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# GLOBAL ATTRACTOR FOR WEAKLY DAMPED NONLINEAR SCHRÖDINGER EQUATIONS IN $L^2(\mathbb{R})$

O. GOUBET AND L. MOLINET

ABSTRACT. We prove that the weakly damped nonlinear Schrödinger flow in  $L^2(\mathbb{R})$  provides a dynamical system which possesses a global attractor. The proof relies on the continuity of the Schrödinger flow for the weak topology in  $L^2(\mathbb{R})$ .

## 1. INTRODUCTION

Nonlinear Schrödinger (NLS) and Korteweg-de Vries equations are asymptotical models for the waterwave propagation. These models supplemented with a damping and an external force provide examples of infinite-dimensional dynamical systems, in the framework described in [13], [7], [12], [9]. We focus here on the cubic NLS equation

$$(1) \quad u_t + \gamma u + iu_{xx} + i|u|^2u = f,$$

where  $\gamma > 0$  is the damping parameter and where the external forcing  $f(x)$ , that is independent of  $t$ , belongs to  $L^2(\mathbb{R})$ .

To define an infinite-dimensional dynamical system from this evolution equation, we supplement (1) with an initial data  $u_0$  in some Sobolev space  $X$  (or in some complete metric space) such that the corresponding initial value problem is well posed (in the Hadamard sense: existence and uniqueness of trajectories  $u(t) = S(t)u_0$  in  $X$ , continuity of  $S(t) : u_0 \mapsto u(t)$  in  $X$ ). This short article is concerned with the existence of a global attractor of the NLS flow with low regular initial data in  $L^2(\mathbb{R})$ . Recall that a global attractor is a compact set, invariant by the flow, that attracts all trajectories uniformly on bounded sets. Note that the existence of a such set for the NLS equations in more regular function spaces, as for instance  $H^1(\mathbb{R})$ , is well known (see [1] and the references therein).

Our main result states as follows

**Theorem 1.1.** *The semi-group  $S(t)$  provides an infinite-dimensional dynamical system in  $L^2(\mathbb{R})$  that has a global attractor  $\mathcal{A}$ .*

Let us describe the strategy of the proof. As in [4], we first prove that the NLS flow features a weak global attractor in  $L^2(\mathbb{R})$ , that is a global attractor

for the weak topology of  $L^2(\mathbb{R})$ . For that purpose, we need to establish the following result that is interesting on its own :

**Theorem 1.2.** *The semi-group  $S(t)$  is a continuous mapping for the weak topology of  $L^2(\mathbb{R})$ .*

The usual arguments to prove this kind of result in a function space  $X$  make use of the fact that the initial value problem is well-posed in a function space where  $X$  is locally compactly embedded (cf. [4]). Here such an argument can not be invoked since there is no available result on the well-posedness of the (1) in a space where  $L^2(\mathbb{R})$  is locally compactly embedded (cf. [8]). Note that the situation for KdV equations is easier according to this last point (see [6], [14]).

Therefore this result is new and can be outlined as follows. Calling  $U(t)u_0$  the solution of

$$(2) \quad u_t + iu_{xx} = 0; \quad u(0) = u_0,$$

it is well-known that the *linear* Schrödinger equation features the so-called Kato's smoothing effect that reads:

$$(3) \quad \|D_x^{1/2}U(t)u_0\|_{L_x^\infty L_t^2} \leq c\|u_0\|_{L_x^2}.$$

Above,  $D_x = \sqrt{-\Delta}$  stand for the operator with Fourier symbol  $|\xi|$ . Using the Christ-Kiselev theorem [3] (as in [11] in another context), we are able to prove that this smoothing effect is also valid for the nonlinear Schrödinger equation. Then the weak continuity is valid due to some compactness argument that allow us to pass to the limit in the nonlinear term.

These arguments are developed in Section 1 below. In Section 2 we complete the proof of Theorem 1.1. First we prove the existence of a weak attractor. Then, using the famous J. Ball's argument (see [2], [15], [10]), we establish that the weak attractor is actually a global attractor in the usual sense.

## 2. CONTINUITY OF THE FLOW FOR THE WEAK TOPOLOGY

To begin with, we observe that for finite time results the damping parameter and the external forcing do not play a role. Then in this section we may assume for the sake of simplicity that  $\gamma = 0$  and  $f = 0$ .

The usual way to solve the IVP problem associated to

$$(4) \quad u_t + iu_{xx} + i|u|^2u = 0,$$

supplemented with initial data  $u_0$  is to perform a fixed point argument for the Duhamel's form of (4) that reads

$$(5) \quad u(t) = U(t)u_0 - i \int_0^t U(t-s)|u(s)|^2u(s)ds.$$

Thanks to well-known Strichartz inequalities, we usually perform a fixed point into the space  $C([0, T], L^2(\mathbb{R})) \cap L_{T,x}^6$ , where  $L_{T,x}^6 = L^6([0, T] \times \mathbb{R}_x)$ . We first state and prove

**Proposition 2.1.** *There exists a numerical constant  $c$  such that for a solution to (5)*

$$(6) \quad \|D_x^{1/2}u\|_{L_x^\infty L_T^2} \leq c(\|u_0\|_{L_x^2} + \|u_0\|_{L_x^2}^3).$$

Proof of the Proposition: the key point is to estimate the nonlinear term in (5). For that purpose, we first recall the dual estimate to (3) that reads

$$(7) \quad \left\| \int_{\mathbb{R}} U(-s)D_x^{1/2}Gds \right\|_{L_x^2} \leq c\|G\|_{L_x^1 L_t^2}.$$

We now prove

$$(8) \quad \left\| \int_{\mathbb{R}_s} U(t-s)D_x^{1/2}f ds \right\|_{L_x^\infty L_t^2} \leq c\|f\|_{L_{t,x}^{\frac{6}{5}}}.$$

Actually, following P. Tomas duality argument, it is equivalent to prove that for any smooth function  $G$  that satisfies  $\|G\|_{L_x^1 L_t^2} \leq 1$ , it holds

$$(9) \quad \left| \int_{\mathbb{R}^3} U(t-s)D_x^{1/2}f(s,x)\overline{G}(t,x)dt dx ds \right| \leq c\|f\|_{L_{t,x}^{\frac{6}{5}}}.$$

Note that the left-hand side member of the above estimate can be rewritten as

$$(10) \quad \left| \int_{\mathbb{R}} \left( \int_{\mathbb{R}} U(-s)D_x^{1/2}f(s,x) ds \right) \left( \int_{\mathbb{R}} \overline{U(-t)G(t,x)} dt \right) dx \right|,$$

Hence, applying Cauchy-Schwarz in  $x$  and using (7), it finally suffices to check that

$$(11) \quad \left\| \int_{\mathbb{R}} U(-s)f ds \right\|_{L_x^2} \leq c\|f\|_{L_{t,x}^{\frac{6}{5}}},$$

Since this is nothing else but the dual form of the classical Strichartz estimate for the Schrödinger group on  $\mathbb{R}$ :

$$(12) \quad \|U(t)u_0\|_{L_{t,x}^6} \leq c\|u_0\|_{L_x^2},$$

we are done.

Recall now from [11]

**Lemma 2.2.** *(Christ-Kiselev) Consider a linear operator defined on space-time functions  $f(t, x)$  by*

$$(13) \quad Tf(t) = \int_{\mathbb{R}_s} K(t, s)f(s)ds.$$

*Assume*

$$(14) \quad \|Tf\|_{L_x^\infty L_t^2} \leq c \|f\|_{L_{t,x}^{\frac{6}{5}}},$$

*then*

$$(15) \quad \left\| \int_0^t K(t,s)f(s)ds \right\|_{L_x^\infty L_t^2} \leq c \|f\|_{L_{t,x}^{\frac{6}{5}}}.$$

According to [11], this is valid since  $\min(+\infty, 2) > \max(\frac{6}{5}, \frac{6}{5})$ .

We then apply this argument to the nonlinear term in (5), for  $t \in [0, T]$ . This leads to

$$(16) \quad \begin{aligned} \left\| \int_0^t D_x^{1/2} U(t-s) \bar{u}^2 u ds \right\|_{L_x^\infty L_T^2} &\leq c \|u^3\|_{L_{T,x}^{\frac{6}{5}}} \\ &\leq c \|u\|_{L_{T,x}^{\frac{18}{5}}}^3 \leq c \|u\|_{L_{T,x}^2} \|u\|_{L_{T,x}^6}^2. \end{aligned}$$

We conclude the proof of the proposition using that  $u$  is bounded in  $C([0, T], L^2(\mathbb{R})) \cap L_{T,x}^6$ .  $\square$

At this stage we complete the proof of Theorem 1.2. Consider  $u_{0,\varepsilon} \rightharpoonup u_0$  in  $L_x^2$ . Due to the previous proposition, we know that, for any  $K$  compact subset of  $\mathbb{R}_x$ , the sequence  $u_\varepsilon$  remains in a bounded set of  $C([0, T], L^2(\mathbb{R})) \cap L_{T,x}^6 \cap L_T^2 H_x^{\frac{1}{2}}(K)$ . Going back to the equation, we observe that  $\partial_t u_\varepsilon$  remains in a bounded set of  $L_T^2 H_x^{-2}$ . Hence, due to a standard compactness argument, the sequence  $u_\varepsilon$ , up to a subsequence extraction, converges towards some function  $v$  strongly in  $L_T^2 L^2(K)$ . By interpolation, the strong convergence is also valid in  $L_T^4 L^4(K)$ . This allows us to pass to the limit in the equation and to conclude that the limit  $v$  is a solution of (4) belonging to the class of uniqueness  $L_{T,x}^6$ . Set  $(\cdot, \cdot)$  for the  $L_x^2$  scalar product. By (4) and the bounds above, it is easy to check that, for any smooth space function  $\phi$  with compact support, the family  $\{t \mapsto (u_\varepsilon(t), \phi)\}$  is uniformly equi-continuous on  $[0, T]$ . Ascoli's theorem then ensures that  $(u_\varepsilon(\cdot), \phi)$  converges to  $(v(\cdot), \phi)$  uniformly on  $[0, T]$  and thus  $v(0) = u_0$ . By uniqueness, it follows that  $v \equiv u$  and from the above convergence result, it results that  $u_\varepsilon(t) \rightharpoonup u(t)$  in  $L_x^2$  for all  $t \in [0, T]$ .

$\square$

### 3. PROOF OF THE MAIN THEOREM

To begin with, we prove the existence of an absorbing ball for the semi-group; multiplying (1) by  $\bar{u}$  and integrating in  $x$  the real part of the resulting equation

$$(17) \quad \frac{1}{2} \frac{d}{dt} \|u\|_{L_x^2}^2 + \gamma \|u\|_{L_x^2}^2 = \operatorname{Re} \int f \bar{u} dx \leq \frac{1}{2} \gamma \|u\|_{L_x^2}^2 + \frac{1}{2\gamma} \|f\|_{L_x^2}^2.$$

This implies

$$(18) \quad \|u(t)\|_{L_x^2}^2 \leq e^{-\gamma t} \|u_0\|_{L_x^2}^2 + \frac{1 - e^{-\gamma t}}{\gamma^2} \|f\|_{L_x^2}^2.$$

**Proposition 3.1.** *The ball  $X$  of radius  $M_0 = 2\frac{\|f\|_{L_x^2}}{\gamma}$  is an absorbing set for the dynamical system under consideration.*

We endow then this absorbing ball with the weak topology of  $L_x^2$ .  $X$  is then a compact metric space and  $S(t)$  acts continuously on  $X$  according to Theorem 1.2. Therefore, using Theorem I.1.1 in [13] the  $\omega$ -limit set  $\mathcal{A} = \bigcup_{s>0} \bigcap_{t>s} \overline{S(t)X}$  is a global attractor. In fact

$$(19) \quad \mathcal{A} = \{a \in X; \exists b_n \in X, t_n \rightarrow +\infty, S(t_n)b_n \rightharpoonup a.\}$$

We plan to transform this weak convergence into a strong convergence. We use the famous J. Ball's argument. We begin with the energy equation that asserts that for any  $\tau > 0$ , due to (17),

$$(20) \quad \|S(t_n)b_n\|_{L_x^2}^2 = e^{-2\gamma\tau} \|S(t_n-\tau)b_n\|_{L_x^2}^2 - 2\operatorname{Re} \int_0^\tau \int_{\mathbb{R}_x} e^{-2\gamma s} \bar{f}(x) S(t_n-s)b_n ds dx.$$

According to the weak convergence, we have

$$(21) \quad \lim_{n \rightarrow +\infty} 2\operatorname{Re} \int_0^\tau \int_{\mathbb{R}_x} e^{-2\gamma s} \bar{f}(x) S(t_n-s)b_n ds dx = 2\operatorname{Re} \int_0^\tau \int_{\mathbb{R}_x} e^{-2\gamma s} \bar{f}(x) S(-s)a ds dx.$$

Using once again the energy equality (17) we also have that

$$(22) \quad \|a\|_{L_x^2}^2 = e^{-2\gamma\tau} \|S(-\tau)a\|_{L_x^2}^2 - 2\operatorname{Re} \int_0^\tau \int_{\mathbb{R}_x} e^{-2\gamma s} \bar{f}(x) S(-s)a ds dx.$$

Therefore

$$(23) \quad \limsup_n \|S(t_n)b_n\|_{L_x^2}^2 \leq \|a\|_{L_x^2}^2 + 2e^{-2\gamma\tau} M_0^2,$$

since for  $n > \tau$   $S(t_n - \tau)b_n$  is in  $X$  and  $S(-\tau)a$ , that belongs to the weak attractor, remains trapped in  $X$ . Letting  $\tau \rightarrow +\infty$  implies that  $\mathcal{A}$  attracts the bounded sets for the  $L_x^2$  strong topology. To prove that  $\mathcal{A}$  is compact is very similar and then omitted.  $\square$

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