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Analytical Calculation of Cuboidal Magnet Interactions in 3D

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Topics : Analyses of EM field and force field – Material: Permanent Magnets

Abstract : A synthesis of all the analytical expressions of the interaction energy, force components and torque components is presented. It allows the analytical calculation of all the interactions when the magnetizations are in any direction.

The 3D analytical expressions are difficult to obtain, but the torque and force expressions are very simple to use.

1. Introduction

The analytical expressions are a very powerful and a very fast method to calculate magnetic interactions. It is why the analytical expressions of all the interactions, energy, forces and torques between two cuboidal magnets are very important results [1] – [7]. Many problems can be solved by the addition of elementary interactions. The simpler shape of elementary volume is the parallelepiped, with its cuboidal volume. It is why many 3D calculations can be made by the way of 3D interactions between two elementary magnets of cuboidal shape.

Until a recent time, only the force components between two magnets with their magnetization direction parallel to one edge of the parallelepipeds have been analytically solved [1]. Two new results have been recently obtained:

- the torque between two magnets [8],
- the force components and torque when the directions of the magnetization are perpendicular [9] – [12].

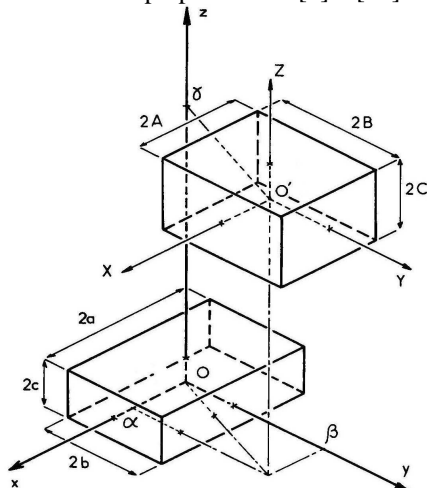


Figure 1: basic geometrical disposition

Consequently, by combining parallel and perpendicular magnetization directions, the interaction energy and all the components of force and torque can be calculated by fully analytical expressions, for any magnetization direction, and for any relative position

between the two magnets. The only two hypotheses are that the magnets own a cuboidal shape (Figure 1), and they are uniformly magnetized.

To complete the results, the field and induction analytical calculation are given in Annex by using the same parameters.

2. The basic mathematical model

The interactions between two parallelepiped magnets are studied. Their edges are respectively parallel (Fig. 1). The magnetizations J and J' are supposed to be rigid and uniform in each magnet. The dimensions of the first magnet are $2a \times 2b \times 2c$, and its polarization is J . Its center is O , the origin of the axes $Oxyz$. For the second magnet, the dimensions are $2A \times 2B \times 2C$, its polarization is J' , and the coordinates of its centre O' are (α, β, γ) . The side $2a$ is parallel to the side $2A$, and so on. The magnet dimensions are given on Table 1.

Table 1: Magnet dimensions and position

Axis	Ox	Oy	Oz
First Magnet (J)	2a	2b	2c
Second Magnet (J')	2A	2B	2C
Second Magnet Position O'	α	β	γ

The magnetization directions shown on Figure 2 correspond to the case when the polarizations J and J' have the same direction, parallel to the side $2c$. Note that the calculation stays valid when they are in opposite direction; only the expression sign is reversed. The polarizations J and J' are supposed to be rigid and uniform. They can be replaced by distributions of magnetic charges on the poles. It is the coulombian representation of the magnetization.

Their density σ is defined by $\sigma = \vec{J} \cdot \vec{n}$. On the example of Figure 2, since J is perpendicular to the surfaces $2a \times 2b$ and oriented to the top, these faces wear the density $\sigma = +J$ on the upper face (North Pole), and $\sigma = -J$ on the lower face (South Pole). All the analytical calculations have been made by successive integrals. We have determined the scalar potential created by one charged surface. From this scalar potential, the induction components can be obtained by derivation. For the whole magnets, we have calculated the interaction energy. The forces and the torques can be deduced by linear and angular derivation. The most difficult analytical calculation is the interaction energy in 3D. It is made by four successive integrations. The first one gives a logarithm function. In the second one, you have two logarithm and two arc-tangent functions. The last one owns many complex functions based on logarithm and arc-tangent functions.

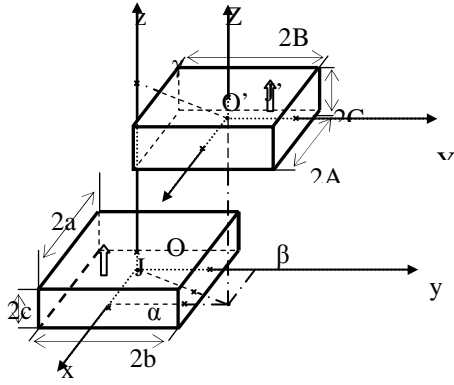


Figure 2 – Magnet configuration with parallel magnetization directions

3. Interaction energy calculation for parallel magnetization directions

The interaction energy in a system of two magnets with parallel magnetization directions (Fig. 2) is given by:

$$E = \frac{J \cdot J'}{4\pi\mu_0} \sum_{p=0}^1 \sum_{q=0}^1 (-1)^{p+q} \int_{-C}^C dY \int_{-A}^A dX \int_{-b}^b dy \int_{-a}^a \frac{1}{r} dx$$

$$\text{with } r = \sqrt{(\alpha + X - x)^2 + (\beta + Y - y)^2 + (\gamma + (-1)^l C - (-1)^p c)^2}$$

The obtained expressions of the interaction energy are:

$$E = \frac{J \cdot J'}{4\pi\mu_0} \sum_{i=0}^1 \sum_{j=0}^1 \sum_{k=0}^1 \sum_{l=0}^1 \sum_{p=0}^1 \sum_{q=0}^1 (-1)^{i+j+k+l+p+q} \psi(U_{ij}, V_{kl}, W_{pq}, r)$$

with

$$\psi(U, V, W, r) = \frac{U(V^2 - W^2)}{2} \ln(r - U) + \frac{V(U^2 - W^2)}{2} \ln(r - V) + UVW \cdot \text{tg}^{-1}\left(\frac{UV}{rW}\right) + \frac{r}{6}(U^2 + V^2 - 2W^2)$$

The secondary variables are:

$$U_{ij} = \alpha + (-1)^j A - (-1)^i a$$

$$V_{kl} = \beta + (-1)^l B - (-1)^k b$$

$$W_{pq} = \gamma + (-1)^q C - (-1)^p c$$

$$\text{with } r = \sqrt{U_{ij}^2 + V_{kl}^2 + W_{pq}^2}$$

4. Force calculation for parallel magnetization directions

From the interaction energy, the force components can be obtained by $\vec{F} = -\text{grad } E$. Consequently the force components are:

$$F = \frac{J \cdot J'}{4\pi\mu_0} \sum_{i=0}^1 \sum_{j=0}^1 \sum_{k=0}^1 \sum_{l=0}^1 \sum_{p=0}^1 \sum_{q=0}^1 (-1)^{i+j+k+l+p+q} \phi(U_{ij}, V_{kl}, W_{pq}, r)$$

The components F_x , F_y and F_z are respectively given by using ϕ_x , ϕ_y and ϕ_z :

$$\phi_x(U, V, W, r) = \frac{(V^2 - W^2)}{2} \ln(r - U) + UV \ln(r - V) + VW \cdot \text{tg}^{-1}\left(\frac{UV}{W \cdot r}\right) + \frac{1}{2} U \cdot r$$

$$\phi_y(U, V, W, r) = \frac{(U^2 - W^2)}{2} \ln(r - V) + UV \ln(r - U) + UW \cdot \text{tg}^{-1}\left(\frac{UV}{W \cdot r}\right) + \frac{1}{2} V \cdot r$$

$$\phi_z(U, V, W, r) = -UW \ln(r - U) - VW \ln(r - V) + UW \cdot \text{tg}^{-1}\left(\frac{UV}{W \cdot r}\right) - W \cdot r$$

5. Interaction energy calculation for perpendicular magnetization directions

With applying the same procedure to magnets with perpendicular magnetization (Fig. 3), we obtain for energy:

$$E = \frac{J \cdot J'}{4\pi\mu_0} \sum_{l=0}^1 \sum_{p=0}^1 \int_{-C}^C dZ \int_{-A}^A dX \int_{-b}^b dy \int_{-a}^a \frac{1}{r} dx$$

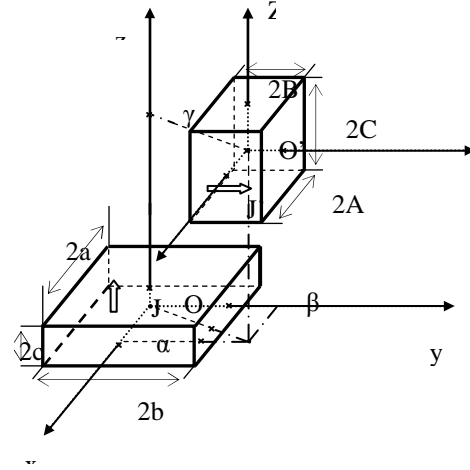


Figure 3 – Magnet configuration with perpendicular magnetization direction

The obtained expressions of the interaction energy are:

$$E = \frac{J \cdot J'}{4\pi\mu_0} \sum_{i=0}^1 \sum_{j=0}^1 \sum_{k=0}^1 \sum_{l=0}^1 \sum_{p=0}^1 \sum_{q=0}^1 (-1)^{i+j+k+l+p+q} \psi(U_{ij}, V_{kl}, W_{pq}, r)$$

with

$$\psi(U, V, W, r) = \frac{V(V^2 - 3U^2)}{6} \ln(W + r) + \frac{W(W^2 - 3U^2)}{6} \ln(V + r) + \frac{U}{6} \left(3V^2 \text{tg}^{-1}\left(\frac{UV}{V \cdot r}\right) + 3W^2 \text{tg}^{-1}\left(\frac{UV}{W \cdot r}\right) + U^2 \text{tg}^{-1}\left(\frac{VW}{U \cdot r}\right) \right) + UVW \cdot \ln(-U + r) + \frac{V \cdot W \cdot r}{3}$$

The four secondary variables U, V, W, r are the same

6. Force calculation for perpendicular magnetization directions

From the gradient of energy the force components are:

$$F = \frac{J \cdot J'}{4\pi\mu_0} \sum_{i=0}^1 \sum_{j=0}^1 \sum_{k=0}^1 \sum_{l=0}^1 \sum_{p=0}^1 \sum_{q=0}^1 (-1)^{i+j+k+l+p+q} \phi(U_{ij}, V_{kl}, W_{pq}, r)$$

The components F_x , F_y and F_z are respectively given by using ϕ_x , ϕ_y and ϕ_z :

$$\phi_x(U, V, W, r) = -VW \ln(r - U) + VU \ln(r + W) + WU \ln(r + V) - \frac{U^2}{2} \text{tg}^{-1}\left(\frac{VW}{U \cdot r}\right) - \frac{V^2}{2} \text{tg}^{-1}\left(\frac{UW}{V \cdot r}\right) - \frac{W^2}{2} \text{tg}^{-1}\left(\frac{UV}{W \cdot r}\right)$$

$$\phi_y(U, V, W, r) = \frac{(U^2 - V^2)}{2} \ln(r + W) - UW \ln(r - U) - UV \cdot \text{tg}^{-1}\left(\frac{UW}{V \cdot r}\right) - \frac{1}{2} W \cdot r$$

$$\phi_z(U, V, W, r) = \frac{(U^2 - W^2)}{2} \ln(r + V) - UV \ln(r - U) - UW \cdot \text{tg}^{-1}\left(\frac{UV}{W \cdot r}\right) - \frac{1}{2} V \cdot r$$

7. Inclined magnetization

When one of the two magnetization directions is inclined (Fig.4), it can be decomposed as the sum of parallel and perpendicular case. The whole energy and the forces can be calculated by addition of the interactions between the magnetization components.

All the energy and force components can be obtained with the previous expressions.

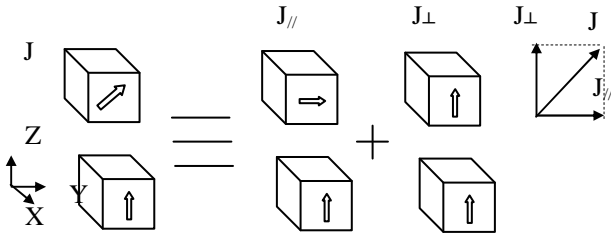


Figure 4 – Inclined case configuration

8. Torque component calculation for parallel magnetization directions

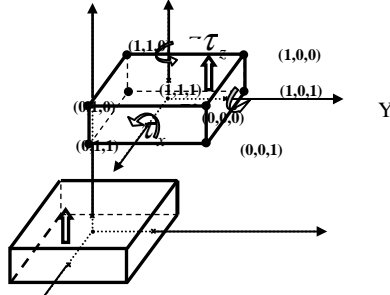


Fig. 5: Torque components and corner position

The first magnet of centre O is supposed to be fixed. The second magnet is submitted to a torque T for a movement around its centre O', which components are τ_x , τ_y and τ_z (Fig. 5).

By starting directly from the magnet dimensions and positions, the torque is given by:

$$T = \frac{J \cdot J'}{4\pi\mu_0} \sum_{i=0}^1 \sum_{j=0}^1 \sum_{k=0}^1 \sum_{l=0}^1 \sum_{p=0}^1 \sum_{q=0}^1 (-1)^{i+j+k+l+p+q} \tau(U_{ij}, V_{lk}, W_{pq})$$

For each torque component the expression is given by:

$$\begin{aligned} \tau_x = & \frac{1}{2} \left((-1)^q C(U^2 - W^2) + 2(-1)^j BVW \right) \ln(r - V) \\ & - U \left((-1)^q CV + (-1)^j BW \right) \ln(r - U) \\ & + U \left((-1)^q CW + (-1)^j BV \right) \cdot \operatorname{tg}^{-1} \left(\frac{UV}{rW} \right) + \frac{r}{2} \left((-1)^q CV - 2(-1)^j BW \right) \end{aligned}$$

$$\begin{aligned} \tau_y = & \frac{1}{2} \left((-1)^q C \cdot (V^2 - W^2) + 2(-1)^j AUW \right) \ln(r - U) \\ & + V \left((-1)^q CU + (-1)^j AW \right) \ln(r - V) \\ & + V \left((-1)^q CW - (-1)^j AU \right) \cdot \operatorname{tg}^{-1} \left(\frac{UV}{W \cdot r} \right) + \frac{r}{2} \left((-1)^q C \cdot U + 2(-1)^j AW \right) \end{aligned}$$

$$\begin{aligned} \tau_z = & \frac{1}{2} \left((-1)^j B(V^2 - W^2) + 2(-1)^j AUW \right) \ln(r - U) \\ & - \frac{1}{2} \left((-1)^j A(U^2 - W^2) + (-1)^j A(U^2 - W^2) \right) \ln(r - V) \\ & + \left((-1)^j BVW + (-1)^j UW \right) \cdot \operatorname{tg}^{-1} \left(\frac{UV}{W \cdot r} \right) - \frac{1}{2} \left((-1)^j B \cdot U + (-1)^j A \cdot V \right) \cdot r \end{aligned}$$

with always:

$$U_{ij} = \alpha + (-1)^j A - (-1)^i a$$

$$V_{kl} = \beta + (-1)^l B - (-1)^k b$$

$$W_{pq} = \gamma + (-1)^q C - (-1)^p c$$

$$\text{and } r = \sqrt{U_{ij}^2 + V_{kl}^2 + W_{pq}^2}$$

9. Torque component calculation for perpendicular magnetization directions, and inclined magnetization

For the torque calculation, it is important to observe the energy expression. The interaction energy E depends on the value of the polarization of each magnet and on a geometrical analytical expression which is only function of the magnet corner position. Consequently each corner wears a part of the global interaction energy. The scalar value of the energy of each corner E_c can be easily calculated; the global interaction energy on the magnet is the sum of the 8 contributions of the magnet corners. On Fig 4 each corner is defined by its coordinates for the E_c calculation, for example (1,0,0) means $j=1, l=0$ and $q=0$.

The torque expressions can be derived from the force components, exactly from the corner force F_c . The result can be written as a vector product:

$$\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \sum_{j=0}^1 \sum_{l=0}^1 \sum_{q=0}^1 \begin{pmatrix} A(-1)^j \\ B(-1)^l \\ C(-1)^q \end{pmatrix} \times \begin{pmatrix} F_x(j, l, q) \\ F_y(j, l, q) \\ F_z(j, l, q) \end{pmatrix}$$

For all the cases (for any magnetization direction in the space, and for any relative position between the two magnets) the torque can be obtained by the same method. The only two hypotheses remain that the magnets own a cuboidal shape, and they are uniformly magnetized.

10. Stiffnesses

In many systems, the operation is characterized by the force variation or by the torque variation. As example in a magnetic coupling, the angular stiffness K_θ is an important parameter. In a permanent magnet radial bearing, the restoring force is proportional to the radial displacement and the proportionality coefficient is the radial stiffness K_r .

All these stiffnesses can be easily calculated from the force and torque analytical expressions. The stiffnesses are the first derivatives as function of the linear or angular displacement; they can be easily expressed by analytical formulation.

11. Conclusion

All the analytical expressions of the interaction energy between two cuboidal magnets and all the force components and torque components are given in the paper. The only hypotheses are that the magnets own a parallelepiped shape with parallel edges. All the interactions between two elementary magnets represented by the 6 components (force F_x, F_y, F_z and torque τ_x, τ_y and τ_z) can be calculated by simple analytical expressions.

Many problems can be solved by these results. The simpler shape of elementary volume is the parallelepiped, with its cuboidal volume. By the superposition of 3D interactions between elementary magnets, many 3D calculations can be made. These results allow calculating not only the direct interaction of permanent magnets, but also many other devices including soft magnetic materials, and currents. As example, iron yokes can be taken into account by image effect.

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Annex: Magnetic induction created by a permanent magnet

Let us consider a rectangular surface $2a \times 2b$, wearing a uniform pole density σ (Fig. A1). We will calculate the scalar potential in the point P, which coordinates are (X, Y, Z).

The scalar potential V is given by:

$$V = \frac{1}{4\pi\mu_0} \iint_S \frac{\sigma \cdot dS}{\|\vec{r} - \vec{r}'\|}$$

By using the cartesian coordinates, it is equivalent to:

$$V = \frac{\sigma}{4\pi\mu_0} \int_{-b}^b dy' \int_{-a}^a \frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2 + z^2}} dx'$$

After two analytical integrations, we obtain:

$$V = \frac{\sigma}{4\pi\mu_0} \sum_{i=0}^1 \sum_{j=0}^1 (-1)^{i+j} \phi(U_i, V_j, W)$$

The function ϕ is given by:

$$\phi(U, V, W, r) = -U \ln(r-V) - V \ln(r-U) - W \cdot \text{tg}^{-1} \left(\frac{UV}{rW} \right)$$

Intermediary variables are:

$$U_i = x - (-1)^i a, \quad V_j = y - (-1)^j b \quad \text{and} \quad W = z$$

$$r = \sqrt{U_i^2 + V_j^2 + W^2}$$

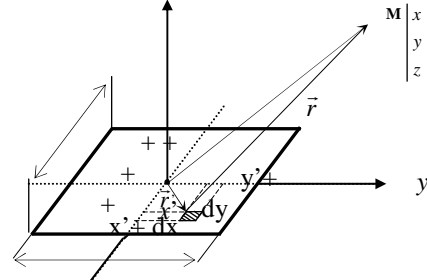


Figure A1 – Geometrical disposition for the potential and the field calculation

From the analytical expression of the scalar potential, the magnetic field H can be easily calculated by derivation.

$$H = -\text{grad}(V),$$

H can be expressed as:

$$H = \frac{\sigma'}{4\pi\mu_0} \sum_{i=0}^1 \sum_{j=0}^1 (-1)^{i+j} \varepsilon(U_i, V_j, W)$$

$$\varepsilon_x = \ln(r-V) \quad \varepsilon_y = \ln(r-U) \quad \varepsilon_z = \text{tg}^{-1} \left(\frac{UV}{Wr} \right)$$

By adding the field created by the two surfaces, we obtain the field created by a cuboidal magnet (Fig. A2).

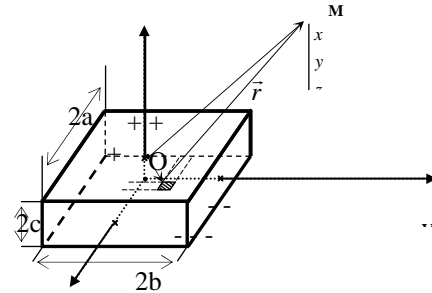


Figure A2 – Magnetic induction created by a magnet

The geometric parameters are shown on Fig. A2. Intermediary variables are:

$$U_i = x - (-1)^i a; \quad V_j = y - (-1)^j b \quad \text{and} \quad W_k = z - (-1)^k c$$

$$r = \sqrt{U_i^2 + V_j^2 + W_k^2}$$

The magnetic field and induction components are given by :

$$H = \frac{\sigma}{4\pi\mu_0} \sum_{k=0}^1 \sum_{i=0}^1 \sum_{j=0}^1 (-1)^{i+j+k} \varepsilon(U_i, V_j, W_k)$$

$$B = \mu_0 H = \frac{\sigma}{4\pi} \sum_{k=0}^1 \sum_{i=0}^1 \sum_{j=0}^1 (-1)^{i+j+k} \varepsilon(U_i, V_j, W_k)$$