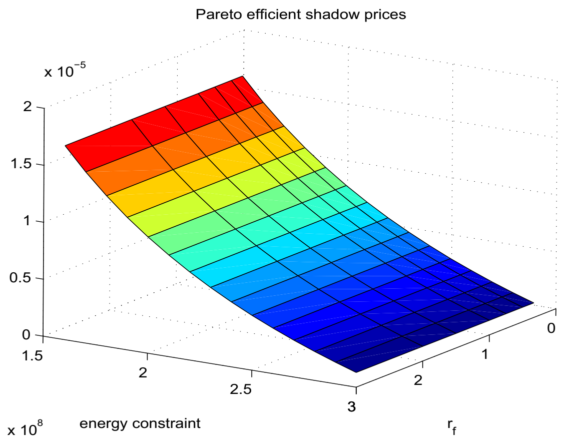

Figure 7: The symmetric Pareto optimal shadow prices in 3D.

This surface could be added as a "ceiling" to the following 3D graphs in Figure 8. However, we will not "cover" this figure, neither will add ceilings to the other graphs, as this could blur the analysis. Instead, we will make occasional references to the respective symmetric Pareto solutions, represented by the dotted lines in Figures 2- 6.

What we show in Figure 8 is how the equilibrium shadow prices λ^* change in r_f, and in E, for Wallonia (left panel) and Flanders (right panel). The red dotted lines represent the symmetric equilibrium shadow prices (*i.e.*, $r_f = 1$ as in Figure 6).

The shadow price for Flanders is $\frac{\lambda^*}{r_f}$ while for Wallonia it is just λ^*. We can clearly see that λ^* increases as r_f rises. A comparison between the values reached by $\bar{\lambda}$ in Figure 7 and λ^* in Figure 8 (left panel) suggests that varying r_f might diminish the difference $\bar{\lambda} - \lambda^*$. Consequently, some equilibria might be socially more desirable than some other equilibria. We will verify this conjecture by examining the total revenue for the country as a function of r_f (see Figure 9).

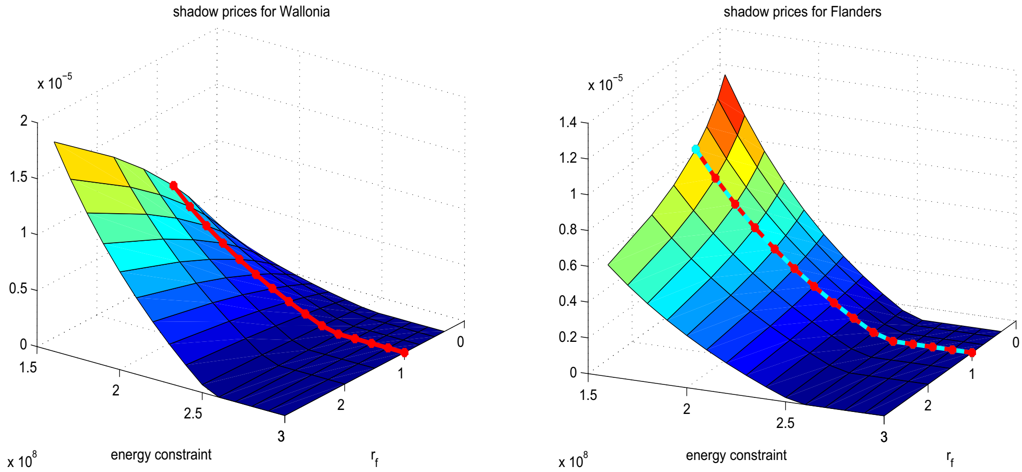

Figure 8: The equilibrium shadow prices as functions of the weight r_f and the energy constraints.

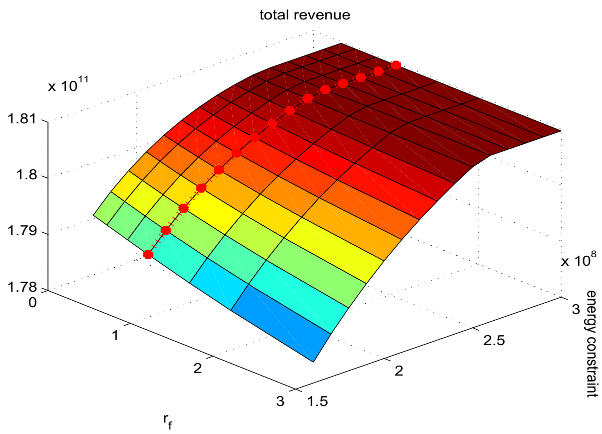

Figure 9: Total revenue of the country as a function of the weight r_f and the energy constraints.

We can see in Figure 9 that the total revenue increases in $\dfrac{1}{r_f}$. This corresponds to shifting the responsibility for the constraint's satisfaction away from Wallonia and allowing it to consume more energy. In the context of higher energy efficiency of Flanders *vis-a-vis* Wallonia, this might be a surprising conclusion.

However, Figure 10 shows that decreasing $r_f \in [0.005, 3]$ improves total revenue[24] $\Pi_w(\cdot) + \Pi_f(\cdot)$ of the entire country. Indeed, this revenue achieved as the sum of the regional equilibrium revenues for $r_f = 0.05$ is "efficient" *i.e.*, equal to

[24]Remember that $\Pi_w(\cdot), \Pi_f(\cdot)$ are revenues per unit of labour, see the explanation after item (e.) on page 6.

the symmetric Pareto optimal revenue (*i.e.*, achieved for $\alpha = 0.5$). We conjecture that while there may be pairs of $(\Pi_w^*(r_f), \Pi_f^*(r_f))$ that correspond to other Pareto-efficient outcomes, the map $\alpha \rightsquigarrow r_f$ is not necessarily 1:1, see [15].

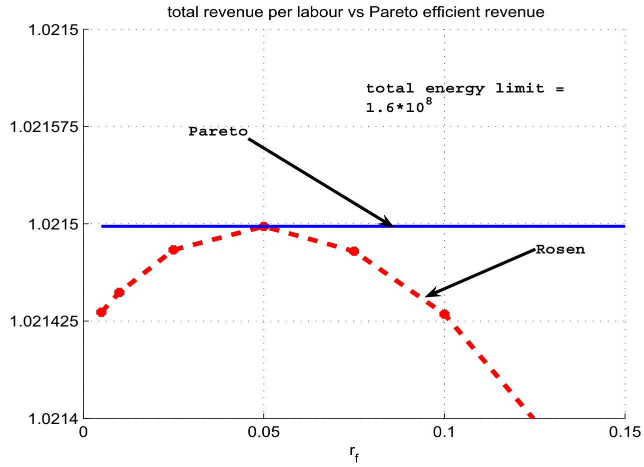

Figure 10: Total revenue per unit of labour as a function of the weight r_f.

Let us now examine what outcomes are caused by the variation of r_f at the regional level.

Figure 11 shows the regional revenues' dependence on r_f and E. We observe that the preferential treatment of Wallonia (small r_f) suits both regions well, albeit Wallonia appears to gain more.

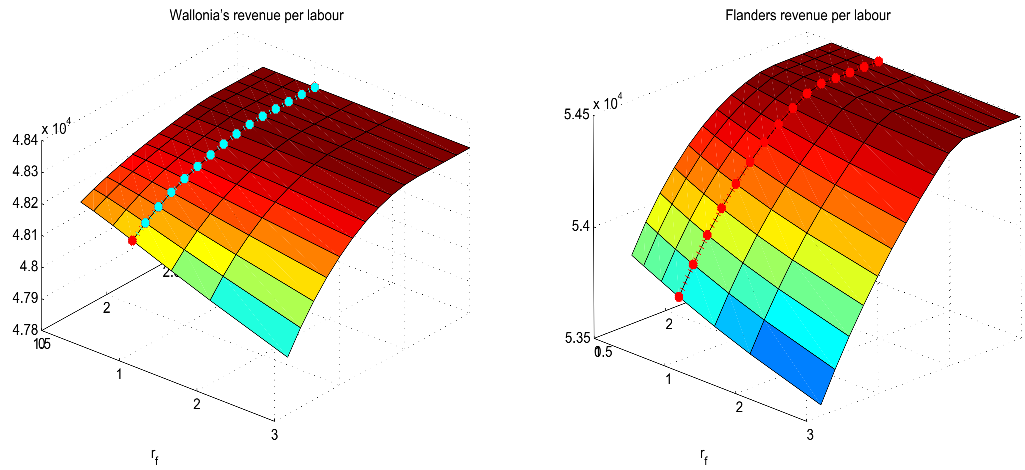

Figure 11: The regional revenues per unit of labour as functions of the weight r_f and the energy constraints.

Of great interest is to examine the regional equilibrium strategies, which lead to the above revenue outcomes. Figure 12 shows the strategic decisions of how much energy should be consumed per *unit of labour* in regions.

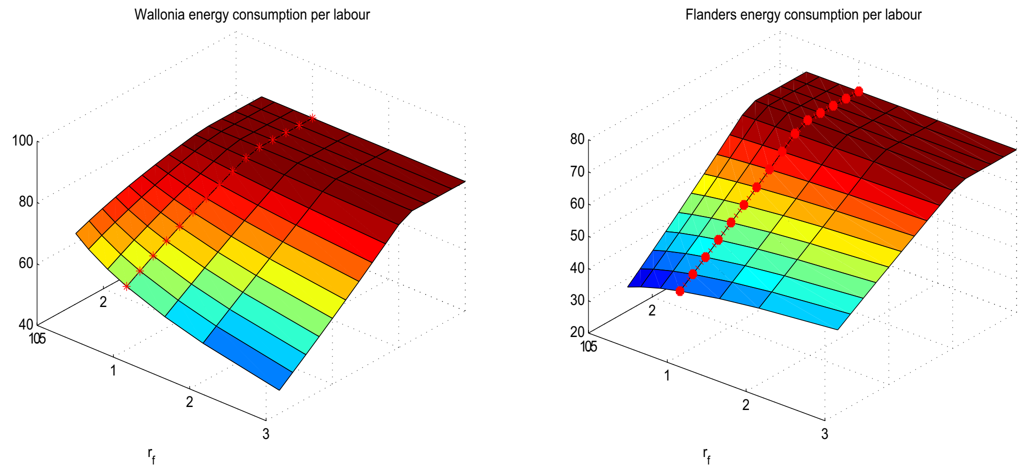

Figure 12: The regional strategies boe/*labour* as functions of the weight r_f and the energy constraints.

Here we can see that increasing weight r_f encourages Flanders to use more energy while lowering it pushes Wallonia to consume more. These tendencies are even more visible when we observe the energy consumption shares in Figure 13.

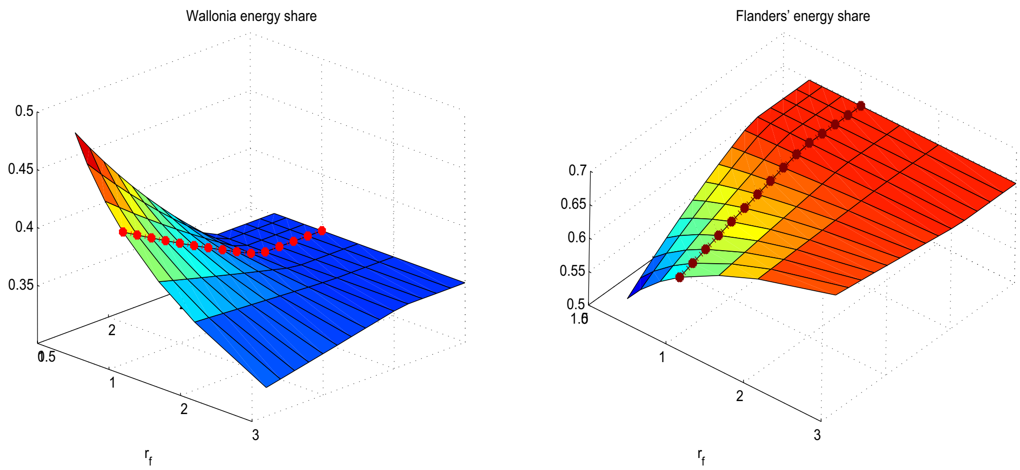

Figure 13: The regional energy consumption shares as functions of the weight r_f and the energy constraints.

Finally we can remark that the coupled constrained equilibria are "efficient" in that the slack on the constraint is zero, see Figure 14.

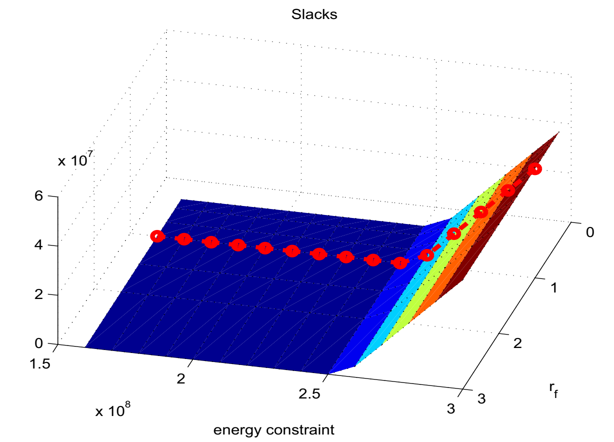

Figure 14: Constraint slack as a function of the weight r_f and the energy constraints.

5.3.4 A variation in the reliance on externality

In our model, the existence of inter-regional externalities is a fundamental ingredient of the problem: depending on the size of the positive externality exerted by the energy-less-efficient region on the more efficient one, the latter should be apportioned a higher or a lower energy share.

Compare the following figures (Figures 15 and 16), obtained for a decreased reliance of Flanders on the positive externality produced by Wallonia ($\delta_f = .002$), with Figures 9 and 11.

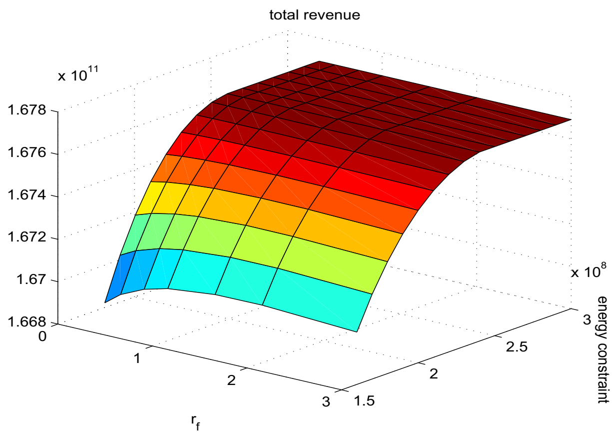

Figure 15: Total revenue of the country as a function of the weight r_f and the energy constraints.

We can clearly see that, now, a preferential treatment of Flanders (*i.e.*, r_f increases) results in higher energy consumption by this region. In consequence, the

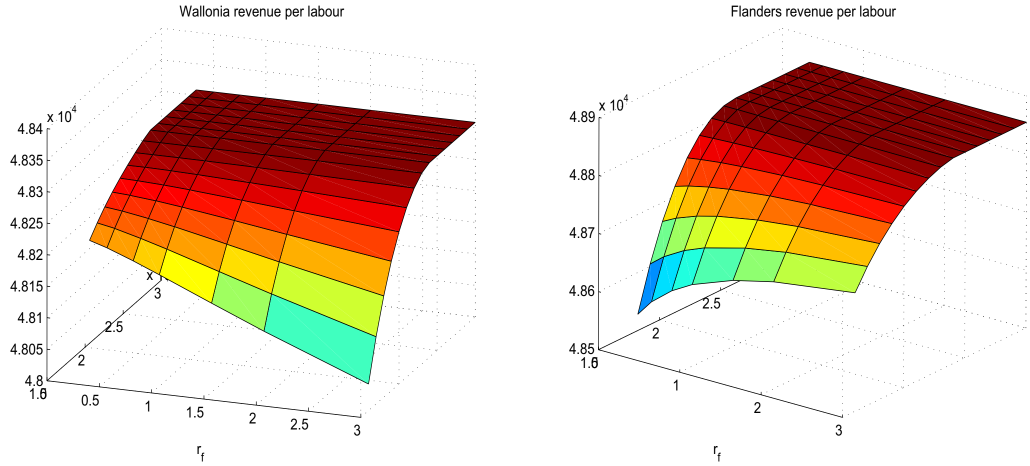

Figure 16: The regional revenues per unit of labour as functions of the weight r_f and the energy constraints.

maximum of $\Pi_w(\cdot) + \Pi_f(\cdot)$ in r_f is not so heavily skewed in favour of Wallonia; we refer to Figures 11 and 10 where the maximum is achieved for $r_f \approx 0.05$.

Figure 17 shows that $r_f = 1$ maximises total revenue $\Pi_w(\cdot)+\Pi_f(\cdot)$. Indeed, this revenue is "efficient" *i.e.*, equal to a Pareto optimal revenue, maximised for $\alpha = 0.5$ (symmetric).

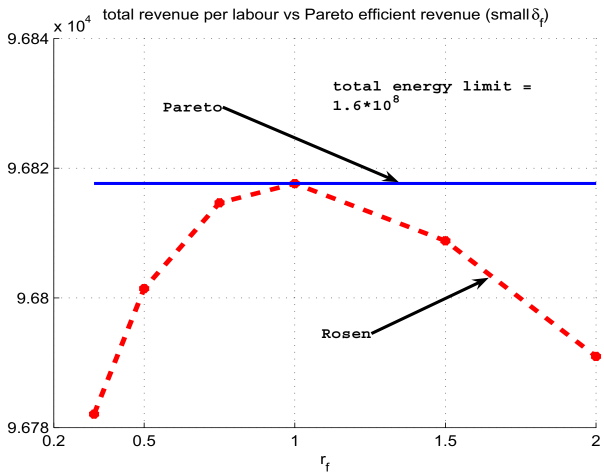

Figure 17: Total revenue per unit of labour as a function of the weight r_f.

We wrap up this section by saying that the regulator can produce an array of equilibria, which depend on r_f, for a given level of externality reliance. For the considered model, these equilibria include one that generates the revenues, which equal the symmetric (*i.e.*, $\alpha = 0.5$) Pareto efficient ones.

6 Concluding remarks

We have proposed a novel methodological approach to regional cost sharing of environmental regulation. The framework is game theoretic, based on the concept of coupled constraint equilibrium, allowing us to formulate *naturally* an important policy problem of national governments in multi-regional countries. The problem is timely and concerns the implementation of international agreements like the Kyoto Protocol. In particular, there are two specific questions that our model helps answer: how to efficiently share the burden of environment regulations (like emissions quotas) across regions? And, how to enforce such a sharing?

The problem is particularly acute when there exist significant structural differences across regions. In the case considered in our paper, regions may differ in their energy efficiency. For example, a region may be (for many good reasons) much more energy intensive than another region(s). If the national government has to allocate emission permits across the regions, what could be the most efficient sharing rule for the country?

In order to give substance to this discussion, we have considered the case Wallonia *vs.* Flanders. Wallonia is traditionally significantly more energy intensive than Flanders while the contribution of the latter to Belgian GDP is clearly larger.

It could be thought that having to enforce a national pollution norm, in accordance with international agreements, the regulator should penalise the more polluting, or *deviating*, region, especially if its contribution to national wealth is markedly lower than that of the less polluting region(s). This is clearly the case of Wallonia in Belgium. Our paper makes a point in this respect: the reasoning that leads to limiting Wallonia's energy use, does not take into account the fact that regions do interact in several meaningful ways such that penalising the more deviating region (from an energy-efficiency norm) may turn out to be inefficient in terms of the joint production maximisation. In our model, the existence of inter-regional externalities is a fundamental ingredient of the story. We surmise that the decision of apportioning the higher, or the lower, energy share to the more efficient, or *disciplined*, region must depend on the size of the positive externality exerted by the more deviating region on the former.

Hence, there is no simple theorem for efficient regulation of cost sharing across regions. One has not only to look at the differences in factor intensity but also to scrutinise the economic interactions between regions, which is far from easy. Even if one restricts these interactions to inter-regional technological spillovers, the issue is not so simple since a substantial part of these spillovers is intangible. Our analysis points at a further and more political ingredient: the government may choose an uneven distribution (across regions) of the responsibility for the joint constraint satisfaction to force a particular outcome.

Our paper shows clearly that the shape of equilibria identified and the corresponding national revenues tightly depend on the parameter r_f. This opens a further important line of research: what could be an *optimal*[25] value of r_f? Our numerical analysis sheds light on some particular properties of our model in this

[25]In "real-life", the government might add other criteria to maximisation of the *current* total revenue when choosing a value of r_f. For example, r_f may need to be greater than one resulting from Figures 2 and 9, if the government wanted region w to *restructure*.

respect. Our ambition is to provide a more general appraisal using less specific models, which seems to us crucial in the design of environmental regulation policies.

Acknowledgment

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Appendix

A Convergence of NIRA

A weakly convex-concave function is a bivariate function that exhibits weak convexity in its first argument and weak concavity in its second argument. The next three definitions (see [18] or [22]) formalise this notion.[26] As Theorem A.1 (the convergence theorem) will document, weak convex-concavity of a function is a crucial assumption needed for convergence of a relaxation algorithm to a coupled constraints equilibrium.

Let X be a convex closed subset of the Euclidean space $\mathbb{R}^m$ and f a continuous function $f : X \to \mathbb{R}$.

Definition A.1. *A function of one argument $f(\mathbf{x})$ is* weakly convex *on X if there exists a function $r(\mathbf{x}, \mathbf{y})$ such that $\forall \mathbf{x}, \mathbf{y} \in X$*

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \geq f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) r(\mathbf{x}, \mathbf{y}) \tag{44}$$
$$0 \leq \alpha \leq 1, \ \text{ and } \ \frac{r(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \ \text{ as } \ \|\mathbf{x}-\mathbf{y}\| \to 0 \qquad \forall \mathbf{x} \in X.$$

Definition A.2. *A function of one argument $f(\mathbf{x})$ is* weakly concave *on X if there exists a function $\mu(\mathbf{x}, \mathbf{y})$ such that, $\forall \mathbf{x}, \mathbf{y} \in X$*

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \leq f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha(1-\alpha) \mu(\mathbf{x}, \mathbf{y}) \tag{45}$$
$$0 \leq \alpha \leq 1, \ \text{ and } \ \frac{\mu(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \to 0 \quad \text{ as } \quad \|\mathbf{x}-\mathbf{y}\| \to 0 \qquad \forall \mathbf{x} \in X.$$

Example: *The convex function $f(x) = x^2$ is weakly concave (see [16]) but the convex function $f(x) = |x|$ is not.*

Now take a bivariate function $\Psi : X \times X \to \mathbb{R}$ defined on a product $X \times X$, where X is a convex closed subset of the Euclidean space $\mathbb{R}^m$.

Definition A.3. *A function of two vector arguments, $\Psi(\mathbf{x}, \mathbf{y})$ is referred to as* weakly convex-concave *if it satisfies weak convexity with respect to its first argument and weak concavity with respect to its second argument.*

The functions $r(\mathbf{x}, \mathbf{y}; \mathbf{z})$ and $\mu(\mathbf{x}, \mathbf{y}; \mathbf{z})$ were introduced with the concept of weak convex-concavity and are called the *residual terms*. Notice that smoothness of $\Psi(\mathbf{z}, \mathbf{y})$ is not required. However, if $\Psi(\mathbf{x}, \mathbf{y})$ is twice continuously differentiable with respect to both arguments on $X \times X$, the residual terms satisfy (see [16])

$$r(\mathbf{x}, \mathbf{y}; \mathbf{y}) = \tfrac{1}{2}\langle A(\mathbf{x}, \mathbf{x})(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + o_1(\|\mathbf{x}-\mathbf{y}\|^2) \tag{46}$$

and

$$\mu(\mathbf{y}, \mathbf{x}; \mathbf{x}) = \tfrac{1}{2}\langle B(\mathbf{x}, \mathbf{x})(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + o_2(\|\mathbf{x}-\mathbf{y}\|^2) \tag{47}$$

[26] Recall the following elementary definition: a function is "just" *convex* $\iff$

$$\alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) \ \geq \ f\Big(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}\Big), \quad \alpha \in [0,1].$$

where $A(\mathbf{x},\mathbf{x}) = \Psi_{\mathbf{xx}}(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido-Isoda function with respect to the first argument and $B(\mathbf{x},\mathbf{x}) = \Psi_{\mathbf{yy}}(\mathbf{x},\mathbf{y})|_{\mathbf{y}=\mathbf{x}}$ is the Hessian of the Nikaido-Isoda function with respect to the second argument, both evaluated at $\mathbf{y} = \mathbf{x}$.

To prove the inequality of condition (e) of Theorem A.1 (the convergence theorem, below) under the assumption that $\Psi(\mathbf{x},\mathbf{y})$ is twice continuously differentiable, it suffices to show that

$$Q(\mathbf{x},\mathbf{x}) = A(\mathbf{x},\mathbf{x}) - B(\mathbf{x},\mathbf{x}) \tag{48}$$

is strictly positive definite.

Theorem A.1 (Convergence theorem)**.** *There exists a unique normalised Nash equilibrium point to which the algorithm* (43) *converges if:*

a. *X is a convex, compact subset of $I\!R^m$,*

b. *the Nikaido-Isoda function $\Psi : X \times X \to I\!R$ is a weakly convex-concave function and $\Psi(\mathbf{x},\mathbf{x}) = 0$ for $\mathbf{x} \in X$,*

c. *the optimum response function $Z(\mathbf{x})$ is single valued and continuous on X,*

d. *the residual term $r(\mathbf{x},\mathbf{y};\mathbf{z})$ is uniformly continuous on X w.r.t. $\mathbf{z}$ for all $\mathbf{x},\mathbf{y} \in X$,*

e. *the residual terms satisfy*

$$r(\mathbf{x},\mathbf{y};\mathbf{y}) - \mu(\mathbf{y},\mathbf{x};\mathbf{x}) \geq \beta(\|\mathbf{x}-\mathbf{y}\|)\,, \qquad \mathbf{x},\mathbf{y} \in X \tag{49}$$

where $\beta(0) = 0$ and β is a strictly increasing function (i.e., $\beta(t_2) > \beta(t_1)$ if $t_2 > t_1$),

f. *the relaxation parameters α_s satisfy*

- *either (non-optimised step)*

  (a) $\alpha_s > 0$,
  (b) $\sum_{s=0}^{\infty} \alpha_s = \infty$,
  (c) $\alpha_s \to 0$ *as* $s \to \infty$.

- *or (optimised step)*

$$\alpha_s = \arg\min_{\alpha\in[0,1)} \left\{ \max_{\mathbf{y}\in X} \Psi(\mathbf{x}^{(s+1)}(\alpha),\mathbf{y}) \right\}. \tag{50}$$

Proof. See [16] for a proof. □

B Rosen's weights in $\mathbb{R}^2_+$

Consider a game with payoffs $\Pi_1(e), \Pi_2(e)$ that satisfy (12) (diagonal strict concavity). So, we know that this game has a unique equilibrium for a choice of r_1, r_2. The equilibrium fist order conditions are

$$\left.\begin{aligned} \frac{\partial \Pi_1(e)}{\partial e_1} &= -\frac{\lambda(r_1, r_2)}{r_1} \\ \frac{\partial \Pi_2(e)}{\partial e_2} &= -\frac{\lambda(r_1, r_2)}{r_2} \end{aligned}\right\} \tag{51}$$

where $\lambda \geq 0$ is the shadow price of the common constraint of type (7). We notice that conditions (51) are equivalent to

$$\left.\begin{aligned} \frac{\partial \Pi_1(e)}{\partial e_1} &= -\frac{\lambda(r, 1)}{r} \\ \frac{\partial \Pi_2(e)}{\partial e_2} &= -\lambda(r, 1) \end{aligned}\right\} \tag{52}$$

if

$$\left.\begin{aligned} \frac{\lambda(r_1, r_2)}{r_1} &= \frac{\lambda'(r, 1)}{r} \\ \frac{\lambda(r_1, r_2)}{r_2} &= \lambda'(r, 1) \end{aligned}\right\}. \tag{53}$$

The above is true if

$$r \equiv \frac{r_1}{r_2}. \tag{54}$$