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ON THE GEOMETRY OF ALGEBRAIC GROUPS AND HOMOGENEOUS SPACES

MICHEL BRION

ABSTRACT. Given a connected algebraic group G over an algebraically closed field and a G -homogeneous space X , we describe the Chow ring of G and the rational Chow ring of X , with special attention to the Picard group. Also, we investigate the Albanese and the “anti-affine” fibrations of G and X .

1. INTRODUCTION

Linear algebraic groups and their homogeneous spaces have been thoroughly investigated; in particular, the Chow ring of a connected linear algebraic group G over an algebraically closed field k was determined by Grothendieck (see [Gr58, p. 21]), and the rational Chow ring of a G -homogeneous space admits a simple description via Edidin and Graham’s equivariant intersection theory (see [Br98, Cor. 12]). But arbitrary algebraic groups have attracted much less attention, and very basic questions about their homogeneous spaces appear to be unanswered: for example, a full description of their Picard group (although much information on that topic may be found in work of Raynaud, see [Ra70]).

The present paper investigates several geometric questions about such homogeneous spaces. Specifically, given a connected algebraic group G over k , we determine the Chow ring $A^*(G)$ and obtain two descriptions of the Picard group $\text{Pic}(G)$. For a G -homogeneous space X , we also determine the rational Chow ring $A^*(X)_{\mathbb{Q}}$ and rational Picard group $\text{Pic}(X)_{\mathbb{Q}}$. Furthermore, we study local and global properties of two homogeneous fibrations of X : the Albanese fibration, and the less known “anti-affine” fibration.

Quite naturally, our starting point is the Chevalley structure theorem, which asserts that G is an extension of an abelian variety A by a connected linear (or equivalently, affine) group G_{aff} . The corresponding G_{aff} -torsor $\alpha_G : G \rightarrow A$ turns out to be locally trivial for the Zariski topology (Proposition 2.2); this yields a long exact sequence which determines $\text{Pic}(G)$ (Proposition 2.12).

Also, given a Borel subgroup B of G_{aff} , the induced morphism $G/B \rightarrow A$ (a fibration with fibre the flag variety G_{aff}/B) turns out to be trivial (Lemma 2.1). It follows that $A^*(G) = (A^*(A) \otimes A^*(G_{\text{aff}}/B))/I$, where I denotes the ideal generated by the image of the characteristic homomorphism $X(B) \rightarrow \text{Pic}(A) \times \text{Pic}(G_{\text{aff}}/B)$ (Theorem 2.7). This yields in turn a presentation of $\text{Pic}(G)$. As a consequence, the “Néron-Severi” group $\text{NS}(G)$, consisting of algebraic equivalence classes of line bundles, is isomorphic to $\text{NS}(A) \times \text{Pic}(G_{\text{aff}})$ (Corollary 2.13); therefore, $\text{NS}(G)_{\mathbb{Q}} \cong \text{NS}(A)_{\mathbb{Q}}$.

Thus, α_G (the Albanese morphism of G) behaves in some respects as a trivial fibration. But it should be emphasized that α_G is almost never trivial, see Proposition 2.2. Also, for a G -homogeneous space X , the Albanese morphism α_X , still a homogeneous fibration, may fail to be Zariski locally trivial (Example 3.4). So, to describe $A^*(X)_{\mathbb{Q}}$ and $\text{Pic}(X)_{\mathbb{Q}}$, we rely on other methods, namely, G_{aff} -equivariant intersection theory (see [EG98]). Rather than giving the full statements of the results (Theorem 3.8 and Proposition 3.10), we point out two simple consequences: if $X = G/H$ where $H \subset G_{\text{aff}}$, then $A^*(X)_{\mathbb{Q}} = (A^*(A)_{\mathbb{Q}} \otimes A^*(G_{\text{aff}}/H)_{\mathbb{Q}})/J$, where the ideal J is generated by certain algebraically trivial divisor classes of A (Corollary 3.9). Moreover, $\text{NS}(X)_{\mathbb{Q}} \cong \text{NS}(A)_{\mathbb{Q}} \times \text{Pic}(G_{\text{aff}}/H)_{\mathbb{Q}}$, as follows from Corollary 3.11.

Besides α_G , we also consider the natural morphism $\varphi_G : G \rightarrow \text{Spec } \mathcal{O}(G)$ that makes G an extension of a connected affine group by an “anti-affine” group G_{ant} (as defined in [Br09a]). We show that the anti-affine fibration φ_G may fail to be locally trivial, but becomes trivial after an isogeny (Propositions 2.3 and 2.4).

For any G -homogeneous space X , we define an analogue φ_X of the anti-affine fibration as the quotient map by the action of G_{ant} . It turns out that φ_X only depends on the variety X (Lemma 3.1), and that α_X, φ_X play complementary roles. Indeed, the product map $\pi_X = (\alpha_X, \varphi_X)$ is the quotient by a central affine subgroup scheme (Proposition 3.2). If the variety X is complete, π_X yields an isomorphism $X \cong A \times Y$, where A is an abelian variety, and Y a complete homogeneous rational variety (a result of Sancho de Salas, see [Sa01, Thm. 5.2]). We refine that result by determining the structure of the subgroup schemes $H \subset G$ such that the homogeneous space G/H is complete (Theorem 3.5). Our statement can be deduced from the version of [Sa01, Thm. 5.2] obtained in [Br09b], but we provide a simpler argument.

The methods developed in [Br09a] and the present paper also yield a classification of those torsors over an abelian variety that are homogeneous, i.e., isomorphic to all of their translates; the total spaces of such torsors give interesting examples of homogeneous spaces under non-affine algebraic groups. This will be presented in detail elsewhere.

An important question, left open by the preceding developments, asks for descriptions of the (integral) Chow ring of a homogeneous space, and its higher Chow groups. Here the approach via equivariant intersection theory raises difficulties, since the relation between equivariant and usual Chow theory is only well understood for special groups (see [EG98]). In contrast, equivariant and usual K -theory are tightly related for the much larger class of factorial groups, by work of Merkur’ev (see [Me97]); this suggests that the higher K -theory of homogeneous spaces might be more accessible.

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Notation and conventions. Throughout this article, we consider algebraic varieties, schemes, and morphisms over an algebraically closed field k . We follow the conventions of the book [Ha77]; in particular, a *variety* is an integral separated scheme of finite type over k . By a point, we always mean a closed point.

An *algebraic group* G is a smooth group scheme of finite type; then each connected component of G is a nonsingular variety. We denote by e_G the neutral element and by G^0 the neutral component of G , i.e., the connected component containing e_G .

Recall that every connected algebraic group G has a largest connected affine algebraic subgroup G_{aff} . Moreover, G_{aff} is a normal subgroup of G , and the quotient $G/G_{\text{aff}} =: \text{Alb}(G)$ is an abelian variety. In the resulting exact sequence

$$(1.1) \quad 1 \longrightarrow G_{\text{aff}} \longrightarrow G \xrightarrow{\alpha_G} \text{Alb}(G) \longrightarrow 1,$$

the homomorphism α_G is the *Albanese morphism* of G , i.e., the universal morphism to an abelian variety (see [Co02] for a modern proof of these results).

Also, recall that G admits a largest subgroup scheme G_{ant} which is *anti-affine*, i.e., such that $\mathcal{O}(G_{\text{ant}}) = k$. Moreover, G_{ant} is smooth, connected and central in G , and $G/G_{\text{ant}} =: \text{Aff}(G)$ is the largest affine quotient group of G . In the exact sequence

$$(1.2) \quad 1 \longrightarrow G_{\text{ant}} \longrightarrow G \xrightarrow{\varphi_G} \text{Aff}(G) \longrightarrow 1,$$

the homomorphism φ_G is the *affinization morphism* of G , i.e., the natural morphism $G \rightarrow \text{Spec } \mathcal{O}(G)$ (see [DG70, Sec. III.3.8]). The structure of anti-affine algebraic groups is described in [Sa01] (see also [Br09a, SS08] for a classification of these groups over an arbitrary field).

Finally, recall the *Rosenlicht decomposition*:

$$(1.3) \quad G = G_{\text{aff}} G_{\text{ant}},$$

and $G_{\text{aff}} \cap G_{\text{ant}}$ contains $(G_{\text{ant}})_{\text{aff}}$ as an algebraic subgroup of finite index (see [Ro56, Cor. 5, p. 440]). As a consequence, we have

$$G_{\text{ant}}/(G_{\text{ant}} \cap G_{\text{aff}}) \cong G/G_{\text{aff}} \cong \text{Alb}(G)$$

and also

$$G_{\text{aff}}/(G_{\text{ant}} \cap G_{\text{aff}}) \cong G/G_{\text{ant}} \cong \text{Aff}(G).$$

2. ALGEBRAIC GROUPS

2.1. Albanese and affinization morphisms. Throughout this subsection, we fix a connected algebraic group G , and choose a Borel subgroup B of G , i.e., of G_{aff} . We begin with some easy but very useful observations:

Lemma 2.1. (i) B contains $G_{\text{aff}} \cap G_{\text{ant}}$.

(ii) The product $BG_{\text{ant}} \subset G$ is a connected algebraic subgroup. Moreover, $(BG_{\text{ant}})_{\text{aff}} = B$, and the natural map $\text{Alb}(BG_{\text{ant}}) \rightarrow \text{Alb}(G)$ is an isomorphism.

(iii) The multiplication map $\mu : G_{\text{ant}} \times G_{\text{aff}} \rightarrow G$ yields an isomorphism

$$(2.1) \quad \text{Alb}(G) \times G_{\text{aff}}/B = G_{\text{ant}}/(G_{\text{ant}} \cap G_{\text{aff}}) \times G_{\text{aff}}/B \xrightarrow{\cong} G/B.$$

Proof. (i) Note that $G_{\text{aff}} \cap G_{\text{ant}}$ is contained in the scheme-theoretic centre $C(G_{\text{aff}})$. Next, choose a maximal torus $T \subset B$. Then $C(G_{\text{aff}})$ is contained in the centralizer $C_{G_{\text{aff}}}(T)$, a Cartan subgroup of G_{aff} , and hence of the form TU where $U \subset R_u(G_{\text{aff}})$. Thus, $C_{G_{\text{aff}}}(T) \subset TR_u(G_{\text{aff}}) \subset B$.

(ii) The first assertion holds since G_{ant} centralizes B .

Clearly, $(BG_{\text{ant}})_{\text{aff}}$ contains B . Moreover, we have

$$BG_{\text{ant}}/B \cong G_{\text{ant}}/(B \cap G_{\text{ant}}) = G_{\text{ant}}/(G_{\text{aff}} \cap G_{\text{ant}}) \cong G/G_{\text{aff}},$$

which yields the second assertion.

(iii) In view of the Rosenlicht decomposition, μ is the quotient of $G_{\text{ant}} \times G_{\text{aff}}$ by the action of $G_{\text{ant}} \cap G_{\text{aff}}$ via $z \cdot (x, y) := (zx, z^{-1}y)$. We extend this action to an action of $(G_{\text{ant}} \cap G_{\text{aff}}) \times B$ via $(z, b) \cdot (x, y) := (zx, z^{-1}yb^{-1}) = (zx, yz^{-1}b^{-1})$. By (i), the quotient of $G_{\text{ant}} \times G_{\text{aff}}$ by the latter action exists and is isomorphic to $G_{\text{ant}}/(G_{\text{ant}} \cap G_{\text{aff}}) \times G_{\text{aff}}/B$; this yields the isomorphism (2.1). \square

Next, we study the Albanese map $\alpha_G : G \rightarrow \text{Alb}(G)$, a G_{aff} -torsor (or principal homogeneous space; see [Gr60] for this notion):

Proposition 2.2. (i) α_G is locally trivial for the Zariski topology.

(ii) α_G is trivial if and only if the extension (1.1) splits.

Proof. (i) The map

$$\alpha_{BG_{\text{ant}}} : BG_{\text{ant}} \longrightarrow \text{Alb}(BG_{\text{ant}}) \cong \text{Alb}(G)$$

is a torsor under the connected solvable affine algebraic group B , and hence is locally trivial (see e.g. [Se58, Prop. 14]). By Lemma 2.1 (ii), it follows that α_G has local sections. Thus, this torsor is locally trivial.

(ii) We may identify α_G with the natural map

$$(G_{\text{ant}} \times G_{\text{aff}})/(G_{\text{aff}} \cap G_{\text{ant}}) \longrightarrow G_{\text{ant}}/(G_{\text{aff}} \cap G_{\text{ant}}),$$

a homogeneous bundle associated with the torsor $G_{\text{ant}} \rightarrow G_{\text{ant}}/(G_{\text{aff}} \cap G_{\text{ant}})$ and with the $G_{\text{aff}} \cap G_{\text{ant}}$ -variety G_{aff} . Thus, the sections of α_G are identified with the morphisms (of varieties)

$$f : G_{\text{ant}} \longrightarrow G_{\text{aff}}$$

which are $G_{\text{aff}} \cap G_{\text{ant}}$ -equivariant. But any such morphism is constant, as G_{aff} is affine and $\mathcal{O}(G_{\text{ant}}) = k$. Thus, if α_G has sections, then $G_{\text{aff}} \cap G_{\text{ant}}$ is trivial, since this group scheme acts faithfully on G_{aff} . By the Rosenlicht decomposition, it follows that $G \cong G_{\text{aff}} \times G_{\text{ant}}$ and $G_{\text{ant}} \cong \text{Alb}(G)$; in particular, (1.1) splits. The converse is obvious. \square

Similarly, we consider the affinization map $\varphi_G : G \rightarrow \text{Aff}(G)$, a torsor under G_{ant} .

Proposition 2.3. (i) φ_G is locally trivial (for the Zariski topology) if and only if the group scheme $G_{\text{aff}} \cap G_{\text{ant}}$ is smooth and connected. Equivalently, $G_{\text{aff}} \cap G_{\text{ant}} = (G_{\text{ant}})_{\text{aff}}$.

(ii) φ_G is trivial if and only if the torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}})$ is trivial. Equivalently, $G_{\text{aff}} \cap G_{\text{ant}} = (G_{\text{ant}})_{\text{aff}}$ and any character of $G_{\text{aff}} \cap G_{\text{ant}}$ extends to a character of G_{aff} .

Proof. (i) We claim that φ_G is locally trivial if and only if it admits a rational section. Indeed, if $\sigma : G/G_{\text{ant}} \dashrightarrow G$ is such a rational section, defined at some point $x_0 = \varphi(g_0)$, then the map $x \mapsto g\sigma(g^{-1}x)$ is another rational section, defined at gx_0 , where g is an arbitrary point of G . (Alternatively, the claim holds for any torsor over a nonsingular variety, as follows by combining [Se58, Lem. 4] and [CO92, Thm. 2.1]).

We now argue as in the proof of Proposition 2.2, and identify φ_G with the natural map

$$(G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{aff}} \cap G_{\text{ant}}) \longrightarrow G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}}).$$

This identifies rational sections of φ_G with rational maps

$$f : G_{\text{aff}} - \longrightarrow G_{\text{ant}}$$

which are $G_{\text{aff}} \cap G_{\text{ant}}$ -equivariant. Such a rational map descends to a rational map

$$\bar{f} : G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}}) - \longrightarrow G_{\text{ant}}/(G_{\text{aff}} \cap G_{\text{ant}}) = \text{Alb}(G).$$

But $G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}})$ is an affine algebraic group, and hence is rationally connected. Since $\text{Alb}(G)$ is an abelian variety, it follows that \bar{f} is constant; we may assume that its image is the neutral element. Then f is a rational $G_{\text{aff}} \cap G_{\text{ant}}$ -equivariant map $G_{\text{aff}} - \rightarrow G_{\text{aff}} \cap G_{\text{ant}}$, i.e., a rational section of the torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}})$. Clearly, this is only possible if $G_{\text{aff}} \cap G_{\text{ant}}$ is smooth and connected.

Conversely, if the (affine, commutative) group scheme $G_{\text{aff}} \cap G_{\text{ant}}$ is smooth and connected, then the torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}})$ has rational sections. By the preceding argument, the same holds for the torsor φ_G .

(ii) By the same argument, the triviality of φ_G is equivalent to that of the torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}})$, and this implies the equality $G_{\text{aff}} \cap G_{\text{ant}} = (G_{\text{ant}})_{\text{aff}}$. Write $(G_{\text{ant}})_{\text{aff}} = TU \cong T \times U$, where T is a torus and U a connected commutative unipotent algebraic group. Then a section of the torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/TU$, being a TU -equivariant map $G_{\text{aff}} \rightarrow TU$, yields a T -equivariant map $f : G_{\text{aff}} \rightarrow T$. We may assume that $f(e_{G_{\text{aff}}}) = e_T$. Then f is a homomorphism by rigidity, and restricts to the identity on T . Thus, each character of T extends to a character of G_{aff} . Any such character must be U -invariant, and hence each character of $G_{\text{aff}} \cap G_{\text{ant}}$ extends to a character of G_{aff} .

Conversely, assume that each character of $G_{\text{aff}} \cap G_{\text{ant}}$ extends to a character of G_{aff} . Then there exists a homomorphism $f : G_{\text{aff}} \rightarrow T$ that restricts to the identity of T . Let H denote the kernel of f . Then the multiplication map $H \times T \rightarrow G_{\text{aff}}$ is an isomorphism; in particular, the torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/T \cong H$ is trivial. Moreover, H contains U , and the torsor $H \rightarrow H/U$ is trivial, since U is connected and unipotent, and H/U is affine. Thus, the torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/TU$ is trivial. \square

Next, we show the existence of an isogeny $\pi : \tilde{G} \rightarrow G$ such that the torsor $\varphi_{\tilde{G}}$ is trivial.

Following [Me97], we say that a connected affine algebraic group H is *factorial*, if $\text{Pic}(H)$ is trivial; equivalently, the coordinate ring $\mathcal{O}(H)$ is factorial. By [Me97, Prop. 1.10], this is equivalent to the derived subgroup of $G/R_u(G)$ (a connected semi-simple group) being simply connected.

Proposition 2.4. *There exists an isogeny $\pi : \tilde{G} \rightarrow G$, where \tilde{G} is a connected algebraic group satisfying the following properties:*

- (i) π restricts to an isomorphism $\tilde{G}_{\text{ant}} \cong G_{\text{ant}}$.
- (ii) $\tilde{G}_{\text{aff}} \cap \tilde{G}_{\text{ant}}$ is smooth and connected.
- (iii) $\text{Aff}(\tilde{G})$ is factorial.

Then \tilde{G}_{aff} is factorial as well. Moreover, the \tilde{G}_{ant} -torsor $\varphi_{\tilde{G}}$ is trivial.

Proof. Consider the connected algebraic group

$$\tilde{G} := (G_{\text{aff}} \times G_{\text{ant}}) / (G_{\text{ant}})_{\text{aff}},$$

where $(G_{\text{ant}})_{\text{aff}}$ is embedded in $G_{\text{aff}} \times G_{\text{ant}}$ via $z \mapsto (z, z^{-1})$. By the Rosenlicht decomposition, the natural map $\tilde{G} \rightarrow G$ is an isogeny, and induces an isomorphism $\tilde{G}_{\text{ant}} \rightarrow G_{\text{ant}}$. Moreover, $\tilde{G}_{\text{aff}} \cap \tilde{G}_{\text{ant}} \cong (G_{\text{ant}})_{\text{aff}}$ is smooth and connected. Replacing G with \tilde{G} , we may thus assume that (i) and (ii) already hold for G .

Next, since $\text{Aff}(G)$ is a connected affine algebraic group, there exists an isogeny $p : H \rightarrow \text{Aff}(G)$, where H is a connected factorial affine algebraic group (see [FI73, Prop. 4.3]). The pull-back under p of the extension (1.2) yields an extension

$$1 \longrightarrow G_{\text{ant}} \longrightarrow \tilde{G} \longrightarrow H \longrightarrow 1,$$

where \tilde{G} is an algebraic group equipped with an isogeny $\pi : \tilde{G} \rightarrow G$. Clearly, \tilde{G} is connected and satisfies (i) and (iii) (since $\text{Aff}(\tilde{G}) = H$). To show (ii), note that

$$\text{Alb}(G_{\text{ant}}) = G_{\text{ant}} / (G_{\text{ant}})_{\text{aff}} = G_{\text{ant}} / (G_{\text{ant}} \cap G_{\text{aff}})$$

and hence the natural map $\text{Alb}(G_{\text{ant}}) \rightarrow \text{Alb}(G)$ is an isomorphism. Since that map is the composite

$$\text{Alb}(G_{\text{ant}}) \longrightarrow \text{Alb}(\tilde{G}) \longrightarrow \text{Alb}(G)$$

induced by the natural maps $G_{\text{ant}} \rightarrow \tilde{G} \rightarrow G$, and the map $\text{Alb}(\tilde{G}) \rightarrow \text{Alb}(G)$ is an isogeny, it follows that the map $\text{Alb}(G_{\text{ant}}) \rightarrow \text{Alb}(\tilde{G})$ is an isomorphism; this is equivalent to (ii).

To show that \tilde{G}_{aff} is factorial, note that $\tilde{G}_{\text{aff}} \cap \tilde{G}_{\text{ant}}$ is a connected commutative affine algebraic group, and hence is factorial. Moreover, the exact sequence

$$1 \longrightarrow \tilde{G}_{\text{aff}} \cap \tilde{G}_{\text{ant}} \longrightarrow \tilde{G}_{\text{aff}} \longrightarrow \text{Aff}(\tilde{G}) \longrightarrow 1$$

yields an exact sequence of Picard groups

$$\text{Pic}(\tilde{G}_{\text{aff}} \cap \tilde{G}_{\text{ant}}) \longrightarrow \text{Pic}(\tilde{G}_{\text{aff}}) \longrightarrow \text{Pic}(\text{Aff}(\tilde{G}))$$

(see e.g. [FI73, Prop. 3.1]) which implies our assertion.

Finally, to show that $\varphi_{\tilde{G}}$ is trivial, write $\tilde{G}_{\text{aff}} \cap \tilde{G}_{\text{ant}} = TU$ as in the proof of Proposition 2.3. Arguing in that proof, it suffices to show that the T -torsor $\tilde{G}_{\text{aff}}/U \rightarrow \text{Aff}(\tilde{G})$ is trivial. But this follows from the factoriality of $\text{Aff}(\tilde{G})$. \square

Remarks 2.5. (i) The commutative group $H^1(\text{Aff}(G), G_{\text{ant}})$, that classifies the isotrivial G_{ant} -torsors over $\text{Aff}(G)$, is torsion. Indeed, there is an exact sequence

$$H^1(\text{Aff}(G), (G_{\text{ant}})_{\text{aff}}) \longrightarrow H^1(\text{Aff}(G), G_{\text{ant}}) \longrightarrow H^1(\text{Aff}(G), \text{Alb}(G_{\text{ant}}))$$

(see [Se58, Prop. 13]). Moreover, $H^1(\text{Aff}(G), \text{Alb}(G_{\text{ant}}))$ is torsion by [Se58, Lem. 7], and $H^1(\text{Aff}(G), (G_{\text{ant}})_{\text{aff}})$ is torsion as well, since $\text{Aff}(G)$ is an affine variety with finite Picard group.

(ii) One may ask whether there exists an isogeny $\pi : \tilde{G} \rightarrow G$ such that the torsor $\alpha_{\tilde{G}}$ is trivial. The answer is affirmative when k is the algebraic closure of a finite field: indeed, by a theorem of Arima (see [Ar60]), there exists an isogeny $G_{\text{aff}} \times A \rightarrow G$ where A is an abelian variety. However, the answer is negative over any other field: indeed, there exists an anti-affine algebraic group G , extension of an elliptic curve

by \mathbb{G}_m (see e.g. [Br09a, Ex. 3.11]). Then \tilde{G} is anti-affine as well, for any isogeny $\pi : \tilde{G} \rightarrow G$ (see [Br09a, Lem. 1.4]) and hence the map $\alpha_{\tilde{G}}$ cannot be trivial.

2.2. Chow ring. The aim of this subsection is to describe the Chow ring of the connected algebraic group G in terms of those of $A := \text{Alb}(G)$ and of $\mathcal{B} := G_{\text{aff}}/B$, the flag variety of G_{aff} . For this, we need some preliminary results on characteristic homomorphisms.

We denote the character group of G_{aff} by $X(G_{\text{aff}})$. The G_{aff} -torsor $\alpha_G : G \rightarrow A$ yields a *characteristic homomorphism*

$$(2.2) \quad \gamma_A : X(G_{\text{aff}}) \longrightarrow \text{Pic}(A)$$

which maps any character to the class of the associated line bundle over A . Likewise, we have the characteristic homomorphism

$$(2.3) \quad c_A : X(B) \longrightarrow \text{Pic}(A)$$

associated with the B -torsor $\alpha_{BG_{\text{ant}}} : BG_{\text{ant}} \rightarrow A$.

Lemma 2.6. *The image of c_A is contained in $\text{Pic}^0(A)$, and contains the image of γ_A as a subgroup of finite index.*

Proof. The first assertion is well known in the case that B is a torus, i.e., BG_{ant} is a semi-abelian variety; see e.g. [Se59, VII.3.16]. The general case reduces to that one as follows: we have $B = TU$, where U denotes the unipotent part of B , and T is a maximal torus. Then U is a normal subgroup of BG_{ant} and the quotient group $H := (BG_{\text{ant}})/U$ is a semi-abelian variety. Moreover, $\alpha_{BG_{\text{ant}}}$ factors as the U -torsor $BG_{\text{ant}} \rightarrow H$ followed by the T -torsor $\alpha_H : H \rightarrow A$, and c_A has the same image as the characteristic homomorphism $X(T) \rightarrow \text{Pic}(A)$ associated with the torsor α_H , under the identification $X(B) \cong X(T)$.

To show the second assertion, consider the natural map $G_{\text{ant}} \rightarrow A$, a torsor under $G_{\text{ant}} \cap G_{\text{aff}}$, and the associated homomorphism

$$\sigma_A : X(G_{\text{ant}} \cap G_{\text{aff}}) \longrightarrow \text{Pic}(A).$$

Then γ_A is the composite map

$$X(G_{\text{aff}}) \xrightarrow{u} X(G_{\text{ant}} \cap G_{\text{aff}}) \xrightarrow{\sigma_A} \text{Pic}(A)$$

and likewise, c_A is the composite map

$$X(B) \xrightarrow{v} X(G_{\text{ant}} \cap G_{\text{aff}}) \xrightarrow{\sigma_A} \text{Pic}(A)$$

where u, v denote the restriction maps. Moreover, v is surjective, and u has a finite cokernel since $G_{\text{ant}} \cap G_{\text{aff}} \subset C(G_{\text{aff}})$. \square

Similarly, the B -torsor $G_{\text{aff}} \rightarrow G_{\text{aff}}/B$ yields a homomorphism

$$c_{\mathcal{B}} : X(B) \longrightarrow \text{Pic}(\mathcal{B})$$

that fits into an exact sequence

$$(2.4) \quad 0 \longrightarrow X(G_{\text{aff}}) \longrightarrow X(B) \xrightarrow{c_{\mathcal{B}}} \text{Pic}(\mathcal{B}) \longrightarrow \text{Pic}(G_{\text{aff}}) \longrightarrow 0$$

(see [FI73, Prop. 3.1]). More generally, the Chow ring $A^*(G_{\text{aff}})$ is the quotient of $A^*(\mathcal{B})$ by the ideal generated by the image of $c_{\mathcal{B}}$ (see [Gr58, p. 21]). We now generalize this presentation to $A^*(G)$:

Theorem 2.7. *With the notation and assumptions of this subsection, there is an isomorphism of graded rings*

$$(2.5) \quad A^*(G) \cong (A^*(A) \otimes A^*(\mathcal{B}))/I,$$

where I denotes the ideal generated by the image of the map

$$(c_A, c_{\mathcal{B}}) : X(B) \longrightarrow \text{Pic}(A) \times \text{Pic}(\mathcal{B}) \cong A^1(A) \otimes 1 + 1 \otimes A^1(\mathcal{B}).$$

Proof. Let $c_{G/B} : X(B) \rightarrow \text{Pic}(G/B)$ denote the characteristic homomorphism. Then, as in [Gr58, p. 21], we obtain that $A^*(G) \cong A^*(G/B)/J$, where the ideal J is generated by the image of $c_{G/B}$. But $G/B \cong A \times \mathcal{B}$ by Lemma 2.1 (iii). Moreover, the natural map $A^*(A) \otimes A^*(\mathcal{B}) \rightarrow A^*(A \times \mathcal{B})$ is an isomorphism, as follows e.g. from [FMSS95, Thm. 2]). This identifies $\text{Pic}(G/B)$ with $\text{Pic}(A) \times \text{Pic}(\mathcal{B})$, and $c_{G/B}$ with $(c_A, c_{\mathcal{B}})$. \square

The rational Chow ring $A^*(G)_{\mathbb{Q}}$ admits a simpler presentation, which generalizes the isomorphism $A^*(G_{\text{aff}})_{\mathbb{Q}} \cong \mathbb{Q}$:

Proposition 2.8. *With the notation and assumptions of this subsection, the pull-back under α_G yields an isomorphism*

$$A^*(G)_{\mathbb{Q}} \cong A^*(A)_{\mathbb{Q}}/J_{\mathbb{Q}},$$

where J denotes the ideal of $A^*(A)$ generated by the image of γ_A , i.e., by Chern classes of G_{aff} -homogeneous line bundles.

Proof. Choose again a maximal torus $T \subset B$, with Weyl group

$$W := N_{G_{\text{aff}}}(T)/C_{G_{\text{aff}}}(T).$$

Let S denote the symmetric algebra of the character group $X(T) \cong X(B)$. Then Theorem 2.7 yields an isomorphism

$$A^*(G) \cong (A^*(A) \otimes A^*(\mathcal{B})) \otimes_S \mathbb{Z},$$

where $A^*(A)$ is an S -module via c_A , and likewise for $A^*(\mathcal{B})$; the map $S \rightarrow \mathbb{Z}$ is of course the quotient by the maximal graded ideal. Moreover, $c_{\mathcal{B}}$ induces an isomorphism $A^*(\mathcal{B})_{\mathbb{Q}} \cong S_{\mathbb{Q}} \otimes_{S_{\mathbb{Q}}^W} \mathbb{Q}$, where S^W denotes the ring of W -invariants in S . This yields in turn an isomorphism

$$A^*(G)_{\mathbb{Q}} \cong A^*(A)_{\mathbb{Q}} \otimes_{S_{\mathbb{Q}}^W} \mathbb{Q} \cong A^*(A)_{\mathbb{Q}}/K,$$

where K denotes the ideal of $A^*(A)_{\mathbb{Q}}$ generated by the image of the maximal homogeneous ideal of $S_{\mathbb{Q}}^W$. In view of [Vi89, Lem. 1.3], K is also generated by Chern classes of G_{aff} -homogeneous vector bundles on A . Any such bundle admits a filtration with associated graded a direct sum of B -homogeneous line bundles, since any finite-dimensional G_{aff} -module has a filtration by B -submodules, with associated graded a direct sum of one-dimensional B -modules. Furthermore, by Lemma 2.6, the Chern class of any B -homogeneous line bundle is proportional to that of a G_{aff} -homogeneous line bundle; this completes the proof. \square

Remark 2.9. Assume that G_{aff} is special, i.e., that any G_{aff} -torsor is locally trivial; equivalently, the characteristic homomorphism $S \rightarrow A^*(\mathcal{B})$ is surjective (see [Gr58, Thm. 3]). Then, by the preceding argument, $A^*(G) = A^*(A)/K$, where K is generated by Chern classes of G_{aff} -homogeneous vector bundles. Clearly, $K \supset J$; this inclusion may be strict, as shown by the following example.

Let L be an algebraically trivial line bundle on an abelian variety A ; then there is an extension

$$1 \longrightarrow \mathbb{G}_m = \text{Aut}_A(L) \longrightarrow G_L \longrightarrow A \longrightarrow 1,$$

where G_L is a connected algebraic group contained in $\text{Aut}(L)$ (the automorphism group of the variety L). Consider the vector bundle $E := L \oplus L$ over A ; then we have an exact sequence

$$1 \longrightarrow \text{Aut}_A(E) \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where G is a connected algebraic subgroup of $\text{Aut}(E)$. Hence $G_{\text{aff}} = \text{Aut}_A(E) \cong \text{GL}(2)$. Thus, the subring of $A^*(A)$ generated by Chern classes of G_{aff} -homogeneous vector bundles is also generated by the Chern classes of E , that is, by $2c_1(L)$ and $c_1(L)^2$.

If that ring is generated by $2c_1(L)$ only, then $c_1(L)^2$ is an integral multiple of $4c_1(L)^2$, and hence is torsion in $A^2(A)$. But this cannot hold for all algebraically trivial line bundles on a given abelian surface A over the field of complex numbers. Otherwise, the products $c_1(L)c_1(M)$, where $L, M \in \text{Pic}^0(A)$, generate a torsion subgroup of $A^2(A) = A_0(A)$. But by [BKL76, Thm. A2], that subgroup equals the kernel $T(A)$ of the natural map $A_0(A) \rightarrow \mathbb{Z} \times A$ given by the degree and the sum. Moreover, $T(A)$ is non-zero by [Mu69], and torsion-free by [Ro80].

Corollary 2.10. *Let $g := \dim(A)$, then $A^i(G)_{\mathbb{Q}} = 0$ for all $i > g$, and $A^g(G)_{\mathbb{Q}} \neq 0$.*

Proof. Proposition 2.8 yields readily the first assertion; it also implies that $A^g(G)_{\mathbb{Q}}$ is the quotient of $A^g(A)_{\mathbb{Q}} = A_0(A)_{\mathbb{Q}}$ by a subspace consisting of algebraically trivial cycle classes. \square

Remark 2.11. Likewise, $A_i(G) = 0$ for all $i > \dim(B)$, in view of Theorem 2.7. This vanishing result also follows from the fact that the abelian group $A_*(G)$ is generated by classes of B -stable subvarieties (see [FMSS95, Thm. 1]).

2.3. Picard group. By Theorem 2.7, the Picard group of G admits a presentation

$$(2.6) \quad X(B) \xrightarrow{(c_A, c_B)} \text{Pic}(A) \times \text{Pic}(B) \longrightarrow \text{Pic}(G) \longrightarrow 0.$$

Another description of that group follows readily from the exact sequence of [FI73, Prop. 3.1] applied to the locally trivial fibration $\alpha_G : G \rightarrow A$ with fibre G_{aff} :

Proposition 2.12. *There is an exact sequence*

$$(2.7) \quad 0 \longrightarrow X(G) \longrightarrow X(G_{\text{aff}}) \xrightarrow{\gamma_A} \text{Pic}(A) \longrightarrow \text{Pic}(G) \longrightarrow \text{Pic}(G_{\text{aff}}) \longrightarrow 0,$$

where γ_A is the characteristic homomorphism (2.2), and where all other maps are pull-backs.

Next, we denote by $\text{Pic}^0(G) \subset \text{Pic}(G)$ the group of algebraically trivial divisors modulo rational equivalence, and we define the “Néron-Severi” group of G by

$$\text{NS}(G) := \text{Pic}(G) / \text{Pic}^0(G).$$

Corollary 2.13. *The exact sequence (2.7) induces an exact sequence*

$$(2.8) \quad 0 \longrightarrow X(G) \longrightarrow X(G_{\text{aff}}) \xrightarrow{\gamma_A} \text{Pic}^0(A) \longrightarrow \text{Pic}^0(G) \longrightarrow 0$$

and an isomorphism

$$(2.9) \quad \text{NS}(G) \cong \text{NS}(A) \times \text{Pic}(G_{\text{aff}}).$$

In particular, the abelian group $\text{NS}(G)$ is finitely generated, and the pull-back under α_G yields an isomorphism

$$(2.10) \quad \text{NS}(G)_{\mathbb{Q}} \cong \text{NS}(A)_{\mathbb{Q}}.$$

Proof. The image of γ_A is contained in $\text{Pic}^0(A)$ by Lemma 2.6. Also, note that the pull-back under α_G maps $\text{Pic}^0(A)$ to $\text{Pic}^0(G)$; similarly, the pull-back under the inclusion $G_{\text{aff}} \subset G$ maps $\text{Pic}^0(G)$ to $\text{Pic}^0(G_{\text{aff}}) = 0$. In view of this, the exact sequence (2.8) follows from (2.7).

Together with (2.6), it follows in turn that $\text{NS}(G)$ is the quotient of $\text{NS}(A) \times \text{Pic}(\mathcal{B})$ by the image of $c_{\mathcal{B}}$; this implies (2.9). \square

Also, note that a line bundle M on A is ample if and only if $\alpha_G^*(M)$ is ample, as follows from [Ra70, Lem. XI 1.11.1]. In other words, the isomorphism (2.10) identifies both ample cones.

3. HOMOGENEOUS SPACES

3.1. Two fibrations. Throughout this section, we fix a *homogeneous variety* X , i.e., X has a transitive action of the connected algebraic group G . We choose a point $x \in X$ and denote by $H = G_x$ its stabilizer, a closed subgroup scheme of G . This identifies X with the *homogeneous space* G/H ; the choice of another base point x replaces H with a conjugate.

Since G/G_{aff} is an abelian variety, the product $G_{\text{aff}}H \subset G$ is a closed normal subgroup scheme, independent of the choice of x . Moreover, the homogeneous space $G/G_{\text{aff}}H$ is an abelian variety as well, and the natural map

$$\alpha_X : X = G/H \longrightarrow G/G_{\text{aff}}H = X/G_{\text{aff}}$$

is the Albanese morphism of X . This is a G -equivariant fibration with fibre

$$G_{\text{aff}}H/H \cong G_{\text{aff}}/(G_{\text{aff}} \cap H).$$

If G acts faithfully on X , then H is affine in view of [Ma63, Lemma, p. 154]. Hence G_{aff} has finite index in $G_{\text{aff}}H$. In other words, the natural map

$$G/G_{\text{aff}} = \text{Alb}(G) \longrightarrow \text{Alb}(X) = G/G_{\text{aff}}H \cong G_{\text{ant}}/(G_{\text{ant}} \cap G_{\text{aff}}H)$$

is an isogeny.

We may also consider the natural map

$$\varphi_X : X = G/H \longrightarrow G/G_{\text{ant}}H = X/G_{\text{ant}} \cong G_{\text{aff}}/(G_{\text{aff}} \cap G_{\text{ant}}H).$$

Note that the central subgroup $G_{\text{ant}} \subset G$ acts on X via its quotient

$$G_{\text{ant}}/(G_{\text{ant}} \cap H) \cong G_{\text{ant}}H/H,$$

an anti-affine algebraic group. Moreover, φ_X is a torsor under that group. In particular, if G acts faithfully on X , then $G_{\text{ant}} \cap H$ is trivial, and hence φ_X is a G_{ant} -torsor.

Like the Albanese morphism α_X , the map φ_X only depends on the abstract variety X : this follows from our next result, which generalizes [BP08, Lem. 2.1] (about actions of abelian varieties) to actions of anti-affine groups.

Lemma 3.1. *Given a variety Z , there exists an anti-affine algebraic group $\text{Aut}_{\text{ant}}(Z)$ of automorphisms of Z , such that every action of an anti-affine algebraic group Γ on Z arises from a unique homomorphism $\Gamma \rightarrow \text{Aut}_{\text{ant}}(Z)$. Moreover, $\text{Aut}_{\text{ant}}(Z)$ centralizes any connected group scheme of automorphisms of Z .*

For a homogeneous variety X as above, we have

$$\text{Aut}_{\text{ant}}(X) = G_{\text{ant}}/(G_{\text{ant}} \cap H).$$

Proof. Consider an anti-affine algebraic group Γ and a connected group scheme G , both acting faithfully on Z . Arguing as in the proof of [BP08, Lem. 2.1] and replacing the classical rigidity lemma for complete varieties with its generalization to anti-affine varieties (see [SS08, Thm. 1.7]), we obtain that Γ centralizes G .

Next, we claim that $\dim(\Gamma) \leq 3 \dim(Z)$. To see this, let $\Gamma_{\text{aff}} = U \times T$ where U is a connected commutative unipotent group and T is a torus, and put $A := \Gamma/\Gamma_{\text{aff}}$ so that $\dim(\Gamma) = \dim(T) + \dim(U) + \dim(A)$. Then $\dim(T) \leq \dim(Z)$ since the torus T acts faithfully on Z . Moreover, $\dim(U) \leq \dim(A)$ by [Br09a, Thm. 2.7]. Finally, $\dim(A) \leq \dim(Z)$ since the action of Γ on Z induces an action of $A = \text{Alb}(\Gamma)$ on the Albanese variety of the smooth locus of Z , and that action has a finite kernel by a theorem of Nishi and Matsumura (see [Ma63], or [Br09b] for a modern proof). Putting these facts together yields the claim.

In turn, the claim implies the first assertion, in view of the connectedness of anti-affine groups.

For the second assertion, we may assume that G acts faithfully on X . Then $G_{\text{ant}} \subset \text{Aut}_{\text{ant}}(X) =: \Gamma$, and Γ centralizes G . Thus, Γ acts on X/G_{ant} so that φ_X is equivariant. But X/G_{ant} is homogeneous under G_{aff} , and hence has a trivial Albanese variety. By the Nishi-Matsumura theorem again, it follows that every connected algebraic group acting faithfully on X/G_{ant} is affine. Since Γ is anti-affine, it must act trivially on X/G_{ant} . In particular, each orbit of Γ in X is an orbit of G_{ant} . But Γ acts freely on X (since the product $\Gamma G \cong (\Gamma \times G)/G_{\text{ant}}$ is a connected algebraic group acting faithfully on X , and $(\Gamma G)_{\text{ant}} = \Gamma$.) It follows that $\Gamma = G_{\text{ant}}$. \square

Proposition 3.2. *Assume that G acts faithfully on X . Then the product map*

$$(3.1) \quad \pi_X := (\alpha_X, \varphi_X) : X \longrightarrow X/G_{\text{aff}} \times X/G_{\text{ant}}$$

is a torsor under $G_{\text{aff}}H \cap G_{\text{ant}}$, an affine commutative group scheme which contains $(G_{\text{ant}})_{\text{aff}}$ as an algebraic subgroup of finite index.

Proof. The map π_X is identified with the natural map

$$G/H \longrightarrow G/G_{\text{aff}}H \times G/G_{\text{ant}}H.$$

By the Rosenlicht decomposition (1.3), the right-hand side is homogeneous under G ; it follows that we may view π_X as the natural map

$$G/H \longrightarrow G/(G_{\text{aff}}H \cap G_{\text{ant}}H).$$

But H is a normal subgroup scheme of $G_{\text{ant}}H$, and hence of $G_{\text{aff}}H \cap G_{\text{ant}}H$. Moreover,

$$G_{\text{aff}}H \cap G_{\text{ant}}H = (G_{\text{aff}}H \cap G_{\text{ant}})H \cong (G_{\text{aff}}H \cap G_{\text{ant}}) \times H,$$

since $G_{\text{aff}}H \cap G_{\text{ant}}$ centralizes H , and $(G_{\text{aff}}H \cap G_{\text{ant}}) \cap H = G_{\text{ant}} \cap H$ is trivial by the faithfulness assumption. It follows that π_X is a torsor under $G_{\text{aff}}H \cap G_{\text{ant}}$. The latter group scheme contains $G_{\text{aff}} \cap G_{\text{ant}}$ as a normal subgroup scheme, and

$$(G_{\text{aff}}H \cap G_{\text{ant}})/(G_{\text{aff}} \cap G_{\text{ant}}) \cong (G_{\text{aff}}H \cap G_{\text{ant}})G_{\text{aff}}/G_{\text{aff}} = G_{\text{aff}}H/G_{\text{aff}}$$

where the latter equality follows again from (1.3). As a consequence, the quotient $(G_{\text{aff}}H \cap G_{\text{ant}})/(G_{\text{aff}})_{\text{ant}}$ is finite; this completes the proof. \square

We now obtain a criterion for the local triviality of φ_X , which generalizes Proposition 2.3 (i) with a different argument. The map α_X is not necessarily locally trivial, as shown by Example 3.4.

Proposition 3.3. *Assume that G acts faithfully on X . Then the G_{ant} -torsor φ_X is locally trivial if and only if $G_{\text{aff}} \cap G_{\text{ant}} = (G_{\text{ant}})_{\text{aff}}$ and $H \subset G_{\text{aff}}$. Under these assumptions, π_X is locally trivial as well.*

Proof. If φ_X is locally trivial, then X contains an open G_{ant} -stable subset, equivariantly isomorphic to $G_{\text{ant}} \times Y$, where Y is an open subset of X/G_{ant} . It follows that

$$\text{Alb}(X) \cong \text{Alb}(G_{\text{ant}} \times Y) \cong \text{Alb}(G_{\text{ant}}) \times \text{Alb}(Y) \cong \text{Alb}(G_{\text{ant}}) \times \text{Alb}(X/G_{\text{ant}}).$$

But $\text{Alb}(X/G_{\text{ant}})$ is trivial, since X/G_{ant} is homogeneous under G_{aff} . As a consequence, the natural map

$$\text{Alb}(G_{\text{ant}}) = \text{Alb}(G_{\text{ant}}H/H) \longrightarrow \text{Alb}(G/H) = \text{Alb}(X)$$

is an isomorphism. Now recall that

$$\text{Alb}(G_{\text{ant}}) = G_{\text{ant}}/(G_{\text{ant}})_{\text{aff}} \quad \text{and} \quad \text{Alb}(X) = G_{\text{ant}}/(G_{\text{ant}} \cap G_{\text{aff}}H).$$

It follows that

$$(3.2) \quad (G_{\text{ant}})_{\text{aff}} = G_{\text{ant}} \cap G_{\text{aff}}H.$$

In particular, $(G_{\text{ant}})_{\text{aff}} = G_{\text{ant}} \cap G_{\text{aff}}$, and

$$G_{\text{aff}}H = G_{\text{aff}}(G_{\text{ant}} \cap G_{\text{aff}}H) = G_{\text{aff}}(G_{\text{ant}})_{\text{aff}} = G_{\text{aff}},$$

i.e., $H \subset G_{\text{aff}}$.

Conversely, if $(G_{\text{ant}})_{\text{aff}} = G_{\text{ant}} \cap G_{\text{aff}}$ and $H \subset G_{\text{aff}}$, then (3.2) clearly holds. As a consequence, π_X is locally trivial. Moreover, $X/G_{\text{ant}} \cong G_{\text{aff}}/(G_{\text{ant}})_{\text{aff}}H$ and $(G_{\text{ant}})_{\text{aff}} \cap H$ is trivial. Thus, the natural map $G_{\text{aff}}/H \longrightarrow G_{\text{aff}}/(G_{\text{ant}})_{\text{aff}}H$ is a $(G_{\text{ant}})_{\text{aff}}$ -torsor, and hence has local sections. Since G_{aff}/H is a closed subvariety of X , this yields local sections of φ_X . \square

Example 3.4. Let $G := A \times \mathrm{SL}(2)$, where A is an abelian variety. Denote by T the diagonal torus of $\mathrm{SL}(2)$, and let H be the subgroup of G generated by T and (a, n) , where $a \in A$ is a point of order 2, and n is any point of $N_{\mathrm{SL}(2)}(T) \setminus T$. Then the Albanese morphism of G/H is not locally trivial.

Otherwise, the pull-back

$$i^* : \mathrm{Pic}(G/H) \longrightarrow \mathrm{Pic}(G_{\mathrm{aff}}H/H)$$

(under the inclusion $i : G_{\mathrm{aff}}H/H \rightarrow G/H$) is surjective, since $G_{\mathrm{aff}}H/H$ is a fibre of $\alpha_{G/H}$. We now show that $\mathrm{Pic}(G_{\mathrm{aff}}H/H) \cong \mathbb{Z}$, and

$$\alpha_{G/H}^* : \mathrm{Pic}(\mathrm{Alb}(G/H)) \longrightarrow \mathrm{Pic}(G/H)$$

is an isomorphism over the rationals. Since the composite map $i^* \alpha_{G/H}^*$ is zero, this yields a contradiction.

Clearly, $G_{\mathrm{aff}} = \mathrm{SL}(2)$ and $G_{\mathrm{aff}} \cap H = H^0 = T$, and hence

$$G_{\mathrm{aff}}H/H \cong G_{\mathrm{aff}}/(G_{\mathrm{aff}} \cap H) \cong \mathrm{SL}(2)/T.$$

As a consequence, $\mathrm{Pic}(G_{\mathrm{aff}}H/H) \cong \mathrm{X}(T) \cong \mathbb{Z}$. Moreover, $G/H^0 \cong A \times \mathrm{SL}(2)/T$ equivariantly for the action of H/H^0 . The latter group has order 2, and its non-trivial element acts via translation by a on A , and via right multiplication by n on $\mathrm{SL}(2)/T$. Since the natural map $G/H^0 \rightarrow G/H$ is the quotient by H/H^0 acting via right multiplication, we obtain by [Fu98, Ex. 1.7.6]:

$$\mathrm{Pic}(G/H)_{\mathbb{Q}} \cong \mathrm{Pic}(G/H^0)_{\mathbb{Q}}^{H/H^0} \cong \mathrm{Pic}(A \times \mathrm{SL}(2)/T)^{(a,n)}$$

Moreover, the natural map $\mathrm{Pic}(A) \times \mathrm{Pic}(\mathrm{SL}(2)/T) \rightarrow \mathrm{Pic}(A \times \mathrm{SL}(2)/T)$ is an isomorphism, since the variety $\mathrm{SL}(2)/T$ is rational. Also, n acts via multiplication by -1 on $\mathrm{Pic}(\mathrm{SL}(2)/T) \cong \mathbb{Z}$. This yields an isomorphism

$$(3.3) \quad \mathrm{Pic}(G/H)_{\mathbb{Q}} \cong \mathrm{Pic}(A)_{\mathbb{Q}}^a \cong \mathrm{Pic}(A/a)_{\mathbb{Q}},$$

where $A/a = G/G_{\mathrm{aff}}H$ is the Albanese variety of G/H (this description of the rational Picard group will be generalized to all homogeneous spaces in the final subsection). The isomorphism (3.3) is obtained from the pull-back $\mathrm{Pic}(A) \rightarrow \mathrm{Pic}(A \times \mathrm{SL}(2)/T)$ under α_{G/H^0} , and hence is the pull-back under $\alpha_{G/H}$.

3.2. Complete homogeneous spaces. In this subsection, we describe the subgroup schemes $H \subset G$ such that G/H is complete, in terms of the Rosenlicht decomposition.

Theorem 3.5. *Let G be a connected algebraic group, and H a closed subgroup scheme. Then the homogeneous space G/H is complete if and only if*

$$(3.4) \quad H = (H \cap G_{\mathrm{aff}})(H \cap G_{\mathrm{ant}})$$

where both $G_{\mathrm{aff}}/(H \cap G_{\mathrm{aff}})$ and $G_{\mathrm{ant}}/(H \cap G_{\mathrm{ant}})$ are complete; equivalently, $H \cap G_{\mathrm{aff}}$ contains a Borel subgroup of G_{aff} , and $H \cap G_{\mathrm{ant}}$ contains $(G_{\mathrm{ant}})_{\mathrm{aff}}$.

Under these assumptions, the map (3.1),

$$\pi_{G/H} : G/H \longrightarrow G/G_{\mathrm{aff}}H \times G/G_{\mathrm{ant}}H \cong G_{\mathrm{ant}}/(H \cap G_{\mathrm{ant}}) \times G_{\mathrm{aff}}/(H \cap G_{\mathrm{aff}})$$

is an isomorphism, $G_{\mathrm{ant}}/(H \cap G_{\mathrm{ant}})$ is an abelian variety, and $G_{\mathrm{aff}}/(H \cap G_{\mathrm{aff}})$ is a (complete, homogeneous) rational variety.

Proof. If (3.4) holds, then the Rosenlicht decomposition yields a surjective morphism

$$G_{\text{aff}}/(H \cap G_{\text{aff}}) \times G_{\text{ant}}/(H \cap G_{\text{ant}}) \longrightarrow G/H,$$

and hence G/H is complete.

To show the converse, we first argue along the lines of the proof of [Br09b, Thm. 4]. If G/H is complete, then H contains a Borel subgroup $B \subset G_{\text{aff}}$ by Borel's fixed point theorem. Hence $G_{\text{aff}} \cap G_{\text{ant}}$ fixes the base point of G/H , by Lemma 2.1. But $G_{\text{aff}} \cap G_{\text{ant}}$ is contained in the centre of G , and hence acts trivially on G/H . Thus, we may replace G with $G/(G_{\text{aff}} \cap G_{\text{ant}})$, and hence assume that

$$G \cong A \times G_{\text{aff}},$$

where A is an abelian variety. In particular, $G_{\text{ant}} = A$.

Also, note that the radical $R(G_{\text{aff}})$ fixes a point of G/H , and hence is contained in H . Thus, we may assume that G_{aff} is semi-simple.

We may further assume that A acts faithfully on G/H ; then the (scheme theoretic) intersection $A \cap H$ is just the neutral element e_A . Thus, the second projection

$$p_2 : G \longrightarrow G_{\text{aff}}$$

restricts to a closed immersion $H \hookrightarrow G_{\text{aff}}$. In particular, H is affine; therefore, the reduced neutral component H_{red}^0 is contained in G_{aff} . Clearly, H_{red}^0 contains B , and hence is a parabolic subgroup of G that we denote by P . Moreover, $p_2(H)$ contains P as a subgroup of finite index; thus, $p_2(H)_{\text{red}} = P$. It follows that $H_{\text{red}} = P$.

We now diverge from the proof in [Br09b, Thm. 4], which relies on the Bialynicki-Birula decomposition. Choose a parabolic subgroup $P^- \subset G_{\text{aff}}$ opposite to P . Then the product

$$P R_u(P^-) \cong P \times R_u(P^-)$$

is an open neighborhood of P in G_{aff} , and hence of $p_2(H)$ as well. Thus, $p_2(H) = P (R_u(P^-) \cap H)$ where $R_u(P^-) \cap H$ is a finite (local) group scheme, normalized by the Levi subgroup $L = P \cap P^-$ of P . It follows that

$$H = P \Gamma,$$

where $\Gamma := (A \times R_u(P^-)) \cap H$ is isomorphic to $R_u(P^-) \cap H$ via p_2 ; note that L normalizes Γ . Next, choose a maximal torus $T \subset L$. Then T acts on $A \times R_u(P^-)$ by conjugation, and the quotient (in the sense of geometric invariant theory) is the first projection

$$p_1 : A \times R_u(P^-) \longrightarrow A$$

with image the T -fixed point subscheme. Thus, for the closed T -stable subscheme $\Gamma \subset A \times R_u(P^-)$, the quotient is the restriction of p_1 with image $A \cap \Gamma$. But $A \cap \Gamma \subset A \cap H$ is trivial. Hence so is $p_1(\Gamma) = p_1(H)$, i.e., $H \subset G_{\text{aff}}$. Thus,

$$G/H = A \times (G_{\text{aff}}/H)$$

with projections $\alpha_{G/H}$ and $\varphi_{G/H}$. This proves all our assertions. \square

Remarks 3.6. (i) Theorem 3.5 gives back the isomorphism $G/B \cong A \times G_{\text{aff}}/B$, obtained in Lemma 2.1 via a more direct argument.

(ii) It is easy to describe the affine or quasi-affine homogeneous spaces in terms of the Rosenlicht decomposition. Specifically, G/H is affine (resp. quasi-affine) if and only if G contains G_{ant} and $G_{\text{aff}}/(H \cap G_{\text{aff}})$ is affine (resp. quasi-affine). Indeed, G_{ant} acts trivially on any quasi-affine variety, as follows e.g. from [Br09a, Lem. 1.1].

3.3. Rational Chow ring. In this subsection, we describe the rational Chow ring $A^*(G/H)_{\mathbb{Q}}$, where G/H is as in Subsection 3.1. We may assume that G acts faithfully on G/H , and hence that H is affine; in particular, $H^0 \subset G_{\text{aff}}$. We may assume in addition that H is reduced. Indeed, for an arbitrary subgroup scheme H , the natural map $\pi : G/H_{\text{red}} \rightarrow G/H$ is a torsor under the infinitesimal group scheme H/H_{red} ; thus, π is finite and bijective, and

$$\pi^* : A^*(G/H_{\text{red}}) \longrightarrow A^*(G/H)$$

is an isomorphism over \mathbb{Q} .

To state our result, we need some notation and preliminaries. Let $T \subset G_{\text{aff}}$ be a maximal torus, and W its Weyl group. Denote by $S = S_T$ the symmetric algebra of the character group $X(T)$; then W acts on S , and the invariant ring S^W is independent of the choice of T ; we denote that graded ring by $S_{G_{\text{aff}}}$.

Lemma 3.7. (i) *The restriction to T induces an injective homomorphism $X(G_{\text{aff}}) \rightarrow X(T)^W$ with finite cokernel. In particular,*

$$(3.5) \quad X(G_{\text{aff}})_{\mathbb{Q}} \cong S_{G_{\text{aff}}, \mathbb{Q}}^1$$

(the subspace of homogeneous elements of degree 1).

(ii) *Choose a maximal torus T_H of H , and a maximal torus T of G containing T_H . Then the restriction to T_H induces a homomorphism of graded rings*

$$(3.6) \quad r_{H^0} : S_{G_{\text{aff}}} \longrightarrow S_{H^0}.$$

Moreover, the quotient H/H^0 acts on S_{H^0} , and the image of r_{H^0} is contained in the invariant subring.

Proof. (i) The assertion is well known if G_{aff} is reductive. The general case reduces to that one by considering $\bar{G}_{\text{aff}} := G_{\text{aff}}/R_u(G_{\text{aff}})$, a connected reductive group with character group isomorphic to $X(G_{\text{aff}})$. Indeed, the image of T in \bar{G}_{aff} is a maximal torus \bar{T} , isomorphic to T . Moreover, the corresponding Weyl group \bar{W} satisfies $X(T)^W \cong X(\bar{T})^{\bar{W}}$: to see this, it suffices to show that the map $N_{G_{\text{aff}}}(T) \rightarrow N_{\bar{G}_{\text{aff}}}(\bar{T})$ is surjective. Let $g \in G_{\text{aff}}$ such that its image \bar{g} normalizes \bar{T} . Then gTg^{-1} is a maximal torus of $R_u(G_{\text{aff}})T$, a connected solvable subgroup of G_{aff} . Thus, $gTg^{-1} = \gamma^{-1}T\gamma$ for some $\gamma \in R_u(G_{\text{aff}})$. Replacing g with $g\gamma$ (which leaves \bar{g} unchanged), we obtain that $g \in N_{G_{\text{aff}}}(T)$.

(ii) We claim that the restriction $S_T \rightarrow S_{T_H}$ maps S^W to the subring of invariants of $N_{G_{\text{aff}}}(T_H)$. Indeed, given $g \in N_{G_{\text{aff}}}(T_H)$, the conjugate $g^{-1}Tg$ contains T_H , and hence is a maximal torus of $C_{G_{\text{aff}}}(T_H)$. Thus, there exists $\gamma \in C_{G_{\text{aff}}}(T_H)$ such that $g^{-1}Tg = \gamma T \gamma^{-1}$. Replacing again g with $g\gamma$, we may assume that $g \in N_{G_{\text{aff}}}(T)$; this yields our claim.

By that claim, r_{H^0} is well defined. To prove the final assertion, we may replace H with $H \cap G_{\text{aff}}$, since each element of the quotient $H/(H \cap G_{\text{aff}}) \cong G_{\text{aff}}H/G_{\text{aff}}$ has

a representative in the centre of G . Using the conjugacy of maximal tori in H^0 , we obtain as above that $H \cap G_{\text{aff}} = H^0 N_{H \cap G_{\text{aff}}}(T_H)$. In other words,

$$H/H^0 \cong N_{H \cap G_{\text{aff}}}(T_H)/N_{H^0}(T_H)$$

which yields the desired invariance. \square

We may now formulate our description of $A^*(G/H)_{\mathbb{Q}}$:

Theorem 3.8. *Let G be a connected algebraic group with Albanese variety A , and let $H \subset G$ be an affine algebraic subgroup. Consider the action of the finite group H/H^0 on $A^*(A)$ via the action of its quotient $H/(H \cap G_{\text{aff}})$ on $A = G/G_{\text{aff}}$ by translations, and its action on $S_{H^0, \mathbb{Q}}$ as in Lemma 3.7. Then*

$$A^*(G/H)_{\mathbb{Q}} = (A^*(A)_{\mathbb{Q}} \otimes S_{H^0, \mathbb{Q}})^{H/H^0} / I,$$

where I denote the ideal generated by the image of

$$(\gamma_A, r_{H^0}^+) : X(G_{\text{aff}})_{\mathbb{Q}} \times S_{G_{\text{aff}}, \mathbb{Q}}^+ \longrightarrow \text{Pic}(A)_{\mathbb{Q}} \times S_{H^0, \mathbb{Q}}^+ \cong A^1(A)_{\mathbb{Q}} \otimes 1 + 1 \otimes S_{H^0, \mathbb{Q}}^+.$$

Here γ_A denotes the characteristic homomorphism (2.2), and $r_{H^0}^+ : S_{G_{\text{aff}}, \mathbb{Q}}^+ \rightarrow S_{H^0, \mathbb{Q}}^+$ denotes the restriction of the map (3.6) to the maximal graded ideals.

Proof. Note first that the image of γ_A does consist of invariants of H/H^0 , since it is contained in $\text{Pic}^0(A)$, the invariant subgroup of $\text{Pic}(A)$ for the action of A on itself by translations. Likewise, the image of $r_{H^0}^+$ consists of invariants by Lemma 3.7.

We now employ arguments of equivariant intersection theory (see [EG98]); for later use, we briefly review its construction.

Given a linear algebraic group H and an integer $i \geq 0$, there exist an H -module V and an open H -stable subset $U \subset V$ such that the quotient $U \rightarrow U/H$ exists and is an H -torsor, and $\text{codim}_V(V \setminus U) > i$ (this may be seen as an approximation of the classifying bundle $EH \rightarrow BH$). For any H -variety X , we may form the “mixed quotient”

$$X \times^H U := (X \times U)/H,$$

where H acts diagonally on $X \times U$. The Chow group $A^i(X \times^H U)$ turns out to be independent of the choices of V and U ; this defines the equivariant Chow group $A_H^i(X)$. If X is smooth, then

$$A_H^*(X) := \bigoplus_i A_H^i(X)$$

is equipped with an intersection product which makes it a graded algebra. In particular, the equivariant Chow ring of the point is a graded algebra denoted by $A^*(BH)$, and $A^*(BH)_{\mathbb{Q}} \cong S_{H, \mathbb{Q}}$ if H is connected. Moreover, $A_H^*(X)$ is a graded algebra over $A^*(BH)$.

We now prove four general facts of equivariant intersection theory which are variants of known results, but for which we could not find any appropriate references.

Step 1. For any smooth variety X equipped with an action of a connected linear algebraic group G_{aff} , we have an isomorphism of graded rings

$$A^*(X)_{\mathbb{Q}} \cong A_{G_{\text{aff}}}^*(X)_{\mathbb{Q}} \otimes_{S_{G_{\text{aff}}, \mathbb{Q}}} \mathbb{Q},$$

where the map $S_{G_{\text{aff}}, \mathbb{Q}} \rightarrow \mathbb{Q}$ is the quotient by the maximal graded ideal.

Indeed, if G_{aff} is a torus, then the statement holds in fact over the integers, by [Br97, 2.3 Cor. 1]. For an arbitrary G_{aff} with maximal torus T and Weyl group W , the definition of equivariant Chow groups combined with [Vi89, Thm. 2.3] yields a natural isomorphism

$$(3.7) \quad A_{G_{\text{aff}}}^*(X)_{\mathbb{Q}} \otimes_{S_{G_{\text{aff}}, \mathbb{Q}}} S_{T, \mathbb{Q}} \cong A_T^*(X)_{\mathbb{Q}}$$

which in turn implies our assertion.

Step 2. There is an isomorphism of graded rings

$$A_{G_{\text{aff}}}^*(G/H) \cong A_H^*(A).$$

Indeed, for a fixed degree i and an approximation $V \supset U \rightarrow U/G_{\text{aff}}$ as above, we have

$$A_{G_{\text{aff}}}^i(G/H) = A^i(G/H \times^{G_{\text{aff}}} U) = A^i((G \times U)/(G_{\text{aff}} \times H)).$$

This yields isomorphisms

$$A_{G_{\text{aff}}}^i(G/H) \cong A_{G_{\text{aff}} \times H}^i(G \times U) \cong A_{G_{\text{aff}} \times H}^i(G),$$

where the first one follows from [EG98, Prop. 8], and the second one holds since $G \times U$ is open in $G \times V$ and the complement has codimension $> i$. By symmetry, this implies our assertion.

Step 3. For any smooth variety X equipped with an action of the linear algebraic group H , the finite group H/H^0 acts on $A_{H^0}^*(X)$, and we have an isomorphism

$$A_H^*(X)_{\mathbb{Q}} \cong A_{H^0}^*(X)_{\mathbb{Q}}^{H/H^0}.$$

Indeed, by the definition of equivariant Chow groups, we may reduce to the case where the quotient $X \rightarrow X/H$ exists and is an H -torsor. Then

$$A_H^*(X)_{\mathbb{Q}} \cong A^*(X/H)_{\mathbb{Q}} \cong A^*(X/H^0)_{\mathbb{Q}}^{H/H^0} \cong A_{H^0}^*(X)_{\mathbb{Q}}^{H/H^0},$$

where the first and last isomorphism follow from [EG98, Prop. 8] again, and the middle one from [Fu98, Ex. 1.7.6].

Step 4. In the situation of Step 3, assume in addition that H^0 acts trivially on X . Then there is an isomorphism

$$A_H^*(X)_{\mathbb{Q}} \cong (A^*(X)_{\mathbb{Q}} \otimes S_{H^0, \mathbb{Q}})^{H/H^0}.$$

Indeed, by Step 3, we may reduce to the case where H is connected. Then $X \times^H U \cong X \times U/H$; this defines a homomorphism of graded rings $A^*(X) \rightarrow A_H^*(X)$ and, in turn, a homomorphism of graded $A^*(BH)$ -algebras

$$f : A^*(X) \otimes A^*(BH) \longrightarrow A_H^*(X).$$

If H is a torus, then f is an isomorphism, as follows from [Br97, Thm. 2.1]. For an arbitrary H with maximal torus T , (3.7) implies that f is an isomorphism after tensor product with $S_{T, \mathbb{Q}}$ over $S_{H, \mathbb{Q}}$. Since $S_{T, \mathbb{Q}}$ is faithfully flat over $S_{H, \mathbb{Q}}$, this yields our assertion.

We may now prove Theorem 3.8: by Steps 1 and 2,

$$A^*(G/H)_{\mathbb{Q}} \cong A^*_{G_{\text{aff}}}(G/H)_{\mathbb{Q}} \otimes_{S_{G_{\text{aff}}, \mathbb{Q}}} \mathbb{Q} \cong A^*_H(A)_{\mathbb{Q}} \otimes_{S_{G_{\text{aff}}, \mathbb{Q}}} \mathbb{Q},$$

where H acts on A via its quotient $H/(H \cap G_{\text{aff}})$. Since $H \cap G_{\text{aff}}$ contains H^0 , we obtain by Steps 3 and 4:

$$A^*(G/H)_{\mathbb{Q}} \cong (A^*(A)_{\mathbb{Q}} \otimes S_{H^0, \mathbb{Q}})^{H/H^0} \otimes_{S_{G_{\text{aff}}, \mathbb{Q}}} \mathbb{Q},$$

where $S_{G_{\text{aff}}, \mathbb{Q}}$ acts on $A^*(A)_{\mathbb{Q}}$ via the characteristic homomorphism, and on $S_{H^0, \mathbb{Q}}$ via the restriction. This yields our assertion. \square

This description of $A^*(G/H)_{\mathbb{Q}}$ takes a much simpler form in the case that $H \subset G_{\text{aff}}$:

Corollary 3.9. *Let G be a connected algebraic group with Albanese variety A , and let H be an algebraic subgroup of G_{aff} . Then*

$$(3.8) \quad A^*(G/H)_{\mathbb{Q}} \cong A^*(A)_{\mathbb{Q}}/J \otimes A^*(G_{\text{aff}}/H)_{\mathbb{Q}},$$

where J denotes the ideal of $A^*(A)_{\mathbb{Q}}$ generated by $\gamma_A(\ker(r_H))$.

Proof. Note that H/H^0 acts trivially on A , and hence on $A^*(A)$. Also, by Theorem 3.8 applied to G_{aff}/H , the ring $A^*(G_{\text{aff}}/H)_{\mathbb{Q}}$ is the quotient of $(S_{H^0, \mathbb{Q}})^{H/H^0}$ by the ideal generated by the image of γ_A . This implies (3.8) in view of Theorem 3.8 again. \square

3.4. Rational Picard group. In this subsection, we obtain an analogue of the exact sequence (2.7) for homogeneous spaces. To formulate the result, we construct a “restriction map”

$$(3.9) \quad r_H : X(G_{\text{aff}})_{\mathbb{Q}} \longrightarrow X(H)_{\mathbb{Q}}$$

(although H is not necessarily contained in G_{aff}). The image of the restriction $X(G_{\text{aff}}) \rightarrow X(H^0)$ consists of invariants of H/H^0 , since any character of G_{aff} is invariant under the action of G by conjugation. Moreover,

$$(3.10) \quad X(H^0)^{H/H^0}_{\mathbb{Q}} \cong X(H)_{\mathbb{Q}}$$

(this isomorphism may be obtained by viewing $X(H)$ as the group of algebraic classes $H^1_{\text{alg}}(H, k^*)$; alternatively, it follows from the isomorphism $X(H) \cong A^1(BH)$, a consequence of [EG98, Thm. 1], together with Step 3 where X is a point). This yields the desired map, and we may now state:

Proposition 3.10. *There is an exact sequence*

$$(3.11) \quad 0 \longrightarrow X(G/H)_{\mathbb{Q}} \longrightarrow X(G_{\text{aff}})_{\mathbb{Q}} \xrightarrow{(\gamma_A, r_H)} \text{Pic}(A/H)_{\mathbb{Q}} \times X(H)_{\mathbb{Q}} \longrightarrow \text{Pic}(G/H)_{\mathbb{Q}} \longrightarrow 0,$$

where $X(G/H)$ denotes the group of characters of G which restrict trivially to H (so that $X(G/H) \cong \mathcal{O}(G/H)^*/k^*$), γ_A is the characteristic homomorphism (2.2), and r_H is the restriction map (3.9).

If H is contained in G_{aff} (e.g., if H is connected), then there is an exact sequence

$$(3.12) \quad 0 \longrightarrow X(G/H) \longrightarrow X(G_{\text{aff}}) \xrightarrow{(\gamma_A, r_H)} \text{Pic}(A) \times X(H) \longrightarrow \text{Pic}(G/H) \longrightarrow \text{Pic}(G_{\text{aff}}),$$

where r_H denotes the (usual) restriction to H .

Proof. By Theorem 3.8, $\text{Pic}(G/H)_{\mathbb{Q}}$ is the quotient of $(\text{Pic}(A)_{\mathbb{Q}} \times S_{H^0, \mathbb{Q}}^1)^{H/H^0}$ by the image of (γ_A, r_{H^0}) . Moreover,

$$(\text{Pic}(A)_{\mathbb{Q}})^{H/H^0} \cong \text{Pic}(A/H)_{\mathbb{Q}}$$

by [Fu98, Ex. 1.7.6] again, since H acts on A via its finite quotient $H/(H \cap G_{\text{aff}})$. Also, $S_{H^0, \mathbb{Q}}^1 \cong X(H^0)_{\mathbb{Q}}$ by (3.5), and hence

$$(S_{H^0, \mathbb{Q}}^1)^{H/H^0} \cong X(H)_{\mathbb{Q}}$$

in view of (3.10). This yields the exact sequence (3.11), except for the description of the kernel of (γ_A, r_H) . But the kernel of γ_A is $X(G)_{\mathbb{Q}}$ by Proposition 2.12, and the kernel of the induced map $X(G)_{\mathbb{Q}} \rightarrow X(H)_{\mathbb{Q}}$ is $X(G/H^0)_{\mathbb{Q}}$ in view of the definition of r_H . Moreover, there is an exact sequence

$$0 \longrightarrow X(G/H) \longrightarrow X(G/H^0) \longrightarrow X(H/H^0)$$

and hence $X(G/H^0)_{\mathbb{Q}} = X(G/H)_{\mathbb{Q}}$; this completes the proof of (3.11).

To show (3.12), we use the exact sequence (see [KKV89, Sec. 2])

$$X(G_{\text{aff}}) \longrightarrow \text{Pic}_{G_{\text{aff}}}(G/H) \longrightarrow \text{Pic}(G/H) \longrightarrow \text{Pic}(G_{\text{aff}}),$$

where $\text{Pic}_{G_{\text{aff}}}(G/H)$ denotes the group of isomorphism classes of G_{aff} -linearized line bundles on G/H (so that $\text{Pic}_{G_{\text{aff}}}(G/H) = A_{G_{\text{aff}}}^1(G/H)$ in view of [EG98, Thm. 1]). Also,

$$\text{Pic}_{G_{\text{aff}}}(G/H) \cong \text{Pic}_{G_{\text{aff}} \times H}(G) \cong \text{Pic}_H(G/G_{\text{aff}}) \cong \text{Pic}_H(A) \cong \text{Pic}(A) \times X(H),$$

where the last isomorphism holds since H acts trivially on A . As above, this yields (3.12) except for the description of the kernel of (γ_A, r_H) , which follows again from Proposition 2.12. \square

This yields the following descriptions of $\text{Pic}^0(G/H)$ and of the “Néron-Severi group” $\text{NS}(G/H) := \text{Pic}(G/H)/\text{Pic}^0(G/H)$, by arguing as in the proof of Corollary 2.13:

Corollary 3.11. *The pull-back under the quotient map $G \rightarrow G/H$ yields an isomorphism*

$$\text{Pic}^0(G/H)_{\mathbb{Q}} \cong \text{Pic}^0(G)_{\mathbb{Q}}.$$

Moreover,

$$\text{NS}(G/H)_{\mathbb{Q}} \cong \text{NS}(A/H)_{\mathbb{Q}} \times X(H)_{\mathbb{Q}}/r_H(X(G_{\text{aff}})_{\mathbb{Q}}).$$

In particular, $\text{NS}(G/H)_{\mathbb{Q}}$ is a finite-dimensional vector space.

If $H \subset G_{\text{aff}}$, then $\text{Pic}^0(G/H) \cong \text{Pic}^0(G)$; if in addition G_{aff} is factorial, then

$$\text{NS}(G/H) \cong \text{NS}(A) \times X(H)/r_H(X(G_{\text{aff}})) \cong \text{NS}(A) \times \text{Pic}(G_{\text{aff}}/H).$$

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