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## RESEARCH ARTICLE

# On the typical rank of real binary forms

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We determine the rank of a general real binary form of degree $d = 4$ or $d = 5$. In the case $d = 5$, the possible values of the rank of such general forms are $3, 4, 5$. This is the first reported case, to our knowledge, where more than two typical ranks have been found. We prove that a real binary form of degree $d$ with $d$ real roots has rank $d$.

**Keywords:** tensor rank ; binary form ; symmetric tensor ; polynomial ; genericity

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## 1. Introduction

**Motivation.** The decomposition of tensors into a minimal sum of rank-1 terms, sometimes referred to as the *Canonical Polyadic decomposition* (CP), has raised a great interest in several quite different branches of engineering [1–7] as well as in mathematics [8–14]. More precisely, the symmetric case of the CP decomposition is dealt with in [9, 12, 13], and the general case in [11, 14], while [10] is a general thorough reference focused on complexity. If we take the example of tensors of order 4 to fix the ideas, the CP decomposition of a tensor $T$ is defined as:

$$T_{ijk\ell} = \sum_{p=1}^{r} \alpha_i(p)\, \beta_j(p)\, \gamma_k(p)\, \delta_\ell(p)$$

The minimal value of $r$ such that the equality holds is called the *rank* of tensor $T$.

In applications, these tensors are often corrupted by measurement or estimation noises, which can be modeled as additive random variables with continuous probability distribution. The consequence is that tensors to be decomposed are *generic*.

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A key question is then to know what is the rank of such tensors as a function of order and dimensions, and when they admit a *finite number* of CP decompositions. For real or complex *symmetric* tensors decomposed in the *complex* field, this question has been answered, and enters the framework of the *Waring problem* [9, 15]. The Theorem of Alexander and Hirschowitz (see [16] and also [17]) answers that the generic rank is always the expected one, with a finite list of exceptions. However, it has not yet received any response either for non symmetric tensors, or for decompositions in the real field. ten Berge showed that real third order generic tensors can assume two typical ranks, $r$ and $r+1$ [18]. But he could not find generic tensors assuming more than two typical ranks. Regarding general complex tensors, some partial results are contained in [19], while [20] provides a complete answer for tensors of format $2 \times \ldots \times 2$. We point out also the recent preprint [21], where the related concept of signature is studied.

**Contribution.** In order to exhibit more than two typical ranks, it seems necessary to consider tensors of order higher than 3. The simplest case we can think of is that of real symmetric tensors of dimension 2 and order 4 or 5. Yet, such tensors are bijectively associated with homogeneous polynomials in two variables [22], namely binary quartics and quintics, respectively.

The previous definition translates to binary forms as follows. The rank of a binary form $f(x,y) = \sum_{i=0}^{d} \binom{d}{i} a_i \, x^{d-i} y^i$ of degree $d$ defined over the field $\mathbb{K}$ is the minimum integer $r$ such that there is a decomposition $f = \sum_{j=1}^{r} c_j \, (l_j)^d$ where $l_j$ are linear forms and $c_j \in \mathbb{K}$. Note that the rank is called "length" in [23]. If $\mathbb{K}$ is the real field, coefficients $c_j$ can be imposed to belong to $\{-1, 1\}$. If $\mathbb{K}$ is the complex field, one can impose $c_j = 1$ for all coefficients. In the complex field, the rank of a general binary form $f$ of odd degree $d = 2n+1$ is $n+1$. The Sylvester Theorem asserts that the decomposition of such general form $f$ as a sum of $n+1$ powers of linear forms is unique (up to a multiplicative factor for each $c_j$) , and gives also a way to determine it. The *rank* of a general binary form $f$ of even degree $d = 2n$ is $n+1$, but in this case such decompositions form an infinite set, which can be identified with the projective line.

It is interesting that no semicontinuity holds for the rank. Indeed there are forms such that their rank is bigger than the general value and other such that the rank is lower than the general value. Indeed it is well known (see for example Comas and Seiguer [24]) that on the complex numbers the rank can attain all the values between 1 and $d$.

Define $S^{\mathbb{K}}_{d,r} = \{f \in Sym^d(\mathbb{K}^2) | \text{rank } f = r\}$, the set of binary forms of rank exactly $r$. On complex numbers, $S^{\mathbb{C}}_{d,r}$ has a non empty interior only for the pairs $(d,r)$ where $r = \lfloor \frac{d}{2} \rfloor + 1$. That is, for every $d$, there is a unique $r$ such that $S^{\mathbb{C}}_{d,r}$ is dense. The latter values of $r$ are called *generic ranks* in $\mathbb{C}$. On the real numbers we are intested in the pairs $(d,r)$ such that $S^{\mathbb{R}}_{d,r}$ has non empty interior. If for a given value of $d$ there exist several values of $r$, the latter are called *typical ranks*. In other words, if $r_i$ are typical, none of the sets $S^{\mathbb{R}}_{d,r_i}$ is dense, but $\cup_i S^{\mathbb{R}}_{d,r_i}$ is. For $d = 2, 3$ the situation is well known and we recall it in the next section. Our main results are the following, for $d = 4, 5$.

**Main Theorem**

(i) $S^{\mathbb{R}}_{4,r}$ has non empty interior only for $r = 3, 4$. More precisely, assume that $f$ has distinct roots, then

- (i1) if $f$ has four real roots then the rank of $f$ is 4.
- (i2) if $f$ has zero or two real roots then the rank of $f$ is 3.

(ii) $S^{\mathbb{R}}_{5,r}$ has non empty interior only for $r = 3, 4, 5$. More precisely, assume that $f$ has distinct roots, then

- (ii1) if $f$ has five real roots then the rank of $f$ is 5.
- (ii2) if $f$ has one or three real roots then the rank of $f$ is 3 or 4, according to the sign of an invariant of degree 12 defined in section 4, where we provide an algorithm.

Very probably, $S^{\mathbb{R}}_{d,r}$ has a non empty interior if and only if $\lfloor \frac{d}{2} \rfloor + 1 \leq r \leq d$. We leave this as a conjecture to the readers (proved here for $d \leq 5$), as the intriguing problem of the relations between $S^{\mathbb{R}}_{d,r}$ and the loci where the number of real roots is fixed. We prove also in Prop. 3.1 that if a form of degree $d$ has the maximal number $d$ of real roots, then its rank attains the maximum possible value which is again $d$.

There is always an open subset where the general rank is the same as the complex one. In other words, for given order and dimension, the smallest typical rank in the real field coincides with the generic rank in the complex field (see [25], [26]).

## 2. Preliminaries: quadratic and cubic forms

An elementary example illustrating the difference between real and complex rank is given by the equality

$$2x^3 - 6xy^2 = (x + \sqrt{-1}y)^3 + (x - \sqrt{-1}y)^3 = (2^{2/3}x)^3 - (x + y)^3 - (x - y)^3$$

In this case the complex rank is 2 and the real rank is 3.

Let's begin with a known general statement.

Proposition 2.1 Any binary real form of degree $d$ has real rank $\leq d$.

*Proof:* The points of the projective space $\mathbf{P}^d = \mathbf{P}(S^d\mathbb{R}^2)$ correspond to forms $f = \sum_{i=0}^{d} \binom{d}{i} a_i \, x^{d-i}y^i$ , which have coordinates $(a_0, \ldots, a_d)$ . The rational normal curve $C_d$, corresponds to polynomials which are $d$-th powers of linear forms. From the expansion $(t_0x + t_1y)^d = \sum_{i=0}^{d} \binom{d}{i} t_0^{d-i}t_1^i \, x^{d-i}y^i$ we get that the curve $C_d$ can be parametrized by $a_i = t_0^{d-i}t_1^i$. Pick $d - 1$ general points on $C_d$ corresponding to $l_i^d = (l_{i,0}x + l_{i,1}y)^d$ for $i = 1, \ldots, d - 1$. The linear span of $f$ and these points is a hyperplane, whose equation $\sum \binom{d}{i} a_i c_i = 0$ restricts to $C_d$ to the binary form $\sum \binom{d}{i} c_i t_0^{d-i} t_1^i$ of degree $d$ with the $d - 1$ real roots $(t_0, t_1) = (l_{i,0}, l_{i,1})$ (because $\sum \binom{d}{i} c_i l_{i,0}^{d-i} l_{i,1}^i = 0$) hence also the last root is real, corresponding to a last linear form $l_d^d$. This means that $f$ is a projective linear combination of the the powers $l_i^d$ for $i = 1, \ldots, d$, or equivalently, $f$ has rank $\leq d$. □

Because of the bijection mentioned earlier, $f \in Sym^2(\mathbb{K}^2)$ corresponds to a symmetric $2 \times 2$ matrix, and its rank is the same as the rank of the associated matrix. Hence $f$ has rank 2 if and only if $f$ has distinct roots, both for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. So $S^{\mathbb{R}}_{2,r}$ has non empty interior only if $r = 2$.

For any field of characteristic zero, the dual ring of differential operators $D = \mathbb{K}[\partial_x, \partial_y]$ acts on the polynomial ring $R = \mathbb{K}[x, y] = \oplus_{d\geq 0} Sym^d(\mathbb{K}^2) = \oplus_{d\geq 0} R_d$ with the usual rules of differentiations. This action preserves the degrees, in the

sense that we have the pairing

$$R_d \otimes D_k \to R_{d-k}$$

which means that the homogeneous differential operators of degree $k$ takes homogeneous polynomials of degree $d$ to homogeneous polynomials of degree $d-k$.

If $l = ax + by \in R_1$, we denote by $l^\perp = b\partial_x - a\partial_y \in D_1$ its orthogonal (apolar) operator, such that $l^\perp(l) = 0$. The main remark, which goes back to the XIXth century, is that the form $f = \sum_{j=1}^r c_j (l_j)^d$ is killed by the operator $\prod_{j=1}^r l_j^\perp$.

The space of operators of degree $k$ which kill a given form $f$ of degree $d$ is the kernel of the linear map $A_f \colon D_k \to R_{d-k}$; see [22] for instance.

In the basis $\{\partial_x^{k-i}\partial_y^i\}_{i=0,\dots k}$ and $\{\binom{d-k}{j} x^{d-k-j}y^j\}_{j=0,\dots,d-k}$ the matrix of $A_f$ is (up to scalar multiples) the well known Hankel matrix (called catalecticant by Sylvester) of size $(d-k+1)\times(k+1)$

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_k \\ a_1 & a_2 & a_3 & \dots & \\ \vdots & & & & \vdots \\ a_{d-k} & & & \dots & a_d \end{bmatrix}$$

For general forms $f$, the map $A_f$ has maximal rank. We may assume $2k \le d$, so that its kernel has dimension $d-2k$. This holds on any field $\mathbb{K}$.

In particular the general form of degree $d = 2n+1$ is killed by a unique operator of degree $n$. This gives a proof of Sylvester Theorem over $\mathbb{C}$. The kernel of the Hankel matrix of size $n \times (n+1)$ is an operator $g$ of degree $n$, and decomposing it over $\mathbb{C}$ as $g = \prod_{j=1}^n l_j^\perp$ gives the decomposition of $f$ as sum of $n$ $d$-th powers of the $l_j$. If $f$ is real, the kernel $g$ is real, but the decomposition is possible only if all the roots of $g$ (thought as a polynomial) are real.

Comon and Mourrain explored further this situation [27]. If it is not possible to find a kernel of degree $n$ such that all its roots are real, we consider operators of degree $(n+1)$ and we look for such an operator in the kernel such that all the roots are real. If this is not possible we go further, and so on.

We use this technique by reproving the classically known criterion to determine the rank in the case $d = 3$. For $f \in Sym^3(\mathbb{R}^2)$, consider the discriminant

$$\Delta(f) = 4 \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}^2$$

Proposition 2.2 $S^{\mathbb{R}}_{3,r}$ has non empty interior only for $r = 2, 3$. Precisely, let $f$ be a polynomial of third degree without multiple roots. Then

(i) $f$ has rank two if and only if $\Delta(f) < 0$, or equivalently, if and only if $f$ has one real root

(ii) $f$ has rank three if and only if $\Delta(f) > 0$, or equivalently, if and only if $f$ has three real roots.

*Proof:*

The differential operators of degree two which annihilate $f$ consist of the kernel of the matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{bmatrix}$$

The discriminant of the quadratic generator of the kernel coincides with $-\Delta(f)$; thus the operators have two real roots if $\Delta(f) < 0$ and this means that the rank-2 complex decomposition is actually real. Note also that a cubic of real rank two can have only one real root. Indeed the equation $l_1^3 + l_2^3 = 0$ reduces to the three linear equations $l_1 - e^{\frac{n\pi i}{3}} l_2 = 0$ for $n = 0, 1, 2$. This proves (i). If $\Delta(f) > 0$, the quadratic generator has no real root and (ii) follows from Prop. 2.1. □

**Remark** The complete description of rank of tensors of dimension 2 and order 3 can be found in [28](ch. 14, ex. 4.5) in the complex case and in [29] in the real case.

## 3. Forms with all real roots, with an application to quartics

Proposition 3.1 Let $f$ be a real binary form of degree $d$ with $d$ real distinct roots. Then the (real) rank of $f$ is equal to $d$.

*Proof:* The proof is by induction on $d$. By the previous section, let $d \geq 3$. Assume the rank is $\leq d-1$. Then we get $f = \sum_{i=1}^{d-1} l_i^d$ and since there are infinitely many such decompositions we may assume that $l_{d-1}$ does not divide $f$. Consider the rational function

$$F = \frac{f}{l_{d-1}}$$

Under a linear (real) change of projective coordinates $\phi(x, y) = (x', y')$ with $y' = l_{d-1}$ we get $G(x', y') = F(\phi^{-1}(x', y')) = \frac{f(\phi^{-1}(x', y'))}{y'^d}$

Then the polynomial $G(x', 1) = \sum_{i=1}^{d-2} n_i(x')^d + 1$ has $d$ distinct real roots since $f$ had, (where $\deg n_i = 1$) and its derivative $\frac{d}{dx'} G(x', 1) = \sum_{i=1}^{d-2} d n_i(x')^{d-1} \frac{d}{dx'}(n_i(x'))$ has $d-1$ distinct real roots. Now $\frac{d}{dx'} G(x', 1)$ has rank $\leq d-2$, indeed $\frac{d}{dx'} n_i(x')$ are constants. This contradicts the inductive assumption. Hence the assumption was false and the rank of $f$ must exceed $d-1$. The rank of $f$ must eventually be equal to $d$ from Proposition 2.1. □

Lemma 3.2 It is easy to check that under a general Möbius transformation $x \mapsto \frac{ax+b}{cx+d}$ with $ad - bc \neq 0$ both the number of real roots and the rank of a binary form are invariant.

Lemma 3.3 The following are canonical forms under the action of the Möbius transformation group for general forms.
Case $d = 4$:

- $(x^2 + y^2)(x^2 + ay^2)$ ($a > 0$ no real root, $a < 0$ two real roots)
- $(x^2 - y^2)(x^2 + ay^2)$ ($a < 0$ four real roots)

Case $d = 5$:

- $x(x^2 + y^2)(x^2 + 2axy + by^2)$ ($b - a^2 > 0$ one real root, $b - a^2 < 0$ three real roots)

- $x(x^2-y^2)(x^2+2axy+by^2)$ $\quad (b-a^2<0$ five real roots)

*Proof:* We prove just the first case for $d=4$, the other ones being analogous. When there are two pairs of conjugate roots, they lie in the complex plane on a circle with real center, then a convenient circle inversion makes the four roots on a vertical line. A translation and a homothety centered at zero conclude the argument.

When there is one pair of conjugate roots, assume that they are $\pm\sqrt{-1}$. Then consider the tranformations $x \mapsto \frac{x+c}{-cx+1}$, which preserve $\pm\sqrt{-1}$ and it is easy to show that a convenient choice of $c$ makes the sum of the other two roots equal to zero. □

Proposition 3.4 Let $f$ be a real binary form of degree $d$ with distinct roots.

(i) If $f$ has $d$ real roots then for every $(a,b)\neq(0,0)$ the binary form $af_x+bf_y$ has $d-1$ real roots.

(ii) Conversely, if for every $(a,b)\neq(0,0)$ the binary form $af_x+bf_y$ has $d-1$ real roots and $3\leq d\leq 5$, then $f$ has $d$ real roots.

*Proof:* (i) Consider that for any substitution $x=at+c, y=bt+d$ with $ad-bc\neq 0$ we have that $F(t)=f(at+c,bt+d)$ has $d$ real roots, then $\frac{d}{dt}f(at+c,bt+d)=af_x+bf_y$ has $d-1$ real roots corresponding to the $d-1$ extremal points of $F$.

(ii) Assume that $f$ has $\leq d-1$ (hence $\leq d-2$) real roots, and let us show that there exist $(a,b)$ such that $af_x+bf_y$ has $\leq d-2$ (hence $\leq d-3$) real roots.

For $d=3$, after a Möbius transformation, we may assume that $f=x^3+3xy^2$. Then $f_x=3(x^2+y^2)$ has no real roots.

For $d=4$ we may assume by the Lemma 3.3 that $f=(x^2+y^2)(x^2+ay^2)$. For $a>-1$ we consider $f_x=x\cdot(4x^2+2(a+1)y^2)$ which has only one real root. For $a<-1$ we consider $f_y=y\cdot(4ay^2+2(a+1)x^2)$ which has only one real root. For $a=-1$ then $f_x-f_y$ has only one real root.

For $d=5$ we may assume by the Lemma 3.3 that $f=x(x^2+y^2)(x^2+2axy+by^2)$. The discriminant of $f_x$ is (up to a positive scalar multiple) $D(a,b):=-540a^2-1584a^4+830b^3-180b^4-180b^2-8192a^6+405b^5+405b-7476a^2b^2+1548a^2b+14784a^4b-396a^2b^3+576a^4b^2-432b^4a^2$. It can be shown that $f_x$ has zero real roots if $D(a,b)>0$ and two real roots if $D(a,b)<0$. This concludes the proof. □

Corollary 3.5 Let $f$ be a real binary form of degree $d$ with distinct roots. If $f$ has rank $d$ and $3\leq d\leq 5$ then it has $d$ real roots.

*Proof:* The proof is by induction on $d$. For $d=3$ it follows from the Prop. 2.2. Let $4\leq d\leq 5$. If $f$ has $\leq d-2$ real roots then by Prop. 3.4(ii), there exists $(a,b)\neq(0,0)$ such that the binary form $af_x+bf_y$ has $\leq d-3$ real roots. Otherwise, if for every $(a,b)\neq(0,0)$ the form $af_x+bf_y$ has $d-1$ real roots, it follows from Prop. 3.4(ii) that $f$ has $d$ real roots. Then by the inductive assumption $af_x+bf_y$ has rank $\leq d-2$, otherwise, if the rank is $d-1$, it follows that $af_x+bf_y$ has $d-1$ real roots. So we get $af_x+bf_y=\sum_{i=1}^{d-2} l_i^{d-1}$. Choose $c,d$ such that $ad-bc\neq 0$. Let $F(t)=f(at+c,bt+d)$. We get that $F'(t)=\sum_{i=1}^{d-2} n_i(t)^{d-1}$ for some degree one polynomials $n_i$ and by integration there is a constant $K$ and degree one polynomials $m_i$ such that $\frac{F(t)}{(bt+d)^d}=\sum_{i=1}^{d-2}\frac{m_i(t)^d}{(bt+d)^d}+\frac{K}{(bt+d)^d}$.

With the substitution $t=\frac{dx-yc}{-bx+ay}$ we get that the rank of $f$ is $\leq d-1$, which is against the assumption. □

**Remark** We do not know if the part (ii) of the Prop. 3.4 holds for $d\geq 6$. If this is true, also the Cor. 3.5 can be extended.

Now we can give our promised application to quartics.

*Proof of the Main Theorem in the case* $d = 4$. Prop. 3.1 and Cor. 3.5 show that the rank 4 is attained if and only if $f$ has four real roots. By the results quoted at the end of the introduction, the smallest typical rank in the real field coincides with the generic rank in the complex field , which in this case is 3. By the Prop. 2.1 no other typical ranks are possible. □

We end this section with a last general result about forms such that their complex rank is smaller than the general one.

Proposition 3.6 Let $f$ be a real binary form of degree $d$ and complex rank $k$. Assume that $2 \leq k < \lfloor \frac{d}{2} \rfloor + 1$. Then there are only two possibilities:

(i) The real rank is equal to the complex rank

(ii) The real rank of $f$ is $\geq d - k + 2$ (when $k = 2$ the equality holds).

*Proof:*

Assume that (i) does not hold. This means that the contraction

$$D_k \to R_{d-k}$$

has rank $k$ and that the one dimensional kernel is generated by one operator with at least two complex conjugate roots. It follows that also the transpose operator

$$D_{d-k} \to R_k$$

has rank $k$, and the operators in the kernel are given exactly by the previous operator times every operator of degree $d - 2k$. In particular no operator in the kernel has all real roots.

This argument works also for the next contraction

$$D_{d-k+1} \to R_{k-1}$$

which has again rank $k$. At the next step it is possible to find an operator in the kernel with all real roots. This concludes the proof. When $k = 2$ the equality holds by Prop. 2.1. □

## 4. Quintics

It is relatively easy to check if a real quintic form has rank three. Indeed we have to compute the kernel $[\beta_0, \beta_1, \beta_2, \beta_3]$ of the matrix

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{bmatrix}$$

The coefficients $\beta_i$ are given by the $3 \times 3$ minors of the above matrix. Now in order to check if $\beta = \sum_{i=0}^{3} x^{3-i} y^i \beta_i$ has three real roots we consider the invariant

of degree 12

$$I_{12}(\alpha_0, \ldots, \alpha_5) := \Delta(\beta)$$

where $\Delta$ is the discriminant considered before. A nickname for $I_{12}$ could be the "apple invariant", as we will se in the next pictures. We have that the general $f$ has rank three if and only if $\beta$ has three real roots and by Prop. 2.2 (ii) this happens if and only if $I_{12} > 0$.

*Proof of the Main Theorem in the case* $d = 5$

By Prop. 3.1 and Cor. 3.5 we get that, for $d = 5$, a form $f$ with distinct roots has rank 5 if and only if it has 5 real roots. Hence, we have just to prove that both rank three and four have nonempty interior in the other cases.

For quintics with less than 5 real roots, it is enough to consider the canonical form $f = x(x^2 + y^2)(x^2 + 2axy + by^2)$ of Lemma 3.3, and compute for this form the sign of the invariant $I_{12}$. Now $f$ has rank three if and only if $I_{12} > 0$, and rank four if and only if $I_{12} < 0$.

This sign can be checked explicitly from the expression (up to a positive scalar multiple) $I_{12}(a, b) = 2b^3 - 142b^4 - 64a^6 + 3752b^5 - 83a^2b^2 + 144a^4b + 3960a^2b^3 - 11840a^4b^2 - 62900b^4a^2 - 311952b^5a^4 - 593208b^7a^2 + 375552b^3a^6 + 287488ba^8 + 348552b^5a^2 + 243056a^4b^3 + 12096a^6b - 243410b^6a^2 - 673952a^4b^4 - 392736a^6b^2 + 99840a^8b^2 + 700160a^6b^4 + 700160a^4b^6 + 92940a^2b^8 + 7680a^8b^4 + 19712a^{10}b^2 - 336640a^8b^3 - 6560a^6b^6 - 287040a^6b^5 - 115712a^{10}b + 480a^4b^8 - 44400a^4b^7 - 4680a^2b^9 + 77a^2b^{10} - 4096a^8 - 43192b^9 + 170652b^7 - 43192b^6 + 170652b^8 - 78848a^{10} + 3752b^{10} + 2b^{12} - 142b^{11} + 8192a^{12}$.

Yet, two regions of the affine space separated by a variety have a non empty interior. This completes the proof. □

The region where $I_{12} > 0$ intersects in a set of positive measure the two regions where $f$ has one or three real roots, separated by the curve $b - a^2 = 0$. The same holds true for the region where $I_{12} < 0$. As an illustration, these regions are depicted in Figures 1 and 2. For real quintics in the canonical form $f = x(x^2 + y^2)(x^2 + 2axy + by^2)$ the ones with rank four consist of the shaded area in the $(a, b)$-plane in Figure 1, that is, the area where $I_{12} < 0$. $I_{12}$ contains only even powers of $a$, hence the pictures are symmetric with respect the $b$-axis. It follows that the "apple" is all over the parabola (depicted in dashed line in the figures).

The figure 2 shows the quintics with one real root (over the parabola) which have rank four; we get a bounded region.

We get that most of quintics with one real root have rank three, but both rank three and four have nonempty interior, although the rank four part is bounded. In the same way most of quintics with three real roots (under the parabola) have rank three, but both ranks three and four have nonempty interior, and even both regions are unbounded.

We conclude this paper with an algorithm which summarizes our results on real quintics.

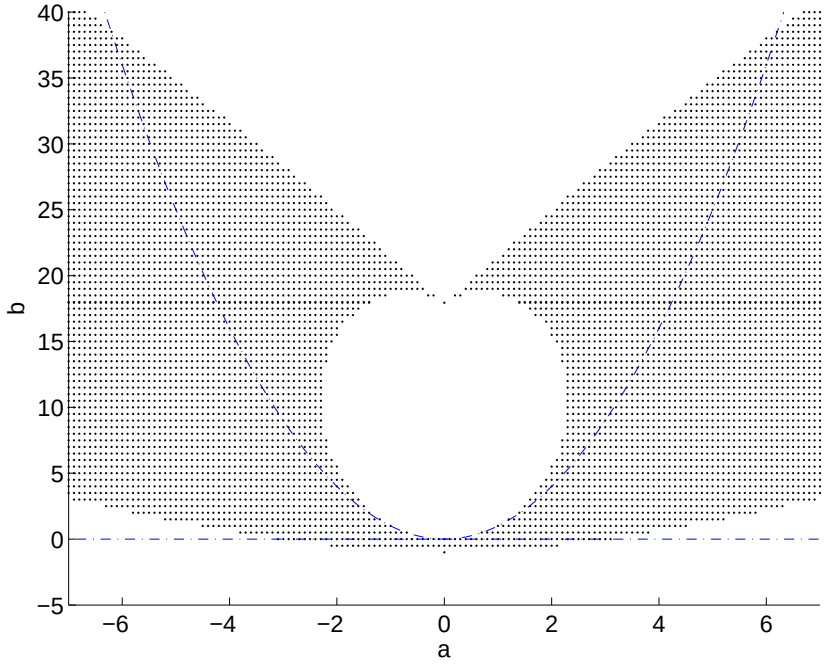


Figure 1. In the shaded area, $I_{12} < 0$, and the quintics considered have rank 4. In the area where $I_{12} > 0$, they have rank 3.

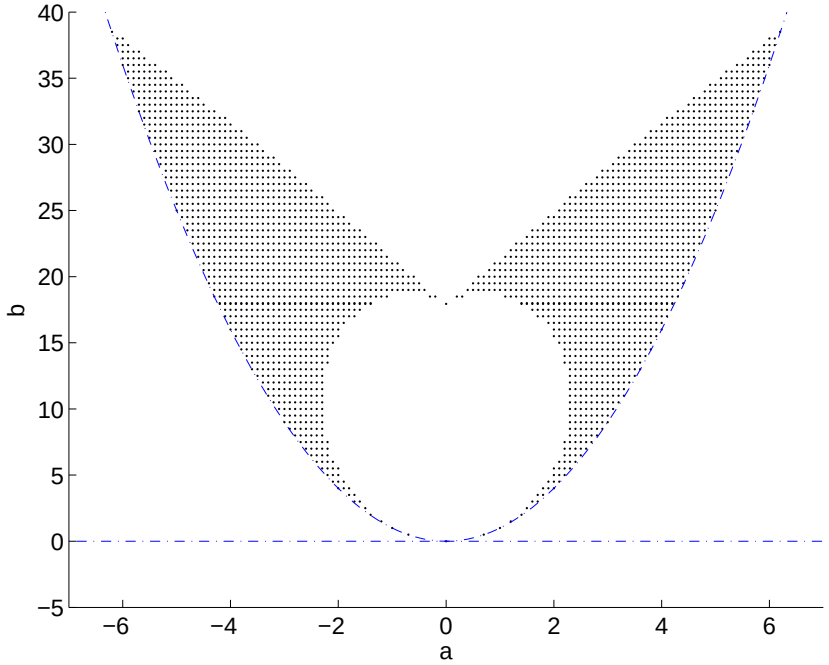


Figure 2. In the shaded area, the quintics considered have rank four and have only one real root; the parabola $b - a^2 = 0$ is represented in dashed line.

***Algorithm to compute the rank of a generic quintic***

Let

$$f(x) = \alpha_0 x^5 + 5\alpha_1 x^4 + 10\alpha_2 x^3 + 10\alpha_3 x^2 + 5\alpha_4 x + \alpha_5$$

(1) If $f$ has five (distinct) real roots then the real rank of $f$ is 5.
(2) Otherwise consider the cubic

$$g(x) = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_3 & \alpha_4 & \alpha_5 \end{vmatrix} x^3 - \begin{vmatrix} \alpha_0 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_4 & \alpha_5 \end{vmatrix} x^2 + \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_5 \end{vmatrix} x - \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix}$$

a) If $g$ has three (distinct) real roots then the real rank of $f$ is 3.
b) If $g$ has one real root and a pair of conjugate (distinct) roots then the real rank of $f$ is 4.

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