Reconstruction of the equilibrium of the plasma in a Tokamak and identification of the current density profile in real time

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Abstract. The reconstruction of the equilibrium of a plasma in a Tokamak is a free boundary problem described by the Grad-Shafranov equation in axisymmetric configuration. The right-hand side of this equation is a nonlinear source, which represents the toroidal component of the plasma current density. This paper deals with the identification of this nonlinearity source from experimental measurements in real time. The proposed method is based on a fixed point algorithm, a finite element resolution, a reduced basis method and a least-square optimization formulation. This is implemented in a software called Equinox with which several numerical experiments are conducted to explore the identification problem. It is shown that the identification of the current density averaged over the magnetic surfaces is very robust.

PACS numbers: 02.30.Zz 02.60.-x 52.55.-s 52.55.Fa 52.65.-y

Submitted to: *Inverse Problems*

1. Introduction

In fusion experiments a magnetic field is used to confine a plasma in the toroidal vacuum vessel of a Tokamak [1]. The magnetic field is produced by external coils surrounding the vacuum vessel and also by a current circulating in the plasma itself. The resulting magnetic field is helicoidal.

Let us denote by $\mathbf{j}$ the current density in the plasma, by $\mathbf{B}$ the magnetic field and by p the kinetic pressure. The momentum equation for the plasma is

$$\rho\frac{d\mathbf{u}}{dt} + \nabla p = \mathbf{j} \times \mathbf{B}$$

where $\mathbf{u}$ represents the mean velocity of particles and ρ the mass density. At the slow resistive diffusion time scale [2] the term $\rho\dfrac{d\mathbf{u}}{dt}$ can be neglected compared to ∇p and the equilibrium equation for the plasma simplifies to

$$\mathbf{j} \times \mathbf{B} = \nabla p$$

meaning that at each instant in time the plasma is at equilibrium and the Lorentz force $\mathbf{j} \times \mathbf{B}$ balances the force ∇p due to kinetic pressure. Taking into account the magnetostatic Maxwell equations which are satisfied in the whole space (including the plasma) the equilibrium of the plasma in presence of a magnetic field is described by

$$\mu_0\mathbf{j} = \nabla \times \mathbf{B}, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\mathbf{j} \times \mathbf{B} = \nabla p, \tag{3}$$

where μ_0 is the magnetic permeability of the vaccum. Ampere's theorem is expressed by Eq. (1) and Eq. (2) represents the conservation of magnetic induction. From the equilibrium equation (3) it is clear that

$$\mathbf{B} \cdot \nabla p = 0 \text{ and } \mathbf{j} \cdot \nabla p = 0.$$

Therefore field lines and current lines lie on isobaric surfaces. These isosurfaces form a family of nested tori called magnetic surfaces which enable to define the magnetic axis and the plasma boundary. On the one hand the innermost magnetic surface degenerates into a closed curve and is called magnetic axis and on the other hand the plasma boundary corresponds to the surface in contact with a limiter or to a magnetic separatrix (hyperbolic line with an X-point).

The Grad-Shafranov equation [3, 4, 5] is a rewriting of Eqs. (1-3) under the axisymmetric assumption. Consider the cylindrical coordinate system $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$. The magnetic field $\mathbf{B}$ is supposed to be independent of the toroidal angle ϕ. Let us decompose it in a poloidal field $\mathbf{B}_p = B_r\mathbf{e}_r + B_z\mathbf{e}_z$ and a toroidal field $\mathbf{B}_\phi = B_\phi\mathbf{e}_\phi$ (see Fig. 1).

Let us also introduce the poloidal flux

$$\psi(r, z) = \frac{1}{2\pi}\int_D \mathbf{B}\mathbf{ds} = \int_0^r B_z r dr$$

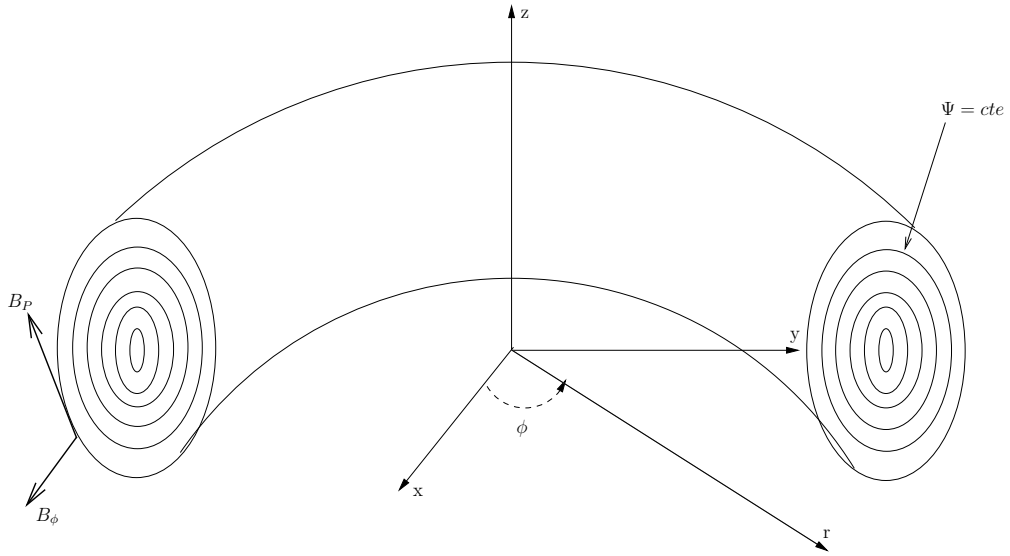

Figure 1. Toroidal geometry.

where D is the disc having as circumference the circle centered on the Oz axis and passing through a point (r, z) in a poloidal section. From Eq. (2) one deduces that $\mathbf{B}_P = \frac{1}{r}[\nabla\psi \times \mathbf{e}_\phi]$. Therefore $\mathbf{B}.\nabla\psi = 0$ meaning that ψ is a constant on each magnetic surface and that $p = p(\psi)$.

The same poloidal-toroidal decomposition can be applied to $\mathbf{j}$. From Eq. (1) it is clear that $\nabla \cdot \mathbf{j} = 0$. As for $\mathbf{B}_p$ it is shown that there exists a function f, called the diamagnetic function, such that $\mathbf{j}_p = \frac{1}{r}[\nabla(\frac{f}{\mu_0}) \times \mathbf{e}_\phi]$. Since $\mathbf{j}.\nabla p = 0$ then $\nabla f \times \nabla p = 0$ and f is constant on the magnetic surfaces, $f = f(\psi)$.

From Eq. (1) one also deduces that $\mathbf{B}_\phi = \frac{f}{r}\mathbf{e}_\phi$ and $\mathbf{j}_\phi = (-\Delta^*\psi)\mathbf{e}_\phi$ where

$$\Delta^*. = \frac{\partial}{\partial r}(\frac{1}{\mu_0 r}\frac{\partial .}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{\mu_0 r}\frac{\partial .}{\partial z}).$$

To sum up

$$\begin{cases} \mathbf{B} = \mathbf{B}_P + \mathbf{B}_\phi \\ \mathbf{B}_P = \frac{1}{r}[\nabla\psi \times \mathbf{e}_\phi] \\ \mathbf{B}_\phi = \frac{f}{r}\mathbf{e}_\phi \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{j} = \mathbf{j}_P + \mathbf{j}_\phi \\ \mathbf{j}_P = \frac{1}{r}[\nabla\frac{f}{\mu_0} \times \mathbf{e}_\phi] \\ \mathbf{j}_\phi = -\Delta^*\psi\mathbf{e}_\phi \end{cases}$$

From Eq. (3) one deduces that

$$(\mathbf{j}_p + j_\phi\mathbf{e}_\phi) \times (\mathbf{B}_p + B_\phi\mathbf{e}_\phi) = -\frac{1}{\mu_0 r}B_\phi\nabla f + j_\phi\frac{1}{r}\nabla\psi = \nabla p$$

and since

$$\nabla p = p'(\psi)\nabla\psi \text{ and } \nabla f = f'(\psi)\nabla\psi$$

the Grad-Shafranov equation valid in the plasma reads

$$-\Delta^*\psi = rp'(\psi) + \frac{1}{\mu_0 r}(ff')(\psi) \tag{4}$$

Thus under the axisymmetric assumption, the three dimensional equilibrium Eqs. (1 - 3) reduce to a two dimensional non linear problem. Note that the right-hand side of Eq. (4) represents the toroidal component j_ϕ of the current density in the plasma which is determined by the unknown functions p' and ff'. In the vaccum there is no current and the poloidal flux statisfies

$$-\Delta^*\psi = 0$$

In this paper, we are interested in the numerical reconstruction of the equilibrium i.e of the poloidal flux ψ and in the identification of the unknown plasma current density [6, 7, 8]. In a control perspective this reconstruction has to be achieved in real time from experimental measurements. The main difficulty consists in identifying the functions p' and ff' in the non linear right-hand side source term in Eq. (4). An iterative strategy involving a finite element method for the resolution of the direct problem and a least square optimisation procedure for the identification of the non linearity using a decomposition basis is proposed.

Section 2 is devoted to the statement of the mathematical problem and to the description of the experimental measurements avalaible. The proposed algorithm is described in Section 3. This methodology has been implemented in a software called Equinox and numerical results using synthetic and real measurements are presented in Section 4.

2. Setting of the direct and inverse problems

2.1. Experimental measurements

Although the unknown functions $p'(\psi)$ and $(ff')(\psi)$ cannot be directly measured in a Tokamak several measurements are available:

- Magnetic measurements: they represent the basic information on which any equilibrium reconstruction relies. Flux loops provide measurements of ψ and magnetic probes provide measurements of the poloidal field $\mathbf{B}_P$ at several points around the vacuum vessel. Let Ω be the domain representing the vacuum vessel and $\partial\Omega$ its boundary. In what follows we assume that we are able to obtain the Dirichlet boundary conditions $\psi = g_D$ and the Neumann boundary conditions $\frac{1}{r}\frac{\partial\psi}{\partial n} = g_N$ at any points of the contour $\partial\Omega$ thanks to a preprocessing of the magnetic measurements. This preprocessing can either be a simple interpolation between real measurements or be the result of some boundary reconstruction algorithm which computes ψ outside the plasma satisfying $\Delta^*\psi = 0$ under the constraint of the measurements [9, 10, 11].

A second set of measurements which can be used as a complement to magnetic measurements are internal measurements:

- Interferometric measurements: they give the values of the integrals along a family of chords c_i of the electronic density $n_e(\psi)$ which is approximately constant on each flux line $\displaystyle\int_{c_i} n_e(\psi)\, dl = \gamma_i$.
- Polarimetric measurements: they give the value of the integrals

$$\int_{c_i} \frac{n_e(\psi)}{r} \frac{\partial \psi}{\partial n} dl = \alpha_i.$$

  $\dfrac{\partial \psi}{\partial n}$ is the normal derivative of ψ along the chord c_i.

Even when using magnetic measurements only for the equilibrium reconstruction the numerical algorithm presented in this paper also uses:

- Current measurement: it gives the value of the total plasma current I_p defined by

$$I_p = \int_{\Omega_p} j_\phi dx.$$

  Ampere's theorem shows that this quantity can be deduced from magnetic measurements.
- Toroidal field measurement: it gives the value B_0 of the toroidal component of the field in the vacuum at the point $(r_0, 0)$ where r_0 is the major radius of the Tokamak. This is used for the integration of ff' into f and for the computation of the safety factor q (see Appendix A).

2.2. Direct problem

The equilibrium of a plasma in a Tokamak is a free boundary problem. The plasma boundary is determined either as being the last flux line in a limiter L or as being a magnetic separatrix with an X-point (hyperbolic point). The region $\Omega_p \subset \Omega$ containing the plasma is defined by

$$\Omega_p = \{\mathbf{x} \in \Omega,\ \psi(\mathbf{x}) \geq \psi_b\}$$

where $\psi_b = \max_L \psi$ in the limiter configuration or $\psi_b = \psi(X)$ when an X-point exists.

In the vacuum region, the right-hand side of Eq. 4 vanishes and the equilibrium equation reads

$$\Delta^* \psi = 0 \text{ in } \Omega \setminus \Omega_p$$

Let us introduce the normalized flux $\bar{\psi} = \dfrac{\psi - \psi_a}{\psi_b - \psi_a} \in [0, 1]$ in Ω_p with $\psi_a = \max_{\Omega_p} \psi$, $A(\bar{\psi}) = \dfrac{r_0}{\lambda} p'(\psi)$ and $B(\bar{\psi}) = \dfrac{1}{\lambda \mu_0 r_0} (ff')(\psi)$. This is introduced so that the non dimensional and unknown functions A and B are defined and identified on

the fixed interval $[0,1]$. Imposing Dirichlet boundary conditions the final equilibrium equation is expressed as the boundary value problem:

$$\left\{\begin{array}{rcl} -\Delta^*\psi & = & \lambda[\dfrac{r}{r_0}A(\bar{\psi})+\dfrac{r_0}{r}B(\bar{\psi})]\chi_{\Omega_p} \quad \text{in } \Omega \\ \psi & = & g_D \quad \text{on } \partial\Omega \end{array}\right. \tag{5}$$

The free boundary aspect of the problem reduces to the particular non linearity appearing through χ_{Ω_p} the characteristic function of Ω_p. The parameter λ is a scaling factor used to ensure that the given total current value I_p is satisfied

$$I_p = \lambda \int_{\Omega_p} [\frac{r}{r_0}A(\bar{\psi})+\frac{r_0}{r}B(\bar{\psi})]dx. \tag{6}$$

2.3. Inverse problem

The inverse problem consists in the identification of functions A and B from the measurements available. It is formulated as a least-square minimization problem

$$\left\{\begin{array}{l} \text{Find } A^*,\ B^*,\ n_e^* \text{ such that :} \\ \\ J(A^*,B^*,n_e^*) = \inf J(A,B,n_e). \end{array}\right. \tag{7}$$

If magnetic measurements only are used the formulation only needs the A and B variables and the J_1 and J_2 terms in Eq. (8) below are not needed. When polarimetric and interferometric measurements are used, the electronic density $n_e(\bar{\psi})$ also has to be identified even if it does not appear in Eq. (5). The cost function J is defined by

$$J(A,B,n_e) = J_0 + J_1 + J_2 + J_\varepsilon \tag{8}$$

J_0 describes the misfit between computed and measured tangential component of $\mathbf{B_p}$

$$J_0 = \frac{1}{2}\sum_{k=1}^{N}(w_k)^2(\frac{1}{r}\frac{\partial\psi}{\partial n}(M_k) - g_N)^2$$

where N is the number of points M_k of the boundary $\partial\Omega$ where the magnetic measurements are given.

$$J_1 = \frac{1}{2}\sum_{k=1}^{N_c}(w_k^{polar})^2(\int_{C_k}\frac{n_e(\bar{\psi})}{r}\frac{\partial\psi}{\partial n}dl - \alpha_k)^2$$

and

$$J_2 = \frac{1}{2}\sum_{k=1}^{N_c}(w_k^{inter})^2(\int_{C_k} n_e(\bar{\psi})dl - \gamma_k)^2$$

N_c is the number of chords over which interferometry and polarimetry measurements are given. The weights w give the relative importance of the different measurements used. As a consequence of the ill-posedness of the identification of A, B and n_e, a Tikhonov regularization term J_ε is introduced [12] where

$$J_\varepsilon = \frac{\varepsilon_A}{2}\int_0^1 [A''(x)]^2dx + \frac{\varepsilon_B}{2}\int_0^1 [B''(x)]^2dx + \frac{\varepsilon_{n_e}}{2}\int_0^1 [n_e''(x)]^2dx$$

and ε_A, ε_B and ε_{n_e} are the regularization parameters.

The values of the different weights and parameters introduced in the cost function are discussed in Section 4.

It should be noticed here that magnetic measurements provide Dirichlet and Neumann boundary conditions. The choice was made to use the Dirichlet boundary conditions in the resolution of direct problem and to include the Neumann boundary conditions in the cost function formulated to solve the inverse problem. This is arbitrary and another solution could have been chosen.

3. Algorithm and numerical resolution

3.1. Overview of the algorithm

The aim of the method is to reconstruct the equilibrium and the toroidal current density in real time. At each time step determined by the availability of new measurements during a discharge, the algorithm consists in constructing a sequence $(\psi^n, \Omega_p^n, A^n, B^n, \lambda^n)$ converging to the solution vector $(\psi, \Omega_p, A, B, \lambda)$. The unknown function n_e may be added too if interferometry and polarimetry measurements are used. The sequence is obtained through the following iterative loop:

- Starting guess: ψ^0, Ω_p^0, A^0, B^0 and λ^0 known from the previous time step solution.
- Step 1 - Optimisation step: compute λ^{n+1} satisfying (6)

$$\lambda^{n+1} = I_p / \int_{\Omega_p^n} [\frac{r}{R_0} A^n(\bar{\psi}^n) + \frac{R_0}{r} B^n(\bar{\psi}^n)] dx$$

  then compute $A^{n+1}(\bar{\psi}^n)$ and $B^{n+1}(\bar{\psi}^n)$ using the least square procedure detailed in Section 3.2.2.

- Step 2 - Direct problem step: compute of ψ^{n+1} solution to

$$\left\{ \begin{array}{rcll} -\Delta^* \psi^{n+1} & = & \lambda^{n+1} [\frac{r}{R_0} A^{n+1}(\bar{\psi}^n) + \frac{R_0}{r} B^{n+1}(\bar{\psi}^n)] \chi_{\Omega_p^n} & \text{in } \Omega \\ \psi^{n+1} & = & g_D & \text{on } \partial\Omega. \end{array} \right. \quad (9)$$

  and the new plasma domain Ω_p^{n+1}.

- $n := n+1$. If the process has not converged return to Step 1 else $(\psi, \Omega_p, A, B, \lambda) = (\psi^n, \Omega_p^n, A^n, B^n, \lambda^n)$. The process is supposed to have converged when the relative residu $\dfrac{||\psi^{n+1} - \psi^n||}{||\psi^n||}$ is small enough.

At each iteration of the algorithm, an inverse problem corresponding to the optimization step and an approximated direct Grad-Shafranov problem have to be solved successively. In Eq. (9), $\bar{\psi}^n$ is known and since the right-hand side does not depend on ψ^{n+1} the boundary value problem (9) is linear.

In the next section the numerical methods used to solve the two problems corresponding to step 1 and step 2 are detailed.

3.2. Numerical resolution

3.2.1. The finite element method for the direct problem The resolution of the direct problem is based on a classical P^1 finite element method [13]. Let us consider the family of triangulation τ_h of Ω, and V_h the finite dimensional subspace of $H^1(\Omega)$ defined by

$$V_h = \{v_h \in H^1(\Omega), v_{h|T} \in P^1(T), \forall T \in \tau_h\}.$$

and introduce $V_h^0 = V_h \cap H_0^1(\Omega)$. The discrete variational formulation of the boundary value problem 9 reads

$$\begin{cases} \text{Find } \psi_h \in V_h \text{ with } \psi_h = g_D \text{ on } \partial\Omega \text{ such that} \\ \forall v_h \in V_h^0, \int_\Omega \frac{1}{\mu_0 r} \nabla\psi_h \cdot \nabla v_h dx = \int_{\Omega_p} \lambda[\frac{r}{R_0} A(\bar{\psi^*}) + \frac{R_0}{r} B(\bar{\psi^*})] v_h dx \end{cases} \quad (10)$$

where ψ^* represents the known value of ψ at the previous iteration. Numerically the Dirichlet boundary conditions are imposed using the method consisting in computing the stiffness matrix $\hat{K}$ of the Neumann problem and modifying it. Consider (v_i) a basis of V_h then $\hat{K}_{ij} = \int_\Omega \frac{1}{\mu_0 r} \nabla v_i \nabla v_j dx$ The modifications consist in replacing the rows corresponding to each boundary node setting 1 on the diagonal terms and 0 elsewhere. At each iteration only the right-hand side of the linear system in which the Dirichlet boundary conditions appear has to be modified. The linear system corresponding to Eq. 10 can be written in the form

$$K.\Psi = y + g \quad (11)$$

where K is the $n \times n$ modified stiffness matrix, Ψ is the unknown vector of size n the number of nodes of the finite elements mesh, y is the vector associated with the modified right-hand side of Eq. (10) and g is the vector corresponding to the Dirichlet boundary conditions.

The matrix K is sparse and so is its LU decomposition. The inverse matrix K^{-1} however is not sparse. The linear system (11) is inverted using the LU decomposition since it is computationally cheaper than using the full inverse matrix K^{-1} which is nevertheless needed for the optimization step of the algorithm in Eq. (15) below.

The vector y depends on functions A and B which are determined in the optimization step. Functions A, B and n_e are decomposed on a finite dimensional basis $(\Phi_i)_{i=1,\dots,m}$ of functions defined on $[0,1]$

$$A(x) = \sum_i^m a_i \Phi_i(x),\ B(x) = \sum_i^m b_i \Phi_i(x) \text{ and } n_e(x) = \sum_i^m c_i \Phi_i(x).$$

The vector y reads

$$y = Y(\bar{\psi^*})u \quad (12)$$

where $u = (a_1, ..., a_m, b_1, ..., b_m) \in \mathbb{R}^{2m}$ is the vector of the components of functions A and B in the basis (Φ_i). The matrix Y of size $n \times 2m$ is defined as follows. Each row i of Y is decomposed as

$$Y_{ij}(\bar{\Psi}^*) = \begin{cases} \displaystyle\int_{\Omega_p} \lambda \frac{r}{R_0} \Phi_j(\bar{\psi^*}) v_i dx & \text{if } 1 \leq j \leq m \\ \displaystyle\int_{\Omega_p} \lambda \frac{R_0}{r} \Phi_{j-m}(\bar{\psi^*}) v_i dx & \text{if } m+1 \leq j \leq 2m. \end{cases}$$

3.2.2. Detailed numerical algorithm One equilibrium computation corresponds to one instant in time during a pulse. The quasi-static approximation consists in considering that at each instant the Grad-Shafranov equation is satisfied. During a pulse successive equilibrium configurations are computed with a time resolution Δt corresponding to the acquisition time of measurements:

- Initialization before the discharge: the modified stiffness matrix K, its LU decomposition as well its inverse K^{-1} are computed once for all and stored.
- Consider that the equilibrium at time $t - \Delta t$ is known and that a new set of measurements is acquired at time t.
- Computation of the new equilibrium at time t through the iterative loop briefly described in the previous Section and detailed below:

The equilibrium from the previous time step is used as a first guess in the iterative loop.

Step 1 - Optimization step During the optimisation step, n_e is first estimated from interferometric measurements and A and B are computed in a second time.

- Compute the electronic density n_e based on the equilibrium of the previous iteration $\bar{\psi}^*$ using a least square formulation for the minimun of J_2 with Tikhonov regularization and solving the associated normal equation: The flux $\bar{\psi}^*$ is given.

$$n_e(x) = \sum_{j=1}^{m} v_j \phi_j(x)$$

The interferometric measurements for $i = 1 \ldots n_c$ are

$$\gamma_i \approx \int_{C_i} n_e(\bar{\psi^*}) dl = \sum_j v_j \int_{C_i} \phi_j(\bar{\psi^*}) dl = \sum_j v_j B_{ij}$$

The cost functional reads

$$\begin{aligned} J(v) &= \frac{1}{2} \sum_i (w_i^{inter})^2 (\sum_j B_{ij} v_j - \gamma_i)^2 + \frac{\varepsilon}{2} v^T \Lambda v \\ &= \frac{1}{2} ||D^{1/2}(Bv - \gamma)||^2 + \frac{\varepsilon}{2} v^T \Lambda v \end{aligned}$$

where $D^{1/2} = diag(w_i^{inter})$ and the regularization matrix Λ is defined by

$$\Lambda_{ij} = \int_0^1 \Phi"_i(x)\Phi"_j(x)dx$$

and $\Phi_i^"$ is the second derivatives of the basis function Φ_i.
It is minimized solving the associated normal equation

$$(\alpha^2(D^{1/2}B)^T(D^{1/2}B) + \hat{\varepsilon}\Lambda)\hat{v} = \alpha(D^{1/2}B)^T D^{1/2}\gamma \tag{13}$$

Since $n_e \approx 10^{19}m^{-3}$ an adimensionalizing parameter $\alpha = 10^{19}m^{-3}$, such that $v = \alpha\hat{v}$, is introduced in order to precondition the linear system which is inverted using LU decomposition, as well as a reasonable prescribed value for the non dimensional regularization parameter $\hat{\varepsilon} = \alpha^2\varepsilon$.

- Compute λ^{n+1} satisfying Eq. (6). In the right-hand side y, λ appears in the product λu. In order to avoid any divergence issue due to the non uniqueness of λ (for all α, $\lambda u = (\lambda\alpha)(\frac{u}{\alpha})$) the degrees of freedom (dofs) u are scaled by $m = \max(|a_i|)$, u is replaced by $\frac{1}{m}u$ and λ by $m\lambda$.
- Compute A and B. In order to approximate A and B, suppose n_e is known and consider the discrete approximated inverse problem

$$\begin{cases} \text{Find } u \text{ minimizing :} \\ J(u) = \frac{1}{2}\|C(\psi^*)\Psi - d\|_D^2 + \frac{\varepsilon}{2}u^T\Lambda u \end{cases} \tag{14}$$

where $C(\psi^*)$ is the observation operator and d the vector of experimental measurements. The first term in J is the discrete version of $J_0 + J_1$. The second one corresponds to the first two terms of the Tikhonov regularization J_ε with $\varepsilon_A = \varepsilon_B = \varepsilon$ which will always be assumed in order for functions A and B to play a symmetric role.
Let us denote by l the number of measurements available ($l = N + N_c$ if magnetic and polarimetric measurements are used) and by D the diagonal matrix made of the weights w_k and w_k^{polar}, the norm $\|.\|_D$ is defined by $\forall \mathbf{x} \in \mathbb{R}^l \ \ \|\mathbf{x}\|_D^2 = (D\mathbf{x}, \mathbf{x}) = (D^{1/2}\mathbf{x}, D^{1/2}\mathbf{x})$
$C(\psi^*)$ is a sparse matrix of size $l \times n$ and can be viewed as a vector composed of two blocks C_0 of size $N \times n$ and indepedent of ψ^* and $C_1(\psi^*)$ of size $N_c \times n$ corresponding respectively to J_0 and J_1. That is to say that multiplication of the kth row of C_0 by ψ gives the kth Neumann boundary condition approximation

$$(C_0)_k\Psi \approx (\frac{1}{r}\frac{\partial\psi}{\partial n})(M_k).$$

The block $C_1(\psi^*)$ depends on ψ^* through the $n_e(\psi^*)$ function. The multiplication of the kth row of $C_1(\psi^*)$ by Ψ gives the kth polarimetric measurements approximation

$$(C_1(\psi^*))_k\Psi \approx \int_{c_k} \frac{n_e(\psi^*)}{r}\frac{\partial\psi}{\partial n}dl.$$

The matrix Λ is of size $2m \times 2m$ and is block diagonal composed of two blocks Λ_1 and Λ_2 of size $m \times m$, with

$$(\Lambda_1)_{ij} = (\Lambda_2)_{ij} = \int_0^1 \Phi"_i(x)\Phi"_j(x)dx$$

Using Eqs .(11 - 12) problem (14) becomes

$$\begin{aligned} J(u) &= \frac{1}{2}\|C(\psi^*)\Psi - d\|_D^2 + \frac{\varepsilon}{2}u^T\Lambda u \\ &= \frac{1}{2}\|C(\psi^*)K^{-1}Y(\bar{\psi^*})u + (C(\psi^*)K^{-1}g - d)\|_D^2 + \frac{\varepsilon}{2}u^T\Lambda u \\ &= \frac{1}{2}\|Eu - f\|_D^2 + \frac{\varepsilon}{2}u^T\Lambda u \end{aligned}$$

where $E = C(\psi^*)K^{-1}Y(\bar{\psi^*})$ and $f = -C(\psi^*)K^{-1}h + d$. Setting $\tilde{E} = D^{1/2}E$, problem (14) reduces to solve the normal equation

$$(\tilde{E}^T\tilde{E} + \varepsilon\Lambda)u = \tilde{E}^T f \tag{15}$$

whose solution is denoted by u^*.

Direct problem step Update the dofs u and update the flux ψ by solving the linear system

$$K\psi = Y(\bar{\psi^*})u^* + g$$

using the precomputed LU decomposition of matrix K. Update Ω_p possibly computing the position of the X-point if the plasma is not in a limiter configuration.

4. Numerical results

4.1. Twin experiment with noise free magnetic measurements

In this section we assume that the poloidal flux corresponding to an equilibrium configuration ψ is given on the boundary Γ . These Dirichlet boundary conditions can either be real measurements or can be the output from some equilibrium simulation code. In a first step we also assume to know functions p' and ff' (or A and B). It is then possible to run a direct simulation to compute ψ on Ω (see Fig. 2) and thus $\dfrac{\partial\psi}{r\partial n}$ on Γ which can then be used as measurements in an inverse problem resolution.

In this first experiment the magnetic measurements are free of noise. The first guess unknown functions are $A(x) = B(x) = 1-x$ and $\lambda = 1$. The poloidal flux ψ is initially a constant on Ω. The weights in the misfit part of the cost function J_0 related to magnetic measurements are defined by $w_k = \dfrac{1}{\sqrt{N}\sigma}$. Since the error on magnetic measurements are of about one percent we define $\sigma = 0.01B_m$ where B_m is a mean magnetic field value which thanks to Ampere's theorem can be defined as $B_m = \dfrac{\mu_0 I_p}{|\Gamma|}$.

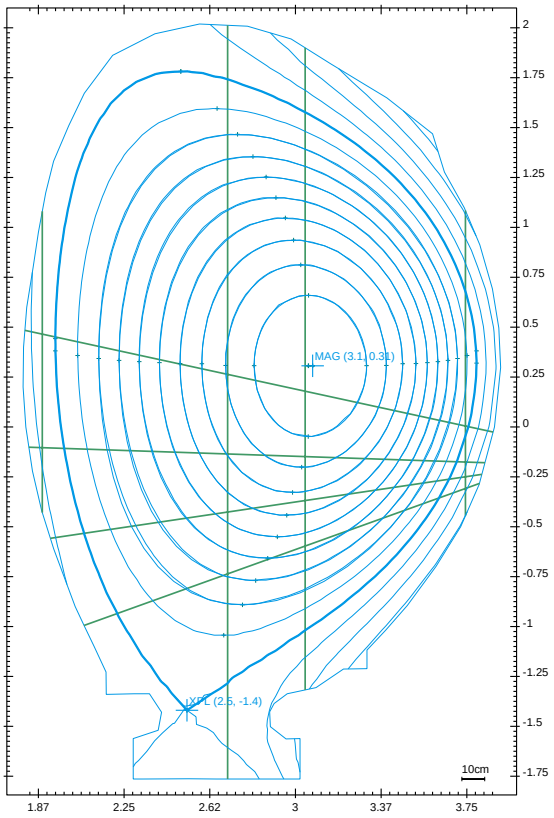

Figure 2. An equilibrium configuration for the tokamak JET from which twin experiments are performed. The domain Ω is shown. Isoflux are plotted from $\bar{\psi} = 0$ (magnetic axis) to $\bar{\psi} = 1$ (plasma boundary defined by the existence of an X-point at point $r = 2.5$ and $z = -1.4$ m) by step of $\Delta\bar{\psi} = 0.1$ Interferometry and polarimetry chords appear in green.

The functions A and B are decomposed in a function basis defined on the interval $[0, 1]$. Several basis have been tested (piecewise affine functions, polynomials, Bplines and wavelets) in order to verify that the result of the identification does not depend on the decomposition basis. This is the case as long as the dimension of the basis is large enough. In the remaining part of this paper each function is decomposed in the same basis of 8 Bsplines [14]. The boundary condition $A(1) = B(1) = 0$ is imposed.

The computations are carried out for several values of the regularization parameters ε ranging from 10^{-10} to 1. We are interested in the ability of the method to recover functions A and B and thus the current density profile averaged over the magnetic surfaces (see Appendix Appendix A):

$$r_0 < \frac{j(r,\bar{\psi})}{r} >= \lambda A(\bar{\psi}) + \lambda r_0^2 < \frac{1}{r^2} > B(\bar{\psi})$$

and the safety factor q (see Appendix B).

As can be seen from Fig. 3 the optimal choice for ε is of about 10^{-5} for which

functions A and B are well recovered. For smaller values some oscillations appear because the regularization is not strong enough and on the contrary greater values lead to less precision in the recovery of the unknown functions since regularization is too strong. In the second column the relative errors on the identified functions are plotted.

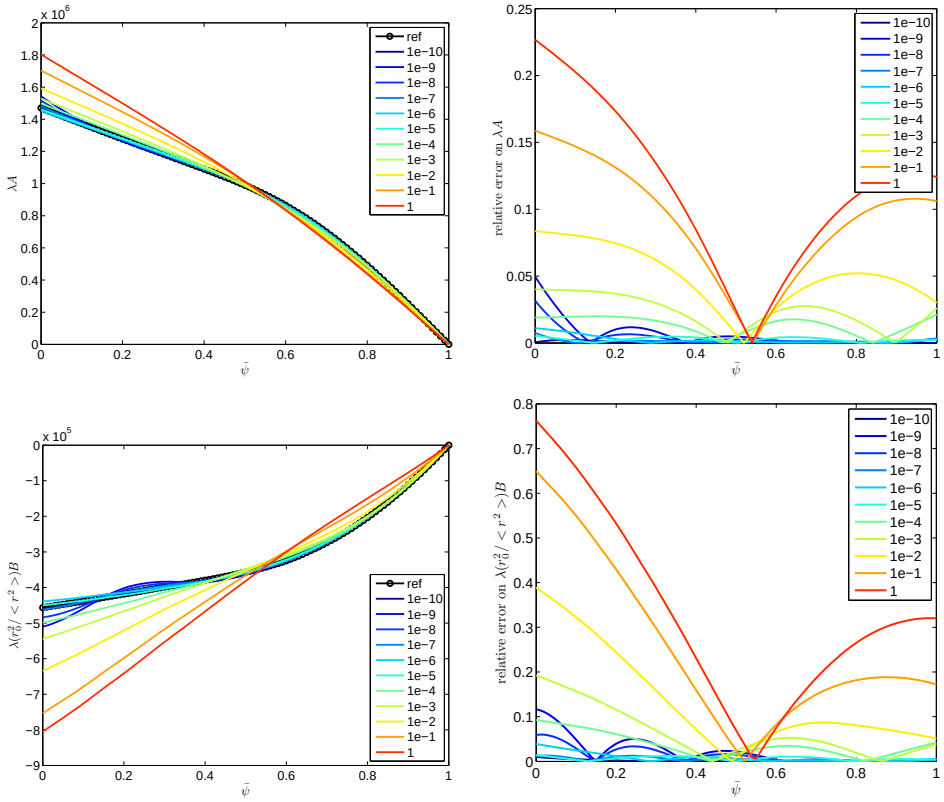

Figure 3. Twin experiment with noise free measurements and different regularization parameters ε ranging from 10^{-10} to 1. Left column: identified functions $\lambda A(\bar{\psi})$, $\lambda r_0^2 < \dfrac{1}{r^2} > \mathbf{B}(\bar{\psi})$ for each different ε value, and the known reference functions. Right column: relative errors.

Figure 4 shows an important point. Almost whatever the chosen value of ε is, i.e. whatever the quality of the identification of A and B is, the identified average current density $r_0 < \dfrac{j(r,\bar{\psi})}{r} >$ as well as the safety factor q are always well recovered and the relative errors are one order of magnitude smaller than for functions A and B. The same kind of observation was made in [8] where the identified functions A and B seemed to be rather sensitive to perturbations whereas the mean current density was very stable.

In Table 1, the evolution of the relative residu on ψ, A, B and λ versus the number of iterations is given. It demonstrates numerically the convergence of the algorithm in this case where a value of 10^{-6} is used as stop condition. The algorithm needs 10 iterations to converge. It is interesting to notice that even though the first guess is not particularly well chosen the relative residu on ψ at the second iteration has already fallen to 4%. In real applications when simulating a whole pulse the first guess for the

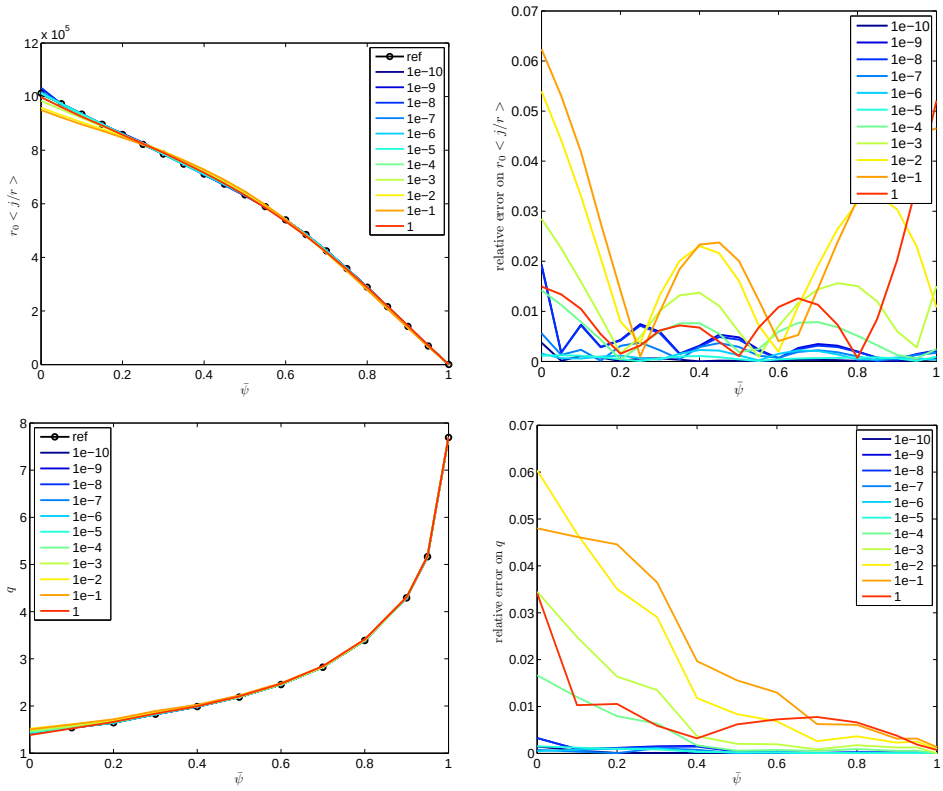

Figure 4. Twin experiment with noise free measurements and different regularization parameters ε ranging from 10^{-10} to 1. Left column: resulting identified mean current density $r_0 < \dfrac{j(r,\bar{\psi})}{r} >$, safety factor q for each ε value and the corresponding known reference values. Right column: relative errors.

computation of the equilibrium at t is the equilibrium computed at $t-\delta t$ and 2 iterations are enough to ensure a good convergence of the algorithm.

4.2. Twin experiment with noisy magnetic measurements

Figures 5 and 6 shows the results of the same type of numerical experiment but with noisy measurements. Each magnetic input, m representing either ψ or $\dfrac{\partial\psi}{r\partial n}$ at a point of the domain boundary Γ is perturbated with a one percent noise normally distributed, $m_\eta = m + \eta$ with $\eta \sim N(m, 0.01m)$. For each chosen value of the regularization parameter the algorithm is run 200 times with measurements randomly perturbated as above. Then for each function λA, $\lambda r_0^2 < \dfrac{1}{r^2} > B$, $r_0 < \dfrac{j(r,\bar{\psi})}{r} >$ and q, a mean function, a median function and a standard deviation function is computed.

In comparison with the noise free case the regularization parameter needs to be significantly increased to values of at least $\varepsilon = 10^{-2}$ and for a safer convergence of the algorithm to $\varepsilon = 10^{-1}$. For smaller values the algorithm either does not converge or gives very oscillating identified functions.

Table 1. Numerical convergence of the algorithm.

Iteration n	$\frac{\|\psi^{n+1}-\psi^n\|}{\|\psi^n\|}$	$\frac{\|A^{n+1}-A^n\|}{\|A^n\|}$	$\frac{\|B^{n+1}-B^n\|}{\|B^n\|}$	$\frac{\|\lambda^{n+1}-\lambda^n\|}{\|\lambda^n\|}$
1	2.64809	6.07599	5.3509	0.100127
2	0.0408642	1.19473	1.42619	9.24968
3	0.0733385	1.83005	1.47338	0.563235
4	0.0404254	0.884617	1.0359	0.108107
5	0.00539736	4.79091	4.37571	0.826455
6	0.000349811	0.127626	0.180449	0.0889022
7	1.58606e-05	0.0262942	0.0246657	0.0263
8	5.67036e-06	0.00294791	0.0024952	0.00315952
9	1.4533e-06	0.000339986	0.000273055	0.000362224
10	6.19066e-07	6.41923e-05	6.51076e-05	6.29838e-05

The mean error on the reconstructed functions is always smaller in the interval $\bar{\psi} \in [0.5, 1]$ than in the interval $[0, 0.5]$. This is due to the fact that magnetic measurements are external to the plasma and do not provide enough information to properly reconstruct the functions in the innermost part of the plasma.

As ε increases the variability or the standard deviation on the identified functions decreases. With small ε the algorithm can find very different functions depending on the perturbations of the measurements. With $\varepsilon = 10^{-2}$ the variability in the identified functions A and B is strong however the mean identified functions are close to the exact reference ones. On the other hand with $\varepsilon = 1$ the variability of the identified functions is strongly reduced but they are quite different from the exact reference functions in the interval $[0, 0.5]$.

It is worth noticing that in all cases the resulting safety factor q and mean current density $r_0 < \dfrac{j(r,\bar{\psi})}{r} >$ are well recovered. The remark of the preceding section on the identifiability of the mean current density still holds: it is quite well recovered even if functions A and B taken separately are not well identified. The mean error on the current density profile is almost always smaller than the mean errors on functions A and B. Moreover this error does not change very much between the different cases and particularly between the $\varepsilon = 10^{-1}$ and the $\varepsilon = 1$ cases. This implies that for a large interval of ε the value of the part of the cost function related to magnetic measurements J_0 is almost constant. Therefore it is difficult to find an optimal value for the regularization parameter. For example the L-curve method [15] for the determination of the regularization parameter can hardly be used and gives some results which are not very reliable since the L-curves are not well behaved and the location of the corner is not clear. The "L" is an almost vertical line. This due to the fact that in a large interval of ε values an increase in ε implies a important decrease in the regularization term $\frac{1}{2}(u^*(\varepsilon))^T \Lambda u^*(\varepsilon)$ but does not lead to a significative increase in the

misfit term $J_0(u^*(\varepsilon))$.

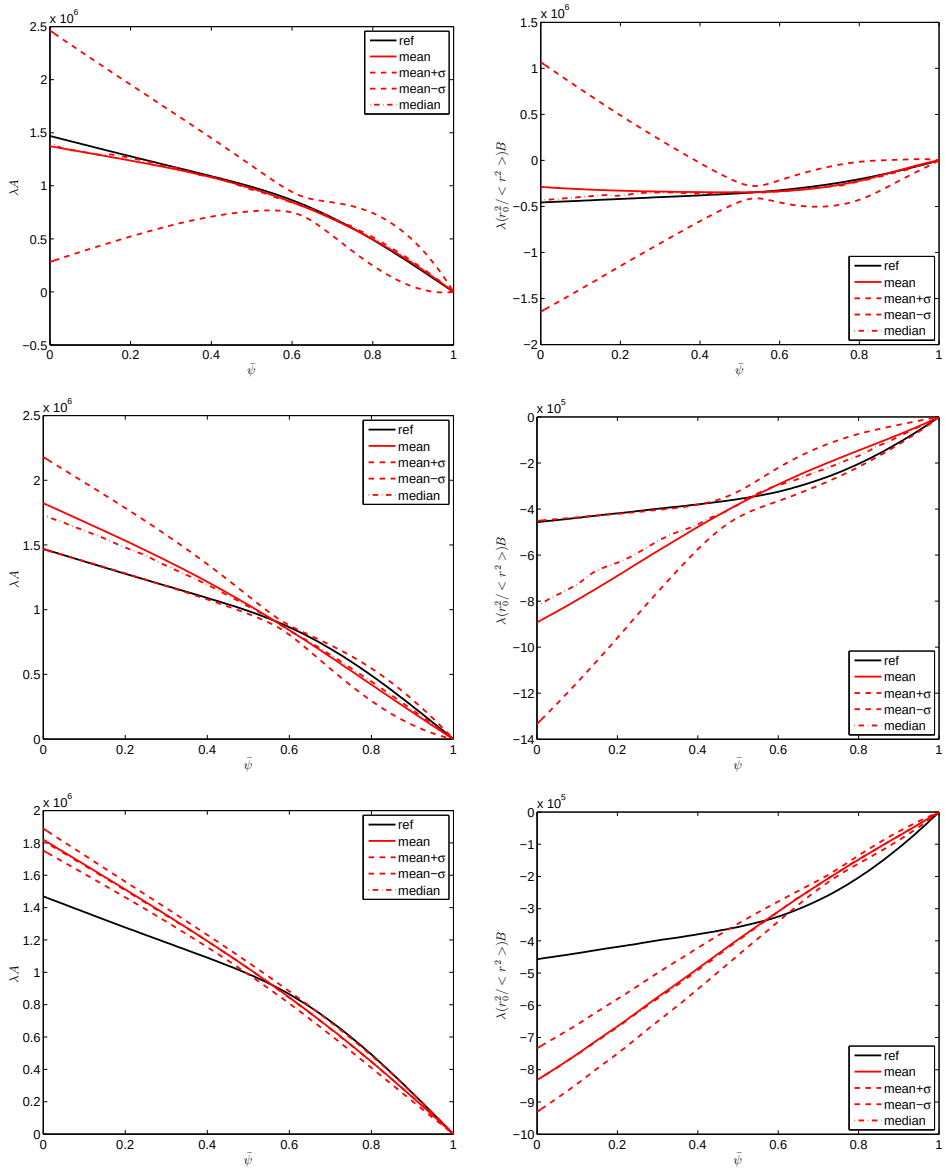

Figure 5. Statistical results of the identification experiments with noisy magnetic measurements. Row 1: $\varepsilon = 10^{-2}$, row 2: $\varepsilon = 10^{-1}$, row 3 $\varepsilon = 1$. Column 1: function $\lambda A(\bar{\psi})$ and column 2: $\lambda r_0^2 < \dfrac{1}{r^2} > B(\bar{\psi})$. For each function the reference value from which the unperturbated measurements were computed is given in black and the mean identified function in red. The mean $\pm$ standard deviation functions are shown in dashed red.

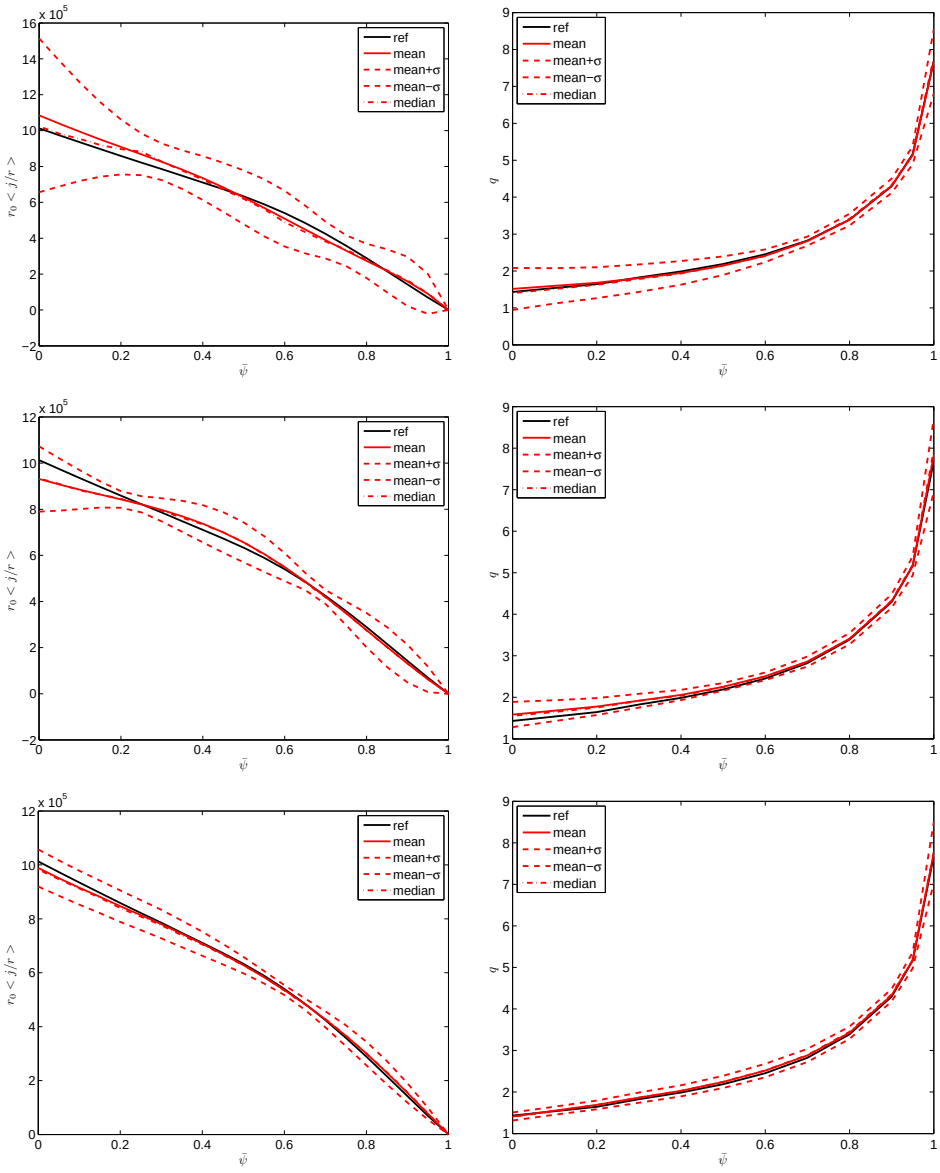

Figure 6. Statistical results of the identification experiments with noisy magnetic measurements. Row 1: $\varepsilon = 10^{-2}$, row 2: $\varepsilon = 10^{-1}$, row 3 $\varepsilon = 1$. Column 1: $r_0 < \dfrac{j(r,\bar{\psi})}{r} >$, and column 2: safety factor q. For each function the reference value is given in black and the mean identified function in red. The mean $\pm$ standard deviation functions are shown in dashed red.

4.3. Twin experiment with noisy magnetic, interferometric and polarimetric measurements

In this last twin experiment interferometric and polarimetric measurements are also used. At first a density profile is prescribed, $n_e(x) = \sqrt{1-x}$ on $[0,1]$, as well as the same A and B functions as in the previous twin experiments. Then similar to the preceding section the equilibrium is computed from given Dirichlet boundary condition. A set of artificial magnetic, interferometric and polarimetric measurements is generated. Finally several twin experiments with a 1% noise are performed and some statistics are computed. The weights related to interferometric and polarimetric measurements in the cost function are defined as

- $w_k^{polar} = \dfrac{1}{\sqrt{N_c}\sigma^{polar}}$, with $\sigma^{polar} = 10^{-1}$ radians
- $w_k^{inter} = \dfrac{1}{\sqrt{N_c}\sigma^{inter}}$, with $\sigma^{inter} = 10^{18}\ m^{-3}$

The determination of the regularization parameter for the density function n_e is far less a problem than for functions A and B since for example the L-curve method works quite well in this case (see Fig. 10 in the next Section) and the n_e function is well recovered as shown on Fig. 9. The regularization parameter for the density function is set to $\varepsilon_{ne} = 10^{-2}$.

The statistical results of the twin experiments are shown on Figs. 7 and 8 for 3 different values of ε. The use of interferometric and polarimetric measurements adds supplementary constraints on the A and B functions. The variability in the recovered functions is less important than in the case where only magnetics are used particularly for $\bar{\psi} \in [0, 0.5]$. This is not surprising since the new measurements are internal and bring some information contained inside the plasma domain. Nevertheless it is not enough to perfectly reconstruct independently the A and B functions. This does not prevent an excellent recovery of the mean current density profile and of the safety factor q. This phenomenum already observed in the magnetics only case is emphasized here where the variability of the recovered profiles has decreased.

4.4. A real pulse

The algorithm detailed in this paper has been implemented in a C++ software called Equinox developed in collaboration with the Fusion Department at Cadarache for Tore Supra and JET. Equinox can be used on the one hand for precise studies in which the computing time is not a limiting factor and on the other hand in a real-time framework to reconstruct the successive plasma equilibrium configurations during a whole pulse. For the time being it is used on JET and ToreSupra pulses but can potentially be used on any Tokamak.

During the real time analysis of a whole pulse an equilibrium is reconstructed from new measurements with a time step of $\Delta t = 100$ ms. For each equilibrium reconstruction the number of iterations of the algorithm is set to 2. This enables fast

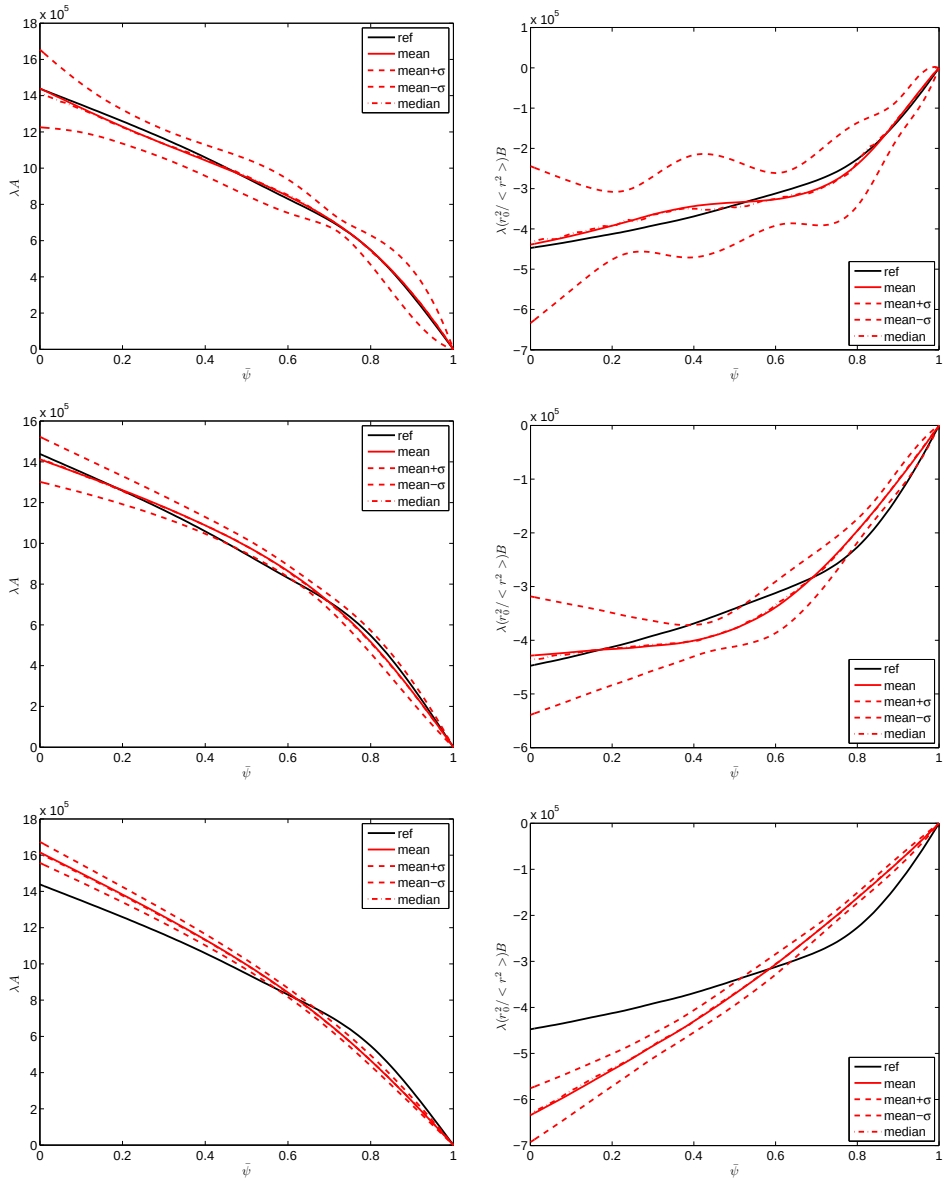

Figure 7. Statistical results of the identification experiments with noisy measurements (magnetics, interferometry and polarimetry). Row 1: $\varepsilon = 10^{-2}$, row 2: $\varepsilon = 10^{-1}$, row 3 $\varepsilon = 1$. Column 1: function $\lambda A(\bar{\psi})$, and column 2: $\lambda r_0^2 < \dfrac{1}{r^2} > B(\bar{\psi})$. For each function the reference value from which the unperturbated measurements were computed is given in black and the mean identified function in red. The mean $\pm$ standard deviation functions are shown in dashed red.

enough computations while a very good precision is achieved since the initial guess for an equilibrium computation at time t is the equilibrium computed at time $t - \Delta t$. After 1 iteration a typical value for the relative residu on ψ is of 10^{-2} and it is of 10^{-3} after 2 iterations. Table 2 gives the size of the finite elements mesh used at ToreSupra and at JET as well as typical computation times on a laptop computer.

During the computations the expensive operations are the updates of matrices C

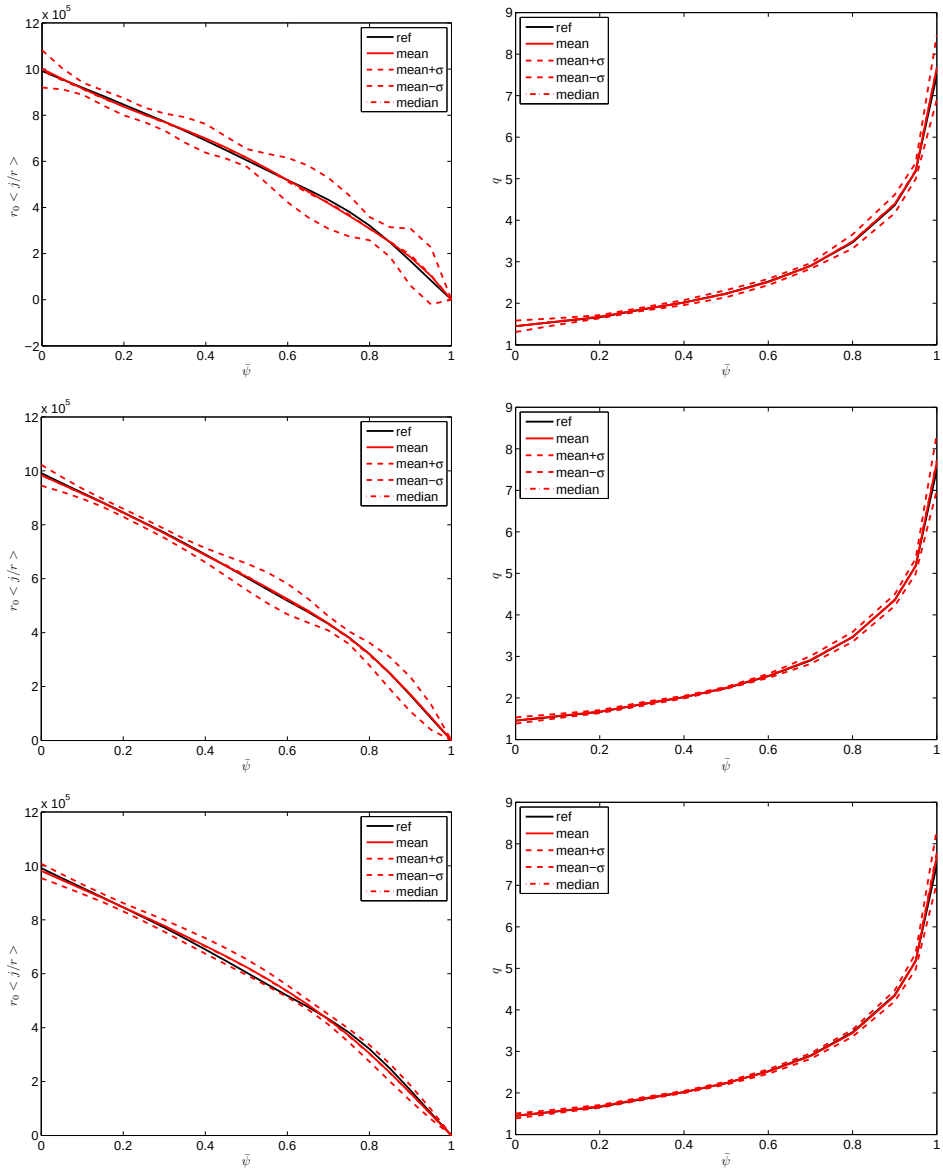

Figure 8. Statistical results of the identification experiments with noisy measurements (magnetics, interferometry and polarimetry). Row 1: $\varepsilon = 10^{-2}$, row 2: $\varepsilon = 10^{-1}$, row 3 $\varepsilon = 1$. Column 1: $r_0 < \dfrac{j(r,\bar{\psi})}{r} >$, and column 2: safety factor q. For each function the reference value is given in black and the mean identified function in red. The mean $\pm$ standard deviation functions are shown in dashed red.

and Y as well as the computation of products CK^{-1} and $CK^{-1}Y$. In order to reduce computation time the K^{-1} matrix is precomputed and only the ψ-dependent part of C is delt with. The resolution of the direct problem, Eq. (11), is cheap since the LU decomposition of the K matrix is also precomputed.

The choice of the regularization parameters is crucial since it determines the balance between the fit to the data and the regularity of the identified functions. It is also difficult as is shown in the preceeding section. Ideally they should be determined for

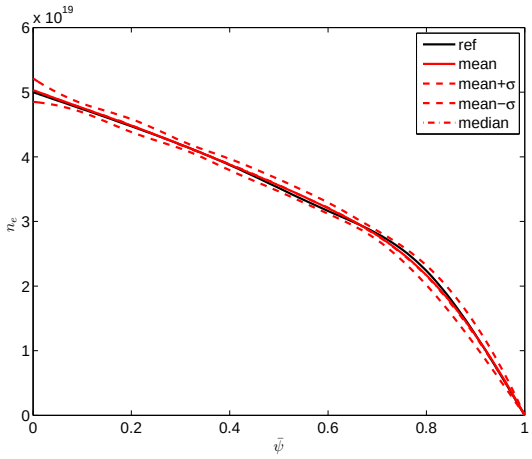

Figure 9. Statistical results for the identification of the denisty function n_e with noisy interferometric measurments.

	ToreSupra	JET
Finite element mesh		
Number of triangles	1382	2871
Number of nodes	722	1470
Computation time (1.80GHz)		
One equilibrium	20 ms	60 ms

Table 2. Typical mesh size and computation time for ToreSupra and JET

each equilibrium reconstruction. However this is not possible in a real-time application and the regularization parameters have to be set apriori to a constant value. From the twin experiments presented in the preceeding sections it is quite clear that a good value for ε the regularization parameter is in the range $[10^{-2}, 1]$. By trial and error on different pulses at JET using magnetics, interferometry and polarimetry, it appeared that a value of $\varepsilon = 5.10^{-2}$ gave good results.

As for the identification of the A and B functions the choice of a good regularization parameter for n_e is crucial. However in this case the L-curve method works quite well and it was used to determine the regularization parameters ε_{n_e} a priori on a number of equilibriums for a few shots. The obtained values showed little variation and the choice of a mean value $\varepsilon = 0.01$ proved to be efficient. Figure 10 shows an example of an L-curve computed for the identification of n_e.

An example of the outputs from Equinox is presented in Figure 11. It is the equilibrium computed at time 44.5 for JET pulse number 77601 using either magnetic measurements only or both magnetic and polarimetric measurements. In both cases the density n_e is computed from interferometric measurements. One can observe the position of the plasma in the vacuum vessel. Isoflux lines are displayed from the magnetic axis to the boundary. The interferometry and polarimetry chords are displayed. At JET the

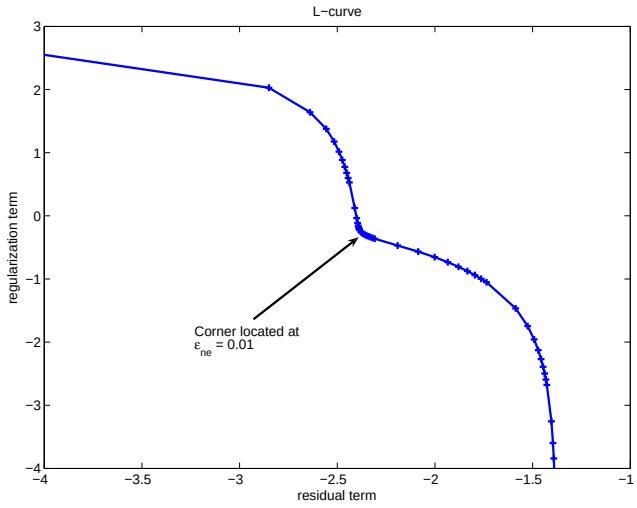

Figure 10. Typical Lcurve for the determination of ε_{n_e}. It is a plot of the parametric curve $x(\varepsilon_{n_e}) = log(\frac{1}{2}||D^{1/2}(Bv^*(\varepsilon_{n_e}) - \gamma)||^2)$, $y(\varepsilon_{n_e}) = log(\frac{1}{2}(v^*(\varepsilon_{n_e}))^T \Lambda v^*(\varepsilon_{n_e}))$ where $v^*(\varepsilon_{n_e})$ is the solution to Eq. (13). Hansen's algorithm [15] locates a corner at $\varepsilon_{n_e} = 0.01$.

chords are ordered as follows: number 1 to 4 are the vertical chords from left to right, and number 5 to 8 are the horizontal ones from bottom to top. The default configuration which is used in this real-time case is to use chords 3 to 8 to identify the density n_e and to use chords 3, 5 and 7 for polarimetry. The other chords are not allways reliable and might give poor quality measurements which can forbid the computation of a satisfying equilibrium. For each chord (used or not) the error between computed and measured interferometry is given in purple. These errors are about 1% for the active chords. The polarimetry absolute errors are given in yellow. Different graphs are plotted on the left of the display. On the first row the identified function A, and corresponding functions p' and p. On the second row the identified function B and corresponding function ff'. The third row gives the toroidal current density j_ϕ in the equatorial plane and the fourth one shows the safety factor. Finally on the fifth row the identified n_e function is plotted.

One should notice that in the particular case shown here the introduction of polarimetry measurements enables the obtention of what is called a reverse q profile (i.e a non monotonic q profile) and the existence of which is confirmed by magnetohydrodynamic analyses (D. Mazon and JET contributors personal communications).

5. Conclusion

We have presented an algorithm for the identification of the current density profile in the Grad-Shafranov equation and the equilibrium reconstruction from experimental measurements in real time. We have shown thanks to several twin experiments that even

though the unknown functions A and B (or p' and ff') taken separately might not be allways exactly identified the resulting mean current density and safety factor seem to be allways well identified. We have also shown that the use of internal polarimetric measurements improves the quality of the identification but is still not enough to perfectly identify both A and B. Finally we have introduced the software Equinox in which this methodology is developed.

Aknowledgements

The authors are grateful to Kristoph Bosak who developped a first version of the code Equinox. Although it has now been thoroughly modified this version was an essential basis to start from.
The authors would also like to thank all colleagues from the CEA at Cadarache in France involved in a collaboration between the University of Nice and the CEA through the LRC (Laboratoire de Recherche Conventionné). Discussions with Francois Saint-Laurent were particularly helpful. Didier Mazon helped introducing us at JET where different people are also involved. In particular Luca Zabeo provided magnetic input data from the boundary code Xloc for Equinox and the work of Robert Felton is essential to implement Equinox on JET real-time system.

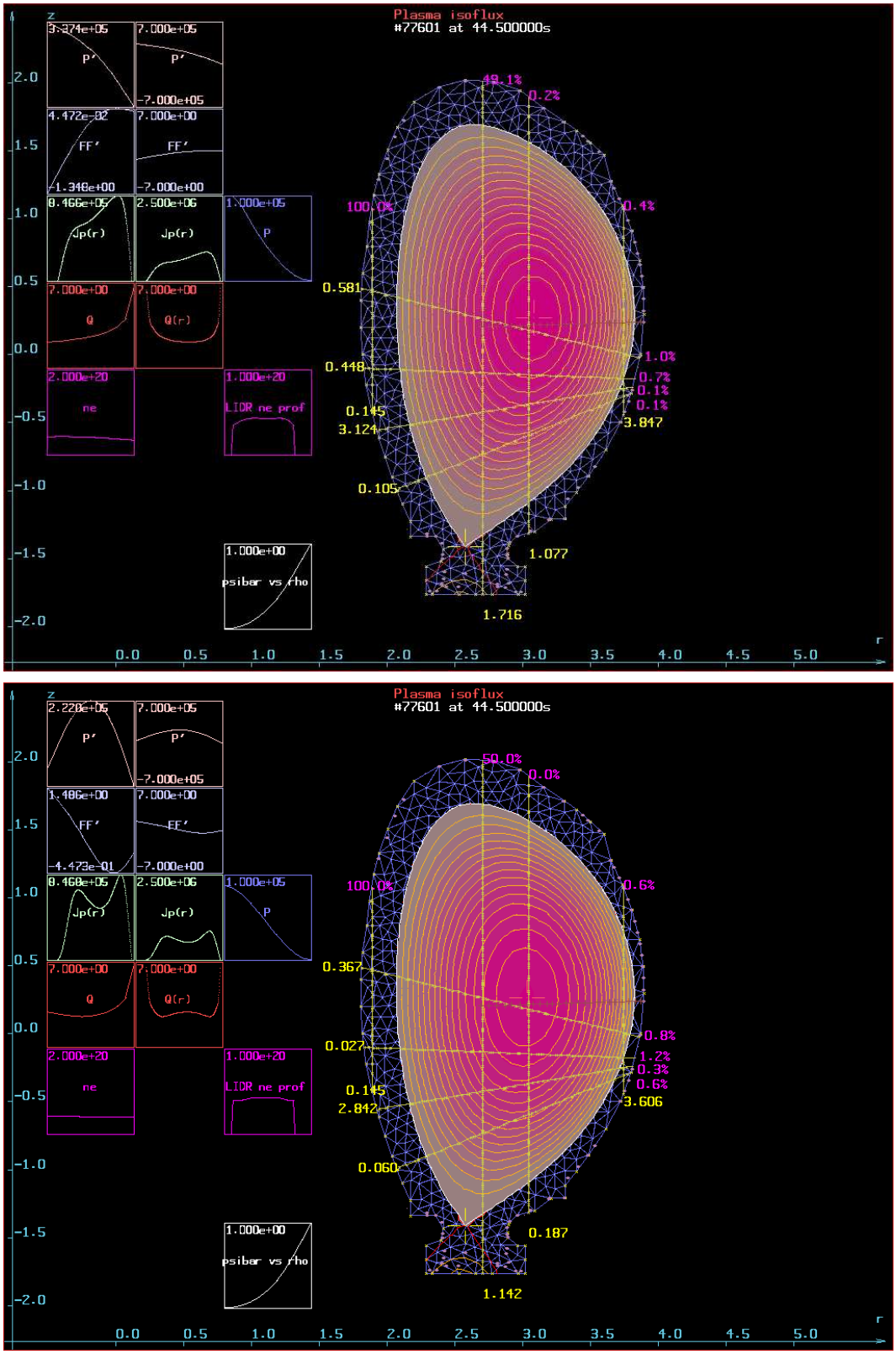

Figure 11. Graphical outputs from Equinox. Reconstructed equilibrium at time 44.5 s for JET pulse number 77601. Top: magnetics only are used. Bottom: magnetics and polarimetry. See text for mor details.

Appendix A. Average over magnetic surfaces

The method of averaging over the magnetic surfaces is detailed in [16] (p 242). The average $< A >$ of an arbitrary quantity A on a magnetic surface S is defined as

$$< A >= \frac{\partial}{\partial V} \int_V A dV$$

where V is the volume inside S. This notion of average has the following property:

$$< A >= \frac{\displaystyle\int_{C_{\bar{\psi}}} \frac{A dl}{B_p}}{\displaystyle\int_{C_{\bar{\psi}}} \frac{dl}{B_p}}$$

where $C_{\bar{\psi}}$ is a closed contour $\bar{\psi} = cte \in (0,1)$ and $B_p = \frac{1}{r}||\nabla\psi||$.

Appendix B. Safety factor q

The safety factor is so called because of the role it plays in determining stability ([1] p 111). It can be seen as the ratio of the variation of the toroidal angle needed for one magnetic field line to perform one poloidal turn.

$$q = \frac{\Delta\phi}{2\pi}$$

Since q is the same for all magnetic field lines on a magnetic surface it is a function of ψ (or $\bar{\psi}$). The expression of q used for computations is the following

$$q(\bar{\psi}) = \frac{1}{2\pi} \int_{C_{\bar{\psi}}} \frac{B_\phi}{r B_p} dl$$

where $C_{\bar{\psi}}$ is a closed contour $\bar{\psi} = cte \in (0,1)$, $B_\phi = \frac{f}{r}$ and

$$f(\psi) = \sqrt{(B_0 r_0)^2 + \int_{\psi_b}^{\psi} (f^2)'(y) dy}$$

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