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A REMARK ON THE SCHRÖDINGER SMOOTHING EFFECT

by

Laurent Thomann

Abstract. — We prove the equivalence between the smoothing effect for a Schrödinger operator and the decay of the associate spectral projectors. We give two applications to the Schrödinger operator in dimension one.

Résumé. — On donne une caractérisation de l'effet régularisant pour un opérateur de Schrödinger par la décroissance de ses projecteurs spectraux. On en déduit deux applications à l'opérateur de Schrödinger en dimension un.

1. Introduction

Let $d \geq 1$, and consider the linear Schrödinger equation

$$(1.1) \quad \begin{cases} i\partial_t u = Hu, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) = f(x) \in L^2(\mathbb{R}^d), \end{cases}$$

where H is a self-adjoint operator on $L^2(\mathbb{R}^d)$.

By the Hille-Yoshida theorem, the equation (1.1) admits a unique solution $u(t) = e^{-itH}f \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$. Under suitable conditions on H , this solution enjoys a local gain of regularity (in the space variable) : For all $T > 0$ there exists $C > 0$ so that

$$\left(\int_0^T \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R}^d)},$$

for some weight Ψ and exponent $\gamma > 0$.

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This phenomenon has been discovered by T. Kato [7] in the context of KdV equations. For the Schrödinger equation in the case $H = -\Delta$, it has been proved by P. Constantin- J.-C. Saut [2], P. Sjölin [11], L. Vega [12] and K. Yajima [13]. The variable coefficients case has been obtained by S. Doi [3, 4, 5, 6].

The more general results are due to L. Robbiano-C. Zuily [9, 10] for equations with obstacles and potentials.

Let H be a self adjoint operator on $L^2(\mathbb{R}^d)$. It can be represented thanks to the spectral measure by

$$H = \int \lambda dE_\lambda.$$

In the sequel we moreover assume that $H \geq 0$. For $N \geq 0$, we can then define the spectral projector P_N associated to H by

$$(1.2) \quad P_N = \mathbf{1}_{[N, N+1[}(H) = \int \mathbf{1}_{[N, N+1[}(\lambda) dE_\lambda.$$

Our main result is a characterisation of the smoothing effect by the decay of the spectral projectors. Denote by $\langle H \rangle = (1 + H^2)^{\frac{1}{2}}$.

Theorem 1.1 (Smoothing effect vs. decay). —

Let $\gamma > 0$ and $\Psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$. Then the following conditions are equivalent

(i) There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$

$$(1.3) \quad \left(\int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}.$$

(ii) There exists $C_2 > 0$ so that for all $N \geq 1$ and $f \in L^2(\mathbb{R}^d)$

$$(1.4) \quad \|\Psi P_N f\|_{L^2(\mathbb{R}^d)} \leq C_2 N^{-\frac{\gamma}{2}} \|P_N f\|_{L^2(\mathbb{R}^d)}.$$

The interesting point is that we can take the same function Ψ and exponent $\gamma > 0$ in both statements (1.3) and (1.4).

By the works cited in the introduction, in the case $H = -\Delta$ on \mathbb{R}^d , (1.3) is known to hold with $\gamma = \frac{1}{2}$ and $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$, for any $\nu > 0$.

There is also a class of operators H on $L^2(\mathbb{R}^d)$ for which (1.3) is well understood. Let $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$, and assume that for $|x|$ large enough $V(x) \geq C \langle x \rangle^k$ and that for any $j \in \mathbb{N}^d$, there exists $C_j > 0$ so that $|\partial_x^j V(x)| \leq C_j \langle x \rangle^{k-|j|}$. Then L. Robbiano and C. Zuily [9] show that the smoothing effect (1.3) holds for the operator $H = -\Delta + V(x)$, with $\gamma = \frac{1}{k}$ and $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$, for any $\nu > 0$.

We now turn to the case of dimension $d = 1$, and consider the operator $H = -\Delta + V(x)$. We make the following assumption on V

Assumption 1. — *We suppose that $V \in C^\infty(\mathbb{R}, \mathbb{R}_+)$, and that there exist $2 < m \leq k$ so that for $|x|$ large enough*

(i) *There exists $C > 1$ so that $\frac{1}{C}\langle x \rangle^k \leq V(x) \leq C\langle x \rangle^k$.*

(ii) *$V''(x) > 0$ and $xV'(x) \geq mV(x) > 0$*

(iii) *For any $j \in \mathbb{N}$, there exists $C_j > 0$ so that $|\partial_x^j V(x)| \leq C_j \langle x \rangle^{k-|j|}$.*

For instance $V(x) = \langle x \rangle^k$ with $k > 2$ satisfies Assumption 1.

It is well known that under Assumption 1, the operator H has a self-adjoint extension on $L^2(\mathbb{R})$ (still denoted by H) and has eigenfunctions $(e_n)_{n \geq 1}$ which form an Hilbertian basis of $L^2(\mathbb{R})$ and satisfy

$$He_n = \lambda_n^2 e_n, \quad n \geq 1,$$

with $\lambda_n \rightarrow +\infty$, when $n \rightarrow +\infty$.

For $N \in \mathbb{N}$ the spectral projector P_N defined in (1.2) can be written in the following way. Let $f = \sum_{n \geq 1} \alpha_n e_n \in L^2(\mathbb{R})$, then

$$P_N f = \sum_{N \leq \lambda_n^2 < N+1} \alpha_n e_n.$$

Observe that we then have $f = \sum_{N \geq 0} P_N f$.

For such a potential, we can remove the spectral projectors in (1.4) and deduce from Theorem 1.1

Corollary 1.2. —

Let $\gamma > 0$ and $\Psi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. Let $H = \Delta + V(x)$ so that $V(x) = x^2$ or $V(x)$ satisfies Assumption 1. Then the following conditions are equivalent

(i) *There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R})$*

$$(1.5) \quad \left(\int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R})}.$$

(ii) *There exists $C_2 > 0$ so that for all $n \geq 1$*

$$(1.6) \quad \|\Psi e_n\|_{L^2(\mathbb{R})} \leq C_2 \lambda_n^{-\gamma}, \quad \forall n \geq 1.$$

The statements (1.5) and (1.6) were obtained by K. Yajima & G. Zhang in [16] when Ψ is the indicator of a compact $K \subset \mathbb{R}$ and with $\gamma = \frac{1}{k}$.

The statement (1.5) holds for $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$, by [9], but as far as we know, the bound (1.6) with this Ψ was unknown.

With Theorem 1.1 we are also able to prove the following smoothing effect for the usual Laplacian Δ on \mathbb{R} .

Proposition 1.3. — *Let $\Psi \in L^2(\mathbb{R})$. Then there exists $C > 0$ so that for all $f \in L^2(\mathbb{R})$*

$$\left(\int_0^{2\pi} \|\Psi(x) \langle \Delta \rangle^{\frac{1}{4}} e^{-it\Delta} f\|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C \|\Psi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

From the works cited in the introduction, we have

$$\left(\int_{\mathbb{R}} \|\Psi(x) \langle \Delta \rangle^{\frac{1}{4}} e^{-it\Delta} f\|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R})},$$

for $\Psi(x) = \langle x \rangle^{\frac{1}{2}-\nu}$, for any $\nu > 0$. Hence Proposition 1.3 shows that we can extend the class of the weights, but we are only able to prove local integrability in time.

Notation. — *We use the notation $a \lesssim b$ if there exists a universal constant $C > 0$ so that $a \leq Cb$.*

2. Proof of the results

We define the self adjoint operator $A = [H]$ (entire part of H) by

$$A = \int [\lambda] dE_\lambda.$$

Notice that we immediately have that $A - H$ is bounded on $L^2(\mathbb{R}^d)$.

The first step in the proof of Theorem 1.1 is to show that we can replace e^{-itH} by e^{-itA} in (1.3)

Lemma 2.1. — *Let $\gamma > 0$ and $\Psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$. Then the following conditions are equivalent*

(i) *There exists $C_1 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$*

$$(2.1) \quad \left(\int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}.$$

(ii) *There exists $C_2 > 0$ so that for all $f \in L^2(\mathbb{R}^d)$*

$$(2.2) \quad \left(\int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C_2 \|f\|_{L^2(\mathbb{R}^d)}.$$

Proof. — We assume (2.1) and we prove (2.2). Let $f \in L^2(\mathbb{R}^d)$ and Define $v = e^{-itH}f$. This function solves the problem

$$(i\partial_t - A)v = (H - A)v, \quad v(0, x) = f(x).$$

Then by the Duhamel formula

$$\begin{aligned} e^{-itH}f = v &= e^{-itA}f - i \int_0^t e^{-i(t-s)A}(H - A)v \, ds \\ &= e^{-itA}f - i \int_0^{2\pi} \mathbf{1}_{\{s < t\}} e^{-i(t-s)A}(H - A)v \, ds. \end{aligned}$$

Therefore by (2.1) and Minkowski

$$\begin{aligned} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itH}v\|_{L_{2\pi}^2 L^2} &\lesssim \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itA}v\|_{L_{2\pi}^2 L^2} \\ &\quad + \int_0^{2\pi} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} \mathbf{1}_{\{s < t\}} e^{-i(t-s)A}(H - A)v\|_{L_t^2 L_x^2} \, ds \\ (2.3) \quad &\lesssim \|f\|_{L^2} + \int_0^{2\pi} \|(H - A)v\|_{L^2} \, ds. \end{aligned}$$

Now use that the operator $(H - A) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is bounded, and by (2.3) we obtain

$$\|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itH}v\|_{L_{2\pi}^2 L^2} \lesssim \|f\|_{L^2},$$

which is (2.2).

The proof of the converse implication is similar. \square

Proof of Theorem 1.1. — The proof is based on Fourier analysis in time. This idea comes from [8] and has also been used in [16], but this proof was inspired by [1].

(i) \implies (ii) : To prove this implication, we use the characterisation (2.1). From (1.2) and the definition of A , $e^{-itA}P_N f = e^{-itN}P_N f$. Hence it suffices to replace f with $P_N f$ in (1.3) and (1.4) follows.

(ii) \implies (i) : Again we will use Lemma 2.1. We assume (2.2) and we first prove that

$$(2.4) \quad \|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA}f\|_{L^2(0, 2\pi; L^2(\mathbb{R}^d))} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Write $f = \sum_{N \geq 0} P_N f$, then

$$\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA}f = \sum_{N \geq 0} e^{-iNt} \langle N \rangle^{\frac{\gamma}{2}} \Psi P_N f.$$

Now by Parseval in time

$$\|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi)}^2 \lesssim \sum_{N \geq 0} \langle N \rangle^\gamma |\Psi P_N f|^2,$$

and by integration in the space variable and (1.4)

$$\begin{aligned} \|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))}^2 &\lesssim \sum_{N \geq 0} \langle N \rangle^\gamma \|\Psi P_N f\|_{L^2(\mathbb{R}^d)}^2 \\ &\lesssim \sum_{N \geq 0} \|P_N f\|_{L^2(\mathbb{R}^d)}^2 = \|f\|_{L^2(\mathbb{R}^d)}^2, \end{aligned}$$

which yields (2.4).

Now since the operator $\langle A \rangle^{-\gamma/2} \langle H \rangle^{\gamma/2}$ is bounded on L^2 and commutes with e^{-itA} , we have by (2.4)

$$\begin{aligned} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))} &= \\ &= \|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} (\langle A \rangle^{-\frac{\gamma}{2}} \langle H \rangle^{\frac{\gamma}{2}} f)\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))} \\ &\lesssim \|\langle A \rangle^{-\frac{\gamma}{2}} \langle H \rangle^{\frac{\gamma}{2}} f\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

which is (2.1). □

Proof of Corollary 1.2. — Let V satisfy Assumption 1. Then by [14, Lemma 3.3] there exists $C > 0$ such that

$$|\lambda_{n+1}^2 - \lambda_n^2| \geq C \lambda_n^{1-\frac{2}{m}},$$

for n large enough. This implies that $[\lambda_n^2] < [\lambda_{n+1}^2]$ for n large enough, because $m > 2$ and $\lambda_n \rightarrow +\infty$. As a consequence

$$P_N f = \alpha_n e_n, \text{ with } n \text{ so that } N \leq \lambda_n^2 < N+1,$$

and this yields the result.

We now consider $V(x) = x^2$. In this case, the eigenvalues are the integers $\lambda_n^2 = 2n+1$, and the claim follows. □

Remark 2.2. — With this time Fourier analysis, we can prove the following smoothing estimate for H which satisfies Assumption 1

$$\|\langle H \rangle^{\frac{\theta(q,k)}{2}} e^{-itH} f\|_{L^p(\mathbb{R}; L^2(0,T))} \lesssim \|f\|_{L^2(\mathbb{R})},$$

where θ is defined by

$$\theta(q, k) = \begin{cases} \frac{2}{k}(\frac{1}{2} - \frac{1}{q}) & \text{if } 2 \leq q < 4, \\ \frac{1}{2k} - \eta \text{ for any } \eta > 0 & \text{if } q = 4, \\ \frac{1}{2} - \frac{2}{3}(1 - \frac{1}{q})(1 - \frac{1}{k}) & \text{if } 4 < q < \infty, \\ \frac{4-k}{6k} & \text{if } q = \infty. \end{cases}$$

This was done in [16] with a slightly different formulation.

Proof of Proposition 1.3. — By Theorem 1.1, we have to prove that the operator T defined by

$$Tf(x) = N^{\frac{1}{4}}\Psi(x)\mathbf{1}_{[N, N+1]}(-\Delta)f(x),$$

is continuous from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ with norm independent of $N \geq 1$. By the usual TT^* argument, it is enough to show the result for TT^* .

The kernel of T is $K(x, y) = N^{\frac{1}{4}}\Psi(x)F_N(x - y)$ where

$$(2.5) \quad F_N(u) = \frac{1}{2\pi} \int e^{iu\xi} \mathbf{1}_{[\sqrt{N}, \sqrt{N+1}]}(|\xi|) d\xi = 4 \cos(D_N u) \frac{\sin(C_N u)}{u},$$

with $C_N = (\sqrt{N+1} - \sqrt{N})/2$ and $D_N = (\sqrt{N+1} + \sqrt{N})/2$.

The kernel of TT^* is given by

$$\Lambda(x, z) = \int K(x, y) \overline{K}(z, y) dy,$$

and by Parseval and (2.5)

$$\begin{aligned} \Lambda(x, z) &= N^{\frac{1}{2}}\Psi(x)\Psi(z) \int F_N(x - y) \overline{F_N}(z - y) dy \\ &= \frac{1}{4} N^{\frac{1}{2}}\Psi(x)\Psi(z) \int e^{i(x-z)\xi} \mathbf{1}_{[\sqrt{N}, \sqrt{N+1}]}(|\xi|) d\xi \\ &= \pi N^{\frac{1}{2}}\Psi(x)\Psi(z) \cos(D_N(x - z)) \frac{\sin(C_N(x - z))}{x - z}. \end{aligned}$$

Now, since $C_N \lesssim 1/\sqrt{N}$ and $|\sin(x)| \leq |x|$, we deduce that $|\Lambda(x, z)| \leq C|\Psi(x)||\Psi(z)|$ (independent of $N \geq 1$), and TT^* is continuous for $\Psi \in L^2(\mathbb{R})$. \square

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