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Laurent Thomann

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# A REMARK ON THE SCHRÖDINGER SMOOTHING EFFECT WITH CONFINING POTENTIAL

by

Laurent Thomann

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**Abstract.** — We prove the equivalence between the smoothing effect for a Schrödinger operator with confining potential and the decay of the associate spectral projectors.

**Résumé.** — On donne une caractérisation de l'effet régularisant pour un opérateur de Schrödinger avec potentiel confinant par la décroissance de ses projecteurs spectraux.

## 1. Introduction

Let  $d \geq 1$ , and consider the linear Schrödinger equation

$$(1.1) \quad \begin{cases} i\partial_t u = Hu, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) = f(x) \in L^2(\mathbb{R}^d), \end{cases}$$

where  $H$  is a self-adjoint operator on  $L^2(\mathbb{R}^d)$ .

By the Hille-Yoshida theorem, the equation (1.1) admits a unique solution  $u(t) = e^{-itH}f \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$ . Under suitable conditions on  $H$ , this solution enjoys a local gain of regularity (in the space variable) : For all  $T > 0$  there exists  $C > 0$  so that

$$\left( \int_0^T \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C \|f\|_{L^2(\mathbb{R}^d)},$$

for some weight  $\Psi$  and exponent  $\gamma > 0$ .

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This phenomenon has been discovered by T. Kato [8] in the context of KdV equations. For the Schrödinger equation in the case  $H = -\Delta$ , it has been proved by P. Constantin- J.-C. Saut [3], P. Sjölin [12], L. Vega [13] and K. Yajima [14]. The variable coefficients case has been obtained by S. Doi [4, 5, 6, 7].

The more general results are due to L. Robbiano-C. Zuily [10, 11] for equations with obstacles and potentials.

In the following,  $H$  will stand for the operator,

$$H = -\Delta + V(x),$$

where  $V$  is a confining potential which satisfies

**Assumption 1.** — *We suppose that  $V \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$ , and that there exists  $k \geq 2$  so that*

(i) *There exists  $C > 1$  so that for all  $x \in \mathbb{R}^d$ ,  $\frac{1}{C}\langle x \rangle^k \leq V(x) \leq C\langle x \rangle^k$ .*

(ii) *For any  $j \in \mathbb{N}^d$ , there exists  $C_j > 0$  so that  $|\partial_x^j V(x)| \leq C_j \langle x \rangle^{k-|j|}$ .*

It is well known that under Assumption 1, the operator  $H$  has a self-adjoint extension on  $L^2(\mathbb{R}^d)$  (still denoted by  $H$ ) and has eigenfunctions  $(e_n)_{n \geq 1}$  which form an Hilbertian basis of  $L^2(\mathbb{R}^d)$  and satisfy

$$He_n = \lambda_n^2 e_n, \quad n \geq 1,$$

with  $\lambda_n \rightarrow +\infty$ , when  $n \rightarrow +\infty$ .

For  $s \in \mathbb{R}$ , we define the (Hilbert) Sobolev spaces based on the operator  $H$

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \langle H \rangle^{\frac{s}{2}} u \in L^2(\mathbb{R}^d)\},$$

where  $\langle H \rangle = (1 + H^2)^{\frac{1}{2}}$ .

For  $N \in \mathbb{N}$ , we define the spectral projector  $P_N$  by the following way. Let  $f = \sum_{n \geq 1} \alpha_n e_n \in L^2(\mathbb{R}^d)$ , then

$$P_N f = \sum_{N \leq \lambda_n^2 < N+1} \alpha_n e_n.$$

Observe that we then have

$$f = \sum_{N \geq 0} P_N f.$$

Our main result is a characterisation of the smoothing effect by the decay of the eigenfunctions.

**Theorem 1.1 (Smoothing effect vs. decay).** —

Let  $\gamma > 0$  and  $\Psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ . Let  $H$  satisfy Assumption 1. Then the following conditions are equivalent

(i) There exists  $C_1 > 0$  so that for all  $f \in L^2(\mathbb{R}^d)$

$$(1.2) \quad \left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R}^d)}.$$

(ii) There exists  $C_2 > 0$  so that for all  $N \geq 1$  and  $f \in L^2(\mathbb{R}^d)$

$$(1.3) \quad \|\Psi P_N f\|_{L^2(\mathbb{R}^d)} \leq C_2 N^{-\frac{\gamma}{2}} \|P_N f\|_{L^2(\mathbb{R}^d)}.$$

The interesting point is that we can take the same function  $\Psi$  and exponent  $\gamma > 0$  in both statements (1.2) and (1.3).

Let  $H$  satisfy Assumption 1 with  $k \geq 2$ . Then L. Robbiano and C. Zuily [10] show that the smoothing effect (1.2) holds with  $\gamma = \frac{1}{k}$  and  $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$ , for any  $\nu > 0$ .

In the special case of the dimension 1, we can remove the spectral projectors and we have the following result.

**Corollary 1.2.** —

Let  $\gamma > 0$  and  $\Psi \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ . Let  $H$  satisfy Assumption 1 in dimension 1. Then the following conditions are equivalent

(i) There exists  $C_1 > 0$  so that for all  $f \in L^2(\mathbb{R})$

$$(1.4) \quad \left( \int_0^{2\pi} \|\Psi(x) \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} f\|_{L^2(\mathbb{R})}^2 dt \right)^{\frac{1}{2}} \leq C_1 \|f\|_{L^2(\mathbb{R})}.$$

(ii) There exists  $C_2 > 0$  so that for all  $n \geq 1$

$$(1.5) \quad \|\Psi e_n\|_{L^2(\mathbb{R})} \leq C_2 \lambda_n^{-\gamma}, \quad \forall n \geq 1.$$

The statements (1.4) and (1.5) were obtained by K. Yajima & G. Zhang in [16] when  $\Psi$  is the indicator of a compact  $K \subset \mathbb{R}$  and with  $\gamma = \frac{1}{k}$ .

The statement (1.4) holds for  $\Psi(x) = \langle x \rangle^{-\frac{1}{2}-\nu}$ , by the work of L. Robbiano & C. Zuily [10], but as far as we know, the bound (1.5) with this  $\Psi$  was unknown.

**Notation.** — We use the notation  $a \lesssim b$  if there exists a universal constant  $C > 0$  so that  $a \leq Cb$ .

## 2. Proof of the results

*Proof of Theorem 1.1.* — The proof is based on Fourier analysis in time. This idea comes from [9] and has also been used in [16], but this proof was inspired by [1].

(i)  $\implies$  (ii) : To prove this implication, it suffices to replace  $f$  with  $P_N f$  in (1.2).

(ii)  $\implies$  (i) : Define the abstract operator  $A$  by

$$Ae_n = [\lambda_n^2]e_n, \quad n \geq 1,$$

where  $[x]$  denotes the integer part of  $x$ .

First we prove the assertion (1.2) with  $A$  instead of  $H$ .

Write  $f = \sum_{N \geq 0} P_N f$ , then

$$\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} f = \sum_{N \geq 0} e^{-iNt} \langle N \rangle^{\frac{\gamma}{2}} \Psi P_N f.$$

Now by Parseval in time

$$\|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi)}^2 \lesssim \sum_{N \geq 0} \langle N \rangle^\gamma |\Psi P_N f|^2,$$

and by integration in the space variable and (1.3)

$$\begin{aligned} \|\Psi \langle A \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi; L^2(\mathbb{R}^d))}^2 &\lesssim \sum_{N \geq 0} \langle N \rangle^\gamma \|\Psi P_N f\|_{L^2(\mathbb{R}^d)}^2 \\ &\lesssim \sum_{N \geq 0} \|P_N f\|_{L^2(\mathbb{R}^d)}^2 = \|f\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

hence the result holds for the operator  $A$ .

Observe that the same computation yields

$$(2.1) \quad \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itA} f\|_{L^2(0,2\pi; L^2(\mathbb{R}))}^2 \lesssim \|f\|_{L^2(\mathbb{R})}^2.$$

Now define  $v = e^{-itH} f$ . This function solves the problem

$$(i\partial_t - A)v = (H - A)v, \quad v(0, x) = f(x).$$

Then by the Duhamel formula

$$\begin{aligned} e^{-itH} f = v &= e^{-itA} f - i \int_0^t e^{-i(t-s)A} (H - A)v \, ds \\ &= e^{-itA} f - i \int_0^{2\pi} \mathbf{1}_{\{s < t\}} e^{-i(t-s)A} (H - A)v \, ds. \end{aligned}$$

Therefore by (2.1) and Minkowski

$$\begin{aligned}
\|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} v\|_{L^2_{2\pi} L^2} &\lesssim \|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itA} v\|_{L^2_{2\pi} L^2} \\
&\quad + \int_0^{2\pi} \|\Psi \langle H \rangle^{\frac{\gamma}{2}} \mathbf{1}_{\{s < t\}} e^{-i(t-s)A} (H - A)v\|_{L^2_t L^2_x} ds \\
(2.2) \qquad \qquad \qquad &\lesssim \|f\|_{L^2} + \int_0^{2\pi} \|(H - A)v\|_{L^2} ds.
\end{aligned}$$

Now observe that the operator  $(H - A) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  is bounded, because  $|\lambda_n^2 - [\lambda_n^2]| \leq 1$ . Finally, by (2.2) we obtain

$$\|\Psi \langle H \rangle^{\frac{\gamma}{2}} e^{-itH} v\|_{L^2_{2\pi} L^2} \lesssim \|f\|_{L^2},$$

which was the claim.  $\square$

*Proof of Corollary 1.2.* — In dimension 1, with a potential  $\frac{1}{C}\langle x \rangle^k \leq V(x) \leq C\langle x \rangle^k$  with  $k \geq 2$ , there exists  $C > 0$  so that

$$\lambda_n^2 \sim Cn^{\frac{2k}{k+2}}, \text{ when } n \rightarrow \infty.$$

Thus,  $[\lambda_n^2] < [\lambda_{n+1}^2]$  for  $n \geq n_0$  large enough, and for  $n \geq n_0$

$$P_N f = \alpha_n e_n, \text{ with } n \text{ so that } N \leq \lambda_n^2 < N + 1.$$

$\square$

**Remark 2.1.** — With this time Fourier analysis, we can prove the following smoothing estimate

$$\|\langle H \rangle^{\frac{\theta(q,k)}{2}} e^{-itH} f\|_{L^p(\mathbb{R}; L^2(0,T))} \lesssim \|f\|_{L^2(\mathbb{R})},$$

where  $\theta$  is defined by

$$\theta(q, k) = \begin{cases} \frac{2}{k}(\frac{1}{2} - \frac{1}{q}) & \text{if } 2 \leq q < 4, \\ \frac{1}{2k} - \eta \text{ for any } \eta > 0 & \text{if } q = 4, \\ \frac{1}{2} - \frac{2}{3}(1 - \frac{1}{q})(1 - \frac{1}{k}) & \text{if } 4 < q < \infty, \\ \frac{4-k}{6k} & \text{if } q = \infty. \end{cases}$$

This was done in [16] with a slightly different formulation.

### References

- [1] N. Burq, G. Lebeau and F. Planchon. Global existence for energy critical waves in 3-D domains. *Journal of the A.M.S.*, (21) 831–845, 2008.
- [2] N. Burq, L. Thomann and N. Tzvetkov. Gibbs measures for the non linear harmonic oscillator. *Preprint*.
- [3] P. Constantin and J.-C. Saut. Local smoothing properties of dispersive equations. *Journal American Mathematical Society* (1988) 413–439.
- [4] S. Doi. Smoothing effects of Schrödinger evolution group on Riemannian manifolds. *Duke Math. J.* 82 (1996) 679–706.
- [5] S. Doi. Smoothing effects for Schrödinger evolution equation and global behavior of geodesic flow. *Math. Ann.* 318 (2000) 355–389.
- [6] S. Doi. Remarks on the Cauchy problem for Schrödinger type equations. *Comm. in PDE*, 21 (1996) 163–178.
- [7] S. Doi. Smoothness of solutions for Schrödinger equations with unbounded potential. *Publ. Res. Inst. Math. Sci* 41 (2005), 1, 175–221.
- [8] T. Kato. On nonlinear Schrödinger equations, II.  $H^s$ -solutions and unconditional well-posedness, *J. Anal. Math.*, 87 (1995) 281–306.
- [9] G. Mockenaupt, A. Seeger and C. Sogge. Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. *J. Amer. Math. Soc.*, 6(1):65–130, 1993.
- [10] L. Robbiano, and C. Zuily. Remark on the Kato smoothing effect for Schrödinger equation with superquadratic potentials. *Comm. Partial Differential Equations*, 33 (2008), no. 4-6, 718–727.
- [11] L. Robbiano, and C. Zuily. The Kato Smoothing Effect for Schrödinger equations with unbounded potentials in exterior domains. *Int. Math. Res. Notices*, 15 (2009), 1636–1698.
- [12] P. Sjölin. Regularity of solution to the Schrödinger equation. *Duke Math. J.* 55 (1987) 699–715.
- [13] L. Vega. Schrödinger equations, pointwise convergence to the initial data. *Proc. Amer. Math. Soc.* 102 (1988) 874–878.
- [14] K. Yajima. On smoothing property of Schrödinger propagator. *Lectures notes in Math.* 1450 Springer Verlag (1990) 20–35.
- [15] K. Yajima, and G. Zhang. Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity. *J. Differential Equations* (2004), no. 1, 81–110.
- [16] K. Yajima, and G. Zhang. Smoothing property for Schrödinger equations with potential superquadratic at infinity. *Comm. Math. Phys.* 221 (2001), no. 3, 573–590.

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LAURENT THOMANN, Université de Nantes, Laboratoire de Mathématiques  
 J. Leray, UMR CNRS 6629, 2, rue de la Houssinière, F-  
 44322 Nantes Cedex 03, France. • *E-mail* : laurent.thomann@univ-nantes.fr  
*Url* : <http://www.math.sciences.univ-nantes.fr/~thomann/>