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Identification of two independent coefficients with one observation for a nonlinear parabolic system

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Résumé

This article is devoted to prove a stability result for two independent coefficients for a 2×2 nonlinear parabolic system with only one observation. The main idea to obtain this result is to use a modified form of the Carleman estimate given in [1].

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1 Introduction

This paper is an improvement of the work [1] in the sense that we determine two independent coefficients with the observation of only one component in a nonlinear 2×2 parabolic system.

Several works concern linear and nonlinear parabolic equations but few concern system of nonlinear parabolic equations. We can cite [6] and the references therein. Such systems arise in biological or ecological system (see [8], [9]).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of \mathbb{R}^n with $n \leq 3$ and $\omega \subset \Omega$ a non empty subset. We denote by ν the outward unit normal to Ω on $\Gamma = \partial\Omega$ assumed to be of class \mathcal{C}^1 . Let $T > 0$ and $t_0 \in (0, T)$. We shall use the following notations $Q_0 = \Omega \times (0, T)$, $Q = \Omega \times (t_0, T)$, $Q_\omega = \omega \times (t_0, T)$, $\Sigma = \Gamma \times (t_0, T)$ and

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$\Sigma_0 = \Gamma \times (0, T)$. We consider the following 2×2 reaction-diffusion system :

$$\begin{cases} \partial_t U = \Delta U + a_{11}(x)U + a_{12}(x)V + a_{13}(x)f(U, V) & \text{in } Q_0, \\ \partial_t V = \Delta V + a_{21}(x)U + a_{22}(x)V & \text{in } Q_0, \\ U(x, t) = k_1(x, t), V(x, t) = k_2(x, t) & \text{on } \Sigma_0, \\ U(x, 0) = U_0 \text{ and } V(x, 0) = V_0 & \text{in } \Omega, \end{cases} \quad (1)$$

where, the function f is assumed to be Lipschitz with respect the two variables U and V .

Uniqueness and existence results for initial boundary value problem for such systems can be found in [7].

Throughout this paper, we consider the following set

$$\Lambda(R) = \{\Phi \in L^\infty(\Omega); \|\Phi\|_{L^\infty(\Omega)} \leq R\},$$

where R is a given positive constant. For $t_0 \in (0, T)$, we denote $T' = \frac{t_0+T}{2}$.

Let (U, V) (resp. (\tilde{U}, \tilde{V})) be solution of (1) associated to $(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0)$ (resp. $(a_{11}, a_{12}, \tilde{a}_{13}, \tilde{a}_{21}, a_{22}, k_1, k_2, U_0, V_0)$) satisfying some regularity and positivity properties :

Assumption 1.1. 1. For $i = 1, 2, j = 1, 2, 3, a_{ij}, \tilde{a}_{13}$ and $\tilde{a}_{21} \in \Lambda(R)$.

2. There exist constants $r_1 > 0$ and $a_0 > 0$ such that

$$\tilde{U}_0 \geq r_1, \tilde{V}_0 \geq 0, a_{21} \geq a_0, \tilde{a}_{21} \geq a_0, a_{11}r_1 + a_{12}\tilde{V}_0 + \tilde{a}_{13}f(r_1, \tilde{V}_0) \geq 0, k_1 \geq r_1 \text{ and } k_2 \geq 0.$$

Such assumptions allow us to state that the function \tilde{U} satisfies $|\tilde{U}(x, T')| \geq r_1 > 0$ in Ω (see [10]).

Assumption 1.2. 1. The function f checks a generalized Lipschitz property in the following sense : $\exists C > 0$, such that $|\partial_t f(U, V) - \partial_t f(\tilde{U}, \tilde{V})| \leq C(|U - \tilde{U}| + |V - \tilde{V}| + |(U - \tilde{U})_t| + |(V - \tilde{V})_t|)$.

2. $\exists r_2 > 0$ such that $f(\tilde{U}, \tilde{V})(T', x) \geq r_2 > 0$ in Ω .

3. $\partial_t f(U, V) \in L^2((0, T); H^2(\Omega))$.

This set of functions is not empty and contains, in particular, a large class of semilinear terms associated with ecological or biological models (e.g. $f(U, V) = U^\alpha V^\beta$ with α and β non negative constants).

The main result is the following Theorem :

Theorem 1.3. Let ω be a subdomain of an open set Ω of \mathbb{R}^n . We suppose that Assumptions 1.1 and 1.2 are checked and $(U, V)(\cdot, T') = (\tilde{U}, \tilde{V})(\cdot, T')$. Furthermore, we assume that \tilde{U}_0, \tilde{V}_0 in $H^2(\Omega)$. Then there exists a constant $C = C(\Omega, \omega, a_0, t_0, T, r_1, r_2, R) > 0$ such that

$$\|a_{21} - \tilde{a}_{21}\|_{L^2(\Omega)}^2 + \|a_{13} - \tilde{a}_{13}\|_{L^2(\Omega)}^2 \leq C\|\partial_t V - \partial_t \tilde{V}\|_{L^2(Q_\omega)}^2.$$

In [1], for a linear reaction diffusion system, we prove a stability result for one coefficient with only one observation. The novelty in this paper is the identification of two coefficients with only one observation for a nonlinear system. The main tool is a Carleman estimate established in [1] which is adapted to recover two independent coefficients, one in each equation of (1).

The paper is organized as follows : In section 2, we give the modified Carleman estimate for a reaction-diffusion system with only one observation. Then using this modified Carleman estimate, we prove in section 3 a stability result for two coefficients with the observation of only one component.

2 Carleman estimate

At first, we recall the general form of the Carleman estimate associated to the operator $\partial_t q - \Delta q$ (see [2], [4], [5]). Let $\omega' \Subset \omega \Subset \Omega$ and let $\tilde{\beta}$ be a $C^2(\Omega)$ function such that

$$\tilde{\beta} > 0, \text{ in } \Omega, \tilde{\beta} = 0 \text{ on } \partial\Omega, \min\{|\nabla\tilde{\beta}(x)|, x \in \overline{\Omega \setminus \omega'}\} > 0 \text{ and } \partial_\nu \tilde{\beta} < 0 \text{ on } \partial\Omega.$$

Then, we define $\beta = \tilde{\beta} + K$ with $K = m\|\tilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (t_0, T)$, we define the following weight functions (see [3])

$$\varphi(x, t) = \frac{e^{\lambda\beta(x)}}{(t - t_0)(T - t)}, \quad \eta(x, t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x)}}{(t - t_0)(T - t)}.$$

We have then the following Carleman estimate :

Theorem 2.1. *Let $\tau \in \mathbb{R}$. Then there exist $\lambda_0 = \lambda_0(\Omega, \omega) \geq 0$, $s_0 = s_0(\lambda_0, T, \tau) > 0$ and a positive constant $C_0 = C_0(\Omega, \omega, \tau)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the following estimate holds :*

$$I(\tau, q) \leq C_0 \left[\iint_{Q_\omega} e^{-2s\eta} \lambda^4 (s\varphi)^{\tau+3} |q|^2 \, dx \, dt + \iint_Q e^{-2s\eta} (s\varphi)^\tau |\partial_t q - \Delta q|^2 \, dx \, dt \right], \quad (2)$$

where

$$I(\tau, q) = \iint_Q e^{-2s\eta} (s\varphi)^{\tau-1} (|\partial_t q|^2 + |\Delta q|^2) \, dx \, dt + \lambda^2 \iint_Q e^{-2s\eta} (s\varphi)^{\tau+1} |\nabla q|^2 \, dx \, dt + \lambda^4 \iint_Q e^{-2s\eta} (s\varphi)^{\tau+3} |q|^2 \, dx \, dt$$

Remark 1. If we denote

$$M_1^{(\tau)} \psi = -\Delta \psi - s^2 \lambda^2 \varphi^2 |\nabla \beta|^2 \psi - \left(\frac{\tau}{2} - s \partial_t \eta\right) \psi, \quad M_2^{(\tau)} \psi = \partial_t \psi + 2s\lambda \left(\varphi + \frac{\tau}{2}\right) \nabla \beta \cdot \nabla \psi,$$

with $\psi = e^{-s\eta} \varphi^{\frac{\tau}{2}} q$, the Carleman estimate (2) also gives an estimation of $\|M_1^{(\tau)} \psi\|_{L^2(Q)}^2 + \|M_2^{(\tau)} \psi\|_{L^2(Q)}^2$.

We assume that $a_{11}, a_{12}, a_{21}, a_{22} \in \Lambda(R)$, $a_{21} \geq a_0 > 0$ and we consider the following system :

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + H_1, & \text{in } Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + H_2 & \text{in } Q_0, \\ Y(x, t) = Z(x, t) = 0 & \text{on } \Sigma_0, \\ Y(x, 0) = K_1, \quad Z(x, 0) = K_2 & \text{in } \Omega, \end{cases} \quad (3)$$

Then we can have, through the result given in [1], the following modified Carleman estimate with a single observation acting on a subdomain ω of Ω for the system (3) :

Theorem 2.2. *There exist $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $s_1 = s_1(\lambda_0, T) > 1$ and a positive constant $C_1 = C_1(\Omega, \omega, R, a_0)$ such that, for any $\lambda \geq \lambda_1$ and any $s \geq s_1$, the following estimate holds :*

$$\begin{aligned} \lambda^{-4}I(-3, Y) + I(0, Z) &\leq C_1 s^4 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \quad (4) \\ + C_1 \left[s^{-3} \lambda^{-4} \iint_Q e^{-2s\eta} \varphi^{-3} |H_1|^2 dx dt + \iint_Q e^{-2s\eta} |H_2|^2 dx dt \right]. \end{aligned}$$

3 Stability result

Let (U, V) (resp. (\tilde{U}, \tilde{V})) be solution of (1) associated to $(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, k_1, k_2, U_0, V_0)$ (resp. $(a_{11}, a_{12}, \tilde{a}_{13}, \tilde{a}_{21}, a_{22}, k_1, k_2, U_0, V_0)$). Then, if we set $u = U - \tilde{U}$, $v = V - \tilde{V}$, $Y = \partial_t u$ and $Z = \partial_t v$, (Y, Z) is solution to the following problem

$$\begin{cases} \partial_t Y = \Delta Y + a_{11}(x)Y + a_{12}(x)Z + \gamma_1 \partial_t f(\tilde{U}, \tilde{V}) + a_{13}(x) \partial_t F(U, V, \tilde{U}, \tilde{V}), & \text{in } Q_0, \\ \partial_t Z = \Delta Z + a_{21}(x)Y + a_{22}(x)Z + \gamma_2 \partial_t \tilde{U} & \text{in } Q_0, \\ Y(x, t) = Z(x, t) = 0 & \text{on } \Sigma_0, \\ Y(x, 0) = \gamma_1 f(U_0, V_0), \quad Z(x, 0) = \gamma_2 U_0 & \text{in } \Omega, \end{cases} \quad (5)$$

where $\gamma_1 = (a_{13} - \tilde{a}_{13})$, $\gamma_2 = (a_{21} - \tilde{a}_{21})$ and $F(U, V, \tilde{U}, \tilde{V}) = f(U, V) - f(\tilde{U}, \tilde{V})$.

If we apply the modified Carleman estimate (4) to the previous system (5), we have

$$\begin{aligned} \lambda^{-4}I(-3, Y) + I(0, Z) &\leq C_1 s^4 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \\ + C_1 \left[s^{-3} \lambda^{-4} \iint_Q e^{-2s\eta} \varphi^{-3} (|\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 + |\partial_t F|^2) dx dt + \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned} \quad (6)$$

Now we shall "absorb" the term $A = s^{-3} \iint_Q e^{-2s\eta} \varphi^{-3} |\partial_t F|^2 dx dt$. So, we need the following lemma (see [6]) :

Lemma 3.1. *There exists a positive constant $C > 0$ such that.*

$$\iint_Q \left| \int_{T'}^t q(x, \xi) d\xi \right|^2 e^{-2s\eta} dx dt \leq \frac{C}{s} \iint_Q |q(x, t)|^2 e^{-2s\eta} dx dt$$

for all large $s > 0$ and $q \in L^2(Q)$.

Since $\varphi^{-3} \leq C \frac{T^6}{4^3}$, $\varphi^{-3} \leq C \frac{T^{12}}{4^6} \varphi^3$ and using Assumption 1.2-(1), the previous Lemma yields

$$A \leq Cs^{-3}(1+s^{-1}) \iint_Q e^{-2s\eta} (|Y|^2 + \varphi^3|Z|^2) dx dt. \quad (7)$$

Therefore, for s and λ large enough, the integral A can be "absorbed" into the left hand side of (6).

Then (6) can be written as follows

$$\begin{aligned} & \lambda^{-4}I(-3, Y) + I(0, Z) \leq C_1 s^4 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \quad (8) \\ & + C_1 \left[s^{-3} \lambda^{-4} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt + \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned}$$

Let us introduce the following integral $\mathcal{I}_1 = \lambda^{-4} \int_{t_0}^{T'} \int_\Omega M_2^{(-3)} \psi_1 \cdot \psi_1 dx dt$, where $\psi_1 = e^{-s\eta} Y \varphi^{-3/2}$. We first estimate \mathcal{I}_1 with the modified Carleman estimate (8) :

$$\begin{aligned} \mathcal{I}_1 & \leq \frac{1}{2} \lambda^{-2} \left[\lambda^{-4} \|M_2^{(-3)} \psi_1\|_{L^2(Q)}^2 + \int_{t_0}^{T'} \int_\Omega e^{-2s\eta} \varphi^{-3} |Y|^2 dx dt \right] \leq \frac{1}{2} \lambda^{-2} C \left[s^4 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \right. \\ & \left. + s^{-3} \lambda^{-4} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt + \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right]. \end{aligned}$$

By computing \mathcal{I}_1 , we obtain

$$\begin{aligned} & \frac{1}{2} \lambda^{-4} \int_\Omega |\psi_1(x, T')|^2 dx \leq 2|\mathcal{I}_1| + Cs \lambda^{-2} \iint_Q e^{-2s\eta} \varphi^{-2} |Y|^2 dx dt \\ & \leq C \lambda^{-2} \left[s^5 \lambda^4 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt + s^{-2} \lambda^{-4} \iint_Q e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt \right. \\ & \quad \left. + s \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \right] \end{aligned}$$

Then, for λ sufficiently large and $\varphi^{-2} \leq CT^4$, we have

$$\lambda^{-4} \int_\Omega e^{-2s\eta(x, T')} \varphi^{-3}(x, T') |Y(x, T')|^2 dx \leq C \left[s^5 \lambda^2 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt \right]$$

$$+ s^{-2}\lambda^{-6} \iint_{\Omega} e^{-2s\eta} \varphi^{-3} |\gamma_1 \partial_t f(\tilde{U}, \tilde{V})|^2 dx dt + s\lambda^{-2} \iint_Q e^{-2s\eta} |\gamma_2 \partial_t \tilde{U}|^2 dx dt \Big]. \quad (9)$$

Moreover, since $(U, V)(\cdot, T') = (\tilde{U}, \tilde{V})(\cdot, T')$, we have $Y(x, T') = \gamma_1 f(\tilde{U}, \tilde{V})(x, T')$. Thus by [7], we can have $\tilde{U} \in H^1((t_0, T); H^2(\Omega))$, so $\partial_t \tilde{U} \in L^2((t_0, T); H^2(\Omega))$. Moreover, by Assumption 1.2-(3) $\partial_t f(\tilde{U}, \tilde{V}) \in L^2((t_0, T); H^2(\Omega))$. Then for $n \leq 3$, $\partial_t \tilde{U}$ and $\partial_t f(\tilde{U}, \tilde{V})$ are in $L^2((t_0, T); L^\infty(\Omega))$ by classical Sobolev imbedding. Thus, using Assumption 1.2-(2) and $|\tilde{U}(x, T')| \geq r_1 > 0$ in Ω , we obtain for λ sufficiently large :

$$\lambda^{-4} \int_{\Omega} e^{-2s\eta(x, T')} \varphi^{-3}(x, T') |\gamma_1|^2 dx \leq C \left[s^5 \lambda^2 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt + s\lambda^{-2} \iint_Q e^{-2s\eta} |\gamma_2|^2 dx dt \right]. \quad (10)$$

In a similar way, we introduce $\mathcal{I}_2 = \int_{t_0}^{T'} \int_{\Omega} M_2^{(0)} \psi_2 \cdot \psi_2 dx dt$, where $\psi_2 = e^{-s\eta} Z$. Thus, using the fact that $Z(x, T') = \gamma_2 \tilde{U}(x, T')$, we obtain for λ sufficiently large

$$\int_{\Omega} e^{-2s\eta(x, T')} |\gamma_2|^2 dx \leq C \left[s^{5/2} \lambda^2 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt + s^{-9/2} \lambda^{-6} \iint_Q e^{-2s\eta} |\gamma_1|^2 dx dt \right]. \quad (11)$$

So, if we add (10) and (11), we have

$$\lambda^{-4} \int_{\Omega} e^{-2s\eta} \varphi^{-3} |\gamma_1|^2 dx + \int_{\Omega} e^{-2s\eta} |\gamma_2|^2 dx \leq C s^5 \lambda^2 \iint_{Q_\omega} e^{-2s\eta} \varphi^4 |Z|^2 dx dt.$$

Then, the proof of Theorem 1.3 is complete.

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