Geometric optics expansions with amplification for hyperbolic boundary value problems: linear problems

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Abstract. We compute and justify rigorous geometric optics expansions for linear hyperbolic boundary value problems that do not satisfy the uniform Lopatinskii condition. We exhibit an amplification phenomenon for the reflection of small high frequency oscillations at the boundary. Our analysis has two important consequences for such hyperbolic boundary value problems. Firstly, we make precise the optimal energy estimate in Sobolev spaces showing that lossesof derivatives must occur from the source terms to the solution. Secondly, we are able to derive a lower bound forthe finite speed of propagation, showing that waves may propagate faster than for the propagation in free space. We illustrate our analysis with some examples.

1. Introduction

The aim of this article is to construct geometric optics expansions of solutions to hyperbolic initial boundary value problems in the high frequency regime. For the linear Cauchy problem, geometric optics expansions are constructed and justified by Lax [13]. The complete justification of weakly nonlinear geometric optics expansions is due to the second author [9] in the case of a single phase and to Joly, Métivier, Rauch [11] in the case of several phases. We refer to these articles for an extensive discussion and further references.

In this article, we are interested with oscillatory initial boundary value problems. This problem is studied by Chikhi [5], Williams [21, 22], Marcou [17] in the noncharacteristic case, and by Lescarret [14] in the characteristic case. Compared to the propagation in free space, the main additional difficulty is the reflection of oscillations at the boundary. In particular, the construction of a formal asymptotic expansion involves a so-called reflection coefficient. All the above mentioned works deal with problems that satisfy either a *dissipation* assumption or a *strong stability* condition. The latter condition is known as the uniform Lopatinskii (or Kreiss-Lopatinskii) condition, see Kreiss [12]. When this condition is satisfied, the reflection coefficient is finite; incident and reflected oscillations have the same amplitude. In this framework, the above mentioned authors are able to construct and justify weakly nonlinear asymptotic expansions.

We investigate here the case where the reflection coefficient may become infinite, namely when the uniform Lopatinskii condition breaks down in the hyperbolic region. To our knowledge, the construction of *formal* geometric optics expansions in this context goes back to the contributions by Majda, Rosales and Artola [16, 1]. The main new feature is the *amplification* of the solution with respect to the oscillatory source terms. More precisely, suitably polarized source terms of frequency $\mathscr{O}(1/\varepsilon)$ and amplitude $\mathscr{O}(\varepsilon)$ give rise to a solution of frequency $\mathscr{O}(1/\varepsilon)$ and amplitude $\mathscr{O}(1)$. This amplification phenomenon justifies the formation of some singularities in fluid dynamics such as Mach stems in reacting gases, see [16]. The main points of the analysis in [16, 1] are summarized and illustrated in the review article [15].

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In this article, we give a *rigorous* justification of such geometric optics expansions with amplification in a general framework. Our work is an extension of [6] where the first author proves the well-posedness of hyperbolic initial boundary value problems when losses of derivatives occur due to the failure of the uniform Lopatinskii condition. Our analysis is restricted here to linear problems in order to underline the structural assumptions that are needed in the symbolic analysis. Weakly nonlinear oscillations will be addressed in a future work. Our results on oscillatory initial boundary value problems have some important consequences. The first application of our work is to make precise the optimal loss of regularity for problems that do not satisfy the uniform Lopatinskii condition in the hyperbolic region; we prove that the loss of one tangential derivative in the main result of [6] is optimal in the scale of Sobolev spaces. In particular, such problems can not have maximally dissipative boundary conditions. The second application of our work deals with the finite speed of propagation. As already shown for some scalar second order hyperbolic equations, see the works by Chazarain, Piriou and Ikawa [4, 10], the speed of propagation for some initial boundary value problems may be greater than the speed of propagation in free space. We prove that this property extends to the case of systems of equations under general conditions. Eventually, we make the regularity of coefficients precise for the theory of such weakly well-posed problems.

Notation. Throughout this article, we let $\mathscr{M}_{n,N}(\mathbb{K})$ denote the set of $n \times N$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and we use the notation $\mathscr{M}_N(\mathbb{K})$ when $n = N$. We let I denote the identity matrix, without mentioning the dimension. The norm of a vector $X \in \mathbb{C}^N$ is $|X| := (X^* X)^{1/2}$, where X^* denotes the conjugate transpose of X. If X, Y are two vectors in $\mathbb{C}^N$, we let $X \cdot Y$ denote the quantity $\sum_j X_j Y_j$, which coincides with the usual scalar product in $\mathbb{R}^N$ when X and Y are real.

The letter C always denotes a constant that may vary from line to line or within the same line.

2. Assumptions and main result

In this article, we are interested in solving oscillatory hyperbolic initial boundary value problems. The space domain is the half-space $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d / x_d > 0\}$. The space variable x is decomposed as $x = (y, x_d)$. We fix once and for all a time $T > 0$, and we define the sets $\Omega_T :=]-\infty, T] \times \mathbb{R}^d_+$ and $\omega_T :=]-\infty, T] \times \mathbb{R}^{d-1}$. We shall study problems of the form:

$$\begin{cases} L(\partial)\, u^\varepsilon := \partial_t u^\varepsilon + \sum_{j=1}^d A_j\, \partial_{x_j} u^\varepsilon + D\, u^\varepsilon = f^\varepsilon\,, & \text{in } \Omega_T\,, \\ B\, u^\varepsilon|_{x_d=0} = g^\varepsilon\,, & \text{on } \omega_T\,, \\ u^\varepsilon|_{t<0} = 0\,. \end{cases} \tag{1}$$

The matrices $A_1, \dots, A_d, D$ belong to $\mathscr{M}_N(\mathbb{R})$, the matrix B belongs to $\mathscr{M}_{p,N}(\mathbb{R})$, and the unknown u^ε takes its values in $\mathbb{R}^N$. The (small) parameter $\varepsilon > 0$ represents the typical wavelength of the oscillatory source terms $f^\varepsilon, g^\varepsilon$. The integer p is made precise below.

Our purpose is to describe the asymptotic behavior of the solution u^ε to (1) as ε tends to zero. The assumptions fall in two categories:

(i) We first make assumptions on the *principal part* of the operator $L(\partial)$ and the boundary conditions encoded by the matrix B. Our goal is to prove results that are *independent* of the zero order term D in the operator $L(\partial)$. This first set of assumptions constitutes our so-called *weak stability condition*.

(ii) Then we describe the oscillations in the source terms f^ε and g^ε.

2.1. The weak stability condition.

In all this article, the matrices $A_1, \dots, A_d$ in (1) are constant and we make the following hyperbolicity assumption.

Assumption 1. *There exist an integer $q \geq 1$, some real functions $\lambda_1, \dots, \lambda_q$ that are analytic on $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree 1, and there exist some positive integers $\nu_1, \dots, \nu_q$ such that:*

$$\forall\, \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \setminus \{0\}\,, \quad \det\Big[\tau\, I + \sum_{j=1}^d \xi_j\, A_j\Big] = \prod_{k=1}^q \big(\tau + \lambda_k(\xi)\big)^{\nu_k}\,.$$

Moreover the eigenvalues $\lambda_1(\xi), \dots, \lambda_q(\xi)$ are semi-simple (their algebraic multiplicity equals their geometric multiplicity) and satisfy $\lambda_1(\xi) < \cdots < \lambda_q(\xi)$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

For simplicity, we restrict our analysis to noncharacteristic boundaries and therefore make the following:

Assumption 2. *The matrix A_d is invertible and the matrix B has maximal rank, its rank p being equal to the number of positive eigenvalues of A_d (counted with their multiplicity). Moreover, the integer p satisfies $1 \leq p \leq N-1$.*

In the normal modes analysis for (1), one first performs a Laplace transform in the time variable t and a Fourier transform in the tangential space variables y, see Benzoni and Serre [3, chapter 4] for a complete description. We let $\tau - i\,\gamma \in \mathbb{C}$ and $\eta \in \mathbb{R}^{d-1}$ denote the dual variables of t and y, and we introduce the symbol

$$\mathscr{A}(\zeta) := -i\,A_d^{-1}\left((\tau - i\gamma)\,I + \sum_{j=1}^{d-1}\eta_j\,A_j\right), \quad \zeta := (\tau - i\gamma, \eta) \in \mathbb{C}\times\mathbb{R}^{d-1}.$$

For future use, we also define the following sets of frequencies:

$$\Xi := \left\{(\tau - i\gamma, \eta) \in \mathbb{C}\times\mathbb{R}^{d-1} \setminus (0,0)/\gamma \geq 0\right\}, \qquad \Sigma := \left\{\zeta \in \Xi/\tau^2 + \gamma^2 + |\eta|^2 = 1\right\},$$
$$\Xi_0 := \left\{(\tau,\eta) \in \mathbb{R}\times\mathbb{R}^{d-1} \setminus (0,0)\right\} = \Xi \cap \{\gamma = 0\}, \qquad \Sigma_0 := \Sigma \cap \Xi_0.$$

Two key objects in our analysis are the hyperbolic region and the glancing set that are defined as follows.

Definition 1.
- *The hyperbolic region $\mathscr{H}$ is the set of all $(\tau,\eta) \in \Xi_0$ such that the matrix $\mathscr{A}(\tau,\eta)$ is diagonalizable with purely imaginary eigenvalues.*
- *Let G denote the set of all $(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d$ such that $\xi \neq 0$ and there exists an integer $k \in \{1,\dots,q\}$ satisfying*
$$\tau + \lambda_k(\xi) = \frac{\partial\lambda_k}{\partial\xi_d}(\xi) = 0.$$
If $\pi(G)$ denotes the projection of G on the d first coordinates (in other words $\pi(\tau,\xi) = (\tau,\xi_1,\dots,\xi_{d-1})$ for all (τ,ξ)), the glancing set $\mathscr{G}$ is $\mathscr{G} := \pi(G) \subset \Xi_0$.

We recall the following result that is due to Kreiss [12] in the strictly hyperbolic case (when all integers ν_j in Assumption 1 equal 1) and to Métivier [18] in our more general framework.

Theorem 1 ([12, 18])**.** *Let Assumptions 1 and 2 be satisfied. Then for all $\zeta \in \Xi\setminus\Xi_0$, the matrix $\mathscr{A}(\zeta)$ has no purely imaginary eigenvalue and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p. Furthermore, $\mathbb{E}^s$ defines an analytic vector bundle over $\Xi\setminus\Xi_0$ that can be extended as a continuous vector bundle over Ξ.*

For all $(\tau,\eta) \in \Xi_0$, we let $\mathbb{E}^s(\tau,\eta)$ denote the continuous extension of $\mathbb{E}^s$ to the point (τ,η). Away from the glancing set $\mathscr{G} \subset \Xi_0$, $\mathbb{E}^s(\zeta)$ depends analytically on ζ, see [18] and the following section. In particular, it follows from the analysis in [18], see similar arguments in [2, 7], that the hyperbolic region $\mathscr{H}$ does not contain any glancing point, and $\mathbb{E}^s(\zeta)$ depends analytically on ζ in the neighborhood of any point $(\underline{\tau},\underline{\eta}) \in \mathscr{H}$. We now make our weak stability condition precise.

Assumption 3.
- *For all $\zeta \in \Xi \setminus \Xi_0$, $\mathrm{Ker}B \cap \mathbb{E}^s(\zeta) = \{0\}$.*
- *The set $\Upsilon := \{\zeta \in \Sigma_0/\mathrm{Ker}B \cap \mathbb{E}^s(\zeta) \neq \{0\}\}$ is nonempty and included in the hyperbolic region $\mathscr{H}$.*
- *There exists a neighborhood $\mathscr{V}$ of Υ in Σ, a real valued $\mathscr{C}^\infty$ function σ defined on $\mathscr{V}$, a basis $E_1(\zeta),\dots,E_p(\zeta)$ of $\mathbb{E}^s(\zeta)$ that is of class $\mathscr{C}^\infty$ with respect to $\zeta \in \mathscr{V}$, and a matrix $P(\zeta) \in \mathrm{GL}_p(\mathbb{C})$ that is of class $\mathscr{C}^\infty$ with respect to $\zeta \in \mathscr{V}$, such that*
$$\forall\,\zeta \in \mathscr{V}, \quad B\begin{pmatrix}E_1(\zeta) & \dots & E_p(\zeta)\end{pmatrix} = P(\zeta)\,\mathrm{diag}\left(\gamma + i\,\sigma(\zeta), 1, \dots, 1\right).$$

As we shall see later on, hyperbolic boundary value problems that satisfy Assumptions 1, 2 and 3 belong to the WR class defined by Benzoni-Gavage, Rousset, Serre and Zumbrun [2]. As shown in [2], this class of problems is *stable* with respect to small perturbations of the coefficients $A_1,\dots,A_d,B$, while other classes of weakly stable problems - those for which the Lopatinskii determinant vanishes

in $\Xi_0 \setminus \mathscr{H}$ or has double roots - are not. We emphasize that the formal geometric optics expansions with amplification derived in [16, 1, 15] correspond to problems in the WR class.

The following paragraph is devoted to the description of the oscillatory source terms f^ε and g^ε in (1).

2.2. **The oscillations.** We consider a planar real phase φ defined on the boundary

$$\forall\, (t,y) \in \omega_T\,, \quad \varphi(t,y) := \underline{\tau}\, t + \underline{\eta}\cdot y\,, \quad (\underline{\tau},\underline{\eta}) \in \Xi_0\,. \tag{2}$$

As follows from earlier works, see for instance [15, section 1], oscillations on the boundary ω_T associated with the phase φ give rise to oscillations in the domain Ω_T associated with some planar phases φ_m. These phases are characteristic for the hyperbolic operator $L(\partial)$ and their trace on ω_T equals φ. For concreteness, we make from now on the following:

Assumption 4. *The phase φ defined by (2) satisfies $(\underline{\tau},\underline{\eta}) \in \Upsilon$. In particular, $(\underline{\tau},\underline{\eta}) \in \mathscr{H}$.*

Thanks to Assumption 4, we know that the matrix $\mathscr{A}(\underline{\tau},\underline{\eta})$ is diagonalizable with purely imaginary eigenvalues. These eigenvalues are denoted $i\,\underline{\omega}_1, \dots, i\,\underline{\omega}_M$, where the $\underline{\omega}_m$'s are real and pairwise distinct. The $\underline{\omega}_m$'s are the roots (and all the roots are real) of the dispersion relation

$$\det\Big[\underline{\tau}\, I + \sum_{j=1}^{d-1} \underline{\eta}_j\, A_j + \omega\, A_d\Big] = 0\,.$$

To each $\underline{\omega}_m$ there corresponds a unique integer $k_m \in \{1,\dots,q\}$ such that $\underline{\tau} + \lambda_{k_m}(\underline{\eta},\underline{\omega}_m) = 0$. We can then define the following real[1] phases and their associated group velocity:

$$\forall\, m = 1,\dots,M\,, \quad \varphi_m(t,x) := \varphi(t,y) + \underline{\omega}_m\, x_d\,, \quad \mathbf{v}_m := \nabla\lambda_{k_m}(\underline{\eta},\underline{\omega}_m)\,. \tag{3}$$

Let us observe that each group velocity $\mathbf{v}_m$ is either incoming or outgoing with respect to the space domain $\mathbb{R}^d_+$: the last coordinate of $\mathbf{v}_m$ is nonzero. This property holds because $(\underline{\tau},\underline{\eta})$ does not belong to the glancing set $\mathscr{G}$. We can therefore adopt the following classification as in [15]:

Definition 2. *The phase φ_m is causal if the group velocity $\mathbf{v}_m$ is incoming ($\partial_{\xi_d}\lambda_{k_m}(\underline{\eta},\underline{\omega}_m) > 0$), and it is noncausal if the group velocity $\mathbf{v}_m$ is outgoing ($\partial_{\xi_d}\lambda_{k_m}(\underline{\eta},\underline{\omega}_m) < 0$).*

In all what follows, we let $\mathscr{C}$ denote the set of indices $m \in \{1,\dots,M\}$ such that φ_m is a causal phase, and $\mathscr{NC}$ denote the set of indices $m \in \{1,\dots,M\}$ such that φ_m is a noncausal phase. We shall show later on that both sets $\mathscr{C}$ and $\mathscr{NC}$ are nonempty.

Eventually, we make the following assumption for the source terms $f^\varepsilon, g^\varepsilon$ in (1):

Assumption 5. *The source term g^ε has the form*

$$g^\varepsilon := \varepsilon\, g(t,y)\, \mathrm{e}^{i\,\varphi(t,y)/\varepsilon}\,,$$

where the amplitude $g \in H^{+\infty}(\omega_T)$ is independent of $\varepsilon \in]0,1]$, and the source term f^ε has the form

$$f^\varepsilon := \varepsilon \sum_{m=1}^{M} f_m(t,x)\, \mathrm{e}^{i\,\varphi_m(t,x)/\varepsilon}\,,$$

where the amplitudes $f_1,\dots,f_M \in H^{+\infty}(\Omega_T)$ are independent of $\varepsilon \in]0,1]$. Moreover, $g, f_1, \dots, f_M$ vanish for $t < 0$.

Given the source terms $(f^\varepsilon, g^\varepsilon)_{\varepsilon\in]0,1]}$, in Assumption 5, we raise the question of the asymptotic behavior of the solution u^ε to (1) as the small parameter ε tends to zero. In particular, what is the asymptotic amplitude of the solution u^ε in L^2 ? Our main result is described in the following paragraph. Observe that more general source terms can be considered by using the linearity of (1).

[1] If $(\underline{\tau},\underline{\eta})$ does not belong to the hyperbolic region $\mathscr{H}$, some of the phases φ_m may be complex, see e.g. [21, 22, 14, 17]. Moreover, glancing phases introduce a new scale $\sqrt{\varepsilon}$ as well as boundary layers, and we do not want to combine this technical difficulty with the amplification phenomenon that is our main point of interest here.

2.3. **Main result.** We recall the following classical definition in geometric optics.

Definition 3. *Let $K \in \mathbb{N}$, and let $(v^\varepsilon)_{\varepsilon\in\,]0,1]}$ denote a family of functions in $H^{+\infty}(\Omega_T)$. We say that $(v^\varepsilon)_{\varepsilon\in\,]0,1]}$ is $\mathscr{O}(\varepsilon^K)$ in $H^{+\infty}_\varepsilon(\Omega_T)$ if for all $\alpha \in \mathbb{N}^{1+d}$, there exists a constant C_α satisfying*

$$\forall\, \varepsilon \in]0,1]\,, \quad \varepsilon^{|\alpha|}\, \left\|\partial^\alpha_{t,x} v^\varepsilon\right\|_{L^2(\Omega_T)} \leq C_\alpha\, \varepsilon^K\,.$$

Our main result is the following:

Theorem 2. *Let Assumptions 1-5 be satisfied, and let $D \in \mathscr{M}_N(\mathbb{R})$. Then there exists a unique family $(u_{n,m})_{n\geq 0, m=1,\dots,M}$ in $H^{+\infty}(\Omega_T)$ such that*

(i) *all functions $u_{n,m}$ vanish for $t < 0$,*
(ii) *the cascade of equations (12), (13) below is satisfied.*

In particular, $u_{0,m} = 0$ for $m \in \mathscr{N}\mathscr{C}$ and the traces $u_{0,m}|_{x_d=0}$ for $m \in \mathscr{C}$ are given by the relation (17) where the function α_0 satisfies the transport equation (19). Given an integer N_0, we can define an approximate solution u^ε_{app,N_0} to (1) by the formula

$$u^\varepsilon_{app,N_0} := \sum_{n=0}^{N_0} \varepsilon^n \sum_{m=1}^{M} u_{n,m}(t,x)\, \mathrm{e}^{i\,\varphi_m(t,x)/\varepsilon}\,.$$

For all $\varepsilon \in]0,1]$, there exists a unique solution $u^\varepsilon \in H^{+\infty}(\Omega_T)$ to (1) that vanishes for $t < 0$. Moreover, for all integer N_0, $(u^\varepsilon - u^\varepsilon_{app,N_0})_{\varepsilon\in\,]0,1]}$ is $\mathscr{O}(\varepsilon^{N_0+1})$ in $H^{+\infty}_\varepsilon(\Omega_T)$.

Theorem 2 shows that u^ε has amplitude $\mathscr{O}(1)$ in L^2, and in L^∞, asymptotically as ε tends to zero. This corresponds to an *amplification* phenomenon of one power of ε from the source terms $f^\varepsilon, g^\varepsilon$ to the solution u^ε.

The rest of this article is organized as follows: in section 3, we prove Theorem 2. As it is rather common in geometric optics, the proof is based on two main steps. In the first step, we determine the so-called WKB expansion of u^ε as a formal series solving (12), (13). In the second step, we show that the exact solution u^ε to (1) is close to the asymptotic expansion. The latter step is a stability problem and is based on the well-posedness result of [6]. In section 4, we prove some results on the nonoscillatory initial boundary value problem (36). Section 5 is devoted to some examples where we clarify the determination of the principal term in the expansion of u^ε, and we also make some comments on the necessity of Assumption 3.

3. Proof of Theorem 2

In all this section, we use the notation

$$L_1(\tau,\xi) := \tau\, I + \sum_{j=1}^{d} \xi_j\, A_j\,,$$

for the symbol of the principal part of $L(\partial)$. For each phase φ_m, $\mathrm{d}\varphi_m$ denotes the differential of the function φ_m with respect to its arguments (t,x). Following [2], we shall say that a complex vector space is of real type if it admits a basis of real vectors.

3.1. **A few preliminary results.** We begin with a first Lemma that gives a decomposition of the extended stable subspace at the hyperbolic frequency $(\underline{\tau},\underline{\eta})$.

Lemma 1. *The stable subspace $\mathbb{E}^s(\underline{\tau},\underline{\eta})$ admits the decomposition*

$$\mathbb{E}^s(\underline{\tau},\underline{\eta}) = \oplus_{m\in\mathscr{C}} \operatorname{Ker} L_1(\mathrm{d}\varphi_m)\,, \tag{4}$$

and each vector space in the decomposition (4) is of real type. In particular, $\mathscr{C}$ is nonempty.

Proof of Lemma 1. The proof follows from some arguments of [18], but we give it here both for the sake of clarity and because we shall use some of the arguments below in our analysis. Using Assumption 1, we know that the real analytic function $\lambda_{k_m}(\eta,\omega)$ admits an extension that is real

analytic in η and holomorphic in ω in a sufficiently small neighborhood of $(\underline{\eta}, \underline{\omega}_m)$. For all $z, \omega \in \mathbb{C}$ and $\eta \in \mathbb{R}^{d-1}$, we have the relation

$$\det\left[\mathscr{A}(z,\eta) - i\,\omega\, I\right] = \det(-i\,A_d)\,\det\left[z\,I + \sum_{j=1}^{d-1} \eta_j\,A_j + \omega\,A_d\right]. \tag{5}$$

For $(z,\eta) = (\underline{\tau}, \underline{\eta})$, the roots in ω of the dispersion relation (5) are the $\underline{\omega}_m$'s and are real. Moreover, we have $\underline{\tau} + \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) = 0$, and the partial derivative $\partial_{\xi_d}\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m)$ is nonzero. The eigenspace of $\mathscr{A}(\underline{\tau}, \underline{\eta})$ associated with the eigenvalue $i\,\underline{\omega}_m$ coincides with the kernel of $L_1(\mathrm{d}\varphi_m)$ and has dimension ν_{k_m}, see Assumption 1.

The Weierstrass preparation Theorem shows that for $(z,\eta,\omega) \in \mathbb{C}\times\mathbb{R}^{d-1}\times\mathbb{C}$ sufficiently close to $(\underline{\tau}, \underline{\eta}, \underline{\omega}_m)$, there holds a factorization

$$z + \lambda_{k_m}(\eta,\omega) = \vartheta(z,\eta,\omega)\,\big(\omega - \omega_m(z,\eta)\big)\,,$$

where ϑ is an analytic function of (z,η,ω) that does not vanish near $(\underline{\tau}, \underline{\eta}, \underline{\omega}_m)$, and ω_m is a function that is holomorphic with respect to z and analytic with respect to η defined on a sufficiently small neighborhood of $(\underline{\tau}, \underline{\eta})$. Moreover, ω_m satisfies $\omega_m(\underline{\tau}, \underline{\eta}) = \underline{\omega}_m$. Consequently, for (z,η) sufficiently close to $(\underline{\tau}, \underline{\eta})$, the eigenvalues of the matrix $\mathscr{A}(z,\eta)$ are the complex numbers $i\,\omega_1(z,\eta), \dots, i\,\omega_M(z,\eta)$, each with algebraic multiplicity $\nu_{k_1}, \dots, \nu_{k_M}$. These eigenvalues are pairwise distinct.

Let $z = \underline{\tau} - i\,\gamma$ with $\gamma > 0$ small enough, and $\eta = \underline{\eta}$. Then a Taylorexpansion shows that the real part of $i\,\omega_m(z,\underline{\eta})$ is negative if and only if $\partial_{\xi_d}\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) > 0$. Moreover, the sign of the real part of $i\,\omega_m(z,\eta)$ does not depend on η as long as γ is positive, see Theorem 1. In other words, $i\,\omega_m(z,\eta)$ is a stable eigenvalue of $\mathscr{A}(z,\eta)$ for $\gamma > 0$ small enough if and only if the phase φ_m is causal.

Following the arguments of [18], which we shall not repeat here, we can show that for all $(z,\eta) \in \Xi$ close to $(\underline{\tau}, \underline{\eta})$, the eigenvalue $i\,\omega_m(z,\eta)$ is semi-simple and the associated eigenspace varies holomorphically with respect to z and analytically with respect to η. Then for all $(z,\eta) \in \Xi \setminus \Xi_0$ close to $(\underline{\tau}, \underline{\eta})$, we have the decomposition

$$\mathbb{E}^s(z,\eta) = \oplus_{m\in\mathscr{C}}\,\mathrm{Ker}\,\big(\mathscr{A}(z,\eta) - i\,\omega_m(z,\eta)\,I\big) = \oplus_{m\in\mathscr{C}}\,\mathrm{Ker}\,\Big(z\,I + \sum_{j=1}^{d-1}\eta_j\,A_j + \omega_m(z,\eta)\,A_d\Big). \tag{6}$$

Using Theorem 1, we can pass to the limit in (6) as γ tends to zero and the claim follows. The vector spaces in the decomposition (4) are of real type because each matrix $L_1(\mathrm{d}\varphi_m)$ has real coefficients and is diagonalizable with real eigenvalues. □

A second preliminary result is the following:

Lemma 2. *The following decompositions hold*

$$\mathbb{C}^N = \oplus_{m=1}^M\,\mathrm{Ker}\;L_1(\mathrm{d}\varphi_m) = \oplus_{m=1}^M\,A_d\,\mathrm{Ker}\;L_1(\mathrm{d}\varphi_m)\,, \tag{7}$$

and each vector space in the decompositions (7) is of real type. In particular, $\mathscr{N}\mathscr{C}$ is nonempty.

We let $P_1, \dots, P_M$, resp. $Q_1, \dots, Q_M$, denote the projectors associated with the first, resp. second, decomposition in (7). Then for all $m = 1, \dots, M$, there holds $\mathrm{Im}\;L_1(\mathrm{d}\varphi_m) = \mathrm{Ker}\;Q_m$.

Proof of Lemma 2. The first decomposition in (7) follows from the diagonalizability of the matrix $\mathscr{A}(\underline{\tau}, \underline{\eta})$:

$$\mathbb{C}^N = \oplus_{m=1}^M\,\mathrm{Ker}\,\big(\mathscr{A}(\underline{\tau}, \underline{\eta}) - i\,\underline{\omega}_m\,I\big) = \oplus_{m=1}^M\,\mathrm{Ker}\;L_1(\mathrm{d}\varphi_m)\,.$$

The second decomposition in (7) follows from the first one because A_d is invertible.

Let $m_0 \in \{1, \dots, M\}$ and let $X \in \mathbb{C}^N$. From the diagonalizability of $\mathscr{A}(\underline{\tau}, \underline{\eta})$, we have

$$\underline{\tau}\,I + \sum_{j=1}^{d-1}\underline{\eta}_j\,A_j = -\sum_{m=1}^{M}\underline{\omega}_m\,A_d\,P_m\,, \quad \sum_{m=1}^{M}P_m = I\,.$$

We can thus write

$$L_1(\mathrm{d}\varphi_{m_0})\,X = \left(\underline{\tau}\,I + \sum_{j=1}^{d-1}\underline{\eta}_j\,A_j + \underline{\omega}_{m_0}\,A_d\right)\,X = \underline{\omega}_{m_0}\,A_d\,X - \sum_{m=1}^{M}\underline{\omega}_m\,A_d\,P_m\,X$$
$$= \sum_{m\neq m_0}(\underline{\omega}_{m_0} - \underline{\omega}_m)\,A_d\,P_m\,X \in \oplus_{m\neq m_0}\,A_d\,\mathrm{Ker}\;L_1(\mathrm{d}\varphi_m) = \mathrm{Ker}\;Q_{m_0}\,.$$

The dimensions of Im $L_1(\mathrm{d}\varphi_{m_0})$ and Ker Q_{m_0} are the same, so we have an equality between these two vector spaces. The proof of Lemma 2 is complete. □

Using the projectors P_m, Q_m, we can define in a unique way the partial inverse R_m of the matrix $L_1(\mathrm{d}\varphi_m)$ by the relations

$$R_m\,L_1(\mathrm{d}\varphi_m) = I - P_m\,,\quad P_m\,R_m = R_m\,Q_m = 0\,. \tag{8}$$

The decompositions (7) involve spaces of real type, so all the matrices P_m, Q_m, R_m have real coefficients[2]. Moreover, each projector Q_m induces an isomorphism from Im P_m to Im Q_m.

Using Assumption 3, we know that the vector space Ker $B \cap \mathbb{E}^s(\underline{\tau},\underline{\eta})$ is one-dimensional, and we also know that this vector space is of real type because B has real coefficients. This vector space is therefore spanned by a vector $e \in \mathbb{R}^N \setminus \{0\}$ that we can decompose in a unique way by using Lemma 1:

$$\mathrm{Ker}\;B \cap \mathbb{E}^s(\underline{\tau},\underline{\eta}) = \mathrm{Span}\;e\,,\quad e = \sum_{m\in\mathscr{C}} e_m\,,\quad P_m\,e_m = e_m\,. \tag{9}$$

Each vector e_m in (9) has real coefficients. We also know that the vector space $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta})$ is $(p-1)$-dimensional and is of real type. We can therefore write it as the kernel of a real linear form

$$B\,\mathbb{E}^s(\underline{\tau},\underline{\eta}) = \left\{X \in \mathbb{C}^p\,,\, b\cdot X = 0\right\}, \tag{10}$$

for a suitable vector $b \in \mathbb{R}^p \setminus \{0\}$.

Eventually, we can introduce the partial inverse of the restriction of B to the vector space $\mathbb{E}^s(\underline{\tau},\underline{\eta})$. More precisely, we choose a supplementary vector space of Span e in $\mathbb{E}^s(\underline{\tau},\underline{\eta})$:

$$\mathbb{E}^s(\underline{\tau},\underline{\eta}) = \mathrm{Span}\;e \oplus \check{\mathbb{E}}^s(\underline{\tau},\underline{\eta})\,. \tag{11}$$

The matrix B then induces an isomorphism from $\check{\mathbb{E}}^s(\underline{\tau},\underline{\eta})$ to the hyperplane $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta})$.

3.2. **Determination of the WKB expansion.**

3.2.1. *The cascade of equations.* We first write the solution u^ε to (1) as a formal series

$$u^\varepsilon(t,x) = \sum_{n\geq 0}\varepsilon^n \sum_{m=1}^{M} u_{n,m}(t,x)\,\mathrm{e}^{i\,\varphi_m(t,x)/\varepsilon}\,.$$

We plug this formal expression of u^ε into the equations (1) and collect the powers of ε. The result is the following cascade of equations in the domain Ω_T, see e.g. Rauch [19]:

$$L_1(\mathrm{d}\varphi_m)\,u_{0,m} = 0\,, \tag{12a}$$
$$i\,L_1(\mathrm{d}\varphi_m)\,u_{1,m} + L(\partial)\,u_{0,m} = 0\,, \tag{12b}$$
$$i\,L_1(\mathrm{d}\varphi_m)\,u_{2,m} + L(\partial)\,u_{1,m} = f_m\,, \tag{12c}$$
$$\forall\, n \geq 2\,,\quad i\,L_1(\mathrm{d}\varphi_m)\,u_{n+1,m} + L(\partial)\,u_{n,m} = 0\,. \tag{12d}$$

[2]As a matter of fact, (7) also holds with $\mathbb{R}^N$ instead of $\mathbb{C}^N$ and if we consider the kernel in $\mathbb{R}^N$ of each matrix instead of the kernel in $\mathbb{C}^N$.

The equations (12) should hold separately for all $m = 1, \dots, M$. The boundary conditions are the following:

$$B \sum_{1 \le m \le M} u_{0,m}|_{x_d=0} = 0\,, \tag{13a}$$

$$B \sum_{1 \le m \le M} u_{1,m}|_{x_d=0} = g\,, \tag{13b}$$

$$\forall\, n \ge 2\,, \quad B \sum_{1 \le m \le M} u_{n,m}|_{x_d=0} = 0\,. \tag{13c}$$

Since u^ε vanishes for $t < 0$, we look for solutions $u_{n,m}$ to (12), (13) that also vanish for $t < 0$.

3.2.2. *The amplitudes for noncausal phases.* The interior equations (12) are sufficient to determine the amplitudes $u_{n,m}$ when φ_m is a noncausal phase. More precisely, we use the projectors P_m, Q_m and the partial inverse R_m satisfying (8) to rewrite the cascade (12) into the equivalent form (see Lax [13] or [19] for similar calculations)

$$u_{0,m} = P_m\, u_{0,m}\,, \tag{14a}$$

$$Q_m\, L(\partial)\, u_{0,m} = 0\,, \tag{14b}$$

$$(I - P_m)\, u_{1,m} = i\, R_m\, L(\partial)\, u_{0,m}\,, \tag{14c}$$

$$Q_m\, L(\partial)\, u_{1,m} = Q_m\, f_m\,, \tag{14d}$$

$$(I - P_m)\, u_{2,m} = i\, R_m \left(L(\partial)\, u_{1,m} - f_m \right), \tag{14e}$$

$$\forall\, n \ge 2\,, \quad Q_m\, L(\partial)\, P_m\, u_{n,m} = -Q_m\, L(\partial)\, (I - P_m)\, u_{n,m}\,, \tag{14f}$$

$$\forall\, n \ge 2\,, \quad (I - P_m)\, u_{n+1,m} = i\, R_m\, L(\partial)\, u_{n,m}\,. \tag{14g}$$

The crucial observation for solving the cascade (14) is the following:

Lemma 3 ([13])**.** *Let $m \in \{1, \dots, M\}$ and let the projectors P_m, Q_m be defined in Lemma 2. Then there holds the relation*

$$Q_m\, L(\partial)\, P_m = (\partial_t + \mathbf{v}_m \cdot \nabla_x)\, Q_m\, P_m + Q_m\, D\, P_m\,,$$

where the group velocity $\mathbf{v}_m$ is defined in (3).

Lemma 3 shows that amplitudes polarized on the kernel of $L_1(\mathrm{d}\varphi_m)$ are propagated at the group velocity $\mathbf{v}_m$. Let us now observe that when φ_m is a noncausal phase, the following initial boundary value problem

$$\begin{cases} (\partial_t + \mathbf{v}_m \cdot \nabla_x)\, Q_m\, P_m\, w + Q_m\, D\, P_m\, w = F\,, & \text{in } \Omega_T\,, \\ P_m\, w|_{t<0} = 0\,, \end{cases} \tag{15}$$

is *strongly* well-posed for any matrix $D \in \mathscr{M}_N(\mathbb{R})$, and any source term $F \in H^{+\infty}(\Omega_T)$ vanishing for $t < 0$ and satisfying $Q_m\, F = F$. Since the group velocity $\mathbf{v}_m$ is outgoing, the initial boundary value problem (15) does not require any boundary condition, see [3, chapter 3]. In this case, there exists a unique solution $P_m\, w \in H^{+\infty}(\Omega_T)$ solution to (15) that vanishes for $t < 0$. This solution can be computed by first decomposing all vectors on a basis of Im Q_m then by integrating along the characteristics defined by $\mathbf{v}_m$.

With this well-posedness result in mind, the equations (14a), (14b) show that the principal term $u_{0,m}$ is zero for $m \in \mathscr{NC}$. Then (14c) gives $u_{1,m} = P_m\, u_{1,m}$. The equation (14d) determines $P_m\, u_{1,m}$ by solving an initial boundary value problem of the form (15) with the source term $Q_m\, f_m$. Observe that $u_{1,m}$ does not vanish because the source term $Q_m\, f_m$ does not necessarily vanish. The component $(I - P_m)\, u_{2,m}$ is then determined by (14e), while again $P_m\, u_{2,m}$ satisfies an initial boundary value problem of the form (15). Inductively, we determine $(I - P_m)\, u_{n,m}$ by using the relation (14g), and we determine $P_m\, u_{n,m}$ by solving an initial boundary value problem of the form (15). The source term for this initial boundary value problem is obtained from (14f). The procedure is entirely analogous to the construction of WKB expansions for the Cauchy problem in free space. Eventually, we have proved:

Proposition 1. *Let $m \in \mathscr{N}\mathscr{C}$, and let $f_m \in H^{+\infty}(\Omega_T)$ vanish for $t < 0$. Then there exists a unique sequence $(u_{n,m})_{n\geq 0}$ in $H^{+\infty}(\Omega_T)$ such that*

(i) *all functions $u_{n,m}$ vanish for $t < 0$,*
(ii) *the cascade (14), or equivalently (12), is satisfied.*

Moreover, there holds $u_{0,m} = 0$ and $u_{1,m} = P_m\, u_{1,m}$. In the particular case $f_m = 0$, all functions $u_{n,m}$ are zero.

3.2.3. *The principal term for causal phases.* For causal phases, the equations in Ω_T are again (14), since the cascade (14) is decoupled for each phase. However, in this case, the group velocity $\mathbf{v}_m$ is incoming and the determination of the amplitudes $u_{n,m}$ in the domain Ω_T requires first to determine the traces $u_{n,m}|_{x_d=0}$ on ω_T. More precisely, when φ_m is a causal phase, the initial boundary value problem

$$\begin{cases} (\partial_t + \mathbf{v}_m \cdot \nabla_x)\, Q_m\, P_m\, w + Q_m\, D\, P_m\, w = F\,, & \text{in } \Omega_T\,, \\ P_m\, w|_{x_d=0} = G\,, & \text{on } \omega_T\,, \\ P_m\, w|_{t<0} = 0\,, \end{cases} \tag{16}$$

is *strongly* well-posed for any matrix D, and for any source terms $(F, G) \in H^{+\infty}(\Omega_T) \times H^{+\infty}(\omega_T)$ vanishing for $t < 0$ and satisfying $Q_m\, F = F$, $P_m\, G = G$. This well-posedness result holds because the Dirichlet boundary conditions are strictly dissipative, see again [3, chapter 3]. We therefore need to determine the trace of the functions $P_m\, u_{n,m}$ on ω_T.

Let us detail how we can determine the trace of each $u_{0,m}$, $m \in \mathscr{C}$. We recall that $u_{0,m} = 0$ if $m \in \mathscr{N}\mathscr{C}$, see Proposition 1. Together with the polarization condition (14a), (13a) reads

$$B \sum_{m\in\mathscr{C}} P_m\, u_{0,m}|_{x_d=0} = 0\,.$$

Using Lemma 1, we know that the vector $\sum_{m\in\mathscr{C}} P_m\, u_{0,m}|_{x_d=0}$ belongs to the stable subspace $\mathbb{E}^s(\underline{\tau}, \underline{\eta})$. Using (9), we obtain that there exists a scalar function α_0 defined on ω_T such that

$$\forall\, m \in \mathscr{C}\,, \quad u_{0,m}|_{x_d=0} = \alpha_0\, e_m\,. \tag{17}$$

Let us now consider the boundary condition (13b), that reads

$$\begin{aligned} B \sum_{m\in\mathscr{C}} P_m\, u_{1,m}|_{x_d=0} &= g - B \sum_{m\in\mathscr{N}\mathscr{C}} u_{1,m}|_{x_d=0} - B \sum_{m\in\mathscr{C}} (I - P_m)\, u_{1,m}|_{x_d=0} \\ &= g - B \sum_{m\in\mathscr{N}\mathscr{C}} u_{1,m}|_{x_d=0} - i\, B \sum_{m\in\mathscr{C}} \big(R_m\, L(\partial)\, u_{0,m}\big)|_{x_d=0}\,, \end{aligned} \tag{18}$$

where we have used (14c) to get the last equality. The vector on the left hand-side of (18) belongs to $B\,\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ thanks to Lemma 1. Consequently, (18) implies a solvability condition: the vector on the right hand-side of (18) should be orthogonal to b, see (10). The following result is the crucial point in our analysis.

Proposition 2. *Let the projectors P_m, Q_m be defined in Lemma 2, and let R_m denote the partial inverse of $L_1(\mathrm{d}\varphi_m)$ satisfying (8). Then we have $R_m\, A_d\, P_m = 0$ for all $m = 1, \dots, M$. Consequently, the operator $\sum_{m\in\mathscr{C}} R_m\, L(\partial)\, P_m$ is tangent to the boundary ω_T.*

Let the vector b satisfy (10). Then there exists a nonzero real number β such that the following relation holds:

$$b \cdot B \sum_{m\in\mathscr{C}} R_m\, L(\partial)\, e_m = \beta \left(\partial_\tau \sigma(\underline{\tau}, \underline{\eta})\, \partial_t + \sum_{j=1}^{d-1} \partial_{\eta_j} \sigma(\underline{\tau}, \underline{\eta})\, \partial_{x_j} \right) + b \cdot B \sum_{m\in\mathscr{C}} R_m\, D\, e_m\,.$$

Moreover, the coefficient $\partial_\tau \sigma(\underline{\tau}, \underline{\eta})$ equals 1.

Let us first admit the result of Proposition 2, and see how we can determine the function α_0 in (17). If we apply the first result in Proposition 2, (18) reads

$$B \sum_{m\in\mathscr{C}} P_m\, u_{1,m}|_{x_d=0} = g - B \sum_{m\in\mathscr{N}\mathscr{C}} u_{1,m}|_{x_d=0} - i\, B \sum_{m\in\mathscr{C}} R_m \left(\partial_t + \sum_{j=1}^{d-1} A_j\, \partial_{x_j} \right) (u_{0,m}|_{x_d=0})\,.$$

We multiply the latter relation by b and use (17). Applying Proposition 2, we obtain a first order equation for α_0 that reads

$$\partial_t \alpha_0 + \sum_{j=1}^{d-1} \partial_{\eta_j} \sigma(\underline{\tau}, \underline{\eta})\, \partial_{x_j} \alpha_0 + \mathscr{D}\, \alpha_0 = \frac{-i}{\beta}\, b \cdot \left(g - B \sum_{m \in \mathscr{N}\mathscr{C}} u_{1,m}|_{x_d=0} \right) . \tag{19}$$

The real number $\mathscr{D}$ in (19) is defined as

$$\mathscr{D} := \frac{1}{\beta}\, b \cdot B \sum_{m \in \mathscr{C}} R_m\, D\, e_m\, .$$

The equation (19) is a Cauchy problem that determines a unique $\alpha_0 \in H^{+\infty}(\omega_T)$ that vanishes for $t < 0$. The Cauchy problem (19) is well-posed because σ is a real valued function so (19) is a scalar transport equation that can be integrated along the characteristics.

Once we have determined α_0, the function $u_{0,m} = P_m\, u_{0,m}$ is obtained by solving an initial boundary value problem of the form (16), see (14b) and Lemma 3:

$$\begin{cases} (\partial_t + \mathbf{v}_m \cdot \nabla_x)\, Q_m\, P_m\, u_{0,m} + Q_m\, D\, P_m\, u_{0,m} = 0\,, & \text{in } \Omega_T\,, \\ P_m\, u_{0,m}|_{x_d=0} = \alpha_0\, e_m\,, & \text{on } \omega_T\,, \\ P_m\, u_{0,m}|_{t<0} = 0\,. \end{cases}$$

Before proving Proposition 2, let us observe that generically, the function α_0 is nonzero, and therefore the $u_{0,m}$'s are nonzero. If we anticipate a little and take for granted that the WKB expansion of u^ε is a good approximation of u^ε as ε tends to zero, we observe that the amplitude of u^ε is asymptotically $\mathscr{O}(1)$ as ε tends to zero. This is the main *amplification* phenomenon that we exhibit in this article. We refer to section 4 for further discussions on this subject.

Proof of Proposition 2. The proof splits in several steps.

• Let us first prove the relation $R_m\, A_d\, P_m = 0$ for all $m = 1, \dots, M$. Let $X \in \mathbb{C}^N$. We have

$$A_d\, P_m\, X \in A_d \operatorname{Ker} L_1(\mathrm{d}\varphi_m) = \operatorname{Im} Q_m\,,$$

see Lemma 2. We thus have $R_m\, A_d\, P_m\, X = R_m\, Q_m\, A_d\, P_m\, X = 0$ where we use (8) to conclude. We have thus proved that the operator $\sum_{m \in \mathscr{C}} R_m\, L(\partial)\, P_m$ is tangent to the boundary ω_T. In particular, we have

$$b \cdot B \sum_{m \in \mathscr{C}} R_m\, L(\partial)\, e_m = \left(b \cdot B \sum_{m \in \mathscr{C}} R_m\, e_m \right) \partial_t + \sum_{j=1}^{d-1} \left(b \cdot B \sum_{m \in \mathscr{C}} R_m\, A_j\, e_m \right) \partial_{x_j} + b \cdot B \sum_{m \in \mathscr{C}} R_m\, D\, e_m\,. \tag{20}$$

It remains to make the coefficients in the transport operator (20) more explicit.

• We now give two possible definitions of the so-called Lopatinskii determinant near $(\underline{\tau}, \underline{\eta})$. As shown in the proof of Lemma 1, we know that for (z, η) close to $(\underline{\tau}, \underline{\eta})$, the eigenvalues $i\, \omega_m(z, \eta)$ of $\mathscr{A}(z, \eta)$ are determined by solving (5). They depend holomorphically on z and analytically on η. These eigenvalues are semi-simple and the corresponding eigenspaces also depend holomorphically on z and analytically on η. Moreover, the decomposition (6) holds. We can therefore construct a basis $F_1(z, \eta), \dots, F_p(z, \eta)$ of the stable subspace $\mathbb{E}^s(z, \eta)$ such that the vectors $F_j(z, \eta)$ depend holomorphically on z and analytically on η in a neighborhood of $(\underline{\tau}, \underline{\eta})$. There is no loss of generality in assuming $F_1(\underline{\tau}, \underline{\eta}) = e$ where e satisfies (9). The basis $F_1, \dots, F_p$ of $\mathbb{E}^s$ allows us to define a first Lopatinskii determinant by the formula

$$\Delta_1(z, \eta) := \det \Big(B\, F_1(z, \eta), \dots, B\, F_p(z, \eta) \Big)\,. \tag{21}$$

Using Assumption 3, we can define a second Lopatinskii determinant by using the basis $E_1, \dots, E_p$ of $\mathbb{E}^s$ that is defined in a neighborhood of $(\underline{\tau}, \underline{\eta})$:

$$\Delta_2(z, \eta) := \det \Big(B\, E_1(z, \eta), \dots, B\, E_p(z, \eta) \Big)\,. \tag{22}$$

Assumption 3 shows that the Lopatinskii determinant Δ_2 satisfies

$$\Delta_2(z,\eta) = \big(\gamma + i\,\sigma(z,\eta)\big)\,\det P(z,\eta)\,. \tag{23}$$

Let us observe that Δ_1 depends holomorphically on z and analytically on η, while Δ_2 is "only" a $\mathscr{C}^\infty$ function of (z,η). Since the E_j's and the F_j's both span the stable subspace $\mathbb{E}^s(z,\eta)$, the Lopatinskii determinants Δ_1 and Δ_2 in (21), (22) are proportional one to the other. Namely, there exists a complex valued $\mathscr{C}^\infty$ function $\vartheta(z,\eta)$, that does not vanish in a neighborhood of $(\underline{\tau},\underline{\eta})$, and that satisfies

$$\Delta_1(z,\eta) = \vartheta(z,\eta)\,\Delta_2(z,\eta)\,. \tag{24}$$

• Differentiating (23) with respect to the real and imaginary parts of z then with respect to the η_j's, we obtain the relations

$$\begin{aligned} \partial_\gamma\Delta_2(\underline{\tau},\underline{\eta}) &= (1 + i\,\partial_\gamma\sigma(\underline{\tau},\underline{\eta}))\,\det P(\underline{\tau},\underline{\eta})\,,\\ \partial_\tau\Delta_2(\underline{\tau},\underline{\eta}) &= i\,\partial_\tau\sigma(\underline{\tau},\underline{\eta})\,\det P(\underline{\tau},\underline{\eta})\,,\\ \forall\, j = 1,\dots,d-1\,,\quad \partial_{\eta_j}\Delta_2(\underline{\tau},\underline{\eta}) &= i\,\partial_{\eta_j}\sigma(\underline{\tau},\underline{\eta})\,\det P(\underline{\tau},\underline{\eta})\,. \end{aligned} \tag{25}$$

In particular, we have $\partial_\gamma\Delta_2(\underline{\tau},\underline{\eta}) \neq 0$ because $\partial_\gamma\sigma(\underline{\tau},\underline{\eta})$ is a real number. The first Lopatinskii determinant Δ_1 depends holomorphically on $z = \tau - i\,\gamma$, so we have

$$\partial_z\Delta_1(\underline{\tau},\underline{\eta}) = \partial_\tau\Delta_1(\underline{\tau},\underline{\eta}) = i\,\partial_\gamma\Delta_1(\underline{\tau},\underline{\eta})\,.$$

We now differentiate (24) and use (25) to obtain

$$\begin{aligned} \partial_\tau\sigma(\underline{\tau},\underline{\eta}) &= 1\,,\\ \partial_z\Delta_1(\underline{\tau},\underline{\eta}) &= i\,\vartheta(\underline{\tau},\underline{\eta})\,\det P(\underline{\tau},\underline{\eta}) \neq 0\,,\\ \forall\, j = 1,\dots,d-1\,,\quad \partial_{\eta_j}\Delta_1(\underline{\tau},\underline{\eta}) &= \partial_z\Delta_1(\underline{\tau},\underline{\eta})\,\partial_{\eta_j}\sigma(\underline{\tau},\underline{\eta})\,. \end{aligned} \tag{26}$$

The only remaining task is to find a relation between the derivatives in (26) and the coefficients of the transport operator in (20).

• Due to our construction of the basis $(F_1,\dots,F_p)$ of $\mathbb{E}^s$, the first column vector in the determinant (21) vanishes for $(z,\eta) = (\underline{\tau},\underline{\eta})$. Moreover, the vector space $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta})$ is spanned by the vectors $B\,F_j(\underline{\tau},\underline{\eta})$, $j = 2,\dots,p$. Let us now observe that the kernel of both linear forms

$$X \in \mathbb{C}^p \longmapsto b\cdot X\,,\quad \text{and}\quad X \in \mathbb{C}^p \longmapsto \det\big(X, B\,F_2(\underline{\tau},\underline{\eta}),\dots,B\,F_p(\underline{\tau},\underline{\eta})\big)\,,$$

is the hyperplane $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta}) \subset \mathbb{C}^p$, see (10). Consequently, there exists a nonzero complex number β_1 such that the following relation holds

$$\forall\, X \in \mathbb{C}^p\,,\quad \det\big(X, B\,F_2(\underline{\tau},\underline{\eta}),\dots,B\,F_p(\underline{\tau},\underline{\eta})\big) = \beta_1\,b\cdot X\,. \tag{27}$$

To complete the proof of Proposition 2, let us differentiate (21) with respect to z and use (27):

$$\partial_z\Delta_1(\underline{\tau},\underline{\eta}) = \det\big(B\,\partial_z F_1(\underline{\tau},\underline{\eta}), B\,F_2(\underline{\tau},\underline{\eta}),\dots,B\,F_p(\underline{\tau},\underline{\eta})\big) = \beta_1\,b\cdot B\,\partial_z F_1(\underline{\tau},\underline{\eta})\,. \tag{28}$$

Using the decomposition (6), we can decompose the vector $F_1(z,\eta)$ as

$$F_1(z,\eta) = \sum_{m\in\mathscr{C}} F_{1,m}(z,\eta)\,,\quad \Big(z\,I + \sum_{j=1}^{d-1}\eta_j\,A_j + \omega_m(z,\eta)\,A_d\Big)\,F_{1,m}(z,\eta) = 0\,.$$

Differentiating the latter relation with respect to z and applying the matrix R_m, we get[3]

$$R_m\,e_m + (I - P_m)\,\partial_z F_{1,m}(\underline{\tau},\underline{\eta}) = 0\,.$$

Summing with respect to $m \in \mathscr{C}$ and using Lemma 1, we obtain

$$\partial_z F_1(\underline{\tau},\underline{\eta}) + \sum_{m\in\mathscr{C}} R_m\,e_m \in \mathbb{E}^s(\underline{\tau},\underline{\eta})\,,$$

so (28) yields

$$b\cdot B\sum_{m\in\mathscr{C}} R_m\,e_m = -\frac{\partial_z\Delta_1(\underline{\tau},\underline{\eta})}{\beta_1} =: \beta\,. \tag{29}$$

[3]Recall the relation $F_1(\underline{\tau},\underline{\eta}) = e$, so $F_{1,m}(\underline{\tau},\underline{\eta}) = e_m$. We also use the relation $R_m\,A_d\,e_m = 0$.

If now we differentiate with respect to η_j instead of differentiating with respect to z, we obtain

$$b \cdot B \sum_{m\in\mathscr{C}} R_m\, A_j\, e_m = -\frac{1}{\beta_1}\,\partial_{\eta_j}\Delta_1(\underline{\tau},\underline{\eta}) = \beta\,\partial_{\eta_j}\sigma(\underline{\tau},\underline{\eta})\,,$$

where we have used (26). We have thus obtained the expression of the coefficients in the transport operator (20). The number β in (29) is necessarily real because the left hand-side of (29) involves only real matrices and real vectors. The proof of Proposition 2 is now complete. □

3.2.4. *The higher order terms for causal phases.* The construction of the amplitudes $u_{n,m}$, $n \geq 1$ and $m \in \mathscr{C}$, follows from an induction argument that we explain in this paragraph. Let us first of all rewrite the cascade of boundary conditions (13) as

$$B \sum_{m\in\mathscr{C}} u_{0,m}|_{x_d=0} = 0\,, \tag{30a}$$

$$B \sum_{m\in\mathscr{C}} P_m\, u_{1,m}|_{x_d=0} = g - B \sum_{m\in\mathscr{N}\mathscr{C}} u_{1,m}|_{x_d=0} - B \sum_{m\in\mathscr{C}} (I-P_m)\, u_{1,m}|_{x_d=0}\,, \tag{30b}$$

$$\forall\, n\geq 2\,,\quad B \sum_{m\in\mathscr{C}} P_m\, u_{n,m}|_{x_d=0} = -B \sum_{m\in\mathscr{N}\mathscr{C}} u_{n,m}|_{x_d=0} - B \sum_{m\in\mathscr{C}} (I-P_m)\, u_{n,m}|_{x_d=0}\,. \tag{30c}$$

We are now going to construct the amplitudes $u_{1,m}$, $m \in \mathscr{C}$. We use the decomposition (11) and write

$$\sum_{m\in\mathscr{C}} P_m\, u_{1,m}|_{x_d=0} = \alpha_1\, e + v_1\,,\quad v_1 \in \check{\mathbb{E}}^s(\underline{\tau},\underline{\eta})\,. \tag{31}$$

The boundary condition (30b) reads

$$B\, v_1 = g - B \sum_{m\in\mathscr{N}\mathscr{C}} u_{1,m}|_{x_d=0} - i\, B \sum_{m\in\mathscr{C}} (R_m\, L(\partial)\, u_{0,m})|_{x_d=0}\,. \tag{32}$$

In the previous paragraph, we have seen that the equation (19) is the compatibility condition that ensures that the right hand-side of (32) belongs to the vector space $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta})$. Since B induces an isomorphism from $\check{\mathbb{E}}^s(\underline{\tau},\underline{\eta})$ to $B\,\mathbb{E}^s(\underline{\tau},\underline{\eta})$, the equation (32) determines a unique $v_1 \in H^{+\infty}(\omega_T)$ that vanishes for $t<0$. We are now going to determine the scalar function α_1 in (31). We use (30c) for $n=2$, and combine with (14c), (14e):

$$\begin{aligned}
B \sum_{m\in\mathscr{C}} P_m\, u_{2,m}|_{x_d=0} &= -B \sum_{m\in\mathscr{N}\mathscr{C}} u_{2,m}|_{x_d=0} - i\, B \sum_{m\in\mathscr{C}} R_m\,(L(\partial)u_{1,m} - f_m)|_{x_d=0} \\
&= -B \sum_{m\in\mathscr{N}\mathscr{C}} u_{2,m}|_{x_d=0} + i\, B \sum_{m\in\mathscr{C}} R_m\, f_m|_{x_d=0} \\
&\quad - i\, B \sum_{m\in\mathscr{C}} R_m\, L(\partial)\,(I-P_m)\, u_{1,m}|_{x_d=0} - i\, B \sum_{m\in\mathscr{C}} R_m\, L(\partial)\, P_m\, u_{1,m}|_{x_d=0} \\
&= -B \sum_{m\in\mathscr{N}\mathscr{C}} u_{2,m}|_{x_d=0} + i\, B \sum_{m\in\mathscr{C}} R_m\, f_m|_{x_d=0} \\
&\quad + B \sum_{m\in\mathscr{C}} R_m\, L(\partial)\, R_m\, L(\partial)\, u_{0,m}|_{x_d=0} - i\, B \sum_{m\in\mathscr{C}} R_m\, L(\partial)\, P_m\, u_{1,m}|_{x_d=0}\,.
\end{aligned}$$

The above equation implies a compatibility condition: the vector on the right hand-side must be orthogonal to b. Applying Proposition 2 and using the decomposition (31), we obtain a transport equation for α_1 of the form

$$\partial_t \alpha_1 + \sum_{j=1}^{d-1} \partial_{\eta_j}\sigma(\underline{\tau},\underline{\eta})\,\partial_{x_j}\alpha_1 + \mathscr{D}\,\alpha_1 = g_1\,, \tag{33}$$

where the source term g_1 belongs to $H^{+\infty}(\omega_T)$ and vanishes for $t<0$, and $\mathscr{D}\in\mathbb{R}$ is the same as in (19). The expression of g_1 involves $u_{0,m}, f_m$'s, $m\in\mathscr{C}$, v_1 etc. and can be deduced from above, but we omit it. We solve (33) and obtain a solution $\alpha_1 \in H^{+\infty}(\omega_T)$ that vanishes for $t<0$.

With v_1 defined by (32) and α_1 satisfying (33), we determine the traces $P_m\, u_{1,m}|_{x_d=0}$, $m\in\mathscr{C}$, in (31). Then $P_m\, u_{1,m}$ satisfies a transport equation in Ω_T that we obtain from (14d). We can

therefore determine the amplitude $u_{1,m}$ in Ω_T by solving an initial boundary value problem of the form (16).

The construction of higher order amplitudes for causal phases follows from a straightforward induction argument that we shall omit. Our construction is summarized in the following:

Proposition 3. *Let the family $(u_{n,m})_{n\geq 0, m\in\mathscr{N}\mathscr{C}}$ in $H^{+\infty}(\Omega_T)$ solve (14), with all functions $u_{n,m}$ vanishing for $t<0$. Then there exists a unique family $(u_{n,m})_{n\geq 0, m\in\mathscr{C}}$ in $H^{+\infty}(\Omega_T)$ such that*

(i) *all functions $u_{n,m}$ vanish for $t<0$,*
(ii) *the cascade (14), (13) is satisfied.*

In particular, the trace of $u_{0,m}$ on ω_T, $m\in\mathscr{C}$, satisfies (17) with α_0 solution to the transport equation (19).

3.3. **Justification of the WKB expansion.** We first recall the following well-posedness result that was proved in [6].

Theorem 3 ([6])**.** *Let Assumptions 1-3 be satisfied and let $T>0$. Then for all functions $f\in L^2(\mathbb{R}^+_{x_d}; H^1_{t,y}(\omega_T))$ and $g\in H^1(\omega_T)$ vanishing for $t<0$, there exists a unique $u\in L^2(\Omega_T)$ that is a weak solution to (36), whose trace on ω_T belongs to $L^2(\omega_T)$, and that vanishes for $t<0$. In addition, there exists a constant C and a parameter $\gamma_0\geq 1$ such that for all $\gamma\geq\gamma_0$, the following estimate holds*

$$\gamma\,\|\mathrm{e}^{-\gamma t}u\|^2_{L^2(\Omega_T)}+\|\mathrm{e}^{-\gamma t}u|_{x_d=0}\|^2_{L^2(\omega_T)}\leq C\left\{\frac{1}{\gamma}\,\|\mathrm{e}^{-\gamma t}f\|^2_{L^2(\Omega_T)}+\frac{1}{\gamma^3}\,\|\mathrm{e}^{-\gamma t}\nabla_{t,y}f\|^2_{L^2(\Omega_T)}\right.$$
$$\left.+\|\mathrm{e}^{-\gamma t}g\|^2_{L^2(\omega_T)}+\frac{1}{\gamma^2}\,\|\mathrm{e}^{-\gamma t}\nabla g\|^2_{L^2(\omega_T)}\right\}. \quad (34)$$

Theorem 3 shows that the initial boundary value problem (1) is well-posed with a loss of one tangential derivative from the source terms $f^\varepsilon, g^\varepsilon$ to the solution u^ε (tangential means with respect to the boundary $\{x_d=0\}$). Theorem 3 holds independently of the zero order term D in the operator $L(\partial)$. We shall use without proof that for smooth source terms, that is when $f^\varepsilon\in H^{+\infty}(\Omega_T)$ and $g^\varepsilon\in H^{+\infty}(\omega_T)$, the solution u^ε to (1) belongs to $H^{+\infty}(\Omega_T)$.

The last thing to prove in Theorem 2 is that the remainder $(u^\varepsilon-u^\varepsilon_{app,N_0})_{\varepsilon\in]0,1]}$ is $\mathscr{O}(\varepsilon^{N_0+1})$ in $H^{+\infty}_\varepsilon(\Omega_T)$. Let us therefore consider an integer N_0. Some computations using (12), (13) show that the remainder $u^\varepsilon-u^\varepsilon_{app,N_0+2}$ is a solution to the system

$$\begin{cases} L(\partial)\,(u^\varepsilon-u^\varepsilon_{app,N_0+2})=-\varepsilon^{N_0+2}\,\sum_{m=1}^M \mathrm{e}^{i\,\varphi_m/\varepsilon}\,L(\partial)\,u_{N_0+2,m}\,, & \text{in } \Omega_T\,,\\ B\,(u^\varepsilon-u^\varepsilon_{app,N_0+2})|_{x_d=0}=0\,, & \text{on } \omega_T\,,\\ (u^\varepsilon-u^\varepsilon_{app,N_0+2})|_{t<0}=0\,. \end{cases} \quad (35)$$

We can then apply the energy estimate (34) of Theorem 3 and obtain

$$\|u^\varepsilon-u^\varepsilon_{app,N_0+2}\|_{L^2(\Omega_T)}\leq C\,\varepsilon^{N_0+1}\,,$$

for a suitable constant C that does not depend on ε. The derivation of energy estimates for higher order derivatives follows the classical procedure described for instance in [3, chapter 9]. We first commute (35) with tangential derivatives $\varepsilon^{|\alpha|}\,\partial^\alpha_{t,y}$ and apply the energy estimate of Theorem 3 to obtain

$$\varepsilon^{|\alpha|}\,\|\partial^\alpha_{t,y}(u^\varepsilon-u^\varepsilon_{app,N_0+2})\|_{L^2(\Omega_T)}\leq C\,\varepsilon^{N_0+1}\,.$$

Then normal derivatives are estimated by using the interior equation in (35) which shows that $\partial_{x_d}(u^\varepsilon-u^\varepsilon_{app,N_0+2})$ is a linear combination of tangential derivatives and other source terms that can be easily estimated. Eventually, we obtain that the remainder $u^\varepsilon-u^\varepsilon_{app,N_0+2}$ is $\mathscr{O}(\varepsilon^{N_0+1})$ in $H^{+\infty}_\varepsilon(\Omega_T)$. The triangle inequality implies that $u^\varepsilon-u^\varepsilon_{app,N_0}$ is also $\mathscr{O}(\varepsilon^{N_0+1})$ in $H^{+\infty}_\varepsilon(\Omega_T)$. This completes the proof of Theorem 2.

4. Applications

In this section, we consider the nonoscillatory initial boundary value problem

$$\begin{cases} L(\partial)\,u := \partial_t u + \sum_{j=1}^d A_j\,\partial_{x_j} u = f\,, & \text{in } \Omega_T\,, \\ B\,u|_{x_d=0} = g\,, & \text{on } \omega_T\,, \\ u|_{t<0} = 0\,. \end{cases} \tag{36}$$

The goal of this section is to give both quantitative and qualitative information on the solution to (36) when Assumptions 1-3 are satisfied by the operator $L(\partial)$ and the boundary condition B. To begin with, we do not consider zero order terms in $L(\partial)$ for simplicity, but the same results hold independently of the zero order term.

4.1. Optimal energy estimates for WR problems and consequences.

Our first result in this paragraph is the following:

Theorem 4. *Let Assumptions 1-3 be satisfied and let $T>0$. Let $s_1, s_2 \geq 0$, and assume that for all functions $f \in L^2(\mathbb{R}^+_{x_d}; H^{s_1}_{t,y}(\omega_T))$ and $g \in H^{s_2}(\omega_T)$ that vanish for $t<0$, there exists a unique $u \in L^2(\Omega_T)$ vanishing for $t<0$ that is a weak solution to (36), and that satisfies an estimate of the form*

$$\|u\|_{L^2(\Omega_T)} \leq C\,\Big(\|f\|_{L^2(\mathbb{R}^+_{x_d}; H^{s_1}(\omega_T))} + \|g\|_{H^{s_2}(\omega_T)} \Big)\,, \tag{37}$$

where the constant $C=C(T)$ depends on T but not on f, g, u. Then $s_1 \geq 1$ and $s_2 \geq 1$.

Theorem 4 shows that the loss of regularity in Theorem 3 from the source terms to the solution is optimal in the scale of Sobolev spaces with tangential regularity.

Proof of Theorem 4. • Let us argue by contradiction and assume $s_2 < 1$. We consider the initial boundary value problem (36) with $f=0$ and a highly oscillatory source term g^ε on the boundary ω_T. We thus consider a source term g^ε satisfying Assumption 5 with a plane phase φ satisfying Assumption 4. We further assume that the amplitude function g in Assumption 5 is of the form

$$g(t,y) = \psi(t,y)\,b\,,$$

where the vector $b \in \mathbb{R}^p$ satisfies (10) and ψ is a real valued nonzero $\mathscr{C}^\infty$ function with compact support in $\omega_T \cap \{t \geq 0\}$.

Since $\|g^\varepsilon\|_{L^2(\omega_T)}$ is $\mathscr{O}(\varepsilon)$ and $\|g^\varepsilon\|_{H^1(\omega_T)}$ is $\mathscr{O}(1)$, interpolation inequalities yield

$$\forall\, \varepsilon \in]0,1]\,, \quad \|g^\varepsilon\|_{H^{s_2}(\omega_T)} \leq C\,\varepsilon^{1-s_2}\,,$$

with a constant C that depends on ψ, φ, T but not on ε. In particular, g^ε tends to 0 in $H^{s_2}(\omega_T)$ as ε tends to 0 since we have assumed $s_2 < 1$. The energy estimate (37) shows that the solution $u^\varepsilon \in L^2(\Omega_T)$ to the oscillatory initial boundary value problem

$$\begin{cases} \partial_t u^\varepsilon + \sum_{j=1}^d A_j\,\partial_{x_j} u^\varepsilon = 0\,, & \text{in } \Omega_T\,, \\ B\,u^\varepsilon|_{x_d=0} = g^\varepsilon\,, & \text{on } \omega_T\,, \\ u^\varepsilon|_{t<0} = 0\,, \end{cases} \tag{38}$$

tends to 0 in $L^2(\Omega_T)$ as ε tends to 0. Theorem 2 shows that (38) has a unique smooth solution $v^\varepsilon \in H^{+\infty}(\Omega_T)$ that vanishes for $t<0$ and that is well approximated by its WKB expansion. Since smooth solutions are weak solutions, v^ε coincides with the weak solution u^ε given by the assumption of Theorem 4. Moreover, we know that the difference

$$u^\varepsilon - \sum_{m=1}^M u_{0,m}\,\mathrm{e}^{i\,\varphi_m/\varepsilon}\,,$$

tends to 0 in $L^2(\Omega_T)$ as ε tends to 0. The triangle inequality then shows that the approximate solution

$$\sum_{m=1}^M u_{0,m}\,\mathrm{e}^{i\,\varphi_m/\varepsilon}\,,$$

tends to 0 in $L^2(\Omega_T)$ as ε tends to 0. It remains to apply the following:

Lemma 4. *Let $v_1, \dots, v_M \in L^2(\Omega_T)$. Then the sum $\sum_{m=1}^M v_m \, \mathrm{e}^{i\,\varphi_m/\varepsilon}$ tends to 0 in $L^2(\Omega_T)$ as ε tends to 0 if and only if all functions v_m vanish.*

Applying Lemma 4, we obtain that all functions $u_{0,m}$ are zero. In particular, the trace of all $u_{0,m}$'s, $m \in \mathscr{C}$, vanish and the function α_0 in (17) is zero. This is obviously in contradiction with the equation (19) since we know that all noncausal amplitudes $u_{n,m}$ are zero (see Proposition 1) so the source term in (19) reduces to a $-i\,|b|^2\,\beta^{-1}\,\psi$ which is not identically zero. We are therefore led to a contradiction and we get $s_2 \geq 1$.

• It remains to show $s_1 \geq 1$. Again we argue by contradition and assume $s_1 < 1$. Then we choose a zero source term on the boundary ω_T in (36) and a highly oscillatory source term f^ε in the domain Ω_T. More precisely, we choose the source term f^ε of the form

$$f^\varepsilon(t,x) = \varepsilon\,\psi_1(t,x)\,\mathrm{e}^{i\,\varphi_1(t,x)/\varepsilon}\,X\,,$$

where $X \in \mathrm{Im}\ P_1$ is a constant vector, and ψ_1 is a real valued nonzero $\mathscr{C}^\infty$ function with compact support in $\Omega_T \cap \{t \geq 0\}$. Up to reordering the phases, we can always assume that φ_1 is a noncausal phase.

Applying the same arguments as above, we obtain that the solution $u^\varepsilon \in L^2(\Omega_T)$ to the problem

$$\begin{cases} \partial_t u^\varepsilon + \sum_{j=1}^d A_j\,\partial_{x_j} u^\varepsilon = f^\varepsilon\,, & \text{in } \Omega_T\,, \\ B\,u^\varepsilon|_{x_d=0} = 0\,, & \text{on } \omega_T\,, \\ u^\varepsilon|_{t<0} = 0\,, \end{cases}$$

tends to zero in $L^2(\Omega_T)$ as ε tends to zero. Lemma 4 then implies that all functions $u_{0,m}$ and α_0 vanish. For the source term f^ε defined above, the right hand-side of equation (19) reduces to

$$b \cdot B\,u_{1,1}|_{x_d=0} = 0\,, \tag{39}$$

since for $m \in \mathscr{N}\mathscr{C}$ with $m \neq 1$, the amplitudes $u_{n,m}$'s are zero (Proposition 1). Observe that the relation (39) holds independently of the function ψ_1 and of the vector X in the definition of the oscillatory source term f^ε. By our previous analysis, we know that the function $u_{1,1}$ satisfies $P_1\,u_{1,1} = u_{1,1}$ and is a solution to the transport equation

$$(\partial_t + \mathbf{v}_1 \cdot \nabla_x)\,Q_1\,u_{1,1} = \psi_1\,Q_1\,X\,.$$

We integrate along the characterstics and derive

$$Q_1\,u_{1,1}(t,x) = \left(\int_0^t \psi_1(s, x + (s-t)\,\mathbf{v}_1)\,\mathrm{d}s \right) Q_1\,X\,.$$

Let us now recall that Q_1 induces an isomorphism from $\mathrm{Im}\ P_1$ to $\mathrm{Im}\ Q_1$, so we get

$$u_{1,1}(t,x) = \left(\int_0^t \psi_1(s, x + (s-t)\,\mathbf{v}_1)\,\mathrm{d}s \right) X\,,$$

because both $u_{1,1}$ and X belong to $\mathrm{Im}\ P_1$. Using (39) we obtain

$$\forall\, X \in \mathrm{Im}\ P_1\,, \quad b \cdot B\,X = 0\,.$$

The same argument can be reproduced for all noncausal phases. In the end, we have proved

$$\forall\, X \in \oplus_{m \in \mathscr{N}\mathscr{C}}\,\mathrm{Im}\ P_m\,, \quad b \cdot B\,X = 0\,.$$

Combining with (10) and Lemma 1, we find that the vector $b \cdot B$ is orthogonal to all vectors of $\mathbb{C}^N$ and is therefore equal to zero. However, this is in contradiction with the result of Proposition 2 which shows in particular that $b \cdot B$ is not zero. We have thus obtained $s_1 \geq 1$. The proof of Theorem 4 is now complete. □

By the way we have proved the following result, which will be useful later.

Lemma 5. *One can always find $m_0 \in \mathscr{N}\mathscr{C}$ and $X \in \ker L_1(d\varphi_{m_0})$ such that $b \cdot B\,X \neq 0$.*

It remains to prove Lemma 4 above.

Proof of Lemma 4. We extend all functions v_m by zero outisde of Ω_T so we consider the v_m's as elements of $L^2(\mathbb{R}^{1+d})$. We have

$$\left\| \sum_{m=1}^{M} v_m \, \mathrm{e}^{i\,\varphi_m/\varepsilon} \right\|^2_{L^2(\mathbb{R}^{1+d})} = \sum_{m=1}^{M} \|v_m\|^2_{L^2(\mathbb{R}^{1+d})} + 2\,\mathrm{Re} \sum_{m_1<m_2} \int_{\mathbb{R}^{1+d}} v_{m_1}(t,x)\cdot\overline{v_{m_2}}(t,x)\,\mathrm{e}^{i\,(\underline{\omega}_{m_1}-\underline{\omega}_{m_2})\,x_d/\varepsilon}\,\mathrm{d}t\,\mathrm{d}x\,.$$

From Fourier's analysis, we know that the scalar products in the right hand-side converge to zero as ε tends to zero because $v_{m_1}\cdot\overline{v_{m_2}}$ belongs to $L^1(\mathbb{R}^{1+d})$. Passing to the limit, we get

$$\sum_{m=1}^{M} \|v_m\|^2_{L^2(\mathbb{R}^{1+d})} = 0\,,$$

and the proof is complete. □

The following result is in the same spirit as Theorem 4.

Theorem 5. *Assume that there exists a symmetric positive definite matrix S such that all matrices $S\,A_j$ are symmetric. If Assumptions 1-3 hold, then the boundary conditions defined by the matrix B are not maximally dissipative. In other words, there exists some vector $X\in\mathbb{R}^N$ such that $B\,X=0$ and $X\cdot S\,A_d\,X>0$.*

Proof of Theorem 5. We argue by contradiction and assume that the boundary conditions are maximally dissipative. Theorem 3.2 in [3] shows that the initial boundary value problem (36) is well-posed for $g=0$. More precisely, for all $f\in L^1([0,T];L^2(\mathbb{R}^d_+))$, there exists a unique $u\in\mathscr{C}([0,T];L^2(\mathbb{R}^d_+))$ solution to the problem (36) with $g=0$, and the solution u satisfies the estimate

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2(\mathbb{R}^d_+)} \le C_T \int_0^T \|f(s)\|_{L^2(\mathbb{R}^d_+)}\,\mathrm{d}s\,.$$

Extending u and f by 0 for negative times, we obtain the estimate

$$\|u\|_{L^2(\Omega_T)} \le C_T\,\|f\|_{L^2(\Omega_T)}\,.$$

Then we can proceed as in the proof of Theorem 4 and get a contradiction. □

An alternative proof of Theorem 5 that uses the result of [2] rather than energy estimates is presented in Appendix A. The consequence of Theorem 5 is that for maximally dissipative problems, the uniform Lopatinskii condition can break down only at glancing points or because of the existence of surface waves. Examples of such problems, for instance the well-known Rayleigh waves in elastodynamics, can be found in Domański [8], see also [3, chapter 7].

4.2. **Lower bound for the finite speed of propagation.** To our knowledge, there is no general result on the finite speed of propagation for initial boundary value problems in the WR class. Here we shall not prove that such problems obey the property of finite speed of propagation. We shall rather assume that the property of finite speed of propagation holds and we shall derive a lower bound for the maximal propagation speed. The result was suggested by the remarks in [3, chapter 8]. Our result is the following:

Theorem 6. *Let Assumptions 1-3 be satisfied. Assume moreover that there exists a constant $V>0$ such that the following property holds: for all $R_1,R_2\ge 0$, for all $x_0\in\mathbb{R}^d_+$ and for all $y_0\in\mathbb{R}^{d-1}$, if the source terms $(f,g)\in L^2(\mathbb{R}^+_{x_d};H^1_{t,y}(\omega_T))\times H^1(\omega_T)$ have compact supports satisfying*

$$\mathbf{supp}\,f \subset \{(t,x)\in\overline{\Omega_T}/\,t\ge 0\,,|x-x_0|\le R_1\}\,,$$
$$\mathbf{supp}\,g \subset \{(t,y)\in\omega_T/\,t\ge 0\,,|y-y_0|\le R_2\}\,,$$

then the solution $u\in L^2(\Omega_T)$ to (36) given by Theorem 3 satisfies

$$\mathbf{supp}\,u \subset \{(t,x)\in\overline{\Omega_T}/\,t\ge 0\,,|x-x_0|\le R_1+V\,t\}\cup\{(t,x)\in\overline{\Omega_T}/\,t\ge 0\,,|x-(y_0,0)|\le R_2+V\,t\}\,.$$

Then we have $V \geq \max(V_{Cauchy}, V_{boundary})$ *where the velocities* $V_{Cauchy}, V_{boundary}$ *are defined by*

$$V_{Cauchy} := \max_{\xi \in \mathbb{R}^d, |\xi|=1} \max\left(|\lambda_1(\xi)|, \dots, |\lambda_q(\xi)|\right), \quad V_{boundary} := \max_{(\tau,\eta)\in\Upsilon} |\nabla_\eta \sigma(\tau,\eta)|\,.$$

There are reasons to believe that the lower bound in Theorem 6 is sharp. When $V_{boundary} > V_{Cauchy}$, the speed of propagation for (36) is greater than the speed of propagation for the Cauchy problem. Examples of this kind already appeared in [4, 10, 3]. We also refer to Section 5 for some examples in two space dimensions.

Proof of Theorem 6. The speed V_{Cauchy} corresponds to the speed of propagation for the Cauchy problem. Choosing first $g = 0$ in (36) and source terms f whose support lies far from the boundary, we can apply the result of finite speed of propagation for the Cauchy problem (see e.g. [3, chapter 2]) and derive the lower bound $V \geq V_{Cauchy}$. It remains to prove the lower bound $V \geq V_{boundary}$ for which we argue by contradiction. We thus assume from now on that V satisfies $V < V_{boundary}$. We consider the initial boundary value problem (36) with $f = 0$, and a highly oscillatory source term g^ε on the boundary. More precisely, we consider a hyperbolic frequency $(\underline{\tau}, \underline{\eta}) \in \Xi_0$ verifying

$$\sigma(\underline{\tau}, \underline{\eta}) = 0\,, \quad V_{boundary} = |\nabla_\eta \sigma(\underline{\tau}, \underline{\eta})|\,.$$

We define the phase φ as in (2). Then we consider a source term g^ε defined as follows:

$$g^\varepsilon(t,y) := \varepsilon\, \psi_1(t)\, \psi_2(y)\, b\, \mathrm{e}^{i\,\varphi(t,y)/\varepsilon}\,,$$

with b as in (10). The functions ψ_1, ψ_2 are nonnegative and $\mathscr{C}^\infty$ with compact support. More precisely, we assume that ψ_1 is supported in $[0,T]$ and is positive in the open interval $]0,T[$. Similarly we assume that ψ_2 is supported in the closed unit ball of $\mathbb{R}^{d-1}$ and is positive in the open unit ball of $\mathbb{R}^{d-1}$.

Let us now state the following Lemma, which will be proved later on.

Lemma 6. *Under the assumptions of Theorem 2 and using the same notation, there exists a constant* $C \geq 0$ *such that*

$$\forall\, \varepsilon \in]0,1]\,, \quad \|u^\varepsilon - u^\varepsilon_{app,N_0}\|_{L^\infty(\Omega_T)} \leq C\, \varepsilon^{N_0+1}\,.$$

Since the source term in the domain Ω_T is zero here, all the amplitudes for noncausal phases in the WKB expansion vanish (Proposition 1). Moreover, the scalar function α_0 in (17) satisfies the transport equation (19). In our case, we have $\mathscr{D} = 0$ and the source term in the right hand-side of (19) reduces here to $-i\,|b|^2\,\psi_1\,\psi_2/\beta$. Integrating along the characteristics, we find

$$\alpha_0(t,y) = \frac{-i\,|b|^2}{\beta} \int_0^t \psi_1(s)\, \psi_2\big(y + (s-t)\,\nabla_\eta \sigma(\underline{\tau}, \underline{\eta})\big)\, \mathrm{d}s\,. \tag{40}$$

Since we have assumed $V < V_{boundary}$, we can consider some constant $\delta > 0$ such that $1 + V\,T + \delta < 1 + V_{boundary}\,T - \delta$. The source term g^ε satisfies the assumption of Theorem 6 with $R_2 = 1$ and $y_0 = 0$. Let us now consider a point $Y \in \mathbb{R}^{d-1}$ with $|Y| \geq 1 + V\,T + \delta$. From the assumption of Theorem 6, we have $u^\varepsilon(T, Y, 0) = 0$ for all $\varepsilon \in]0,1]$. Moreover, the functions u^ε and $u^\varepsilon_{app,0}$ are continuous on $\overline{\Omega_T}$, so we have

$$\|u^\varepsilon - u^\varepsilon_{app,0}\|_{L^\infty(\omega_T)} \leq \|u^\varepsilon - u^\varepsilon_{app,0}\|_{L^\infty(\Omega_T)} \leq C\,\varepsilon\,,$$

where we use Lemma 6. In particular, the pointwise value $u^\varepsilon_{app,0}(T,Y,0)$ tends to zero as ε tends to zero. However, relation (17) shows that we have

$$u^\varepsilon_{app,0}(T,Y,0) = \sum_{m\in\mathscr{C}} u_{0,m}(T,Y,0)\, \mathrm{e}^{i\,\varphi(T,Y)/\varepsilon} = \mathrm{e}^{i\,\varphi(T,Y)/\varepsilon}\, \alpha_0(T,Y)\, e\,.$$

The only possibility for $u^\varepsilon_{app,0}(T,Y,0)$ to tend to zero is that $\alpha_0(T,Y)$ vanishes.

We have therefore proved that $\alpha_0(T,\cdot)$ is identically zero outside the ball of radius $1 + V\,T + \delta$. In particular, $\alpha_0(T,\cdot)$ vanishes on the sphere of radius $1 + V_{boundary}\,T - \delta$, and this is in contradiction with the expression (40). The proof of Theorem 6 is thus complete. □

Let us now prove Lemma 6.

Proof of Lemma 6. We first fix an integer $N_1 > (d+1)/2$. We have

$$u^\varepsilon - u^\varepsilon_{app,N_0} = \big(u^\varepsilon - u^\varepsilon_{app,N_0+N_1}\big) + \varepsilon^{N_0+1} \sum_{n=N_0+1}^{N_0+N_1} \varepsilon^{n-N_0-1} \sum_{m=1}^{M} u_{n,m}\, \mathrm{e}^{i\,\varphi_m/\varepsilon}\,.$$

We recall that all amplitudes $u_{n,m}$ belong to $H^{+\infty}(\Omega_T)$ and are therefore bounded. Using the triangle inequality, it is thus sufficient to prove the estimate

$$\forall\, \varepsilon \in]0,1]\,, \quad \|u^\varepsilon - u^\varepsilon_{app,N_0+N_1}\|_{L^\infty(\Omega_T)} \le C\,\varepsilon^{N_0+1}\,,$$

for a suitable constant C. Let us define $r^\varepsilon := u^\varepsilon - u^\varepsilon_{app,N_0+N_1}$. Theorem 2 shows that r^ε satisfies an estimate of the form

$$\forall\, \varepsilon \in]0,1]\,, \quad \sum_{|\alpha|\le N_1} \varepsilon^{|\alpha|}\, \|\partial^\alpha_{t,x} r^\varepsilon\|_{L^2(\Omega_T)} \le C\,\varepsilon^{N_0+N_1+1}\,,$$

with a constant C that does not depend on ε. We now apply the Sobolev imbedding in the H^s_ε norms, see e.g. [19], and obtain

$$\|r^\varepsilon\|_{L^\infty(\Omega_T)} \le \frac{C}{\varepsilon^{(1+d)/2}} \sum_{|\alpha|\le N_1} \varepsilon^{|\alpha|}\, \|\partial^\alpha_{t,x} r^\varepsilon\|_{L^2(\Omega_T)} \le C\,\varepsilon^{N_0+1}\,.$$

The proof of Lemma 6 is complete. □

4.3. **Reflection of oscillating waves: amplification of initial data.** In this paragraph, we consider a zero order term D in the operator $L(\partial)$. Let $m_0 \in \mathscr{N}\mathscr{C}$ and let φ_{m_0} be the corresponding noncausal phase. Using the classical results of linear geometric optics for the Cauchy problem (Lax [13]), we know there exist on the domain $]-1,+\infty[\times\mathbb{R}^d$ solutions $\underline{u}^\varepsilon(t,x)$ of $L(\partial)\,\underline{u}^\varepsilon = 0$ of the form

$$\underline{u}^\varepsilon(t,x) = \mathrm{e}^{i\varphi_{m_0}/\varepsilon} \sum_{n=1}^{N_0} \varepsilon^n\, \underline{u}_{n,m_0} + \mathscr{O}(\varepsilon^{N_0+1})\,, \tag{41}$$

in the sense of $H^{+\infty}_\varepsilon(]-1,T[\times\mathbb{R}^d)$ for all $T>0$, and such that the restriction of $\underline{u}^\varepsilon$ to $]-1,0[\times\mathbb{R}^d$ is supported in the region $x_d \ge 0$:

$$\mathbf{supp}\,\big(\underline{u}^\varepsilon|_{]-1,0[\times\mathbb{R}^d}\big) \subset \big\{x_d \ge 0\big\}\,.$$

Introduce for any $T \ge 0$, $\tilde\Omega_T := \Omega_T \cap \{-1 < t < T\}$ and $\tilde\omega_T := \omega_T \cap \{-1 < t < T\}$. It follows from the support property that $\underline{u}^\varepsilon$ is a solution of the homogeneous boundary value problem

$$L(\partial)\,\underline{u}^\varepsilon = 0 \,\text{in}\ \tilde\Omega_0, \quad B\,\underline{u}^\varepsilon|_{x_d=0} = 0 \,\text{on}\ \tilde\omega_0\,.$$

Now we consider the following initial boundary value problem for a given $T>0$:

$$\begin{cases} L(\partial)\,u^\varepsilon = 0\,, & \text{in } \tilde\Omega_T\,,\\ B\,u^\varepsilon|_{x_d=0} = 0\,, & \text{on } \tilde\omega_T\,,\\ u^\varepsilon|_{\tilde\Omega_0} = \underline{u}^\varepsilon\,, \end{cases} \tag{42}$$

which is interpreted as an oscillatory high frequency wave defined in the past that hits the boundary $\{x_d = 0\}$ in the future, producing a family of reflected oscillating waves that we want to describe. The goal is to exhibit an *amplified reflected wave*: $\underline{u}^\varepsilon$ has amplitude $\mathscr{O}(\varepsilon)$ in $\tilde\Omega_0$ and we are going to show that the solution u^ε to (42) has amplitude $\mathscr{O}(1)$ in $\tilde\Omega_T$. There are several ways to do the analysis. For instance one can search the solution of the form $u^\varepsilon = \underline{u}^\varepsilon + v^\varepsilon$ and use the previous sections to find a rigorous asymptotic expansion of v^ε. However we prefer a direct approach which can be made rigorous by following the proof of Theorem 2.

We look for an approximate solution to (42) of the form

$$u^\varepsilon_{app} := \sum_{n=0}^{N_0-1} \varepsilon^n \sum_{m=1}^{M} u_{n,m}(t,x)\, \mathrm{e}^{i\,\varphi_m(t,x)/\varepsilon}\,.$$

The expected response to the oscillatory initial condition $\underline{u}^\varepsilon$ has size $\mathscr{O}(1)$ if at least one of the $u_{0,m}$ is not zero, see Lemma 4. In the interior domain $\tilde\Omega_T$, the cascade of BKW equations is exactly the cascade (12) written down in paragraph 3.2.1, for which the analysis is almost done. Only the data

are different: there are no source terms ($f_m = 0$, $g = 0$) but the amplitude u_{1,m_0} is no more null in the past. Let us repeat rapidly the construction of the first $u_{n,m}$ for $n = 0, 1$, the construction of the higher order terms being similar.

Let us consider first the noncausal modes. For $m \in \mathscr{NC}$, $u_{0,m} = P_m\, u_{0,m}$ is null because $\underline{u}^\varepsilon$ has amplitude $\mathscr{O}(\varepsilon)$ so the initial condition for $u_{0,m}$ vanishes. Consequently $u_{1,m} = P_m\, u_{1,m}$ is given by the equation

$$\begin{cases} (\partial_t + \mathbf{v}_m \cdot \nabla_x)\, Q_m\, P_m\, u_{1,m} + Q_m\, D\, P_m\, u_{1,m} = 0\,, \quad \text{in } \tilde\Omega_T\,, \\ P_m\, u_{1,m}|_{\tilde\Omega_0} = \underline{u}_{1,m}\,. \end{cases} \tag{43}$$

The same arguments as in section 3 apply, showing that all the terms $u_{n,m}$ are zero when $m \in \mathscr{NC} \setminus \{m_0\}$ (because the incident oscillatory wave $\underline{u}^\varepsilon$ is polarized on the phase φ_{m_0} and has no component on any other phase). The term u_{1,m_0} is given by equation (43) with $m = m_0$ and is nonzero in general because the data $\underline{u}_{1,m_0}$ is not.

Consider now the causal modes. The polarization $u_{0,m} = P_m\, u_{0,m}$ and the boundary condition

$$B\left(\sum_{m\in\mathscr{C}} P_m u_{0,m}\right)\Big|_{x_d=0} = 0\,,$$

still imply the relation (17). The real function α_0 is determined by the transport equation (19), which now simply reads

$$\begin{cases} \partial_t \alpha_0 + \displaystyle\sum_{j=1}^{d-1} \partial_{\eta_j}\sigma(\underline{\tau},\underline{\eta})\,\partial_{x_j}\alpha_0 + \mathscr{D}\,\alpha_0 = \frac{i}{\beta}\, b\cdot B\, u_{1,m_0}|_{x_d=0}\,, \quad \text{in } \tilde\omega_T\,, \\ \alpha_0|_{t<0} = 0\,. \end{cases} \tag{44}$$

Note that the source term in (44) actually vanishes in $t < 0$ because of the condition on the support of $\underline{u}_{1,m_0}$. Hence the function α_0, or equivalently the trace of $u_{0,m}$, is not identically zero on $\tilde\omega_T$ if and only if there exists $(t, y) \in]0, T[\times\mathbb{R}^{d-1}$ such that

$$b\cdot B\, u_{1,m_0}(t, y, 0) \neq 0\,.$$

Lemma 5 tells that one can always choose m_0 and $X \in \ker L_1(\mathrm{d}\varphi_{m_0})$ such that $b\cdot B\, X \neq 0$. Consequently one has just to choose $\underline{u}_{1,m_0}$ such that the solution u_{1,m_0} to (43) satisfies $u_{1,m_0}(t_0, y_0, 0) = X$ at some point of the boundary with $0 < t_0 < T$. Integrating backwards along the characteristics, the latter condition can be achieved provided the initial condition $\underline{u}_{1,m_0}|_{t=0}$ is suitably chosen. The details are left to the reader. To summarize we can state the following result:

Theorem 7. *One can always find $m_0 \in \mathscr{NC}$, an incident wave of size $\mathscr{O}(\varepsilon)$ of the form (41) such that the solution to the problem (42) is of size $\mathscr{O}(1)$ and of the form*

$$u^\varepsilon = \sum_{m\in\mathscr{C}} u_{0,m}(t,x)\, \mathrm{e}^{i\,\varphi_m(t,x)/\varepsilon} + \mathscr{O}(\varepsilon) \quad \text{in } H^\infty_\varepsilon(\tilde\Omega_T)\,,$$

with at least one nonzero profile $u_{0,m}$ in $t > 0$.

We emphasize that this behavior is very different from what happens for the system of linear elastodynamics with homogenous Neumann boundary conditions where the uniform Lopatinskii condition also fails but in the "elliptic region", see [3, 20, 17]. For this system, an incident oscillatory wave coming from the interior of the domain and hitting the boundary cannot reach the bad frequency and excite the "singular mode"[4]. In other words, an incident oscillatory wave cannot produce a Rayleigh wave. This is not surprising because there is no loss of derivative from the interior source term f to the solution u when the Lopatinskii condition fails in the elliptic region.

As an example, let us consider the case $d = N = 2$ and the operator

$$L_1(\partial) = \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_{x_1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{x_2}\,,$$

[4]This singular mode is responsible for the existence of boundary waves called "Rayleigh waves".

with boundary conditions defined by a matrix $B = \begin{pmatrix} \sqrt{3/2} & -\sqrt{1/2} \end{pmatrix}$. This example is detailed in subsection 5.2 and is adapted from Madja-Artola [15]. We are also preparing for the next subsection where we will consider this example again. The phase on the boundary is $\varphi = 2\,t + x_1$ because the uniform Lopatinskii condition degenerates at the point $(\tau = 2, \eta = 1)$. Assumptions 1, 2, 3, 4, are satisfied. There are two characteristic phases in play ($M = 2$), one causal φ_1 and one noncausal φ_2 that satisfy $\dim\ker L_1(\mathrm{d}\varphi_1) = \dim\ker L_1(\mathrm{d}\varphi_2) = 1$. In this case, the meaning of Lemma 5 is $m_0 = 2$ and

$$\forall\, a \in \ker L_1(\mathrm{d}\varphi_2) \setminus \{0\}\,, \quad b \cdot B\,a \neq 0\,.$$

As a consequence, we see that an incident wave oscillating with respect to φ_2 of the form (41) is *always* amplified by reflection on the boundary for this system.

4.4. **Dependence of energy estimates on zero order terms.** The important estimate (34) in Theorem 3 is proved in [6] under some more general assumptions on the systems and in the case of variable coefficients. More precisely, existence and uniqueness of a weak solution $u \in L^2(\Omega_T)$ to (36) with the continuity estimate (34) can be achieved for zero order coefficients D that are Lipschitzean. This regularity is needed in [6] in order to apply symbolic calculus rules. The problem we raise is to determine whether Lipschitzean regularity for D is necessary for (34) to hold.

We show by a counter-example that the energy estimate (34) is no longer true under the weaker assumption that the matrix D is only bounded. In other words, the well-posedness result with loss of regularity of Theorem 3 is independent of Lipschitzean zero order terms but is *not* independent of bounded zero order terms. This is in sharp contrast with the uniformly stable case where bounded zero order terms are completely harmless. This is also surprising compared with the situation for the Cauchy problem.

For our counter-example we choose $d = N = 2$ and use again the symmetric hyperbolic operator

$$L_1(\partial) = \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_{x_1} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{x_2}\,,$$

with boundary conditions defined by the matrix $B = \begin{pmatrix} \sqrt{3/2} & -\sqrt{1/2} \end{pmatrix}$. If Theorem 3 was independent of bounded zero order terms, there would exist a constant C_0 such that for all $D \in L^\infty(\Omega_T)$ with $\|D\|_{L^\infty(\Omega_T)} \leq 1$, for all source term $f \in L^2(\mathbb{R}^+; H^1(\omega_T))$ vanishing in the past, there exists a unique $u \in L^2(\Omega_T)$ solution to

$$\begin{cases} L_1(\partial)\,u + D(t,x)\,u = f\,, & \text{in } \Omega_T\,, \\ B\,u|_{x_d=0} = 0\,, & \text{on } \omega_T\,, \\ u|_{t<0} = 0\,. \end{cases} \tag{45}$$

Moreover, the energy estimate

$$\|u\|_{L^2(\Omega_T)} \leq C_0\,\|f\|_{L^2(\mathbb{R}^+;H^1(\omega_T))} \tag{46}$$

holds. Let us now prove that the constant C_0 can not be independent of D in the unit ball of L^∞.

Theorem 8. *Let $L_1(\partial)$ and B be fixed as above. Then for all $C_0 > 0$, there exists a matrix valued function $D \in \mathscr{C}^\infty(\mathbb{R}^3, \mathscr{M}_2(\mathbb{R}))$ that is bounded with all derivatives bounded and that satisfies $\|D\|_{L^\infty(\Omega_T)} \leq 1$, and there exists a function $f \in H^\infty(\Omega_T)$ vanishing for $t < 0$ such that the solution $u \in H^\infty(\Omega_T)$ to (45) does not satisfy the inequality (46).*

Proof of Theorem 8. The idea is to introduce a matrix D^ε containing high frequency oscillations in $\varphi_{m_0}/\varepsilon$ with respect to a noncausal mode $m_0 \in \mathscr{NC}$ that is characteristic for $L_1(\partial)$. Oscillations of D^ε will be transmitted to the solution u^ε by resonance. The oscillating wave u^ε will propagate towards the boundary and will be amplified by reflection as in the previous subsection. The reflection creates a wave of size $\mathscr{O}(1/\varepsilon)$ while the source term f of the equation remains bounded in all Sobolev spaces of arbitrarily high order. The energy estimate (46) will collapse as $\varepsilon \to 0$ provided that $\|D^\varepsilon\|_{L^\infty(\Omega_T)} \leq 1$ for all ε.

Let us now detail the construction. We keep the notations of the example at the end of the previous subsection. The boundary phase is $\varphi = 2\,t + x_1$ and the planar phases in the interior are φ_1 which is causal and φ_2 which is noncausal ($M = 2$). Let $C_0 > 0$ be fixed. Fix a real number

$0 < T_1 < T$ and a nonnegative function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}, \mathbb{R})$ supported in $[0, T_1]$ and positive on $[\frac{1}{3}T_1, \frac{2}{3}T_1]$. Take the matrix D^ε of the form

$$D^\varepsilon(t,x) := \chi(t)\, \mathrm{e}^{i\,\varphi_2(t,x)/\varepsilon}\, P_2\,,$$

where P_2 is the projector on $\ker L_1(\mathrm{d}\varphi_2)$ introduced in Lemma 2. Up to multiplying χ by a small positive constant, we can assume that D^ε satisfies $\|D^\varepsilon\|_{L^\infty(\Omega_T)} \leq 1$ for all $\varepsilon \in]0,1]$. Let $f \in H^{+\infty}(\mathbb{R}^3)$ satisfy $f|_{t<0} = 0$ and the support property

$$\mathbf{supp}\, f \subset \big\{(t,x) \in \mathbb{R}^3 \,/\, 0 < t < T_1\,,\ \delta < x_2\big\}\,,$$

for some parameter $\delta > 0$ to be fixed later.

Let v^ε denote the solution of the oscillatory initial value problem

$$\begin{cases} L_1(\partial)\, v^\varepsilon + D^\varepsilon v^\varepsilon = f\,, \quad \text{in }]-\infty, T] \times \mathbb{R}^2\,, \\ v^\varepsilon_{|t<0} = 0\,. \end{cases}$$

The classical results of linear geometric optics [13, 11, 19] show that v^ε admits a WKB expansion at any order of the form

$$v^\varepsilon(t,x) = \sum_{n=0}^{k} \varepsilon^n\, V_n\left(t, x, \frac{\varphi_2(t,x)}{\varepsilon}\right) + \varepsilon^{k+1}\, R_\varepsilon\,, \tag{47}$$

where R_ε is $\mathscr{O}(1)$ in $H^{+\infty}_\varepsilon(]-\infty, T] \times \mathbb{R}^2)$ in the sense of Definition 3. In the expansion (47), the profiles $V_n(t,x,\theta) \in H^{+\infty}(]-\infty, T] \times \mathbb{R}^2 \times \mathbb{R}/2\,\pi\,\mathbb{Z})$ are smooth and $2\,\pi$-periodic with respect to θ. One can always choose f such that $\partial_\theta V_0$ is not identically zero, which means that the first term in the expansion is actually oscillating. These oscillations are created by the oscillations of D, which are transmitted by resonance to v^ε. Every profile splits into its "average part"

$$\overline{V}_n(t,x) := \frac{1}{2\,\pi} \int_0^{2\,\pi} V_n(t,x,\theta)\, \mathrm{d}\theta$$

and its "oscillating part"

$$V_n^*(t,x,\theta) := V_n(t,x,\theta) - \overline{V}_n(t,x)\,.$$

In the region $\{t \leq T_1\}$ the matrix D^ε vanishes, so v^ε is an oscillating solution of a linear hyperbolic system with constant coefficients, in the sense of Lax [13]. In this region, the equation for the profiles V_n decouple into equations for the average part and equations for the oscillating part. For the leading profile V_0 the evolution equations read

$$L_1(\partial)\, \overline{V}_0 = 0 \quad (t > T_1)$$

and

$$P_2 V_0^* = V_0^*\,, \quad (\partial_t + \mathbf{v}_2 \cdot \nabla_x)\, V_0^* = 0 \quad (t > T_1)\,.$$

The function v^ε is solution of a Cauchy problem, but one can choose the parameter δ and the support of f in order that the support of the restriction $v^\varepsilon|_{t<T_1}$ is contained in $\{x_2 > 0\}$ and satisfies

$$\mathbf{dist}\big(\mathbf{supp}(v^\varepsilon|_{]-\infty, T_1[\times\mathbb{R}^2})\,,\,]-\infty, T_1[\times\mathbb{R}^2_-\big) > 0\,, \tag{48}$$

where $\mathbb{R}^2_- := \{x \in \mathbb{R}^2\,, x_2 \leq 0\}$.

Consider then the solution u^ε to the initial boundary value problem

$$\begin{cases} L_1(\partial)\, u^\varepsilon + D^\varepsilon\, u^\varepsilon = f\,, & \text{in } \Omega_T\,, \\ B\, u^\varepsilon|_{x_d=0} = 0\,, & \text{on } \omega_T\,, \\ u^\varepsilon|_{t<0} = 0\,. \end{cases} \tag{49}$$

Local (in time) uniqueness and the support condition (48) imply that there exists T_2, with $T_1 < T_2 \leq T$ such that $u^\varepsilon = v^\varepsilon$ on Ω_{T_2}. Choosing δ small enough, one can assume that the integral curves in $\mathbb{R}^3$ of the field $\partial_t + \mathbf{v}_2 \cdot \nabla_x$ passing through the support of f hit the boundary $\{x_2 = 0\}$ in the region

$\{T_1 < t < T\}$. We know that u^ε is the solution of the following initial boundary value problem for the simple operator $L_1(\partial)$, where we denote $\Omega'_T := \Omega_T \cap \{T_1 < t\}$ and $\omega'_T := \omega_T \cap \{T_1 < t\}$:

$$\begin{cases} L_1(\partial)\, u^\varepsilon = 0\,, & \text{in } \Omega'_T\,, \\ B\, u^\varepsilon|_{x_d=0} = 0\,, & \text{on } \omega'_T\,, \\ u^\varepsilon|_{\Omega_{T_2}} = v^\varepsilon\,. \end{cases} \tag{50}$$

This problem (50) is now similar to the problem of reflection of waves (42) treated in subsection 4.3. The only difference is that the data v^ε of problem (50) has an expansion of the form (47) with general periodic profiles $V_n(t,x,\theta)$, while the data $\underline{u}^\varepsilon$ in the problem (41) has a simpler monochromatic expansion where the profiles are pure exponentials functions $\underline{u}_{n,m_0}\, \mathrm{e}^{i\,\theta}$. However the construction of WKB solutions of geometric optics work as well in this more general case and are directly presented in this form in several articles or lecture notes, see e.g. [19]. We shall not repeat this construction which is completely analogous to the one given in subsections 3.2 and 4.3, leaving the details to the interested reader. Hence the remark at the end of subsection 4.3 still applies in the case of problem (50), yielding a solution u^ε of the form

$$u^\varepsilon(t,x) = \frac{1}{\varepsilon}\, W_{-1}(t,x,\varphi_1/\varepsilon) + \sum_{n=0}^{\ell} \varepsilon^n \left\{ W_n\big(t,x,\frac{\varphi_1}{\varepsilon}\big) + V_n\big(t,x,\frac{\varphi_2}{\varepsilon}\big) \right\} + \varepsilon^{\ell+1}\, R'_\varepsilon\,, \tag{51}$$

for some integer $\ell < k$ that can be taken arbitrarily large. In the formula (51), the profiles $W_n(t,x,\theta)$ belong to the space $H^{+\infty}(\Omega_T \times \mathbb{R}/2\,\pi\,\mathbb{Z})$ and are associated with oscillations on the phase φ_1. They are created by the interaction with the boundary, and satisfy $W_n|_{t<T_2} = 0$. The remainder term R'_ε is $\mathscr{O}(1)$ in $H^{+\infty}_\varepsilon(\Omega_T)$.

The energy inequality (46) for all $\varepsilon \in]0,1]$ only if the principal term W_{-1} in (51) vanishes. Otherwise, the norm $\|u^\varepsilon\|_{L^2(\Omega_T)}$ does not remain bounded as ε goes to zero. However, W_{-1} can not be identically zero due to the result of the previous paragraph. Recall that for our particular example, all incident waves are amplified at the boundary because of the condition

$$\forall\, a \in \ker L_1(\mathrm{d}\varphi_2) \setminus \{0\}\,, \quad b \cdot B\, a \neq 0\,.$$

□

Note that a consequence of the proof of Theorem 8 is the following stronger result:

Theorem 9. *Let $L_1(\partial)$ and B be fixed as above. Let $C_0 > 0$ and let $s \geq 1$. Then there exists a matrix valued function $D \in \mathscr{C}^\infty(\mathbb{R}^3, \mathscr{M}_2(\mathbb{R}))$ that is bounded with all derivatives bounded and that satisfies $\|D\|_{L^\infty(\Omega_T)} \leq 1$, and there exists a function $f \in H^\infty(\Omega_T)$ vanishing for $t < 0$ such that the solution $u \in H^\infty(\Omega_T)$ to (45) does not satisfy the inequality*

$$\|u\|_{L^2(\Omega_T)} \leq C_0\, \|f\|_{L^2(\mathbb{R}^+; H^s(\omega_T))}\,.$$

5. Examples and comments

5.1. **Computation of the transport operator on the boundary.** We begin with a simplification of Proposition 2 in the case $d = 2$.

Lemma 7. *Let $d = 2$. Under the assumptions of Proposition 2, we have $\underline{\eta} \neq 0$ and*

$$b \cdot B \sum_{m \in \mathscr{C}} R_m\, L(\partial)\, e_m = \left(b \cdot B \sum_{m \in \mathscr{C}} R_m\, e_m \right) \left(\partial_t - \frac{\underline{\tau}}{\underline{\eta}}\, \partial_{x_1} \right) + b \cdot B \sum_{m \in \mathscr{C}} R_m\, D\, e_m\,.$$

Lemma 7 shows that in the case $d = 2$, the group velocity $\nabla_\eta \sigma$ coincides with the phase velocity of the oscillations on the boundary. This is no surprise because we consider here a transport equation in one space dimension.

Proof of Lemma 7. First of all, we have $\underline{\eta} \neq 0$ for otherwise, the Lopatinskii condition would break down at some point $(\underline{\tau}, 0) \in \Xi_0$. By homogeneity, this implies that the Lopatinskii condition breaks down at all points $(z,0) \in \Xi$ so there exists a frequency $\zeta \in \Xi \setminus \Xi_0$ where the kernel of B intersects the stable subspace $\mathbb{E}^s(\zeta)$. This is in contradiction with Assumption 3. We thus have $\underline{\eta} \neq 0$.

We recall that for all $m = 1, \dots, M$, we have proved in Proposition 2 the relation $R_m\, A_2\, P_m = 0$. We also recall that for all $m \in \mathscr{C}$, the vector e_m in the decomposition (9) belongs to the kernel of $L_1(\mathrm{d}\varphi_m)$. Starting from the relation

$$L_1(\mathrm{d}\varphi_m)\, e_m = \underline{\tau}\, e_m + \underline{\eta}\, A_1\, e_m + \underline{\omega}_m\, A_2\, e_m = 0\,,$$

we multiply by R_m, sum over $m \in \mathscr{C}$, then multiply by $b \cdot B$, and we obtain

$$b \cdot B \sum_{m \in \mathscr{C}} R_m\, A_1\, e_m = -\frac{\underline{\tau}}{\underline{\eta}}\, b \cdot B \sum_{m \in \mathscr{C}} R_m\, e_m\,.$$

□

We recall that the hyperbolic region $\mathscr{H}$ always contains the projection of the forward cone, see [3, chapter 8]. In particular, $\mathscr{H}$ contains all vectors of the form $(1, \eta)$ with η sufficiently small. The velocity in the transport operator on the boundary can therefore be arbitrarily large if the uniform Lopatinskii condition breaks down at a point $(1, \eta)$ with η arbitrarily small. In particular, we can find examples where the velocity $V_{boundary}$ in Theorem 6 is larger than the speed of propagation for the Cauchy problem V_{Cauchy}, see the following paragraph.

5.2. **A wave-type system.** Our first example is the problem studied by Majda and Artola [15, section 3.C] that we rewrite in our framework. We consider the following system that is equivalent to the 2D wave equation:

$$\begin{cases} \partial_t u^\varepsilon + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_{x_1} u^\varepsilon + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_{x_2} u^\varepsilon = 0\,, & (t, x_1, x_2) \in \Omega_T\,, \\ B\, u^\varepsilon|_{x_2 = 0} = \varepsilon\, g(t, x_1)\, \mathrm{e}^{i\, \varphi(t, x_1)/\varepsilon}\,, & (t, x_1) \in \omega_T\,, \\ u^\varepsilon|_{t<0} = 0\,. \end{cases} \tag{52}$$

For simplicity, we choose a zero source term in the interior equations and no zero order term in the hyperbolic operator. The symmetric hyperbolic operator in (52) has characteristic speeds

$$\lambda_1(\xi_1, \xi_2) := -\sqrt{\xi_1^2 + \xi_2^2}\,, \quad \lambda_2(\xi_1, \xi_2) := \sqrt{\xi_1^2 + \xi_2^2}\,.$$

There are one outgoing characteristic ($\lambda_1(0, 1) < 0$) and one incoming characteristic ($\lambda_2(0, 1) > 0$), so B should be a nonzero row matrix. The precise definition of B will be given later on in order to satisfy some specific requirements. The function g in (52) is assumed to vanish for $t < 0$ and to have $\mathscr{C}^\infty$ regularity with compact support for simplicity. We choose a planar phase φ for the oscillations of the boundary source term g^ε in (52):

$$\varphi(t, x_1) := \underline{\tau}\, t + \underline{\eta}\, x_1\,, \quad (\underline{\tau}, \underline{\eta}) \neq (0, 0)\,.$$

The so-called hyperbolic region $\mathscr{H}$ can be explicitly computed, see e.g. [2, 7]:

$$\mathscr{H} = \big\{(\tau, \eta) \in \mathbb{R} \times \mathbb{R} \,/\, |\tau| > |\eta|\big\}\,.$$

We thus fix from now on a parameter $\mu \in \mathbb{R}$ such that $0 < |\mu| < 1$, and we assume that $(\underline{\tau}, \underline{\eta})$ satisfies $\underline{\tau} > 0$ and $\mu\, \underline{\tau} = \underline{\eta}$. The case $\underline{\tau} < 0$ is entirely similar. The boundary condition B will be required to make the uniform Lopatinskii condition degenerate at $(\underline{\tau}, \underline{\eta})$.

We first determine the planar characteristic phases whose trace on $\{x_2 = 0\}$ equals φ. We thus need to determine the roots ω to the dispersion relation

$$\det\left[\underline{\tau}\, I + \underline{\eta} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0\,.$$

We obtain two real roots[5], that are given by

$$\underline{\omega}_1 := -\sqrt{1 - \mu^2}\, \underline{\tau}\,, \quad \underline{\omega}_2 := -\underline{\omega}_1\,.$$

[5]This is not surprising because $(\underline{\tau}, \underline{\eta})$ belongs to the hyperbolic region.

The associated (real) phases are $\varphi_m(t,x) := \varphi(t,x_1) + \underline{\omega}_m\, x_2$, $m = 1, 2$. The following relations are straightforward[6]:

$$\underline{\tau} + \lambda_1(\underline{\eta}, \underline{\omega}_1) = \underline{\tau} + \lambda_1(\underline{\eta}, \underline{\omega}_2) = 0\,,$$

and we can then compute the group velocity $\mathbf{v}_m := \nabla\lambda_1(\underline{\eta}, \underline{\omega}_m)$ associated with each phase φ_i. A simple calculation shows that the only incoming velocity is $\mathbf{v}_1$, so φ_1 is a causal phase while φ_2 is a noncausal phase.

The following relations are easy to obtain

$$\text{Ker } L_1(\mathrm{d}\varphi_1) = \text{ Span } \underbrace{\begin{pmatrix} -\underline{\omega}_1 \\ \underline{\tau} + \underline{\eta} \end{pmatrix}}_{E_1}, \quad \text{Ker } L_1(\mathrm{d}\varphi_2) = \text{ Span } \underbrace{\begin{pmatrix} -\underline{\omega}_2 \\ \underline{\tau} + \underline{\eta} \end{pmatrix}}_{E_2}.$$

As in Lemma 2, we let $P_1, P_2 \in \mathscr{M}_2(\mathbb{R})$ denote the projectors associated with the decomposition

$$\mathbb{R}^2 = \text{ Span } E_1 \oplus \text{ Span } E_2 = \oplus_{m=1}^2 \text{ Ker } L(\mathrm{d}\varphi_m)\,.$$

Let us now define the vectors $F_i := A_2\, E_i$, $i = 1, 2$, which form a basis of $\mathbb{R}^2$. We let $Q_1, Q_2 \in \mathscr{M}_2(\mathbb{R})$ denote the projectors associated with the decomposition

$$\mathbb{R}^2 = \text{ Span } F_1 \oplus \text{ Span } F_2\,.$$

The reader can check that the image of $L_1(\mathrm{d}\varphi_m)$ is spanned by the vector F_{3-m}, $m = 1, 2$, as shown in Lemma 2. In particular, F_2 coincides with the first column vector of $L_1(\mathrm{d}\varphi_1)$.

We now introduce the partial inverse R_1 of the matrix $L_1(\mathrm{d}\varphi_1)$, that is the unique matrix verifying

$$R_1\, L(\mathrm{d}\varphi_1) = I - P_1\,, \quad R_1\, Q_1 = 0\,.$$

We wish to compute the vector $R_1\, E_1$. To do this, we first decompose E_1 on the basis of the F_m's and obtain

$$E_1 = \mu_1\, F_1 + \mu_2\, F_2\,, \quad \mu_2 := \frac{1}{2}\left(\sqrt{\frac{1-\mu}{1+\mu}} - \sqrt{\frac{1+\mu}{1-\mu}}\right). \tag{53}$$

We compute

$$R_1\, E_1 = \mu_2\, R_1\, F_2 = \mu_2\, R_1\, L(\mathrm{d}\varphi_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mu_2\, (I - P_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu_2 \\ 0 \end{pmatrix} + \theta\, E_1\,, \tag{54}$$

where θ is a suitable real number whose exact value is not useful.

Let us assume from now on that the uniform Lopatinskii condition degenerates at the point $(\underline{\tau}, \underline{\eta})$. This corresponds to a boundary condition B that satisfies $B\, E_1 = 0$. (Recall that B is a 1×2 matrix.) Equivalently, B should be proportional to the matrix B_0 defined by

$$B_0 := \begin{pmatrix} \sqrt{1+\mu} & -\sqrt{1-\mu} \end{pmatrix}.$$

We claim that the vectors $R_1\, E_1$ and E_1 are linearly independent. This can be seen from (54) because the coefficient μ_2 is nonzero[7], see (53). Therefore the row matrix B automatically satisfies $B\, R_1\, E_1 \neq 0$. The transport equation (19) on ω_T is of the form

$$\partial_t \alpha - \frac{1}{\mu}\, \partial_{x_1} \alpha = G\,.$$

We recover the fact that the speed of propagation on the boundary is $1/|\mu|$, which is larger than the maximal speed of propagation for the Cauchy problem, see [4, 10] and the discussion in [3, chapter 8].

The property $B\, R_1\, E_1 \neq 0$ is linked to the size of the system. Here, we can not have simultaneously $B\, E_1 = B\, R_1\, E_1 = 0$. In the next paragraph, we shall see an explicit example of a system of three equations for which $B\, E_1 = B\, R_1\, E_1 = 0$. As predicted from the general theory, this situation occurs only when $(\underline{\tau}, \underline{\eta})$ is a double root of the Lopatinskii determinant. For the problem (52), the boundary conditions defined by B_0 yield boundary value problems for which Assumption 3 is satisfied. The roots of the corresponding Lopatinskii determinant are exactly the points $(\tau, \mu\,\tau) \in \mathscr{H}$

[6] Observe that there is no real root ω to the equation $\underline{\tau} + \lambda_2(\underline{\eta}, \omega) = 0$, which is due to the fact that $(\underline{\tau}, \underline{\eta})$ belongs to the projection of the forward cone, see [7] for more details.

[7] We recall that the parameter μ is nonzero, otherwise the uniform Lopatinskii condition would degenerate at all points $(z, 0)$ with $\mathrm{Im}\ z \leq 0$, which would contradict the weak stability condition.

with $\tau \in \mathbb{R} \setminus \{0\}$. Following our analysis, we can then determine all terms in the WKB expansion and justify that the exact solution u^ε to (52) is close to this expansion when ε goes to zero.

5.3. **The linearized Euler system.** We now consider the linearized isentropic Euler equations in two space dimensions[8]:

$$\begin{cases} \partial_t V^\varepsilon + A_1\,\partial_{x_1} V^\varepsilon + A_2\,\partial_{x_2} V^\varepsilon = 0\,, & (t,x_1,x_2) \in \Omega_T\,, \\ B\,V^\varepsilon|_{x_2=0} = \varepsilon\, g(t,x_1)\,\mathrm{e}^{i\varphi(t,x_1)/\varepsilon}\,, & (t,x_1) \in \omega_T\,, \\ V^\varepsilon|_{t<0} = 0\,, \end{cases} \tag{55}$$

where the 3×3 matrices A_1, A_2 are given by

$$A_1 := \begin{pmatrix} 0 & -v & 0 \\ -c^2/v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad A_2 := \begin{pmatrix} u & 0 & -v \\ 0 & u & 0 \\ -c^2/v & 0 & u \end{pmatrix}\,.$$

The parameters v, u, c are chosen so that

$$v > 0\,, \quad u < 0\,, \quad |u| < c\,.$$

This assumption corresponds to the linearization of the Euler equations at a given specific volume $v > 0$ with sound speed c, and a *subsonic outgoing* velocity $(0, u)$. For such parameters, the operator $\partial_t + A_1\,\partial_{x_1} + A_2\,\partial_{x_2}$ in (55) is strictly hyperbolic with characteristic speeds

$$\lambda_1(\xi_1,\xi_2) := u\,\xi_2 - c\,\sqrt{\xi_1^2+\xi_2^2}\,, \quad \lambda_2(\xi_1,\xi_2) := u\,\xi_2\,, \quad \lambda_3(\xi_1,\xi_2) := u\,\xi_2 + c\,\sqrt{\xi_1^2+\xi_2^2}\,.$$

There are two outgoing characteristics and one incoming characteristic, so B is a nonzero row matrix. We choose a planar phase φ for the oscillations of the boundary source term in (55):

$$\varphi(t,x_1) := \underline{\tau}\,t + \underline{\eta}\,x_1\,, \quad (\underline{\tau},\underline{\eta}) \neq (0,0)\,.$$

The so-called hyperbolic region $\mathscr{H}$ can be explicitly computed, see e.g. [7], and is given by

$$\mathscr{H} = \left\{ (\tau,\eta) \in \mathbb{R}\times\mathbb{R} \,/\, |\tau| > \sqrt{c^2-u^2}\,|\eta| \right\}\,.$$

For concreteness, we fix from now on the parameters $(\underline{\tau},\underline{\eta})$ such that $\underline{\eta} > 0$ and $\underline{\tau} = c\,\underline{\eta}$. In this way, we have[9] $(\underline{\tau},\underline{\eta}) \in \mathscr{H}$.

Our first goal is to determine the planar characteristic phases whose trace on $\{x_2 = 0\}$ equals φ. This amounts to finding the roots ω of the dispersion relation

$$\det\left[\underline{\tau}\,I + \underline{\eta}\,A_1 + \omega\,A_2\right] = 0\,.$$

We obtain three real roots that are given by

$$\underline{\omega}_1 := \frac{2\,M}{1-M^2}\,\underline{\eta}\,, \quad \underline{\omega}_2 := 0\,, \quad \underline{\omega}_3 := -\frac{1}{M}\,\underline{\eta}\,, \quad M := \frac{u}{c} \in]-1,0[\,.$$

The associated (real) phases are $\varphi_i(t,x) := \varphi(t,x_1) + \underline{\omega}_i\,x_2$, $i = 1,2,3$. The relations

$$\underline{\tau} + \lambda_1(\underline{\eta},\underline{\omega}_1) = \underline{\tau} + \lambda_1(\underline{\eta},\underline{\omega}_2) = \underline{\tau} + \lambda_2(\underline{\eta},\underline{\omega}_3) = 0\,,$$

yield the group velocity $\mathbf{v}_i$ associated with each phase φ_i. A simple calculation shows that the only incoming velocity is $\mathbf{v}_1$, so φ_1 is a causal phase while φ_2, φ_3 are noncausal.

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[8] The original equations before linearization are written in the variables $v = 1/\rho, \mathbf{u}$, where ρ denotes the density and $\mathbf{u} \in \mathbb{R}^2$ denotes the velocity.

[9] As a matter of fact, it can even be shown that $(\underline{\tau},\underline{\eta})$ belongs to the projection of the forward cone, see [7], but this will be of no use here.

The following relations are obtained from the definition of the matrices A_1, A_2:

$$\text{Ker } L(\mathrm{d}\varphi_1) = \text{ Span } \underbrace{\begin{pmatrix} \dfrac{1+M^2}{1-M^2} \\ \dfrac{c}{v} \\ \dfrac{2\,M\,c}{(1-M^2)\,v} \end{pmatrix}}_{E_1}, \quad \text{Ker } L(\mathrm{d}\varphi_2) = \text{ Span } \underbrace{\begin{pmatrix} v \\ c \\ 0 \end{pmatrix}}_{E_2},$$

$$\text{Ker } L(\mathrm{d}\varphi_3) = \text{ Span } \underbrace{\begin{pmatrix} 0 \\ 1 \\ M \end{pmatrix}}_{E_3}.$$

We let $P_1, P_2, P_3 \in \mathscr{M}_3(\mathbb{R})$ denote the projectors associated with the decomposition

$$\mathbb{R}^3 = \text{ Span } E_1 \oplus \text{ Span } E_2 \oplus \text{ Span } E_3 = \oplus_i \text{ Ker } L_1(\mathrm{d}\varphi_i)\,.$$

Let us now define $F_i := A_2\,E_i$, $i = 1, 2, 3$, and denote $Q_1, Q_2, Q_3 \in \mathscr{M}_3(\mathbb{R})$ the projectors associated with the decomposition

$$\mathbb{R}^3 = \text{ Span } F_1 \oplus \text{ Span } F_2 \oplus \text{ Span } F_3\,.$$

For future use, we give the expressions

$$F_1 = \begin{pmatrix} -c\,M \\ c^2\,M/v \\ -c^2/v \end{pmatrix}, \quad F_2 = \begin{pmatrix} v\,c\,M \\ c^2\,M \\ -c^2 \end{pmatrix}, \quad F_3 = \begin{pmatrix} -v\,M \\ c\,M \\ c\,M^2 \end{pmatrix}.$$

The reader can check that the image of each matrix $L_1(\mathrm{d}\varphi_i)$ is spanned by the vectors F_j, $j \neq i$, as shown in Lemma 2. In particular, we have the following relations that are useful below:

$$\frac{1}{c}\,F_2 = \frac{1-M^2}{(1+M^2)\,\underline{\eta}}\,L(\mathrm{d}\varphi_1)\begin{pmatrix} 0 \\ M \\ -1 \end{pmatrix}, \quad F_3 = \frac{1-M^2}{(1+M^2)\,\underline{\eta}}\,L(\mathrm{d}\varphi_1)\begin{pmatrix} 0 \\ M \\ M^2 \end{pmatrix}. \tag{56}$$

We now introduce the partial inverse R_1 of the matrix $L(\mathrm{d}\varphi_1)$, that is the unique matrix verifying

$$R_1\,L(\mathrm{d}\varphi_1) = I - P_1\,, \quad P_1\,R_1 = 0\,, \quad R_1\,Q_1 = 0\,.$$

We wish to compute the vector $R_1\,E_1$. To do this, we first decompose E_1 on the basis of the F_i's:

$$E_1 = -\frac{1+M^2}{c\,M\,(1-M^2)}\,F_1 + \frac{1}{v\,M\,(1-M^2)}\left(\frac{1}{c}\,F_2 + F_3\right).$$

Using (56), we obtain

$$\begin{aligned} R_1\,E_1 &= \frac{1}{v\,M\,(1-M^2)}\,R_1\left(\frac{1}{c}\,F_2 + F_3\right) = \frac{1}{v\,M\,(1-M^2)}\,\frac{1-M^2}{(1+M^2)\,\underline{\eta}}\,R_1\,L(\mathrm{d}\varphi_1)\begin{pmatrix} 0 \\ 2\,M \\ M^2-1 \end{pmatrix} \\ &= \frac{1}{v\,M\,(1+M^2)\,\underline{\eta}}\,(I-P_1)\begin{pmatrix} 0 \\ 2\,M \\ M^2-1 \end{pmatrix} = \frac{1}{v\,M\,(1+M^2)\,\underline{\eta}}\begin{pmatrix} 0 \\ 2\,M \\ M^2-1 \end{pmatrix} + \theta\,E_1\,, \end{aligned}$$

where θ is a suitable real number.

The uniform Lopatinskii condition fails at $(\underline{\tau}, \underline{\eta})$ if and only if we have $B\,E_1 = 0$. In this case a degenerate situation occurs when we also have $B\,R_1\,E_1 = 0$. Then the transport equation on the boundary degenerates and does not determine anymore the trace of the main incoming amplitude in the WKB expansion. Observe that the vectors E_1 and $R_1\,E_1$ are linearly independent so we may have $B\,E_1 = B\,R_1\,E_1 = 0$ for a unique nonzero row matrix B_0, up to a multiplicative constant, whose kernel is spanned by E_1 and $R_1\,E_1$. The matrix B_0 can be computed explicitly:

$$B_0 := \begin{pmatrix} -1-M^2 & \dfrac{v}{c}\,(1-M^2) & 2\,M\,\dfrac{v}{c} \end{pmatrix}. \tag{57}$$

We now examine the failure of the uniform Lopatinskii condition for this specific matrix B_0. We first need to compute the stable subspace $\mathbb{E}^s(z,\eta)$ when z has negative imaginary part and η is real. This amounts to finding the roots ω of positive imaginary part to the dispersion relation

$$\det\left[z\,I+\eta\,A_1+\omega\,A_2\right]=0\,.$$

When η is real and z has negative imaginary part, there exists a unique root ω of positive imaginary part to the equation

$$(z+u\,\omega)^2=c^2\,(\omega^2+\eta^2)\,, \tag{58}$$

and the stable subspace $\mathbb{E}^s(z,\eta)$ is the one-dimensional[10] space that is spanned by the eigenvector associated with the eigenmode ω. We obtain

$$\mathbb{E}^s(z,\eta)=\ \mathrm{Span}\ \begin{pmatrix} z+u\,\omega \\ c^2\,\eta/v \\ c^2\,\omega/v \end{pmatrix}\,.$$

The Lopatinskii determinant $\Delta(z,\eta)$ for the boundary condition B_0 in (57) is

$$\Delta(z,\eta)=c\,M\,(1-M^2)\,\omega+c\,(1-M^2)\,\eta-(1+M^2)\,z\,.$$

Let us first check that the Lopatinskii condition is satisfied in $\Xi\setminus\Xi_0$. Assume that thereexists some $(z,\eta)\in\Xi\setminus\Xi_0$ such that $\Delta(z,\eta)=0$. Eliminating z in the polynomial equations

$$\frac{z}{c}=\frac{1-M^2}{1+M^2}\,M\,\omega+\frac{1-M^2}{1+M^2}\,\eta\,,\quad \left(\frac{z}{c}+M\,\omega\right)^2=\omega^2+\eta^2\,.$$

we end up with

$$\omega=\frac{2\,M}{1-M^2}\,\eta\in\mathbb{R}\,,\quad z=c\,\eta\in\mathbb{R}\,.$$

These relations show that the Lopatinskii condition is satisfied in $\Xi\setminus\Xi_0$, and also that the only possible roots of Δ are the points $(c\,\eta,\eta)\in\mathscr{H}$. Extending the stable subspace to all points $(\tau,\eta)\in\Xi_0$, we find indeed that Δ vanishes at points $(c\,\eta,\eta)$ and at no other values of (τ,η). Let us now compute the derivative of Δ with respect to z at a root $(c\,\underline{\eta},\underline{\eta})$. We have

$$\frac{\partial\Delta}{\partial z}(\underline{\tau},\underline{\eta})=c\,M\,(1-M^2)\,\frac{\partial\omega}{\partial z}(\underline{\tau},\underline{\eta})-(1+M^2)\,.$$

The derivative $\partial_z\omega$ is computed by differentiating the equation (58) satisfied by ω (this is possible because $(\underline{\tau},\underline{\eta})$ is not a glancing point so ω depends holomorphically on z in a neighborhood of $\underline{\tau}$):

$$\frac{\partial\omega}{\partial z}(\underline{\tau},\underline{\eta})=\frac{\underline{\tau}+u\,\omega}{(c^2-u^2)\,\omega-u\,\underline{\tau}}=\frac{1+M^2}{c\,M\,(1-M^2)}\,.$$

We obtain that $(\underline{\tau},\underline{\eta})$ is a double root of the Lopatinskii determinant Δ associated with the matrix B_0 in (57). In this case, the transport equation (19) that should determine the trace α_0 of the main incoming amplitude in the WKB expansion degenerates. The correct ansatz for the WKB solution corresponds to an amplification of the boundary source term with a factor $1/\varepsilon^2$.

At this stage, it is not very hard to show the following converse property: let us assume that the boundary condition B is such that $B\,E_1=0$, meaning that the uniform Lopatinskii condition fails at the hyperbolic point $(\underline{\tau},\underline{\eta})$. If $\partial_z\Delta$ vanishes at $(\underline{\tau},\underline{\eta})$, then the matrix B equals B_0 up to a multiplicative constant and $B\,R_1\,E_1=0$.

When the row matrix B satisfies $B\,E_1=0$, and B is not proportional to B_0, it can also be checked (we omit the details here) that the boundary value problem (55) satisfies Assumption 3. The transport equation that determines the trace of the main term in the WKB expansion reads

$$\partial_t\sigma-c\,\partial_{x_1}\sigma=G\,.$$

The speed of propagation on the boundary is c. In this example, it is not larger than the speed of propagation for the Cauchy problem (with our assumption on the parameter u, the speed of propagation for the hyperbolic operator in (55) is $|u|+c$).

[10]Recall that the dimension of $\mathbb{E}^s(z,\eta)$ equals the number of incoming characteristics counted with their multiplicity, which is one here.

On our example, we can formally check that the WR class is *stable* with respect to small perturbations of B. More precisely, the set of row matrices B such that $B\,E_1 = 0$ is a plane. On this plane, there is a straight line spanned by B_0 for which the Lopatinskii determinant has a double root (the set of such matrices is a closed set of zero measure). The complementary set of the straight line spanned by B_0 is the union of two open half-planes, and if B belongs to one of these half-planes, the Lopatinskii determinant has simple roots in the hyperbolic region.

Appendix A. Another proof of Theorem 5

In this appendix, we give an alternative proof of Theorem 5. We know from the result of [2] that problems in the WR class are stable with respect to small perturbations of the coefficients $A_1, \dots, A_d, B$. Moreover problems satisfying Assumptions 1-3 belong to the WR class.

Let us argue by contradition and assume that there exists a Friedrichs symmetrizer S such that the boundary conditions are maximally dissipative for this symmetrizer. For small $\delta > 0$, let us consider the initial boundary value problem

$$\begin{cases} L^\delta(\partial)\,u := \partial_t u + \sum_{j=1}^d A_j^\delta\,\partial_{x_j} u = f\,, & \text{in } \Omega_T\,, \\ B\,u|_{x_d=0} = g\,, & \text{on } \omega_T\,, \\ u|_{t<0} = 0\,, \end{cases} \tag{59}$$

where the matrices A_j^δ are defined as follows:

$$A_j^\delta := \begin{cases} A_j\,, & \text{if } j = 1, \dots, d-1, \\ A_d - \delta\,I\,, & \text{if } j = d. \end{cases}$$

The operator $L^\delta(\partial)$ satisfies Assumptions 1 and 2 for sufficiently small δ, and S is a Friedrichs symmetrizer for $L^\delta(\partial)$. Let $X \in \mathbb{R}^N \setminus \{0\}$ satisfy $B\,X = 0$. Then we have

$$X \cdot S\,A_d^\delta\,X = X \cdot S\,A_d\,X - \delta\,X \cdot S\,X \le -\delta\,X \cdot S\,X < 0\,.$$

Therefore the boundary conditions in (59) are *strictly* dissipative for all $\delta > 0$. From the results in [3, chapter 4], the initial boundary value problem (59) satisfies the uniform Lopatinskii condition. This is in contradiction with the result of [2] since for sufficiently small δ, (59) belongs to the WR class and the Lopatinskii determinant must vanish somewhere in the hyperbolic region. We have thus proved Theorem 5.

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