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Abstract

In this paper we prove the discrete compactness property for a wide class of p finite element approximations of non-elliptic variational eigenvalue problems in two and three space dimensions. In a very general framework, we find sufficient conditions for the p-version of a generalized discrete compactness property, which is formulated in the setting of discrete differential forms of order ℓ on a polyhedral domain in $\mathbb{R}^d$ ($0 < \ell < d$). One of the main tools for the analysis is a recently introduced smoothed Poincaré lifting operator [M. Costabel and A. McIntosh, On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains, Math. Z., (2010)]. In the case $\ell = 1$ our analysis shows that several widely used families of edge finite elements satisfy the discrete compactness property in p and hence provide convergent solutions to the Maxwell eigenvalue problem. In particular, Nédélec elements on triangles and tetrahedra (first and second kind) and on parallelograms and parallelepipeds (first kind) are covered by our theory.

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1 Introduction: Maxwell eigenvalue problem

Maxwell's eigenvalue problem in a closed cavity $\Omega \in \mathbb{R}^3$ with perfectly conducting walls can be written as follows by means of the Maxwell-Ampère and Faraday laws: Find the resonance frequencies $\omega \in \mathbb{R}$ and the electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0, 0)$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{E} &= i\omega\mu\mathbf{H} && \text{in } \Omega \\ \mathbf{curl}\,\mathbf{H} &= -i\omega\epsilon\mathbf{E} && \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} &= 0 && \text{on } \partial\Omega \\ \mathbf{H}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where ϵ and μ denote the dielectric permittivity and magnetic permeability, respectively. The fields $\mathbf{E}$ and $\mathbf{H}$ are sought in $L^2(\Omega)^3$.

For simplicity, we consider now the case of homogeneous isotropic material with normalized material constants ($\epsilon,\ \mu = 1$) — we will come back to the general setting in Remark 6.3. In a classical way, the elimination of the magnetic field from equations (1.1) yields the Maxwell eigenvalue problem with perfectly electrically conducting (PEC) walls in variational form:

$$\begin{aligned} &\text{Seek } \mathbf{u} \in \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \setminus \{0\},\ \omega \in \mathbb{R}_0^+ \ \text{ such that} \\ &(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\Omega)} = \omega^2\,(\mathbf{u}, \mathbf{v})_{L^2(\Omega)} \ \ \forall \mathbf{v} \in \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega)\ . \end{aligned} \tag{1.2}$$

The elimination of the electric field would correspond to the same problem modelled through replacing $\mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega)$ with $\boldsymbol{H}(\mathbf{curl}, \Omega)$[1].

One aim of this paper is to prove the convergence of $H(\mathbf{curl})$-conforming Galerkin discretizations of Maxwell eigenvalue problem (1.2) in the framework of the p-version of the finite element method. The finite element approximation of Maxwell eigenvalues has been the object of intense investigations for more than 20 years. It was soon recognized that the $H(\mathbf{curl})$-conforming Galerkin finite element discretizations need special finite element spaces that are generally termed *edge* finite elements (see [44, 45, 14]).

The first attempts to analyze the discretized eigenvalue problem have been made for the h-version of edge finite elements. We mention [39] as a pioneering work on lowest order edge finite elements, where the *discrete compactness property* (see [2]) has been indicated as a key ingredient for the analysis. Other relevant works on the subject are [13, 8, 19, 43, 40, 24, 9], and we refer the interested reader to [37, 42] and to the references therein for a review on this topic.

In these references, the Maxwell eigenvalue problem is often studied using variational formulations different from (1.2), for example mixed formulations [9],

[1] By and large, we adopt the standard notations for Sobolev spaces, see [34, Ch. 2].

regularized formulations [23, 25] or mixed regularized formulations [5, 17]. With the exception of the method of weighted regularization [23, 25, 17], where H^1-conforming elements can be used, these formulations use the $H(\mathbf{curl})$-conforming edge elements. In their analysis, special conditions implying convergence of the discrete eigenvalue problems are presented, for example the so-called Fortid property [8], or the GAP property [16]. As explained there, these conditions are related to the discrete compactness property. Here we choose to work with the simple variational formulation (1.2) and its generalization to differential forms. The role of the discrete compactness property in this context has been discussed in detail in [19].

The analysis presented in the references above covers the h-version for basically all known families of edge finite elements. It soon turned out, however, that the analysis of the p- and hp-versions of edge finite elements needed tools different from those developed for the h-version. In [12] the two-dimensional triangular case has been studied for the hp-version, but the analysis depends on a conjectured estimate which has only been demonstrated numerically. In [11] a rigorous proof for the hp-version of 2D rectangular edge elements has been proposed (allowing for one-irregular hanging nodes) which, in particular, contains the first proof of eigenvalue/eigenfunction convergence for the pure spectral method (p-version with one element) on a rectangle.

What paved the way for a successful attack on a general p-version analysis was the regularized Poincaré lifting recently introduced in [26]: it enjoys excellent continuity properties and at the same time respects discrete differential forms. In this paper we are going to show how the regularized Poincaré lifting can be combined with another recent invention, the projection based interpolation operators, see [27, 29], to clinch the analysis of the p-version of edge elements. This allows to prove the discrete compactness (and hence the convergence of the discrete eigensolutions) for a wide class of finite elements related to discrete differential forms: for (1.2) this includes, in particular, Nédélec elements on triangles and tetrahedra (first and second kind) and on parallelograms and parallelepipeds (first kind).

As already mentioned, one of the key ingredients for the convergence analysis is the discrete compactness property. Much insight can be gained from investigating it in the more general framework of discrete differential forms (see [4] for a lucid introduction to this subject). In this setting, the proofs are more natural and simultaneously cover, in particular, two- and three-dimensional Maxwell eigenvalue problems.

Plan of the paper. The structure of the paper is as follows. We start in Section 2 with a generalization of (1.2) to eigenvalue problems associated with the de Rham complex on differential forms. Then we define the discrete compactness

property and discuss its significance in the context of Galerkin discretization: in association with two standard completeness properties, it gives a crucial sufficient condition for the convergence of eigenvalues and eigenvectors. Section 3 is the core of our paper and contains the description of our abstract assumptions. Having in mind the p-version of finite elements, we consider a *fixed* mesh $\mathfrak{M}$ of a bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^d$ and a sequence of spaces of discrete differential forms of order ℓ (with $0 < \ell < d$) together with projection operators onto discrete spaces; we prove that our assumptions imply the validity of the discrete compactness property for such a sequence of spaces (Theorem 3.2). The abstract theory relies on the existence of suitable Poincaré lifting operators which are presented in Section 4. The mapping properties of these lifting operators allow to specify some of the function spaces appearing in our abstract assumptions. In Section 5 we recall the classical families of discrete differential forms with high degree polynomial coefficients on simplicial or tensor product elements.

Our abstract theory applies to any dimension d, but for want of suitable regularity results, embeddings, and projection operators, we can give examples satisfying all of its assumptions only in dimensions $d = 2$ and $d = 3$. This is done in Section 6, where we concretize the function spaces and recall embedding results and properties of projection based interpolation operators related to these spaces. All abstract assumptions are then satisfied, leading to the main convergence result stated in Theorem 6.1. The analysis of a p-version edge element discretization of the Maxwell eigenvalue problem (1.2) is covered as case $d = 3$ and $\ell = 1$, see Corollary 6.2.

2 Differential forms and generalized Maxwell eigenvalue problem

The variational eigenvalue problem (1.2) turns out to be a member of a larger family of eigenvalue problems, when viewed from the perspective of differential forms. This more general perspective offers the benefit of a unified theoretical treatment of different kinds of eigenvalue problems, e.g., the scalar Laplace eigenproblem, Maxwell cavity eigenproblems in dimensions 2 and 3, the eigenproblem for the **grad** div-operator in dimension 3. This policy has had remarkable success in numerical analysis recently, *cf.* [3]. Thus, in this section we first recall some basic notions related to differential forms. We refer the interested reader to [4, Sect. 2] for an introduction to this subject.

2.1 Function spaces of differential forms

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, we denote by $C^\infty(\Omega, \Lambda^\ell)$, $0 \le \ell \le d$, the space of smooth differential forms on Ω and by $\mathbf{\Lambda}(\Omega)$ the corresponding anti-commutative graded algebra

$$\mathbf{\Lambda}(\Omega) = \bigoplus_\ell C^\infty(\Omega, \Lambda^\ell).$$

The *exterior derivative* is a graded linear operator of degree one

$$\mathbf{d} : \mathbf{\Lambda}(\Omega) \to \mathbf{\Lambda}(\Omega),$$

that is, for any ℓ it is represented by $\mathsf{d}_\ell : C^\infty(\Omega, \Lambda^\ell) \to C^\infty(\Omega, \Lambda^{\ell+1})$.

We rely on the Hilbert spaces

$$H(\mathsf{d}_\ell, \Omega) := \{\mathbf{v} \in L^2(\Omega, \Lambda^\ell) : \mathsf{d}_\ell\, \mathbf{v} \in L^2(\Omega, \Lambda^{\ell+1})\}\ , \tag{2.1}$$

where $L^2(\Omega, \Lambda^\ell)$ is the space of differential ℓ-forms on Ω with square integrable coefficients in their canonical basis representation, see [26, Sect. 2]. Its inner product can be expressed as

$$(\mathbf{u}, \mathbf{v})_{0,\Omega} := \int_\Omega \mathbf{u} \wedge \star\, \mathbf{v}\ , \quad \mathbf{u}, \mathbf{v} \in L^2(\Omega, \Lambda^\ell)\ , \tag{2.2}$$

with $\star$ the Hodge star operator induced by the Euclidean metric on $\mathbb{R}^d$, which maps ℓ-forms to $(d-\ell)$-forms. As above, a $\circ$ tags the subspaces of forms with vanishing trace $\mathrm{tr}_{\partial\Omega}$ on $\partial\Omega$, which can also be obtained by the completion of compactly supported smooth ℓ-forms with respect to the $H(\mathsf{d}_\ell, \Omega)$-norm:

$$\mathring{H}(\mathsf{d}_\ell, \Omega) := \{\mathbf{v} \in H(\mathsf{d}_\ell, \Omega) : \mathrm{tr}_{\partial\Omega}\, \mathbf{v} = 0\}. \tag{2.3}$$

The subspace of *closed forms* is the kernel of d_ℓ and is denoted by $\mathring{H}(\mathsf{d}_\ell\, 0, \Omega)$:

$$\mathring{H}(\mathsf{d}_\ell\, 0, \Omega) := \{\mathbf{v} \in \mathring{H}(\mathsf{d}_\ell, \Omega) : \mathsf{d}_\ell\, \mathbf{v} = 0\}. \tag{2.4}$$

Finally, we adopt the standard notation for the exterior co-derivative operator $\delta_\ell := \star\, \mathsf{d}_{d-\ell}\, \star$; in particular, we have $\delta_\ell : C^\infty(\Omega, \Lambda^\ell) \to C^\infty(\Omega, \Lambda^{\ell-1})$.

2.2 Variational eigenvalue problems

After choosing bases for the spaces of alternating multilinear forms on $\mathbb{R}^d$, vector fields ("vector proxies") $\Omega \mapsto \mathbb{R}^{\binom{d}{\ell}}$ provide an isomorphic model for differential ℓ-forms on Ω. Choosing the standard "Euclidean basis", the operators $\star, \delta, \mathrm{tr}_{\partial\Omega}$ are

Table 1: Identification between (operators on) differential forms and (operators on) Euclidean vector proxies in $\mathbb{R}^2$ and $\mathbb{R}^3$

Differential form		Proxy representation	
		$d=2$	$d=3$
$\ell=0$	d_0	**grad**	**grad**
	$\mathrm{tr}_{\partial\Omega}\,\phi$	$\phi_{\|\partial\Omega}$	$\phi_{\|\partial\Omega}$
	$\mathring{H}(\mathsf{d}_0,\Omega)$	$\mathring{H}^1(\Omega)$	$\mathring{H}^1(\Omega)$
$\ell=1$	d_1	curl	**curl**
	$\mathrm{tr}_{\partial\Omega}\,\mathbf{u}$	$(\mathbf{u}\times\mathbf{n})_{\|\partial\Omega}$	$(\mathbf{u}\times\mathbf{n})_{\|\partial\Omega}$
	$\mathring{H}(\mathsf{d}_1,\Omega)$	$\mathring{\boldsymbol{H}}(\mathsf{curl},\Omega)$	$\mathring{\boldsymbol{H}}(\mathbf{curl},\Omega)$
	δ_1	div	div
$\ell=2$	d_2	0	div
	$\mathrm{tr}_{\partial\Omega}\,\mathbf{q}$	0	$(\mathbf{q}\cdot\mathbf{n})_{\|\partial\Omega}$
	$\mathring{H}(\mathsf{d}_2,\Omega)$	$L^2(\Omega)$	$\mathring{\boldsymbol{H}}(\mathsf{div},\Omega)$
	δ_2	$\overrightarrow{\mathsf{curl}}$	**curl**

incarnated by familiar operators of classical vector analysis, different for different dimension d and degree ℓ, see Table 1 and [4, Table 2.1].

Hence, the eigenvalue problem (1.2) with $\epsilon,\mu\equiv 1$ is the special case $d=3$, $\ell=1$, of the following variational eigenvalue problem for differential ℓ-forms, $0\le\ell<d$:

$$\begin{aligned}&\text{Seek } \mathbf{u}\in\mathring{H}(\mathsf{d}_\ell,\Omega)\setminus\{0\},\ \omega\in\mathbb{R}_0^+,\text{ such that}\\ &(\mathsf{d}_\ell\,\mathbf{u},\mathsf{d}_\ell\,\mathbf{v})_{0,\Omega}=\omega^2\,(\mathbf{u},\mathbf{v})_{0,\Omega}\quad\forall\mathbf{v}\in\mathring{H}(\mathsf{d}_\ell,\Omega)\ .\end{aligned}\tag{2.5}$$

A key observation is that the bilinear form $(\mathbf{u},\mathbf{v})\mapsto(\mathsf{d}_\ell\,\mathbf{u},\mathsf{d}_\ell\,\mathbf{v})_{0,\Omega}$ has an infinite dimensional kernel $\mathring{H}(\mathsf{d}_\ell\,0,\Omega)$ comprising all closed ℓ-forms. It provides the invariant subspace associated with the essential spectrum $\{0\}$ of (2.5). This essential spectrum can be identified as the main source of difficulties confronted in the Galerkin discretization of (2.5).

On the other hand, any solution $\mathbf{u}$ of (2.5) for $\omega\neq 0$ satisfies $(\mathbf{u},\mathsf{d}_{\ell-1}\,\boldsymbol{\psi})_{0,\Omega}=0$ for all $\boldsymbol{\psi}\in\mathring{H}(\mathsf{d}_{\ell-1}\,0,\Omega)$. Thus the eigenfunctions corresponding to non-zero eigenvalues belong to the subspace

$$\mathring{Y}(\mathsf{d}_\ell,\Omega):=\{\mathbf{v}\in\mathring{H}(\mathsf{d}_\ell,\Omega):(\mathbf{v},\mathsf{d}_{\ell-1}\,\boldsymbol{\psi})_{0,\Omega}=0\ \ \forall\boldsymbol{\psi}\in\mathring{H}(\mathsf{d}_{\ell-1},\Omega)\},\tag{2.6}$$

which means they belong to the kernel of δ_ℓ. This is the generalization of the

divergence free constraint found for electric fields in the Maxwell case. From [46] we learn the following theorem.

Theorem 2.1 *For any $d \in \mathbb{N}$, $0 \leq l \leq d$, the embedding*

$$\mathring{Y}(\mathsf{d}_\ell, \Omega) \hookrightarrow L^2(\Omega, \Lambda^\ell)$$

is compact.

By restricting the eigenvalue problem to $\mathring{Y}(\mathsf{d}_\ell, \Omega)$, we can therefore use Riesz-Schauder theory. This implies that (2.5) gives rise to an unbounded sequence of positive eigenvalues $\lambda^k = (\omega^k)^2$

$$\lambda^0 = 0 < \lambda^1 \leq \lambda^2 \leq \dots , \qquad \lambda^k \to \infty \ (k \to \infty) , \tag{2.7}$$

with associated finite dimensional mutually $L^2(\Omega)$-orthogonal eigenspaces.

Remark 2.2 Owing to the zero trace boundary conditions imposed on the functions in (2.5), it may be called a Dirichlet eigenvalue problem. Using $H(\mathsf{d}_\ell, \Omega)$ as variational space would result in the corresponding Neumann eigenvalue problem. Its analysis runs utterly parallel to the Dirichlet case using the techniques presented below. □

2.3 Approximation of the eigenvalue problem and the role of discrete compactness

In the sequel we fix the degree ℓ, $0 \leq \ell < d$, of the differential forms. Spaces of *discrete differential forms*

$$\mathring{\mathcal{V}}_p^\ell \subset \mathring{H}(\mathsf{d}_\ell, \Omega) , \quad \dim \mathring{\mathcal{V}}_p^\ell < \infty ,$$

lend themselves to a straightforward discretization of (2.5). In this section, $p \in \mathbb{N}$ stands for an abstract discretization parameter, and, sloppily speaking, large values of p hint at trial/test spaces of high resolution.

We consider the approximation of the eigenvalue problem (2.5) by the Galerkin method:

$$\begin{aligned} &\text{Find } \mathbf{u}_p \in \mathring{\mathcal{V}}_p^\ell \setminus \{0\}, \ \omega \in \mathbb{R}_0^+, \text{ such that} \\ &(\mathsf{d}_\ell \, \mathbf{u}_p, \mathsf{d}_\ell \, \mathbf{v}_p)_{0,\Omega} = \omega^2 \, (\mathbf{u}_p, \mathbf{v}_p)_{0,\Omega} \quad \forall \mathbf{v} \in \mathring{\mathcal{V}}_p^\ell . \end{aligned} \tag{2.8}$$

Now, the key issue is convergence of eigenvalues and eigenvectors as $p \to \infty$, rigorously cast into the concept of *spectrally correct, spurious-free approximation* [19, Sect. 4]. Let us recall these notions in a few words for the case of self-adjoint nonnegative operators without continuous spectrum (which is the case here).

The spectral correctness of the approximation of an eigenvalue problem such as (2.5) by a sequence of finite rank eigenvalue problems (2.8) means that all eigenvalues and all eigenvectors of (2.5) are approached by the eigenvalues and eigenvectors of (2.8) as $p \to \infty$. If (2.5) has a compact resolvent (which is the case *only when* $\ell = 0$), the spectral correctness is an optimal notion: It implies that if $\{\lambda^k\}_{k\geq 1}$ and $\{\lambda_p^k\}_{k\geq 1}$ are the increasing eigenvalue sequences of (2.5) and (2.8) (with eigenvalues repeated according to their multiplicities), then

$$\lambda_p^k \to \lambda^k \quad \text{as} \quad p \to \infty \quad \forall k \geq 1, \tag{2.9}$$

and the gaps between eigenspaces (correctly assembled according to multiplicities of the eigenvalues of (2.5)) tend to 0 as $p \to \infty$.

If we face an eigenvalue problem for a self-adjoint non-negative operator with an infinite dimensional kernel, and otherwise discrete positive spectrum (which is the case for (2.5) for all $\ell \geq 1$), the spectral correctness implies the same properties as above with the following modifications of the definitions: Now $\{\lambda^k\}_{k\geq 1}$ is the increasing sequence of *positive* eigenvalues of (2.5) (as specified in (2.7)) and, given a positive number $\varepsilon < \lambda^1$, $\{\lambda_p^k\}_{k\geq 1}$ is the increasing sequence of the eigenvalues of (2.8) larger than ε (still with repetitions according to multiplicities). With such conventions, *spectral correctness* still implies convergence of eigenvalues (2.9) and eigenspaces as above. In this context, *spurious-free approximation* means that there exists $\varepsilon_0 > 0$ such that all eigenvalues of (2.8) less than ε_0 are zero. Therefore, spectrally correct, spurious-free approximation implies the convergence property (2.9) and the corresponding convergence of eigenspaces, if we define $\{\lambda_p^k\}_{k\geq 1}$ as the increasing sequence of the *positive eigenvalues of* (2.8).

There exist several different ways, all well studied and summarized in the literature of the last decade, for proving the convergence of the discrete eigenvalue problem (2.8) to the continuous eigenvalue problem (2.5): One can use a reformulation as an eigenvalue problem in mixed form as analyzed in [9], or one can use a regularization which gives an elliptic eigenvalue problem for the Hodge-Laplace operator as analyzed in [4], or one can follow the arguments of [19] and study the non-elliptic problem (2.5) directly.

Here we outline the latter approach, which employs the analysis of [32] of the approximation of eigenvalue problems of non-compact selfadjoint operators. Since [19] deals only with the Maxwell case, i. e. $d = 3$, $\ell = 1$, we examine the main arguments, in order to verify that they are also valid for the general case. The proofs we give are adaptations of those of [19] to our more general situation.

Let us define the solution operator $A : L^2(\Omega, \Lambda^\ell) \to \mathring{H}(\mathsf{d}_\ell, \Omega)$ of the source problem corresponding to the eigenvalue problem (2.5) and its discrete counter-

part $A_p : \mathring{\mathcal{V}}_p^\ell \to \mathring{\mathcal{V}}_p^\ell$ by

$$\begin{aligned} (\mathsf{d}_\ell\, A\mathbf{f}, \mathsf{d}_\ell\, \mathbf{v})_{0,\Omega} + (A\mathbf{f}, \mathbf{v})_{0,\Omega} &= (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall \mathbf{v} \in \mathring{H}(\mathsf{d}_\ell, \Omega) \\ (\mathsf{d}_\ell\, A_p\mathbf{f}, \mathsf{d}_\ell\, \mathbf{v})_{0,\Omega} + (A_p\mathbf{f}, \mathbf{v})_{0,\Omega} &= (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall \mathbf{v} \in \mathring{\mathcal{V}}_p^\ell\,. \end{aligned} \tag{2.10}$$

Note that the operators A and A_p have the same eigenfunctions and the same eigenvalues (after a transformation) as the eigenvalue problems (2.5) and (2.8). Namely, (2.5) and (2.8) are equivalent to the relations

$$\mathbf{u} = (\omega^2 + 1)A\mathbf{u}\,; \qquad \mathbf{u}_p = (\omega^2 + 1)A_p\mathbf{u}_p\,. \tag{2.11}$$

The infinite-dimensional eigenspace at $\omega = 0$ shows that A is not a compact operator.

Following [19], three conditions are identified that together are necessary and sufficient for a spectrally correct, spurious-free approximation of A by A_p or, equivalently, of the eigenvalue problem (2.5) by the discrete eigenvalue problem (2.8).

The first condition is rather natural. It states that the sequence of discrete spaces $\big(\mathring{\mathcal{V}}_p^\ell\big)_{p\in\mathbb{N}}$ is asymptotically dense in $\mathring{H}(\mathsf{d}_\ell, \Omega)$ (compare [19, Condition (CAS) – completeness of approximating subspaces])

$$\text{(CAS)} \qquad \lim_{p\to\infty} \inf_{\mathbf{v}_p \in \mathring{\mathcal{V}}_p^\ell} \|\mathbf{v} - \mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)} = 0 \quad \forall \mathbf{v} \in \mathring{H}(\mathsf{d}_\ell, \Omega)\,. \tag{2.12}$$

The second condition, only relevant for $\ell > 0$, states that closed forms can be well approximated by discrete closed forms (compare [19, Condition (CDK) – completeness of discrete kernels])

$$\text{(CDK)} \quad \lim_{p\to\infty} \inf_{\mathbf{z}_p \in \mathring{\mathcal{V}}_p^\ell \cap \mathring{H}(\mathsf{d}_\ell\, 0,\Omega)} \|\mathbf{z} - \mathbf{z}_p\|_{L^2(\Omega)} = 0 \quad \forall \mathbf{z} \in \mathring{H}(\mathsf{d}_\ell\, 0, \Omega)\,. \tag{2.13}$$

The third condition is the most intricate one and has been dubbed *discrete compactness*. For its formulation, we introduce the orthogonal complement space of the discrete closed forms:

$$\mathring{\mathcal{Z}}_p^\ell := \{\mathbf{u}_p \in \mathring{\mathcal{V}}_p^\ell :\ (\mathbf{u}_p, \mathbf{z}_p)_{0,\Omega} = 0 \quad \forall \mathbf{z}_p \in \mathring{\mathcal{V}}_p^\ell \cap \mathring{H}(\mathsf{d}_\ell\, 0, \Omega)\}. \tag{2.14}$$

Definition 2.3 *Let us choose $\ell \in \{1, \dots, d-1\}$. The* discrete compactness property *holds for a family $\big(\mathring{\mathcal{V}}_p^\ell\big)_{p\in\mathbb{N}}$ of finite dimensional subspaces of $\mathring{H}(\mathsf{d}_\ell, \Omega)$, if for any subsequence $\mathbb{N}'$ of $\mathbb{N}$, any* bounded *sequence*

$$\big(\mathbf{u}_p\big)_{p\in\mathbb{N}'} \subset \mathring{H}(\mathsf{d}_\ell, \Omega) \quad \textit{with} \quad \mathbf{u}_p \in \mathring{\mathcal{Z}}_p^\ell$$

contains a subsequence that converges in $L^2(\Omega, \Lambda^\ell)$.

The convergence proof is based on two lemmas, the first of which corresponds to [19, Theorem 4.12]. It implies, according to [32, Condition P1) and Theorems 2,4,5,6], the spectral correctness of the approximation.

Lemma 2.4 *If* (2.12) *and the discrete compactness property hold, then*

$$\lim_{p\to\infty} \sup_{\mathbf{v}_p\in\mathring{\mathcal{V}}_p^\ell\,;\,\|\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)}=1} \|A\mathbf{v}_p - A_p\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)} = 0\,. \tag{2.15}$$

Proof. Note first that for $\mathbf{v}_p \in \mathring{\mathcal{V}}_p^\ell \cap \mathring{H}(\mathsf{d}_\ell\, 0,\Omega)$ there holds $A\mathbf{v}_p = \mathbf{v}_p = A_p\mathbf{v}_p$, so that by orthogonal decomposition of $\mathring{\mathcal{V}}_p^\ell$ one gets

$$\sup_{\mathbf{v}_p\in\mathring{\mathcal{V}}_p^\ell\,;\,\|\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)}=1} \|A\mathbf{v}_p - A_p\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)} = \sup_{\mathbf{v}_p\in\mathring{\mathcal{Z}}_p^\ell\,;\,\|\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)}=1} \|A\mathbf{v}_p - A_p\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)}\,.$$

Furthermore, one has by definition of A and A_p

$$\|A\mathbf{v}_p - A_p\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)} = \inf_{\mathbf{w}_p\in\mathring{\mathcal{V}}_p^\ell} \|A\mathbf{v}_p - \mathbf{w}_p\|_{H(\mathsf{d}_\ell,\Omega)}\,.$$

Assume now that (2.15) does not hold. Then there exists $\varepsilon > 0$, a subsequence $\mathbb{N}'$ of $\mathbb{N}$ and a sequence $(\mathbf{v}_p)_{p\in\mathbb{N}'}$ with $\mathbf{v}_p \in \mathring{\mathcal{Z}}_p^\ell$ satisfying $\|\mathbf{v}_p\|_{H(\mathsf{d}_\ell,\Omega)} = 1$ and

$$\|A\mathbf{v}_p - \mathbf{w}_p\|_{H(\mathsf{d}_\ell,\Omega)} \geq \varepsilon \qquad \forall p\in\mathbb{N}',\ \mathbf{w}_p \in \mathring{\mathcal{V}}_p^\ell\,. \tag{2.16}$$

We can apply the discrete compactness property to the sequence $(\mathbf{v}_p)$ and obtain a subsequence converging in $L^2(\Omega,\Lambda^\ell)$ to some $\mathbf{v} \in L^2(\Omega,\Lambda^\ell)$. Since $A : L^2(\Omega,\Lambda^\ell) \to \mathring{H}(\mathsf{d}_\ell,\Omega)$ is continuous, we find $A\mathbf{v} \in \mathring{H}(\mathsf{d}_\ell,\Omega)$, and the approximation property (2.12) provides us with a sequence $(\mathbf{w}_p)$ with $\mathbf{w}_p \in \mathring{\mathcal{V}}_p^\ell$ that converges in $\mathring{H}(\mathsf{d}_\ell,\Omega)$ to $A\mathbf{v}$. Hence for the subsequence we obtain

$$\|A\mathbf{v}_p - \mathbf{w}_p\|_{H(\mathsf{d}_\ell,\Omega)} \leq \|A\mathbf{v}_p - A\mathbf{v}\|_{H(\mathsf{d}_\ell,\Omega)} + \|A\mathbf{v} - \mathbf{w}_p\|_{H(\mathsf{d}_\ell,\Omega)} \to 0\,,$$

in contradiction with (2.16). □

The second lemma corresponds to [19, Corollary 2.20]. It gives the discrete Friedrichs inequality (in [9] also called "ellipticity in the discrete kernel"), and it is easy to see that this implies that $\omega = 0$ is not a limit point of positive discrete eigenvalues, so that the spurious-free property of the approximation follows.

Lemma 2.5 *If* (2.13) *and the discrete compactness property hold, then there exists* $\alpha > 0$ *such that for all* $p \in \mathbb{N}$

$$\|\mathsf{d}_\ell\,\mathbf{v}\|_{L^2(\Omega)} \geq \alpha\,\|\mathbf{v}\|_{L^2(\Omega)} \quad \forall \mathbf{v} \in \mathring{\mathcal{Z}}_p^\ell \tag{2.17}$$

Proof. Assume that (2.17) does not hold. Then there exists a subsequence $\mathbb{N}'$ of $\mathbb{N}$ and a sequence $(\mathbf{v}_p)_{p\in\mathbb{N}'}$ with $\mathbf{v}_p \in \mathring{\mathcal{Z}}_p^\ell$ satisfying

$$\|\mathbf{v}_p\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \lim_{p\to\infty} \|\mathsf{d}_\ell\, \mathbf{v}_p\|_{L^2(\Omega)} = 0\,. \tag{2.18}$$

The discrete compactness property can be applied to this sequence and gives a subsequence converging in $L^2(\Omega,\Lambda^\ell)$ to some $\mathbf{z} \in L^2(\Omega,\Lambda^\ell)$. From (2.18) follows that the convergence actually takes place in $\mathring{H}(\mathsf{d}_\ell,\Omega)$ and that $\mathbf{z} \in \mathring{H}(\mathsf{d}_\ell\, 0,\Omega)$. Therefore the approximation property (2.13) provides us with a sequence $(\mathbf{z}_p)$ with $\mathbf{z}_p \in \mathring{\mathcal{V}}_p^\ell \cap \mathring{H}(\mathsf{d}_\ell\, 0,\Omega)$ that converges in $L^2(\Omega,\Lambda^\ell)$ to $\mathbf{z}$. Hence for the subsequence we find

$$\|\mathbf{v}_p - \mathbf{z}_p\|_{L^2(\Omega)} \le \|\mathbf{v}_p - \mathbf{z}\|_{L^2(\Omega)} + \|\mathbf{z} - \mathbf{z}_p\|_{L^2(\Omega)} \to 0\,.$$

This leads to a contradiction, because $\mathbf{v}_p \in \mathring{\mathcal{Z}}_p^\ell$ and $\mathbf{z}_p \in \mathring{\mathcal{V}}_p^\ell \cap \mathring{H}(\mathsf{d}_\ell\, 0,\Omega)$ are $L^2(\Omega)$-orthogonal, hence for all p

$$\|\mathbf{v}_p - \mathbf{z}_p\|^2_{L^2(\Omega)} = \|\mathbf{v}_p\|^2_{L^2(\Omega)} + \|\mathbf{z}_p\|^2_{L^2(\Omega)} \ge 1\,.$$

□

To summarize, Lemmas 2.4 and 2.5 together prove the following result.

Theorem 2.6 *If the completeness of approximating subspaces* (2.12)*, the completeness of discrete kernels* (2.13) *and the discrete compactness property hold, then* (2.8) *provides a spectrally correct, spurious-free approximation of the eigenvalue problem* (2.5)*.*

Remark 2.7 The main focus of this section is on the *convergence* of the eigenvalues and the eigenfunctions of problem (2.8) to those of (2.5). On the other hand, when considering concrete applications it is crucial to investigate the *order* of convergence. In order to do so, several strategies are available. A straightforward approach which well fits the theory summarized in this section makes use of the results from [33]. Theorem 1 of [33] states in this particular situation that the error in the eigenfunctions (measured as usual by the gap of Hilbert spaces) is bounded by the best approximation, and Theorem 3(c) of [33] states that the eigenvalues achieve double order of convergence since our problem is symmetric. An alternative approach makes use of the equivalence of problems (2.5) and (2.8) with suitable mixed formulations [10, Part 4]; in this case an estimate of the order of convergence can be achieved by the standard Babuška–Osborn theory for the spectral approximation of compact operators applied to the mixed formulations [10, Theorems 13.8, 13.10, 14.9, 14.11]. □

3 An abstract framework implying discrete compactness

In this section we fix a degree of differential forms

$$\ell \in \{1, \dots, d-1\},$$

and we formulate a set of hypotheses which allow us to prove the discrete compactness property. These hypotheses are organized in three groups:

1. standard assumptions related to the finite element spaces $\mathring{\mathcal{V}}_p^\ell$ (Sect. 3.1),
2. assumptions on the existence and key properties of "lifting operators" (Sect. 3.3),
3. hypotheses on projections onto $\mathring{\mathcal{V}}_p^\ell$ complying with the commuting diagram property and satisfying an approximation property (Sect. 3.4).

To state these assumptions we have to introduce intermediate spaces X and S of more regular forms

$$\mathring{\mathcal{V}}_p^\ell \subset X(\mathfrak{M}, \Lambda^\ell) \subset \mathring{H}(\mathsf{d}_\ell, \Omega) \quad \text{and} \quad \mathring{\mathcal{V}}_p^{\ell-1} \subset S(\mathfrak{M}, \Lambda^{\ell-1}) \subset \mathring{H}(\mathsf{d}_{\ell-1}, \Omega)\ ,$$

allowing compact embedding arguments and precise notions of continuity of lifting and projection operators.

3.1 Discrete spaces

Our focus is on finite element spaces. For the sake of simplicity, we restrict ourselves to polyhedral Lipschitz domains Ω. We assume that the finite dimensional trial and test spaces $\mathring{\mathcal{V}}_p^\ell$, $p \in \mathbb{N}$, are based on a *fixed* finite partition $\mathfrak{M}$ of Ω, composed of elements (cells) K:

$$\overline{\Omega} = \bigcup_{K \in \mathfrak{M}} \overline{K} \quad , \quad K \cap K' = \emptyset \text{ , if } K \neq K',\ K, K' \in \mathfrak{M}\ .$$

For a cell $K \in \mathfrak{M}$, let $\mathfrak{F}_m(K)$ designate the set of m-dimensional facets of K: for $m = 0$ these are the vertices, for $m = 1$ the edges, for $m = d-1$ the faces, and $\mathfrak{F}_d(K) = \{K\}$.

We take for granted that the discrete spaces $\mathring{\mathcal{V}}_p^\ell$ can be assembled from local contributions in the sense that for each mesh cell $K \in \mathfrak{M}$ there is a space $\mathcal{V}_p^\ell(K) \subset C^\infty(\overline{K}, \Lambda^\ell)$ of smooth ℓ-forms on K, such that

$$\mathring{\mathcal{V}}_p^\ell = \mathring{\mathcal{V}}_p^\ell(\mathfrak{M}) := \left\{ \mathbf{v} \in \mathring{H}(\mathsf{d}_\ell, \Omega) :\ \mathbf{v}\big|_K \in \mathcal{V}_p^\ell(K)\ \forall K \in \mathfrak{M} \right\}\ . \tag{3.1}$$

In other words, $\mathring{\mathcal{V}}_p^\ell$ can be defined by specifying the local spaces $\mathcal{V}_p^\ell(K)$ and requiring the continuity of traces across inter-element boundaries as well as the boundary conditions on $\partial\Omega$.

In the same fashion, we introduce a corresponding family $\mathring{\mathcal{V}}_p^{\ell-1} \subset \mathring{H}(\mathsf{d}_{\ell-1}, \Omega)$ of spaces of discrete $(\ell-1)$-forms. We will see later on that as a consequence of further hypotheses, the local spaces $\mathcal{V}_p^{\ell-1}(K)$ and $\mathcal{V}_p^\ell(K)$ satisfy an exact sequence property.

3.2 Spaces of more regular forms

We introduce a Hilbert space $X(\mathfrak{M}, \Lambda^\ell) \subset \mathring{H}(\mathsf{d}_\ell, \Omega)$ that captures the extra regularity that distinguishes ℓ-forms in the space $\mathring{Y}(\mathsf{d}_\ell, \Omega)$. We can think of this space as a space of "more regular" ℓ-forms on Ω.

Assumption 1 The space $\mathring{Y}(\mathsf{d}_\ell, \Omega)$ defined in (2.6) is continuously embedded in $X(\mathfrak{M}, \Lambda^\ell)$:

$$\mathring{Y}(\mathsf{d}_\ell, \Omega) \hookrightarrow X(\mathfrak{M}, \Lambda^\ell) \, .$$

This means that with $C > 0$ depending only on Ω

$$\|\mathbf{u}\|_{X(\mathfrak{M},\Lambda^\ell)} \leq C \, \|\mathbf{u}\|_{H(\mathsf{d}_\ell,\Omega)} \quad \forall \mathbf{u} \in \mathring{Y}(\mathsf{d}_\ell, \Omega) \, . \tag{3.2}$$

On the other hand, $X(\mathfrak{M}, \Lambda^\ell)$ has to be small enough to maintain the compact embedding satisfied by $\mathring{Y}(\mathsf{d}_\ell, \Omega)$, *cf.* Thm. 2.1.

Assumption 2 The space $X(\mathfrak{M}, \Lambda^\ell)$ is compactly embedded in $L^2(\Omega, \Lambda^\ell)$:

$$X(\mathfrak{M}, \Lambda^\ell) \overset{\text{comp}}{\hookrightarrow} L^2(\Omega, \Lambda^\ell).$$

As with the discrete spaces, the spaces $X(\mathfrak{M}, \Lambda^\ell)$ are built from local contributions and will therefore depend on the mesh $\mathfrak{M}$. We assume that for each mesh cell $K \in \mathfrak{M}$ there are Hilbert spaces $X(K, \Lambda^\ell)$ so that:

$$X(\mathfrak{M}, \Lambda^\ell) = \left\{ \mathbf{v} \in \mathring{H}(\mathsf{d}_\ell, \Omega) : \, \mathbf{v}\big|_K \in X(K, \Lambda^\ell) \;\; \forall K \in \mathfrak{M} \right\}, \tag{3.3}$$

and, in addition, the norm of $X(\mathfrak{M}, \Lambda^\ell)$ is defined through local contributions:

$$\|\mathbf{u}\|^2_{X(\mathfrak{M},\Lambda^\ell)} = \|\mathbf{u}\|^2_{H(\mathsf{d}_\ell,\Omega)} + \sum_{K\in\mathfrak{M}} \left\|\mathbf{u}\big|_K\right\|^2_{X(K,\Lambda^\ell)} \, . \tag{3.4}$$

Finally, the local spaces have to be large enough to contain the discrete forms for any value of p:

$$\mathcal{V}_p^\ell(K) \subset X(K, \Lambda^\ell) \, . \tag{3.5}$$

Correspondingly, we introduce a space $S(\mathfrak{M}, \Lambda^{\ell-1}) \subset \mathring{H}(\mathsf{d}_{\ell-1}, \Omega)$ of "more regular potentials". Similar to $X(\mathfrak{M}, \Lambda^{\ell})$, the spaces $S(\mathfrak{M}, \Lambda^{\ell-1})$ are mesh-dependent and allow for a characterization through local Hilbert spaces $S(K, \Lambda^{\ell-1})$, $K \in \mathfrak{M}$,

$$S(\mathfrak{M}, \Lambda^{\ell-1}) = \left\{ \boldsymbol{\psi} \in \mathring{H}(\mathsf{d}_{\ell-1}, \Omega) : \boldsymbol{\psi}\big|_K \in S(K, \Lambda^{\ell-1}) \ \forall K \in \mathfrak{M} \right\} . \tag{3.6}$$

They are endowed with the norm

$$\|\boldsymbol{\phi}\|^2_{S(\mathfrak{M},\Lambda^{\ell-1})} = \|\boldsymbol{\phi}\|^2_{H(\mathsf{d}_{\ell-1},\Omega)} + \sum_{K\in\mathfrak{M}} \left\|\boldsymbol{\phi}\big|_K\right\|^2_{S(K,\Lambda^{\ell-1})} . \tag{3.7}$$

The local spaces are large enough to contain the local discrete potential spaces:

$$\mathcal{V}_p^{\ell-1}(K) \subset S(K, \Lambda^{\ell-1}) . \tag{3.8}$$

The following assumption establishes the connection between $X(\mathfrak{M}, \Lambda^{\ell})$ and $S(\mathfrak{M}, \Lambda^{\ell-1})$.

Assumption 3 The exterior derivative maps $S(\mathfrak{M}, \Lambda^{\ell-1})$ continuously into $X(\mathfrak{M}, \Lambda^{\ell})$:

$$S(\mathfrak{M}, \Lambda^{\ell-1}) \subset \{\boldsymbol{\phi} \in \mathring{H}(\mathsf{d}_{\ell-1}, \Omega) : \mathsf{d}_{\ell-1}\, \boldsymbol{\phi} \in X(\mathfrak{M}, \Lambda^{\ell})\},$$

and the image is maximal:

$$\mathsf{d}_{\ell-1}\, S(\mathfrak{M}, \Lambda^{\ell-1}) = \mathsf{d}_{\ell-1}\, \mathring{H}(\mathsf{d}_{\ell-1}, \Omega) \cap X(\mathfrak{M}, \Lambda^{\ell}).$$

To conclude this subsection, note that in the case of an element K touching the boundary $\partial\Omega$, like for the discrete spaces $\mathcal{V}_p^{\ell}(K)$ and $\mathcal{V}_p^{\ell-1}(K)$, the local spaces $X(K, \Lambda^{\ell})$ and $S(K, \Lambda^{\ell-1})$ are not obliged to comply with any boundary conditions.

3.3 Local liftings

A pair of linear mappings $\mathsf{R}_{k,K} : C^{\infty}(K, \Lambda^k) \mapsto C^{\infty}(K, \Lambda^{k-1})$, $k = \ell,\ \ell+1$, is called a *lifting operator* of degree ℓ if it fulfills

$$\mathsf{d}_{\ell-1} \circ \mathsf{R}_{\ell,K} + \mathsf{R}_{\ell+1,K} \circ \mathsf{d}_{\ell} = \mathsf{Id}_{\ell} . \tag{3.9}$$

This relation characterizes a "contracting homotopy" of the de Rham complex [5, Section 5.1.2].

Besides this algebraic relationship, our approach hinges on smoothing properties of the lifting operators, expressed by means of the local spaces $S(K, \Lambda^{\ell-1})$ of

more regular potentials and $X(K,\Lambda^\ell)$ of more regular forms. The next assumption summarizes the continuity expected from the lifting operator.

Assumption 4 For every $K \in \mathfrak{M}$ there is a lifting operator $(\mathsf{R}_{\ell,K}, \mathsf{R}_{\ell+1,K})$ whose components can be extended to continuous mappings

$$\mathsf{R}_{\ell+1,K} : L^2(K,\Lambda^{\ell+1}) \mapsto X(K,\Lambda^\ell) \quad \text{and} \quad \mathsf{R}_{\ell,K} : X(K,\Lambda^\ell) \mapsto S(K,\Lambda^{\ell-1}),$$

and thus identity (3.9) holds on $X(K,\Lambda^\ell)$.

As a consequence, for each cell $K \in \mathfrak{M}$, we have the exact sequence

$$S(K,\Lambda^{\ell-1}) \xrightarrow{\mathsf{d}_{\ell-1}} X(K,\Lambda^\ell) \xrightarrow{\mathsf{d}_\ell} L^2(K,\Lambda^{\ell+1}). \tag{3.10}$$

Finally, the local liftings have to be compatible with the local spaces of discrete differential forms:

Assumption 5 The local operators $\mathsf{R}_{\ell+1,K}$, when applied to exact local discrete $(\ell+1)$-forms, yield local discrete ℓ-forms, *i.e.*,

$$\mathsf{R}_{\ell+1,K} \circ \mathsf{d}_\ell : \quad \mathcal{V}_p^\ell(K) \to \mathcal{V}_p^\ell(K) .$$

3.4 Local projectors

As usual in methods based on discrete commuting diagrams we need projection operators $\pi^k_{p,K}$ onto discrete spaces for $(\ell-1)$-forms and ℓ-forms. For degree $\ell-1$, our local spaces $S(K,\Lambda^{\ell-1})$ of more regular potentials can play the role of domains for the projectors $\pi^{\ell-1}_{p,K}$. For the degree ℓ, by generalization of what we actually need in the case of dimension $d=2$ and $d=3$ for Maxwell, we define our projectors $\pi^\ell_{p,K}$ on smaller spaces than $X(K,\Lambda^\ell)$. We denote these new spaces by $S(K,\Lambda^\ell)$ and require that they contain for all p the p-dependent subspaces

$$\widetilde{X}_p(K,\Lambda^\ell) = \{\mathbf{u} \in X(K,\Lambda^\ell) : \mathsf{d}_\ell\,\mathbf{u} \in \mathsf{d}_\ell\,\mathcal{V}_p^\ell(K)\} . \tag{3.11}$$

On the same model as (3.6)-(3.7), we define the corresponding global spaces $S(\mathfrak{M},\Lambda^\ell)$ and

$$\widetilde{X}_p(\mathfrak{M},\Lambda^\ell) = \{\mathbf{u} \in X(\mathfrak{M},\Lambda^\ell) : \mathsf{d}_\ell\,\mathbf{u} \in \mathsf{d}_\ell\,\mathcal{V}_p^\ell\} \tag{3.12}$$

and we have the continuous embeddings

$$\widetilde{X}_p(\mathfrak{M},\Lambda^\ell) \hookrightarrow S(\mathfrak{M},\Lambda^\ell) \hookrightarrow X(\mathfrak{M},\Lambda^\ell) . \tag{3.13}$$

Assumption 6 There are *local* continuous linear projections

$$\pi_{p,K}^{\ell-1} : S(K,\Lambda^{\ell-1}) \mapsto \mathcal{V}_p^{\ell-1}(K) \quad \text{and} \quad \pi_{p,K}^{\ell} : S(K,\Lambda^{\ell}) \mapsto \mathcal{V}_p^{\ell}(K)$$

for all mesh cells $K \in \mathfrak{M}$.

The standard commuting diagram property is as follows.

Assumption 7 The projectors $\pi_{p,K}^{\ell-1}$ and $\pi_{p,K}^{\ell}$ are compatible with the exterior derivative in the sense that the diagram

$$\begin{array}{ccc} S(K,\Lambda^{\ell-1}) & \xrightarrow{\mathsf{d}_{\ell-1}} & S(K,\Lambda^{\ell}) \\ \Big\downarrow{\scriptstyle \pi_{p,K}^{\ell-1}} & & \Big\downarrow{\scriptstyle \pi_{p,K}^{\ell}} \\ \mathcal{V}_p^{\ell-1}(K) & \xrightarrow{\mathsf{d}_{\ell-1}} & \mathcal{V}_p^{\ell}(K)\,, \end{array}$$

commutes for every $K \in \mathfrak{M}$.

Let us note that, as a consequence of Assumptions 4 and 7, we find that the sequence

$$\mathcal{V}_p^{\ell-1}(K) \xrightarrow{\mathsf{d}_{\ell-1}} \mathcal{V}_p^{\ell}(K) \xrightarrow{\mathsf{d}_{\ell}} \mathsf{d}_{\ell}\left(\mathcal{V}_p^{\ell}(K)\right)$$

is exact.

Besides, the local projections acting on $(\ell-1)$-forms are supposed to enjoy a crucial approximation property using the Hilbert space norms $\|\cdot\|_{S(K,\Lambda^{\ell-1})}$.

Assumption 8 There is a function $\varepsilon_{\ell-1}\colon \mathbb{N} \mapsto \mathbb{R}^+$ with $\lim\limits_{p\to\infty} \varepsilon_{\ell-1}(p) = 0$ so that

$$\left\|\mathsf{d}_{\ell-1}(\boldsymbol{\phi} - \pi_{p,K}^{\ell-1}\boldsymbol{\phi})\right\|_{L^2(K,\Lambda^{\ell})} \leq \varepsilon_{\ell-1}(p)\,\|\boldsymbol{\phi}\|_{S(K,\Lambda^{\ell-1})} \quad \forall \boldsymbol{\phi} \in S(K,\Lambda^{\ell-1})\,.$$

Finally we assume for the projections $\pi_{p,K}^{\ell}$ a natural condition of conformity: For all $\mathbf{u} \in \widetilde{X}_p(K,\Lambda^{\ell})$

$$\operatorname{tr}_F \mathbf{u} = 0 \quad \Rightarrow \quad \operatorname{tr}_F \pi_{p,K}^{\ell}\mathbf{u} = 0 \quad \forall F \in \mathfrak{F}_m(K)\,, \quad \ell \leq m \leq d\,, \tag{3.14}$$

and the corresponding condition for the projections $\pi_{p,K}^{\ell-1}$. This makes it possible to define *global* linear projections

$$\pi_p^{\ell} : S(\mathfrak{M},\Lambda^{\ell}) \mapsto \mathring{\mathcal{V}}_p^{\ell} \quad \text{and} \quad \pi_p^{\ell-1} : S(\mathfrak{M},\Lambda^{\ell}) \mapsto \mathring{\mathcal{V}}_p^{\ell-1}$$

by patching together the local operators

$$(\pi_p^{\ell}\mathbf{u})\big|_K := \pi_{p,K}^{\ell}(\mathbf{u}\big|_K) \quad \text{and} \quad (\pi_p^{\ell-1}\boldsymbol{\phi})\big|_K := \pi_{p,K}^{\ell-1}(\boldsymbol{\phi}\big|_K) \quad \forall K \in \mathfrak{M}\,. \tag{3.15}$$

As a consequence of Assumption 7 and (3.15), the global projectors $\pi_p^{\ell-1}$ and π_p^ℓ inherit the global *commuting diagram property*

$$\begin{array}{ccc} S(\mathfrak{M},\Lambda^{\ell-1}) & \xrightarrow{\mathsf{d}_{\ell-1}} & S(\mathfrak{M},\Lambda^\ell) \\ \Big\downarrow{\scriptstyle \pi_p^{\ell-1}} & & \Big\downarrow{\scriptstyle \pi_p^{\ell}} \\ \mathring{\mathcal{V}}_p^{\ell-1} & \xrightarrow{\mathsf{d}_{\ell-1}} & \mathring{\mathcal{V}}_p^{\ell}. \end{array} \tag{3.16}$$

3.5 Proof of the discrete compactness property

The estimate of Assumption 8 on "potentials" carries over to ℓ-forms with a discrete exterior derivative, that is, the elements of the space $\widetilde{X}_p(\mathfrak{M},\Lambda^\ell)$, see (3.12).

Lemma 3.1 (Global projection error estimate) *Making Assumptions 4 through 8, the estimate*

$$\left\|\mathbf{u}-\pi_p^\ell\mathbf{u}\right\|_{L^2(\Omega,\Lambda^\ell)} \le C\varepsilon_{\ell-1}(p)\,\|\mathbf{u}\|_{X(\mathfrak{M},\Lambda^\ell)} \qquad \forall \mathbf{u}\in\widetilde{X}_p(\mathfrak{M},\Lambda^\ell)$$

holds true, with a constant $C>0$ independent of p.

Proof. Pick any $\mathbf{u}\in\widetilde{X}_p(\mathfrak{M},\Lambda^\ell)$. The locality of the projector π_p^ℓ, *cf.* (3.15), and (3.4) allow purely local considerations. Single out one cell $K\in\mathfrak{M}$, still write $\mathbf{u}=\mathbf{u}\big|_K\in\widetilde{X}_p(K,\Lambda^\ell)$, and split $\mathbf{u}$ on K using (3.9) from Assumption 4:

$$\mathbf{u}=\mathsf{d}_{\ell-1}\,\mathsf{R}_{\ell,K}\mathbf{u}+\mathsf{R}_{\ell+1,K}\,\mathsf{d}_\ell\,\mathbf{u}=\mathsf{d}_{\ell-1}\,\boldsymbol{\phi}+\mathsf{R}_{\ell+1,K}\,\mathsf{d}_\ell\,\mathbf{u}\,. \tag{3.17}$$

with $\boldsymbol{\phi}:=\mathsf{R}_{\ell,K}\mathbf{u}$. The continuity of $\mathsf{R}_{\ell,K}$ from Assumption 4 reveals that

$$\|\boldsymbol{\phi}\|_{S(K,\Lambda^{\ell-1})}\le C\,\|\mathbf{u}\|_{X(K,\Lambda^\ell)}\,, \tag{3.18}$$

where here and below C will denote constants (possibly different at different occurrences) which depend neither on $\mathbf{u}$ nor on p.

Thanks to identity (3.17) and the commuting diagram property from Assumption 7, we have

$$\pi_{p,K}^\ell\mathbf{u}=\mathsf{d}_{\ell-1}\,\pi_{p,K}^{\ell-1}\boldsymbol{\phi}+\pi_{p,K}^\ell\mathsf{R}_{\ell+1,K}\,\mathsf{d}_\ell\,\mathbf{u}\,. \tag{3.19}$$

Recall that $\mathbf{u}\in\widetilde{X}_p(K,\Lambda^\ell)$ belongs to the domain of $\pi_{p,K}^\ell$ by Assumption 6. Further, as $\mathbf{u}\in\widetilde{X}_p(K,\Lambda^\ell)$, from Assumption 5 we infer that

$$\mathsf{R}_{\ell+1,K}\,\mathsf{d}_\ell\,\mathbf{u}\in\mathcal{V}_p^\ell(K)\,. \tag{3.20}$$

Thus, owing to the identities (3.17), (3.19) and the projector property of $\pi_{p,K}^\ell$, the task is reduced to an interpolation estimate for $\pi_{p,K}^{\ell-1}$:

$$(\mathsf{Id}-\pi_{p,K}^\ell)\mathbf{u}=\mathsf{d}_{\ell-1}(\mathsf{Id}-\pi_{p,K}^{\ell-1})\boldsymbol{\phi}+\underbrace{(\mathsf{Id}-\pi_{p,K}^\ell)\mathsf{R}_{\ell+1,K}\,\mathsf{d}_\ell\,\mathbf{u}}_{=0\text{ by (3.20)}}\,. \tag{3.21}$$

As a consequence, invoking Assumption 8,

$$\begin{aligned}\left\|(\mathsf{Id}-\pi^{\ell}_{p,K})\mathbf{u}\right\|_{L^2(K,\Lambda^\ell)} &\overset{(3.21)}{=} \left\|\mathsf{d}_{\ell-1}(\mathsf{Id}-\pi^{\ell-1}_{p,K})\boldsymbol{\phi}\right\|_{L^2(K,\Lambda^\ell)}\\ &\leq \varepsilon_{\ell-1}(p)\left\|\boldsymbol{\phi}\right\|_{S(K,\Lambda^{\ell-1})} \overset{(3.18)}{\leq} C\varepsilon_{\ell-1}(p)\left\|\mathbf{u}\right\|_{X(K,\Lambda^\ell)}\,, \end{aligned} \tag{3.22}$$

which furnishes a local version of the estimate. This estimate is uniform in $K \in \mathfrak{M}$ because $\mathfrak{M}$ is finite. Due to (3.4), squaring (3.22) and summing over all cells finishes the proof. □

We are now in the position to prove the main result of this section.

Theorem 3.2 (Discrete compactness) *Under Assumptions 1 through 8, the discrete compactness property of Definition 2.3 holds for the family* $\left(\mathring{\mathcal{V}}^{\ell}_{p}\right)_{p\in\mathbb{N}}$ *of subspaces of* $\mathring{H}(\mathsf{d}_\ell,\Omega)$.

Proof. The proof resorts to the "standard policy" for tackling the problem of discrete compactness, introduced by Kikuchi [39, 40] for analyzing the h-version of Whitney-1-forms. It forms the core of most papers considering the issue of discrete compactness, see [12, Thm. 2], [11, Thm. 11], [37, Thm. 4.9], [31, Thm. 2], etc.

Let us introduce the discrete analogue of the space $\mathring{Y}(\mathsf{d}_\ell,\Omega)$:

$$\mathring{\mathcal{Y}}^{\ell}_{p} := \{\mathbf{v}_p \in \mathring{\mathcal{V}}^{\ell}_{p} :\ \left(\mathbf{v}_p, \mathsf{d}_{\ell-1}\,\boldsymbol{\psi}_p\right)_{0,\Omega} = 0\ \ \forall \boldsymbol{\psi}_p \in \mathring{\mathcal{V}}^{\ell-1}_{p}\}. \tag{3.23}$$

The space $\mathring{\mathcal{Y}}^{\ell}_{p}$ contains $\mathring{\mathcal{Z}}^{\ell}_{p}$ as a subspace.

We consider a subsequence $\mathbb{N}'$ of $\mathbb{N}$ and a $H(\mathsf{d}_\ell,\Omega)$-bounded sequence $(\mathbf{u}_p)_{p\in\mathbb{N}'}$ with members in $\mathring{\mathcal{Z}}^{\ell}_{p}$. Thus $\mathbf{u}_p$ belongs in particular to $\mathring{\mathcal{Y}}^{\ell}_{p}$ and the sequence $(\mathbf{u}_p)_{p\in\mathbb{N}'}$ satisfies

$$\text{(i)}\quad \mathbf{u}_p \in \mathring{\mathcal{V}}^{\ell}_{p}\,, \tag{3.24}$$

$$\text{(ii)}\quad \left(\mathbf{u}_p, \mathsf{d}_{\ell-1}\,\boldsymbol{\psi}_p\right)_{0,\Omega} = 0 \quad \forall \boldsymbol{\psi}_p \in \mathring{\mathcal{V}}^{\ell-1}_{p}\,, \tag{3.25}$$

$$\text{(iii)}\quad \left\|\mathbf{u}_p\right\|_{H(\mathsf{d}_\ell,\Omega)} \leq 1 \quad \forall p \in \mathbb{N}'\,. \tag{3.26}$$

We have to confirm that it possesses a subsequence that converges in $L^2(\Omega,\Lambda^\ell)$.

We start with the $L^2(\Omega,\Lambda^\ell)$-orthogonal projection of $\mathbf{u}_p$ into $\mathring{Y}(\mathsf{d}_\ell,\Omega)$ and parallel to $\mathsf{d}_{\ell-1}\,\mathring{H}(\mathsf{d}_{\ell-1},\Omega)$: let $\widetilde{\mathbf{u}}_p$ be the unique vector field in $\mathring{H}(\mathsf{d}_\ell,\Omega)$ with

$$\widetilde{\mathbf{u}}_p = \mathbf{u}_p + \mathsf{d}_{\ell-1}\,\widetilde{\boldsymbol{\phi}}_p, \quad \widetilde{\boldsymbol{\phi}}_p \in \mathring{H}(\mathsf{d}_{\ell-1},\Omega)\,, \tag{3.27}$$

$$(\widetilde{\mathbf{u}}_p, \mathsf{d}_{\ell-1}\,\boldsymbol{\psi})_{0,\Omega} = 0 \quad \forall \boldsymbol{\psi} \in \mathring{H}(\mathsf{d}_{\ell-1},\Omega)\,. \tag{3.28}$$

Obviously, the latter condition implies

$$\widetilde{\mathbf{u}}_p \in \mathring{Y}(\mathsf{d}_\ell, \Omega) \; . \tag{3.29}$$

Hence, by virtue of Assumption 1, the fact that $\mathsf{d}_\ell \, \widetilde{\mathbf{u}}_p = \mathsf{d}_\ell \, \mathbf{u}_p$, and (3.12), $\widetilde{\mathbf{u}}_p$ satisfies

$$\widetilde{\mathbf{u}}_p \in \widetilde{X}_p(\mathfrak{M}, \Lambda^\ell), \quad \|\widetilde{\mathbf{u}}_p\|_{X(\mathfrak{M},\Lambda^\ell)} \leq C \, \|\mathbf{u}_p\|_{H(\mathsf{d}_\ell,\Omega)} \; , \tag{3.30}$$

where $C > 0$ does not depend on p.

Since $\mathsf{d}_{\ell-1} \, \widetilde{\boldsymbol{\phi}}_p = \widetilde{\mathbf{u}}_p - \mathbf{u}_p \in X(\mathfrak{M}, \Lambda^\ell)$, Assumption 3 implies that we may assume that $\widetilde{\boldsymbol{\phi}}_p \in S(\mathfrak{M}, \Lambda^{\ell-1})$.

Thus we can use Nédélec's trick [44] to obtain

$$\begin{aligned} \|\widetilde{\mathbf{u}}_p - \mathbf{u}_p\|^2_{L^2(\Omega,\Lambda^\ell)} &= \big(\widetilde{\mathbf{u}}_p - \mathbf{u}_p, \widetilde{\mathbf{u}}_p - \pi_p^\ell \widetilde{\mathbf{u}}_p + \pi_p^\ell \widetilde{\mathbf{u}}_p - \mathbf{u}_p\big)_{0,\Omega} \\ &= \big(\widetilde{\mathbf{u}}_p - \mathbf{u}_p, \widetilde{\mathbf{u}}_p - \pi_p^\ell \widetilde{\mathbf{u}}_p\big)_{0,\Omega} \; . \end{aligned} \tag{3.31}$$

This holds because from (3.27) and the projector property of π_p^ℓ we know

$$\pi_p^\ell \widetilde{\mathbf{u}}_p - \mathbf{u}_p = \pi_p^\ell \mathbf{u}_p + \pi_p^\ell \, \mathsf{d}_{\ell-1} \, \widetilde{\boldsymbol{\phi}}_p - \mathbf{u}_p = \pi_p^\ell \, \mathsf{d}_{\ell-1} \, \widetilde{\boldsymbol{\phi}}_p \, ,$$

and thanks to the commuting diagram property (3.16) (deduced from Assumption 7) combined with the orthogonality conditions (3.25) and (3.28), we find

$$\big(\widetilde{\mathbf{u}}_p - \mathbf{u}_p, \pi_p^\ell \widetilde{\mathbf{u}}_p - \mathbf{u}_p\big)_{0,\Omega} = \Big(\widetilde{\mathbf{u}}_p - \mathbf{u}_p, \mathsf{d}_{\ell-1} \, \pi_p^{\ell-1} \widetilde{\boldsymbol{\phi}}_p\Big)_{0,\Omega} = 0 \; . \tag{3.32}$$

Hence, appealing to Lemma 3.1, with $C > 0$ independent of p, we get

$$\begin{aligned} \|\widetilde{\mathbf{u}}_p - \mathbf{u}_p\|_{L^2(\Omega,\Lambda^\ell)} \leq \big\|\widetilde{\mathbf{u}}_p - \pi_p^\ell \widetilde{\mathbf{u}}_p\big\|_{L^2(\Omega,\Lambda^\ell)} &\leq C \varepsilon_{\ell-1}(p) \, \|\widetilde{\mathbf{u}}_p\|_{X(\mathfrak{M},\Lambda^\ell)} \\ &\overset{(3.30)}{\leq} C \varepsilon_{\ell-1}(p) \, \|\mathbf{u}_p\|_{X(\mathfrak{M},\Lambda^\ell)} \to 0 \quad \text{for } p \to \infty \; . \end{aligned} \tag{3.33}$$

From (3.30) we conclude that the sequence $(\widetilde{\mathbf{u}}_p)_{p\in\mathbb{N}'}$ is uniformly bounded in $X(\mathfrak{M}, \Lambda^\ell)$. By Assumption 2 it has a convergent subsequence in $L^2(\Omega, \Lambda^\ell)$. Owing to (3.33), the same subsequence of $(\mathbf{u}_p)_{p\in\mathbb{N}'}$ will converge in $L^2(\Omega, \Lambda^\ell)$. □

3.6 Approximation of the eigenvalue problem

As discussed in Section 2.3, the discrete compactness property is the cornerstone of the proof of the convergence of the discrete generalized Maxwell eigenvalue problem (2.8).

Corollary 3.3 *In addition to the hypotheses of Theorem 3.2, namely Assumptions 1 through 8, assume that property* (CAS) (2.12) *holds and that the space* $X(\mathfrak{M}, \Lambda^\ell) \cap \mathring{H}(\mathsf{d}_\ell\, 0, \Omega)$ *is dense in* $\mathring{H}(\mathsf{d}_\ell\, 0, \Omega)$. *Then* (2.8) *provides a spectrally correct, spurious-free approximation of the eigenvalue problem* (2.5)*.*

Proof. We use Theorem 2.6 from Section 2.3. Considering that the discrete compactness property is provided by Theorem 3.2, and that we assume the approximation property (CAS) (2.12), we only need to show the approximation property (CDK) (2.13), which concerns the approximation of closed forms by closed discrete forms.

Since we assumed the density of $X(\mathfrak{M}, \Lambda^\ell) \cap \mathring{H}(\mathsf{d}_\ell\, 0, \Omega)$ in $\mathring{H}(\mathsf{d}_\ell\, 0, \Omega)$, it is sufficient to prove (CDK) for $\mathbf{z} \in X(\mathfrak{M}, \Lambda^\ell) \cap \mathring{H}(\mathsf{d}_\ell\, 0, \Omega)$. Such $\mathbf{z}$ belongs to $\widetilde{X}_p(\mathfrak{M}, \Lambda^\ell)$, and we can therefore apply Lemma 3.1, which shows that $\pi_p^\ell \mathbf{z} \to \mathbf{z}$ in $L^2(\Omega, \Lambda^\ell)$. We will have accomplished to show (CDK) with $\mathbf{z}_p = \pi_p^\ell \mathbf{z}$, as soon as we show that $\mathsf{d}_\ell\, \mathbf{z}_p = 0$. Keeping in mind that $\mathbf{z}_p \in \mathring{\mathcal{V}}_p^\ell \subset H(\mathsf{d}_\ell, \Omega)$, we see that it is sufficient to show the local relation $\mathsf{d}_\ell\, \mathbf{z}_p = 0$ in K for every $K \in \mathfrak{M}$. This follows finally as in (3.19) in the proof of Lemma 3.1, because $\mathsf{d}_\ell\, \mathbf{z} = 0$ implies

$$\pi_{p,K}^\ell \mathbf{z} = \mathsf{d}_{\ell-1}\, \pi_{p,K}^{\ell-1} \mathsf{R}_{\ell,K} \mathbf{z}\,.$$

Hence $\mathsf{d}_\ell\, \mathbf{z}_p = \mathsf{d}_\ell\, \pi_{p,K}^\ell \mathbf{z} = \mathsf{d}_\ell\, \mathsf{d}_{\ell-1}\, \pi_{p,K}^{\ell-1} \mathsf{R}_{\ell,K} \mathbf{z} = 0$, which ends the proof. □

Remark 3.4 The abstract theory developed in this section can be applied to the h-version of discrete differential forms, if the dependence of the constants on the size of the cell K is made explicit by means of scaling arguments. Here, we forgo this extra technicality and refer the reader to [37, Sect. 4.4]. □

4 Regularized Poincaré lifting

In this section we describe the construction of a local lifting operator R_ℓ that will satisfy Assumptions 4 and 5 in Section 3.3 for suitable spaces $X(K, \Lambda^\ell)$, $S(K, \Lambda^\ell)$ and $\mathcal{V}_p^\ell(K)$. We follow the presentation in [26], where these operators are analyzed and where it is shown in particular that they are pseudodifferential operators of order -1.

4.1 Definition

We consider a bounded domain $D \subset \mathbb{R}^d$ that is *star-shaped* with respect to some subdomain $B \subset D$, that is,

$$\forall a \in B,\, x \in D: \quad \{(1-t)a + tx,\ 0 < t < 1\} \subset D\,. \tag{4.1}$$

For $a \in B$ and $1 \le \ell \le d$, we define the *Poincaré operator* $\mathsf{R}_{\ell,a}$, acting on a differential form $\mathbf{u} \in C^\infty(D, \Lambda^\ell)$, by the path integral

$$\mathsf{R}_{\ell,a}\mathbf{u}(x) = (x-a) \lrcorner \int_0^1 t^{\ell-1}\, \mathbf{u}\big(a + t(x-a)\big)\, dt\,, \qquad x \in D\,. \tag{4.2}$$

Here the symbol $\lrcorner$ denotes the contraction (also called "interior product") of the vector field $x \mapsto (x-a)$ with the ℓ-form $\mathbf{u}$. It is clear that $\mathsf{R}_{\ell,a}$ maps $C^\infty(D, \Lambda^\ell)$ to $C^\infty(D, \Lambda^{\ell-1})$ and it has been shown (see [35] for proofs in the case $d = 3$) that it can be extended to a bounded operator from $L^2(D, \Lambda^\ell)$ to $L^2(D, \Lambda^{\ell-1})$. In order to define the *regularized Poincaré operator* R_ℓ, we choose a function

$$\theta \in C_0^\infty(\mathbb{R}^d)\,, \quad \operatorname{supp}\theta \subset B\,, \quad \int_B \theta(a)\, da = 1\,,$$

and set

$$\mathsf{R}_\ell \mathbf{u}(x) = \int_B \theta(a) \mathsf{R}_{\ell,a}\mathbf{u}(x)\, da\,. \tag{4.3}$$

4.2 Regularity

The substitution $y = a + t(x-a)$, $\tau = 1/(1-t)$ transforms the double integral in (4.2), (4.3) into

$$\begin{aligned} \mathsf{R}_\ell \mathbf{u}(x) &= \int_{\mathbb{R}^d} \int_1^\infty (\tau-1)^{\ell-1} \tau^{d-\ell} \theta\big(x + \tau(y-x)\big)\, (x-y) \lrcorner \mathbf{u}(y)\, d\tau\, dy \\ &= \int_{\mathbb{R}^d} k(y, y-x) \lrcorner \mathbf{u}(y)\, dy\,, \end{aligned} \tag{4.4}$$

where the kernel $k(y,z)$ has an expansion into quasi-homogeneous terms:

$$\begin{aligned} k(y,z) &= -z \int_0^\infty s^{\ell-1}(s+1)^{d-\ell} \theta\big(y + sz\big)\, ds \\ &= -\sum_{j=0}^{d-\ell} \tbinom{d-\ell}{j} \frac{z}{|z|^{d-j}} \int_0^\infty r^{d-j-1} \theta\big(y + r\frac{z}{|z|}\big)\, dr\,. \end{aligned} \tag{4.5}$$

The operator R_ℓ is therefore a weakly singular integral operator. In [26, Section 3.3], the following result is shown.

Proposition 4.1 *For $1 \le \ell \le d$, the operator R_ℓ is a pseudodifferential operator of order -1 on $\mathbb{R}^d$. It is well defined on $C^\infty(D, \Lambda^\ell)$, it maps $C^\infty(D, \Lambda^\ell)$*

to $C^\infty(D,\Lambda^{\ell-1})$ *and* $C^\infty(\overline{D},\Lambda^\ell)$ *to* $C^\infty(\overline{D},\Lambda^{\ell-1})$*, and for any* $s\in\mathbb{R}$ *it has an extension as a bounded operator*

$$\mathsf{R}_\ell\ :\ H^s(D,\Lambda^\ell)\to H^{s+1}(D,\Lambda^{\ell-1}))\ .$$

Here, $H^s(D,\Lambda^\ell)$ *is the Sobolev space of* ℓ*-forms on* D *of order* s*.*

4.3 Lifting property

The lifting property (3.9) is a consequence of the following identity, which is a special case of "Cartan's magic formula" for Lie derivatives and for a flow field generated by the dilations with center a.

$$\begin{aligned}&\frac{d}{dt}(t^\ell\mathbf{u}\big(a+t(x-a)\big)=\\&\mathsf{d}_{\ell-1}\Big(t^{\ell-1}(x-a)\lrcorner\,\mathbf{u}\big(a+t(x-a)\big)\Big)+t^\ell(x-a)\lrcorner\,\mathsf{d}_\ell\,\mathbf{u}\big(a+t(x-a)\big)\end{aligned}\tag{4.6}$$

Here $\mathbf{u}$ is an ℓ-form. The result is

$$\begin{aligned}\mathsf{d}_{\ell-1}\,\mathsf{R}_\ell\mathbf{u}+\mathsf{R}_{\ell+1}\,\mathsf{d}_\ell\,\mathbf{u}=\mathbf{u}\qquad&(1\le\ell\le d-1)\ ;\\ \mathsf{R}_1\,\mathsf{d}_0\,\mathbf{u}=\mathbf{u}-\big(\theta,\mathbf{u}\big)_{0,D}\qquad&(\ell=0)\ ;\\ \mathsf{d}_{d-1}\,\mathsf{R}_d\mathbf{u}=\mathbf{u}\qquad&(\ell=d)\ .\end{aligned}\tag{4.7}$$

These relations are valid for all $\mathbf{u}\in C_0^\infty(\mathbb{R}^d,\Lambda^\ell)$ and by extension for all $\mathbf{u}\in H^s(D,\Lambda^\ell)$, $s\in\mathbb{R}$.

The perfect match of (4.7) with (3.9) from Assumption 4 suggests that the regularized Poincaré lifting R_ℓ provides suitable local liftings as stipulated in Assumption 4. To this end, we can choose as local spaces of "more regular forms"

$$\begin{aligned}X(K,\Lambda^\ell):=H(\mathsf{d}_\ell,K)\cap H^r(K,\Lambda^\ell)\ ,\\ S(K,\Lambda^{\ell-1}):=H^r(\mathsf{d}_{\ell-1},K)\quad\text{and}\quad S(K,\Lambda^\ell):=H^r(\mathsf{d}_\ell,K)\ ,\end{aligned}\tag{4.8}$$

for some $0<r\le 1$, where we denote by $H^r(\mathsf{d}_k,K)$ the space

$$H^r(\mathsf{d}_k,K):=\{\mathbf{v}\in H^r(K,\Lambda^k):\mathsf{d}_k\,\mathbf{v}\in H^r(K,\Lambda^{k+1})\}\ .$$

All these spaces are equipped with the natural Hilbert space norms. Also keep in mind that the global spaces $X(\mathfrak{M},\Lambda^\ell)$, $S(\mathfrak{M},\Lambda^{\ell-1})$ and $S(\mathfrak{M},\Lambda^\ell)$ are determined by their local definition on the mesh cells K, *cf.* (3.3) and (3.6). For the particular choice (4.8) an assumption of Corollary 3.3 can be verified.

Lemma 4.2 *For* $X(\mathfrak{M},\Lambda^\ell)$ *arising from* (4.8) *the space* $X(\mathfrak{M},\Lambda^\ell)\cap\mathring{H}(\mathsf{d}_\ell\,0,\Omega)$ *is dense in* $\mathring{H}(\mathsf{d}_\ell\,0,\Omega)$*.*

Proof. By [26, Thm. 4.9(c)] we have a direct decomposition

$$\mathring{H}(\mathsf{d}_\ell\, 0, \Omega) = \mathsf{d}_{\ell-1}\, \mathring{H}^1(\Omega, \Lambda^{\ell-1}) \oplus \mathcal{C}_\ell\ , \quad \mathcal{C}_\ell \subset C^\infty_{\overline{\Omega}}(\mathbb{R}^n, \Lambda^\ell)\ , \tag{4.9}$$

where $C^\infty_{\overline{\Omega}}(\mathbb{R}^d, \Lambda^\ell)$ is the space of compactly supported, smooth ℓ-forms on $\mathbb{R}^d$ with support contained in $\overline{\Omega}$ or, equivalently, the space of all smooth ℓ-forms on $\overline{\Omega}$ that vanish on $\partial\Omega$ together with all their derivatives. Since $C^\infty_{\overline{\Omega}}(\mathbb{R}^d, \Lambda^{\ell-1})$ is dense in $\mathring{H}^1(\Omega, \Lambda^{\ell-1})$, we deduce:

$$C^\infty_{\overline{\Omega}}(\mathbb{R}^d, \Lambda^\ell) \cap \mathsf{d}_{\ell-1}\, \mathring{H}^1(\Omega, \Lambda^{\ell-1}) \quad \text{is dense in} \quad \mathsf{d}_{\ell-1}\, \mathring{H}^1(\Omega, \Lambda^{\ell-1})$$

As every $\mathbf{u} \in C^\infty_{\overline{\Omega}}(\mathbb{R}^d, \Lambda^\ell)$ belongs to $X(\mathfrak{M}, \Lambda^\ell)$, the assertion follows. □

We point out that the choice of r in (4.8) is determined by Assumption 1. Also note that whenever we opt for (4.8), Rellich's theorem ensures Assumption 2, because the mesh is kept fixed.

The construction of R_ℓ entails a constraint on the cell shapes. This is satisfied for standard finite element meshes, where the cells usually are convex polyhedra.

Assumption 9 Every cell $K \in \mathfrak{M}$ is a star-shaped polyhedron.

Lemma 4.3 *Assumption 9, the choice* (4.8) *for spaces* $X(K, \Lambda^\ell)$ *and* $S(K, \Lambda^{\ell-1})$ *imply Assumptions 2, 3 and 4.*

Proof. The only fact which remains to be proved is the maximality relation in Assumption 3

$$\mathsf{d}_{\ell-1}\, S(\mathfrak{M}, \Lambda^{\ell-1}) = \mathsf{d}_{\ell-1}\, \mathring{H}(\mathsf{d}_{\ell-1}, \Omega) \cap X(\mathfrak{M}, \Lambda^\ell).$$

The inclusion $\subset$ holds by definition. Let us prove the converse inclusion.
Let $\mathbf{u} \in \mathsf{d}_{\ell-1}\, \mathring{H}(\mathsf{d}_{\ell-1}, \Omega) \cap X(\mathfrak{M}, \Lambda^\ell)$. Thus $\mathbf{u} = \mathsf{d}_{\ell-1}\, \boldsymbol{\phi}$ with $\boldsymbol{\phi} \in \mathring{H}(\mathsf{d}_{\ell-1}, \Omega)$. Since $\mathbf{u} \in L^2(\Omega, \Lambda^\ell)$, using [26, Cor. 4.7] we obtain that there exists $\boldsymbol{\psi} \in \mathring{H}^1(\Omega, \Lambda^{\ell-1})$ such that $\mathbf{u} = \mathsf{d}_{\ell-1}\, \boldsymbol{\psi}$. In particular, $\boldsymbol{\psi}\big|_K$ belongs to $H^r(K, \Lambda^{\ell-1})$ for all K and, since $\mathbf{u}\big|_K$ belongs to $H^r(K, \Lambda^\ell)$, we finally find that $\boldsymbol{\psi}\big|_K \in H^r(\mathsf{d}_{\ell-1}, K)$. □

4.4 Preservation of polynomial forms

Fundamental in finite element methods is the notion of polynomial differential forms. For an ordered ℓ-tuple $I = (i_1, \ldots, i_\ell)$, $i_1 < i_2 < \ldots < i_\ell$, $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, d\}$, let

$$\mathsf{d}x_I := \mathsf{d}x_{i_1} \wedge \cdots \wedge \mathsf{d}x_{i_\ell}\ ,$$

where $\mathsf{d}x_j$, $j = 1, \dots, d$, are the co-ordinate 1-forms in Euclidean space $\mathbb{R}^d$. The space $\mathcal{P}_p(\Lambda^\ell)$ of polynomial ℓ-forms on $\mathbb{R}^d$ is defined as

$$\mathcal{P}_p(\Lambda^\ell) := \left\{ \mathbf{u} = \sum\nolimits_I u_I \, \mathsf{d}x_I : \; u_I \in \mathcal{P}_p(\mathbb{R}^d) \right\} ,$$

where $\sum_I$ indicates summation over all ordered ℓ-tuples, and $\mathcal{P}_p(\mathbb{R}^d)$ is the space of d-variate polynomials of total degree $\leq p$. We remark that for $d \in \{2, 3\}$ polynomial forms possess polynomial vector proxies.

From the definition (4.2) it is clear that the Poincaré operator $\mathsf{R}_{\ell,a}$ maps differential forms with polynomial coefficients to differential forms with polynomial coefficients. The same holds for the regularized Poincaré operator R_ℓ by (4.3). If we want R_ℓ to map a space $P(\Lambda^\ell)$ of differential forms of order ℓ (e.g., with polynomial coefficients) into a space $P(\Lambda^{\ell-1})$ of differential forms of order $\ell - 1$, it is sufficient to require the following two properties, see [26, Proposition 4.2].

Proposition 4.4 *Assume that $P(\Lambda^\ell)$ and $P(\Lambda^{\ell-1})$ are finite-dimensional spaces of differential forms satisfying*
(i) *The space $P(\Lambda^\ell)$ is invariant with respect to dilations and translations, that is*

$$\textit{For any } t \in \mathbb{R}, a \in \mathbb{R}^n : \; \textit{if } \mathbf{u} \in P(\Lambda^\ell), \; \textit{then } \big(x \mapsto \mathbf{u}(tx + a)\big) \in P(\Lambda^\ell) .$$

(ii) *The interior product $x \lrcorner : \mathbf{u} \mapsto x \lrcorner \mathbf{u}$ maps $P(\Lambda^\ell)$ to $P(\Lambda^{\ell-1})$.*
Then R_ℓ maps $P(\Lambda^\ell)$ into $P(\Lambda^{\ell-1})$.

For the compatibility Assumption 5 to hold, it is therefore sufficient to make the following assumption about the local polynomial space $\mathcal{V}_p^\ell(K)$.

Assumption 10

(i) The space $\mathcal{V}_p^\ell(K)$ is invariant with respect to dilations and translations.

(ii) The differential operator $x \lrcorner \mathsf{d}_\ell : \mathbf{u} \mapsto x \lrcorner \mathsf{d}_\ell \, \mathbf{u}$ maps $\mathcal{V}_p^\ell(K)$ into $\mathcal{V}_p^\ell(K)$.

To summarize:

Assumptions 9, 10, and (4.8) $\implies$ Assumptions 2, 3, 4, and 5

5 Discrete differential forms

Now we introduce concrete spaces of discrete differential forms. We merely summarize the constructions that have emerged from research in differential geometry (the "Whitney-forms" introduced in [48]) and finite element theory ("Raviart-Thomas elements" of [47] and "Nédélec finite elements" of [44, 45]). These schemes were later combined into the concept of discrete differential forms [14, 36]. Surveys and many more details can be found in [37, 4, 5, 15].

5.1 Simplicial meshes

Let $\mathfrak{M}$ be a conforming simplicial finite element mesh covering the Lipschitz polyhedron $\Omega \subset \mathbb{R}^d$. As elaborated in [4, Sect. 3 & 4] for $p \in \mathbb{N}$ the following choices

$$\mathcal{V}_p^\ell(K) := \mathcal{P}_{p-1}(\Lambda^\ell)\big|_K + x \lrcorner \mathcal{P}_{p-1}(\Lambda^{\ell+1})\big|_K \tag{5.1}$$

and

$$\mathcal{V}_p^\ell(K) := \mathcal{P}_p(\Lambda^\ell)\big|_K \tag{5.2}$$

of local spaces, through (3.1), gives rise to meaningful global finite elment spaces $\mathring{\mathcal{V}}_\ell^p(\mathfrak{M})$ of discrete differential forms.

By construction both Assumption 9 and Assumption 10 are satisfied for these spaces. The asymptotic density property also holds.

Lemma 5.1 *The spaces $\mathring{\mathcal{V}}_\ell^p(\mathfrak{M})$ of discrete differential forms built from (5.1) or (5.2) meet the requirement (2.12).*

Proof. It is a classical result of finite element theory that the spaces of degree p Lagrangian finite element functions $\mathring{\mathcal{V}}_p^0(\mathfrak{M})$ are asymptotically dense in $\mathring{H}^1(\Omega)$. Thus the space of polynomials ℓ-forms with coefficients in $\mathring{\mathcal{V}}_p^0(\mathfrak{M})$, which is a subspace of $\mathring{\mathcal{V}}_p^\ell(\mathfrak{M})$, is asymptotically dense in $\mathring{H}^1(\Omega,\Lambda^\ell)$. The latter space is obviously dense in $\mathring{H}(\mathsf{d}_\ell,\Omega)$, since this is already true for $C_0^\infty(\Omega,\Gamma^l)$. □

5.2 Tensor product meshes

Let $\mathfrak{M}$ be a conforming finite element mesh of the Lipschitz polyhedron Ω whose cells are affine images of the unit hypercube $\widehat{K}$ in $\mathbb{R}^d$: for $K \in \mathfrak{M}$ the we write $\mathbf{\Phi}_K : \widehat{K} \mapsto K$ for the associated unique affine mapping. We generalize the construction of [44]: on the cube we define (with notations introduced in Section 4.4)

$$\mathcal{V}_p^\ell(\widehat{K}) := \left\{ \widehat{\mathbf{v}} = \sum_I u_I \mathsf{d}x_I,\ u_I(x) = \prod_{j=1}^d u_{I,j}(x_j),\ u_{I,j} \in \begin{cases} \mathcal{P}_{p-1} & \text{if } j \in I \\ \mathcal{P}_p & \text{if } j \notin I \end{cases} \right\}.$$

The local spaces are obtained by affine pullback

$$\mathcal{V}_p^\ell(K) := \left(\mathbf{\Phi}_K^{-1}\right)^* \mathcal{V}_p^\ell(\widehat{K})\ . \tag{5.3}$$

This affine tensor product construction also complies with Assumption 9 and Assumption 10. Completely parallel to Lemma 5.1, one proves the following result.

Lemma 5.2 *The requirement* (2.12) *is satisfied for the spaces* $\mathring{\mathcal{V}}_\ell^p(\mathfrak{M})$ *spawned by* (5.3).

Remark 5.3 For all the above meshes the cells are affine images of a single reference cell, the "unit simplex" or "unit hypercube". We could allow non-affine cells, because, under the assumption that the transformations are "nearly affine", see [22, §4.3], all crucial estimates like Lemma 3.1 can be transferred to the reference cell using the pullback of differential forms. Of course, also the projection operators $\pi_{p,K}^\ell$ would have to be defined by transformation. □

6 Application in dimensions two and three

We adopt the discrete spaces from Sect. 5 along with the regularized Poincaré lifting from Sect. 4. We rely on the choice (4.8) for spaces X and S, with a regularity exponent $r \in (0,1]$ which has to be chosen suitably.

In order to establish the discrete compactness property from Definition 2.3, it remains to verify the regularity Assumption 1 and Assumptions 6, 7, and 8 for convenient local projectors $\pi_{p,K}^\ell$.

Local projectors which make the discrete diagram of Assumption 7 commute do exist in the general framework of differential forms of any degree. They generalize Nédélec edge element projections and can be referred to as *moment based* projection operators. They are suitable for the h-version of finite elements in dimensions 2 and 3. In higher dimensions some of them (for low degree forms) require a higher regularity than H^2 to be defined. In [4, 5], they are modified by an extension-regularization procedure in order to be defined and bounded on L^2. However, such operators cannot be used for the p version of finite elements, because no estimates (stability or error bounds) are known with respect to the polynomial degree p.

The proper projection operators for p-version approximation are so-called *projection based interpolation operators*, see [28, 29, 18, 27, 30]. Variants for any ℓ and d are available and they are designed to commute in the sense of Assumption 7 [37, Sect. 3.5].

At this point we have to abandon the framework of general ℓ and d, because both regularity results (Assumption 1) and the analysis of projection operators (Assumption 8) are not presently available for general ℓ and d. We have to discuss them for special choices of ℓ and d separately, relying on a wide array of sophisticated results from the literature.

Theorem 6.1 (Convergence of Galerkin approximations) *For* $d = 2, 3$, *and* $0 \leq \ell < d$, *the Galerkin discretization of* (2.5) *on a Lipschitz polyhedron based on any*

of the families of discrete differential forms introduced in Sect. 5 offers a spectrally correct, spurious-free approximation.

Proof. We skip the case $\ell = 0$, for which the standard Galerkin approximation theory for operators with compact resolvent can be applied, see [41].

To begin with, we focus on the discrete compactness property and verify the assumptions 1, 6, 7, and 8 for $d = 2$ and $d = 3$ separately.

- $d = 2$, $\ell = 1$: in terms of vector proxies we find the correspondence

$$\mathring{Y}(\mathsf{d}_1, \Omega) \sim \mathring{\boldsymbol{H}}(\mathsf{curl}, \Omega) \cap \boldsymbol{H}(\mathsf{div}\, 0, \Omega) \,. \tag{6.1}$$

Regularity theorems for boundary value problems for $-\Delta$ on the polygon confirm the existence of $\delta = \delta(\Omega) > 0$ such that

$$\mathring{\boldsymbol{H}}(\mathsf{curl}, \Omega) \cap \boldsymbol{H}(\mathsf{div}\, 0, \Omega) \subset \boldsymbol{H}^{\delta+1/2}(\Omega) \,, \tag{6.2}$$

in the sense of continuous embedding, see [34, Sect. 3.2]. This suggests to choose $r = \delta + 1/2$ in (4.8) and Assumption 1 will hold true. Hence, we deal with the concrete spaces

$$X(\mathfrak{M}, \Lambda^1) = \mathring{\boldsymbol{H}}(\mathsf{curl}, \Omega) \cap \prod_{K \in \mathfrak{M}} (H^{\delta+1/2}(K))^2 \,, \tag{6.3}$$

$$S(\mathfrak{M}, \Lambda^0) = \mathring{H}^1(\Omega) \cap \prod_{K \in \mathfrak{M}} H^{\delta+3/2}(K) \,. \tag{6.4}$$

Commuting local projection based interpolation operators $\pi^1_{p,K}$ and $\pi^0_{p,K}$ have been proposed for triangles and for quadrilaterals in [28]. With the choice (6.3) and (6.4) they live up to Assumptions 6 and 7. Assumption 8 holds with $\varepsilon_0(p) = Cp^{-1/2}$ and $C > 0$ depending only on the shape-regularity of the cells, *cf.* [27, Thm. 4.3] and [6, Thm. 4.1]. Finally, these interpolation operators satisfy the natural condition of conformity (3.14) by construction, which makes they meet all our requirements, *cf.* Sect. 3.4.

- $d = 3$, $\ell = 1, 2$: we have the vector proxy incarnation

$$\mathring{Y}(\mathsf{d}_\ell, \Omega) \sim \begin{cases} \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\mathsf{div}\, 0, \Omega) & \text{for } \ell = 1 \,, \\ \mathring{\boldsymbol{H}}(\mathsf{div}, \Omega) \cap \boldsymbol{H}(\mathbf{curl}\, 0, \Omega) & \text{for } \ell = 2 \,. \end{cases} \tag{6.5}$$

Citing results from [1] and [37, Sect. 4.1], we find $\delta = \delta(\Omega) \in]0, \frac{1}{2}]$ and continuous embeddings

$$\mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\mathsf{div}, \Omega), \;\; \mathring{\boldsymbol{H}}(\mathsf{div}, \Omega) \cap \boldsymbol{H}(\mathbf{curl}, \Omega) \subset \boldsymbol{H}^{\delta+1/2}(\Omega) \,. \tag{6.6}$$

Therefore, using the construction (4.8) with $r = \delta + 1/2$, Assumption 1 is satisfied for $\ell \in \{1, 2\}$. The relevant spaces of more regular forms now read

$$X(\mathfrak{M}, \Lambda^1) = \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \cap \prod_{K \in \mathfrak{M}} (H^{\delta+1/2}(K))^3 \,, \tag{6.7}$$

$$X(\mathfrak{M}, \Lambda^2) = \mathring{\boldsymbol{H}}(\mathsf{div}, \Omega) \cap \prod_{K \in \mathfrak{M}} (H^{\delta+1/2}(K))^3 \,, \tag{6.8}$$

$$S(\mathfrak{M}, \Lambda^0) = \mathring{H}^1(\Omega) \cap \prod_{K \in \mathfrak{M}} H^{\delta+3/2}(K) \,, \tag{6.9}$$

$$S(\mathfrak{M}, \Lambda^1) = \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \cap \prod_{K \in \mathfrak{M}} \boldsymbol{H}^{\delta+1/2}(\mathbf{curl}, K) \,. \tag{6.10}$$

The essential commuting local projection based interpolation operators $\pi^m_{p,K}$, $m = 0, 1, 2$, have been introduced in [29] for tetrahedral meshes and in [28] for meshes comprising parallelepipeds. By construction they comply with Assumption 7. Assumption 6 for the spaces $S(\mathfrak{M}, \Lambda^0)$ and $S(\mathfrak{M}, \Lambda^1)$ from (6.9) and (6.10), respectively, and $r = \delta + 1/2$ is a consequence of Sobolev embedding theorems. Relying on [27, Th.5.3] we obtain like in the 2D case that in Lemma 3.1 we can take $\varepsilon_m(p) = Cp^{-1/2}$ for $m = 0$ and $m = 1$.

- Finally, we appeal to Lemmas 5.1, 5.2 together with Lemma 4.2 and apply the abstract theory of Sect. 3 in the form of Corollary 3.3 to conclude the proof of the theorem. □

Corollary 6.2 (Approximation of the Maxwell eigenvalue problem) *The p version finite element discretization of the Maxwell eigenvalue problem* (1.2) *based on edge elements from the first or second Nédélec family on triangles or on tetrahedra, or from the first Nédélec family on parallelograms or on parallelepipeds offers a spectrally correct, spurious-free approximation.*

Remark 6.3 Instead of (1.2) we may consider the variational formulation of the more general Maxwell eigenvalue problem (1.1), corresponding to the case of anisotropic inhomogeneous material:

$$\begin{aligned} &\text{Seek } \mathbf{u} \in \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \setminus \{0\}, \ \omega \in \mathbb{R}^+_0 \ \text{ such that} \\ &\left(\mu^{-1} \,\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v}\right)_{L^2(\Omega)} = \omega^2 \left(\epsilon \mathbf{u}, \mathbf{v}\right)_{L^2(\Omega)} \ \forall \mathbf{v} \in \mathring{\boldsymbol{H}}(\mathbf{curl}, \Omega) \,, \end{aligned} \tag{6.11}$$

with uniformly positive material tensors $\mu = \mu(\mathbf{x})$, $\epsilon = \epsilon(\mathbf{x})$. The same edge element discretizations listed in Corollary 6.2 provide spectrally correct, spurious-free approximations of this problem. This generalization of Corollary 6.2 can be achieved with standard tools (see, in particular, Propositions 2.25, 2.26, and 2.27 of [19], and [38, Sect. 6], [37, Thm. 4.9]). □

Remark 6.4 The restriction on the families of elements mentioned in the Corollary is essentially due to the availability of published results about suitable interpolation operators. Thus, for example, as soon as a generalization of the p version error estimates of [29, 27] for projection-based interpolants to meshes containing prismatic or more general polyhedral elements becomes available, our result about the approximation of the Maxwell eigenvalue problem will apply to such meshes, too. □

Remark 6.5 Several obstacles prevent us from establishing the assumptions of the abstract theory for $d > 3$. On the one hand, continuity properties of projection based interpolation operators have not been investigated for $d > 3$. Also, regularity results along the lines of (6.6) are have not been published for polyhedra in higher dimensions.
On the other hand, the innocuously looking requirement (3.14) for the projection operators — corresponding to the requirement that the global projection operators are constructed elementwise from local degrees of freedom — entails that the trace of forms in $S(K, \Lambda^{\ell-1})$ onto $\ell-1$-dimensional facets in $\mathfrak{F}_{\ell-1}(K)$ must make sense. However, we cannot expect more than H^2 regularity for the space $S(K, \Lambda^{\ell-1})$. Hence, by trace theorems for Sobolev spaces, the spaces $S(K, \Lambda^{\ell-1})$ allow for traces on m-facets for $m > \frac{d}{2} - 2$ at best, which means that $\ell > \frac{d}{2} - 1$ is required to allow for the construction of a local projection based interpolation complying with Assumptions 6 and 8.
Perhaps, an analysis in L^p-spaces as in [1, Lemma 4.7] can make possible an extension of the theory to higher dimensions, but this is beyond the scope of the present article. □

Remark 6.6 Our approach does not cover hp-refinement, for various reasons. One reason is that there exist many variants of hp refinements in 3D, and covering them would in any case require a much longer paper than the present one.
Another reason is technical: The existing convergence proof of the hp approximation of the Maxwell eigenvalue problem in [11] — while also based on the proof of the discrete compactness property — uses a different technical tool, namely an estimate of the L^2 stability of a certain projection operator. This kind of estimate is currently only available for intervals in 1D and for rectangles in 2D.
The technique used in the present paper is based on the regularized Poincaré lifting, and adjusting this to variable polynomial degree poses formidable technical challenges. Only in 2D these could be mastered so far, as was demonstrated in [7] in the context of boundary element analysis. □

7 Conclusion

In this paper we have proved that the p-version of finite elements based on generalized Nédélec edge elements provides a spurious-free spectrally correct approximation of the Maxwell eigenvalue problem. The essential point was the proof of the discrete compactness property. We showed that this property follows from a set of rather natural assumptions about the family of finite element spaces and interpolation operators, and in addition we showed that these assumptions are implied by recently found results on lifting operators and on projection-based interpolants.

In the approach pursued in [4, 5] the discrete compactness property is not addressed directly: in the framework of the h-version for differential forms, modified moment-based projection operators are used. These new operators satisfy the strong property of being uniformly bounded in L^2 and are constructed by means of a delicate extension-regularization procedure, see also [21, 20].

On the one hand this uniform boundedness property is stronger than our assumption 6 and replaces in a certain way the discrete compactness property. But on the other hand, it is currently not known whether a construction of projection operators by extension-regularization could also be employed in the case of the p-version of finite elements, or whether the construction of a p-uniformly L^2-bounded family of cochain projections is even possible.

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