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Quantum phase transition and degeneracy of a circuit-QED vacuum

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We investigate theoretically the quantum vacuum properties of a transmission line resonator inductively coupled to a chain of N superconducting qubits. We derive the quantum Hamiltonian for such circuit-QED system, showing that, due to the type and strength of the interaction, a quantum phase transition occurs in the limit of large N , with a twice degenerate quantum vacuum above a critical coupling. The phase diagram can be fully explored thanks to the controllable ultrastrong coupling of the qubits with the modes of the transmission line resonator. For finite values of N , an energy splitting occurs, which becomes exponentially small with increasing size and coupling.

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Circuit quantum electrodynamics (circuit-QED) is a very fascinating topic for fundamental condensed matter physics, quantum optics and quantum information. In superconducting circuit-QED systems, it has been possible to implement on a chip the celebrated Jaynes-Cummings model by strongly coupling a superconducting qubit to a bosonic mode of a microwave transmission line resonator[1, 2] and also to perform quantum logical operations with two qubits[3]. Up to now, manipulation of quantum states in such circuit-QED systems has dealt with excited states. In fact, in the quantum circuits studied up to now, the quantum optical ground state is non-degenerate (e.g., in the Jaynes-Cummings model the ground state is the vacuum of excitations for the resonator times the ground state of the qubit) and no information can be stored or processed by using only the ground state of the circuit-QED system.

In this letter, we show that by using a chain of qubits embedded in a transmission line resonator, it is possible to obtain a quantum phase transition affecting the quantum vacuum of the system. We have found that a critical coupling occurs in such a circuit-QED system thanks to both the type and ultrastrong strength of the interaction obtainable with inductively coupled qubits. By analytical and numerical calculations, we show that above the critical point, the ground state is twice degenerate and that it is protected with respect to some types of local noise sources. These properties can be achieved even with a moderate number N of qubits.

A sketch of the considered system is depicted in Fig. 1, namely a chain of N identical artificial two-level atoms in a one-dimensional transmission line resonator. Each qubit ('fluxonium' [4]) is made of a Josephson junction coupled to inductances and an external magnetic flux. This artificial atom is inductively coupled to the transmission line resonator. The fluxonium is known to have controllable parameters and to be free from charge offsets[4]; moreover, the inductive coupling can produce extremely large coupling even with a single qubit[5]. In the case of a chain, the Hamiltonian is $H = \sum_j H_j$ where

each cell of size a is labeled by the index j and is located at the position x_j . One can effectively model the resonator as a sequence of inductances $L_r = al_r$ and capacitances $C_r = ac_r$ [6], where l_r (c_r) is the inductance (capacitance) per unit length. In each cell, the Josephson junction, the capacitances, and the inductances of both resonator and fluxonium contribute to the energy as follows:

$$H_j = 4E_{C_r}(\hat{N}_r^{j-1})^2 + \frac{(\hat{\phi}_r^{j-1} - \hat{\phi}_r^j - \hat{\phi}_x^j)^2}{2L_r} + \frac{(\hat{\phi}_x^j)^2}{2L_1} + \frac{(\hat{\phi}_x^j - \hat{\phi}_J^j)^2}{2L_2} + 4E_{C_J}(\hat{N}_J^j)^2 - E_J \cos(\hat{\phi}_J^j + \frac{2e}{\hbar}\Phi_{ext}^j), \quad (1)$$

where the \hat{N} and $\hat{\phi}$ operators represent the number and flux operators for the resonator elements and Josephson junctions (the index 'r' stands for resonator; 'J' for Josephson junction). The charging energies in the Hamiltonian are $E_{C_r} = \frac{e^2}{2C_r}$ and $E_{C_J} = \frac{e^2}{2C_J}$. By applying Kirchoff's laws and by taking $\Phi_{ext} = \pi \frac{\hbar}{2e}$, it is possible to rewrite the global Hamiltonian as $H = H_{res} + H_F + H_{coupling}$ as follows:

$$H_{res} = \sum_{j=1}^N 4E_{C_r}(\hat{N}_r^j)^2 + E_{L_r} \frac{(\hat{\varphi}_r^j - \hat{\varphi}_r^{j-1})^2}{2},$$

$$H_F = \sum_{j=1}^N 4E_{C_J}(\hat{N}_J^j)^2 + E_{L_J} \frac{(\hat{\varphi}_J^j)^2}{2} + E_J \cos(\hat{\varphi}_J^j),$$

$$H_{coupling} = \sum_{j=1}^N G(\hat{\varphi}_r^j - \hat{\varphi}_r^{j-1})\hat{\varphi}_J^j, \quad (2)$$

where we have introduced the dimensionless fluxes $\varphi_r^j = \frac{2e}{\hbar}\hat{\phi}_r^j$, $\varphi_J^j = \frac{2e}{\hbar}\hat{\phi}_J^j$ and the inductance energy constants are $E_{L_r} = (\frac{\hbar}{2e})^2 \frac{L_1+L_2}{L_1L_r+L_1L_2+L_2L_r}$, $E_{L_J} = (\frac{\hbar}{2e})^2 (\frac{L_1+L_r}{L_1L_r+L_1L_2+L_2L_r})$. The magnitude of the coupling constant is $G = (\frac{\hbar}{2e})^2 \frac{L_1}{L_1L_r+L_1L_2+L_2L_r}$.

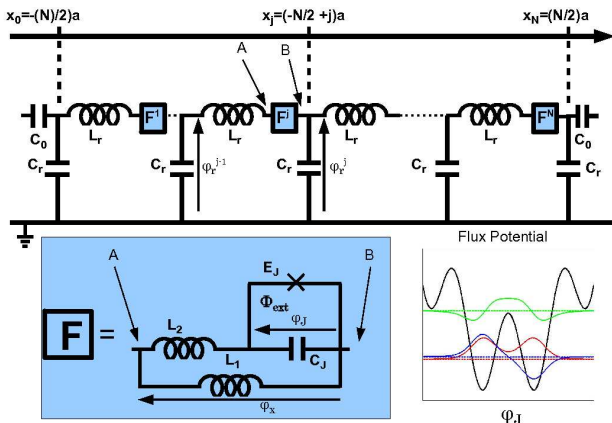


FIG. 1: Description of the system. A chain of N identical, regularly spaced qubits ("F" stands for fluxonium[4]) are embedded in a transmission line resonator. Each fluxonium is coupled inductively to the transmission line resonator. By properly tuning the external magnetic flux, the flux-dependent potential for each fluxonium has a symmetric double well structure with two states, $|0\rangle$ and $|1\rangle$ (with energy difference $\hbar\omega_F$) forming the two-level system artificial atom (parameters used for the inset: $E_J/E_{C_J} = 3$, $E_J/E_{L_J} = 20$; the third level of the fluxonium is well separated in energy).

The Hamiltonian H_{res} describes the transmission line resonator with a renormalized inductance per unit of length $\tilde{l}_r = l_r \frac{L_1 + L_2 + \frac{L_2 L_1}{a l_r}}{L_1 + L_2}$, accounting for the additional inductances in each fluxonium. By following the quantum field treatment in Ref. [6], the position-dependent flux field can be written as:

$$\hat{\phi}(x) = i \sum_{k \geq 1} \frac{1}{\omega_k} \sqrt{\frac{\hbar\omega_k}{2c_r}} f_k(x) (\hat{a}_k - \hat{a}_k^\dagger) \quad (3)$$

where \hat{a}_k^\dagger is the bosonic creation operator of a photon mode with energy $\hbar\omega_k = \frac{k\pi a}{d} \sqrt{8E_c E_{L_r}}$. The spatial profile of the k -th mode is $f_k(x) = -\sqrt{2/d} \sin(\frac{k\pi x}{d})$ for k odd, while $f_k(x) = \sqrt{2/d} \cos(\frac{k\pi x}{d})$ for k even, d being the length of the one-dimensional resonator (in the following, we will consider $d = Na$). The site-dependent fluxes are simply given by the relation $\hat{\phi}_r^j = \hat{\phi}(x_j)$.

The Hamiltonian H_F describes the sum of the bare energies of the artificial fluxonium atoms. By properly tuning the external magnetic flux, it is possible to obtain a symmetric flux-dependent potential energy, as shown in Fig. 1, with a double well structure. Due to the strong anharmonicity of its energy spectrum, the fluxonium can be approximated as a two-level system, when $E_J \gg E_{L_J}$. We call the two first eigenstates of the j -th fluxonium as $|0\rangle_j$ and $|1\rangle_j$ and we introduce the raising operator $\hat{\sigma}_{+,j} = |1\rangle\langle 0|_j$ and $\hat{\sigma}_{-,j} = \hat{\sigma}_{+,j}^\dagger = |0\rangle\langle 1|_j$. By using the Pauli matrix notation, we have $\sigma_{x,j} = \hat{\sigma}_{+,j}^\dagger + \hat{\sigma}_{+,j}$ and $\sigma_{y,j} =$

$-i(\hat{\sigma}_{+,j}^\dagger - \hat{\sigma}_{+,j})$ and $\sigma_{z,j} = 2\hat{\sigma}_{+,j}^\dagger \hat{\sigma}_{+,j} - 1$. Leaving aside a constant term, we then have $H_F = \sum_j \hbar\omega_F \hat{\sigma}_{+,j}^\dagger \hat{\sigma}_{+,j}$, where $\hbar\omega_F$ is the energy splitting between the two states $|0\rangle$ and $|1\rangle$. By considering only the two-level subspace, the Josephson junction flux has the form

$$\hat{\varphi}_J^j \simeq -\varphi_{01}(\hat{\sigma}_{+,j} + \hat{\sigma}_{+,j}^\dagger) = -\varphi_{01}\hat{\sigma}_{x,j}. \quad (4)$$

As it will be clear in the following, it is convenient to introduce excitation creation operators \hat{b}_k^\dagger

$$\hat{b}_k^\dagger = \sqrt{\frac{2}{N}} \sum_{j=1}^N \Delta f_k^*(x_j) \hat{\sigma}_{+,j} \quad (5)$$

for $1 \leq k \leq N-1$, where $\Delta f_k(x_j) = \cos(\frac{k\pi(-\frac{N+1}{2}+j)a}{d})$ for k odd, and $\Delta f_k(x_j) = \sin(\frac{k\pi(-\frac{N+1}{2}+j)a}{d})$ for k even. Note that the collective operator \hat{b}_k^\dagger is a linear superposition of the excitation operators in each fluxonium with an amplitude depending on the spatial dependence of the flux field of the transmission line resonator. In order to get a unitary transformation, it is also necessary to introduce the operator $\hat{b}_N^\dagger = \frac{1}{\sqrt{N}} \sum_j (-1)^j \hat{\sigma}_{+,j}$. In the following, we will consider only the photonic modes $1 \leq k \leq N$ (equivalent to the first Brillouin zone associated to the periodic spatial spacing of the artificial atoms), because, in the conditions we are considering, the higher order (Bragg) modes are energetically well off-resonant. Hence, we get the following effective Hamiltonian:

$$\mathcal{H} = \frac{\hbar}{2} \sum_{1 \leq k \leq N} \hat{\Phi}_k^\dagger \eta \mathcal{M}_k \hat{\Phi}_k \quad (6)$$

where $\hat{\Phi}_k = (\hat{a}_k, \hat{b}_k, \hat{a}_k^\dagger, \hat{b}_k^\dagger)^T$ with the Bogoliubov diagonal metric $\eta = \text{diag}[1, 1, -1, -1]$, and the matrix:

$$\mathcal{M}_k = \begin{pmatrix} \omega_k & -i\Omega_k & 0 & -i\Omega_k \\ i\Omega_k & \omega_F & -i\Omega_k & 0 \\ 0 & -i\Omega_k & -\omega_k & -i\Omega_k \\ -i\Omega_k & 0 & i\Omega_k & -\omega_F \end{pmatrix}. \quad (7)$$

The collective vacuum Rabi frequency reads for $1 \leq k \leq N-1$

$$\hbar\Omega_k = G \frac{4e}{\hbar} \varphi_{01} \sin\left(\frac{k\pi a}{2d}\right) \frac{1}{\omega_k} \sqrt{\frac{\hbar\omega_k N}{2dc_r}}. \quad (8)$$

(and for $k = N$, $\hbar\Omega_N = G \frac{4e}{\hbar} \frac{\varphi_{01}}{\omega_N} \sqrt{\frac{\hbar\omega_N N}{dc_r}}$).

Notice that each k -mode of the resonator is coupled only to the collective matter mode with the same spatial symmetry and $\mathcal{H} = \sum_k \mathcal{H}_k$. Hence, the eigenstates are products of the eigenstates corresponding to the k -subspaces. The effective Hamiltonian in Eq. (6) has been obtained by assuming that the operators \hat{b}_k^\dagger are bosonic, i.e. $[\hat{b}_k, \hat{b}_k^\dagger] \simeq 1$, an approximation working in the limit $N \gg 1$. The excitation spectrum of the collective bosonic

modes depends on the eigenvalues of the matrix \mathcal{M}_k . A very interesting property of the circuit-QED system here considered can be appreciated by inspecting the determinant of \mathcal{M}_k :

$$\text{Det}(\mathcal{M}_k) = \omega_k \omega_F (\omega_k \omega_F - 4\Omega_k^2). \quad (9)$$

Remarkably, it vanishes when the vacuum Rabi frequency is equal to the critical coupling value

$$\Omega_k^c = \frac{\sqrt{\omega_k \omega_F}}{2}, \quad (10)$$

implying that two of the 4 eigenvalues of \mathcal{M}_k are exactly zero. This is reminiscent of quantum phase transitions[7] with Dicke-like Hamiltonians[8], where at the quantum critical point there is a gapless bosonic excitation. For $\Omega_k > \Omega_k^c$, two of the 4 eigenvalues of the matrix \mathcal{M}_k becomes imaginary, manifesting an instability of the normal, non-degenerate, quantum vacuum phase. Note that, depending on the type of interaction, a system with (ultra)strong coupling light-matter regime does not necessarily have a quantum critical point[9].

Above the critical point, the system can not be any longer described by the effective bosonic Hamiltonian \mathcal{H} in Eq. (6) and one has to necessarily consider deviations from bosonicity. Indeed, above the critical coupling, other analytical approaches (e.g., expansions based on the Holstein-Primakoff representation[8]) or exact numerical diagonalizations of the actual Hamiltonian H with fermionic operators are needed. In the limit $N \rightarrow \infty$, it can be shown that above the critical point a symmetry breaking occurs and that the ground state becomes twice degenerate. Indeed, one can define a parity operator $\hat{P} = e^{i\pi\hat{\nu}}$, where $\hat{\nu} = \sum_k \hat{a}_k^\dagger \hat{a}_k + \sum_j \hat{\sigma}_{+,j}^\dagger \hat{\sigma}_{+,j}$ is the operator counting the total number of excitations. The Hamiltonian commutes with such parity operator. In the undercritical normal phase, the vacuum $|G\rangle$ has a well definite parity and it is therefore 'incoherent', i.e. $\langle G|a_k|G\rangle = 0$ and $\langle G|\sigma_{x,j}|G\rangle = 0$. On the other hand, in the overcritical case, the parity symmetry is broken and there is a photon and qubit coherence, namely $\langle G|a_k|G\rangle = \alpha_k \neq 0$ and $\langle G|\sigma_{x,j}|G\rangle = \beta_j \neq 0$. The ground state is twice degenerate, with a second orthogonal ground state such that $\langle G'|a_k|G'\rangle = -\alpha_k$ and $\langle G'|\sigma_{x,j}|G'\rangle = -\beta_j$.

For the case of finite number of fluxonium qubits N and finite number of modes N_m , we have performed numerical diagonalizations of the circuit-QED Hamiltonian (see Fig. 2). In the limit of very large dimensionless coupling ($\frac{\Omega_{k=1}}{\omega_F} \rightarrow \infty$), we have also derived a simple analytical expression for all eigenstates and in particular for the two degenerate ground states by taking into account an arbitrary number N_m of modes for the transmission line resonator. It is convenient to introduce the x -polarized states (eigenstates of $\hat{\sigma}_{x,j}$), namely $|+\rangle_j = \frac{1}{\sqrt{2}}(|1\rangle_j + |0\rangle_j)$ and $|-\rangle_j = \frac{1}{\sqrt{2}}(|1\rangle_j - |0\rangle_j)$. We have found that, in such

infinite coupling limit (where H_F can be neglected with respect to $H_{res} + H_{coupling}$), every eigenstate has the following form: $\Pi_j |\zeta_j\rangle \otimes |\Psi_{res}\rangle$ where $\zeta_j \in \{-, +\}$ and $|\Psi_{res}\rangle$ is a generic quantum state for the transmission line resonator field. In particular, we have analytically found the asymptotic expression for the two ground states $|G_+\rangle$ and $|G_-\rangle$:

$$|G_\pm\rangle \simeq C_G \Pi_j |\pm\rangle_j \otimes \Pi_{k_o} e^{\pm i \left(\frac{g\sqrt{2}}{k_o^{1.5} \sin(\frac{\pi}{2N})} a_{k_o}^\dagger \right)} |0\rangle_{k_o} \otimes \Pi_{k_e} |0\rangle_{k_e} \quad (11)$$

with C_G a normalisation constant, $g = \frac{\Omega_{k=1}}{\sqrt{N}\omega_{k=1}}$ the dimensionless coupling constant per fluxonium, k_o (k_e) standing for the odd (even) k values for the resonator modes.

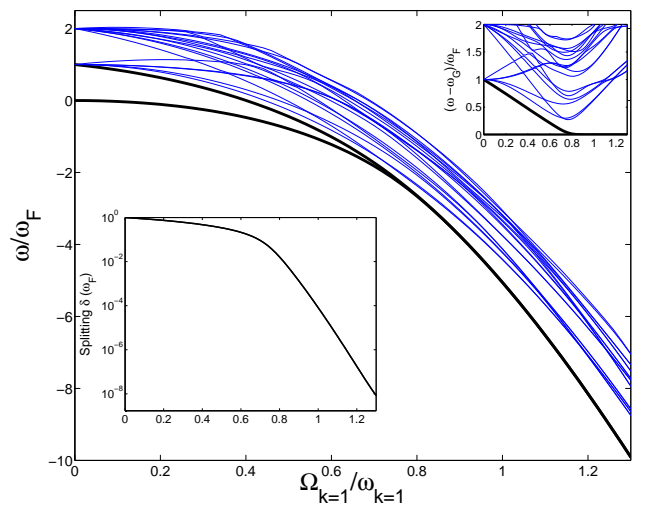


FIG. 2: First 20 energy eigenvalues (normalized to the fluxonium transition energy) versus dimensionless vacuum Rabi coupling for a circuit-QED system with $N = 5$ fluxonium atoms, $N_m = 3$ resonator modes (20 cut-off quanta per mode) and fluxonium frequency $\omega_F = \omega_{k=1}$. Due to the finite value of N , the transition from a non-degenerate to a twice degenerate ground state is not abrupt. Upper inset: the difference between the considered energy eigenvalues and the ground state energy is plotted. Lower inset: normalized energy difference (log scale, over 8 decades) between the first 2 quasi-degenerate levels versus the dimensionless coupling. For large couplings, the two collective ground states are excellently approximated by the analytical formula in Eq. (11).

Eq. (11) shows that the two degenerate ground states are the product of a 'ferromagnetic' state for the chain of qubits times coherent states for the different resonator modes. Importantly, the two orthogonal ground states have opposite polarization of the qubits and opposite phases for the coherent states. Fig. 2 shows the first

20 numerically calculated eigenvalues with $N = 5$ and $N_m = 3$. It is apparent that the transition from a non-degenerate to a twice degenerate ground state is not abrupt, as opposed to the thermodynamic limit of large N . However, even for a relatively small number of qubits and finite coupling g , quasi-degeneracy of the circuit-QED ground state is obtained for coupling slightly larger than the critical value ($\Omega_k^c/\omega_{k=1} = 0.5$) predicted by the effective bosonic Hamiltonian \mathcal{H} in Eq. (6), which is valid for $N \gg 1$. Moreover, we have verified in detail that above the critical value the corresponding states are excellently described by the analytical expression in Eq. (11) (for the strongest coupling considered in Fig. 2, the overlap between the analytical expression for $|G_{\pm}\rangle$ and the numerical counterpart is approximately 95%).

It is important to point out that in presence of a large, but finite coupling, H_F 'perturbs' the effect of $H_{res} + H_{coupling}$ by lifting the degeneracy of the quantum ground state doublet. We have found that for large couplings the corresponding energy splitting decreases exponentially with increasing coupling per fluxonium g and fluxonium number N (see inset in Fig. 2). Clearly, we recover the result of exact degeneracy in the limit of large N . Perturbation theory also allows us to estimate the splitting between the two first states when applying local noise sources proportional to the operators $\sigma_{y,j}$ and $\sigma_{z,j}$. In fact, for the degenerate ground states considered above, we have $\langle G_{\pm}|\hat{\sigma}_{y,j}|G_{\pm}\rangle = \langle G_{\pm}|\hat{\sigma}_{y,j}|G_{\mp}\rangle = 0$ and $\langle G_{\pm}|\hat{\sigma}_{z,j}|G_{\pm}\rangle = \langle G_{\pm}|\hat{\sigma}_{z,j}|G_{\mp}\rangle = 0$. Moreover, in presence of N qubits, the noise effect is zero up to the N -th order perturbation theory[13]. This implies that, by increasing the number N , the noise-induced degeneracy splitting decreases exponentially with N , an issue that we have also verified numerically. The degeneracy is instead not protected with respect to local noise sources proportional to $\sigma_{x,j}$, though different geometrical arrangements in multicavity systems may add further protection.

As we describe in the following, by a judicious tuning of the controllable parameters of the system, it is possible to explore all the phase diagram with a realistic system. Let us suppose that each fluxonium is resonant with the first cavity mode, i.e., ($\omega_F = \omega_{k=1}$). In this resonant case, we find that the dimensionless vacuum Rabi coupling reads

$$\frac{\Omega_{k=1}}{\omega_{k=1}} = \sqrt{N} \sqrt{\frac{Z_{vac}}{2Z_r\alpha}} \mu\nu\gamma \sim 5.7\sqrt{N}\gamma, \quad (12)$$

where $\nu = \frac{1}{4\pi}\varphi_{01} \sim \frac{1}{4}$ for $\frac{E_J}{E_{LJ}} \gg 1$, $\mu = \frac{\sin(\frac{\pi a}{2d})}{\frac{\pi a}{2d}}$. For $\frac{a}{d} \rightarrow 0^+$, we have $\mu \sim 1$. Finally, $\frac{Z_{vac}}{2\alpha} = \frac{\hbar}{e^2} = R_k \sim 25.8k\Omega$ is the impedance quantum, while $Z_r = \sqrt{\frac{L_r}{C_r}} = 50\Omega$ is the standard transmission line impedance. Finally, we have the branching ratio

$$\gamma = \left(\frac{L_r}{L_1L_r + L_1L_2 + L_2L_r}\right)^{\frac{1}{4}} \frac{L_1}{(L_1 + L_2)^{\frac{3}{4}}}. \quad (13)$$

This is the control parameter thanks to which it is possible to tune the dimensionless coupling $\frac{\Omega_{k=1}}{\omega_{k=1}}$. The branching ratio γ is approximately zero for $L_1 \ll L_2$ and tends to 1 when $\frac{L_1}{L_2} \gg 1$. Note that in the resonant case the critical coupling is such that $\frac{\Omega_{k=1}^c}{\omega_{k=1}} = 0.5$, hence by controlling the branching ratio it is indeed possible to explore all the phase diagram.

We wish to point out that the excitations contained in the quantum ground state cannot give rise to extracavity microwave radiation unless a non-adiabatic modulation of the Hamiltonian parameters is applied[11, 14]. The quantum vacuum radiation across the quantum phase transition is an interesting problem to explore in the future. In the opposite limit of adiabatic changes of the Hamiltonian, thanks to the degeneracy, it may be possible in principle to create Berry phases (eventually non-abelian) and conveniently control quantum superpositions in the quantum ground state subspace, another interesting issue to explore. In conclusion, a chain of fluxonium atoms inductively coupled to a transmission line resonator proposed here appears to be a very promising system towards the observation of interesting quantum phase transitions effects and the manipulation of quantum vacua in circuit-QED.

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- [1] A. Wallraff *et al.* Nature **431**, 162 (2004).
- [2] L. S. Bishop *et al.*, Nature Phys. **5**, 105 (2009)
- [3] L. DiCarlo *et al.*, Nature **460**, 240 (2009).
- [4] V. E. Manucharyan, J. Koch, L.I. Glazman, M.H. Devoret, Science in press; preprint arXiv:0906.0831.
- [5] M.H. Devoret, S.M. Girvin, and R.J. Schoelkopf, Ann. Phys. **16**, 767 (2007).
- [6] A. Blais *et al.*, Phys. Rev. A **69**, 062320 (2004).
- [7] See, e.g., Subir Sachdev, *Quantum Phase Transitions*, (Cambridge University Press, 2001).
- [8] C. Emary, T. Brandes, Phys. Rev. Lett. **90**, 166802 (2003) and Phys. Rev. E **67**, 066203 (2003).
- [9] It is important to point out that in the case of photon mode interacting via electric dipole coupling to a large collection of two-level systems, one would find an Hopfield matrix of the form

$$\mathcal{M}_k^{\text{Hopfield}} = \begin{pmatrix} \omega_k + 2D_k & -i\Omega_k & -2D_k & -i\Omega_k \\ i\Omega_k & \omega_{12} & -i\Omega_k & 0 \\ 2D_k & -i\Omega_k & -\omega_k - 2D_k & -i\Omega_k \\ -i\Omega_k & 0 & i\Omega_k & -\omega_{12} \end{pmatrix},$$

where ω_{12} is the two-level transition frequency, C_k is the vacuum Rabi coupling (proportional to the electric dipole matrix element) and D_k is a term arising from the square vector potential in the light-matter interaction. In the

Hopfield matrix[10], $D_k = C_k^2/\omega_{12}$ and $\text{Det}(\mathcal{M}_k^{\text{Hopfield}}) = (\omega_k\omega_{12})^2 \neq 0$ and no quantum critical point exists, even if the coupling can be ultrastrong[11, 12].

- [10] J. J. Hopfield, Phys. Rev. **112**, 1555 (1958).
- [11] C. Ciuti, G. Bastard, I. Carusotto, Phys. Rev. B **72**, 115303 (2005).
- [12] C. Ciuti, I. Carusotto, Phys. Rev. A **74**, 033811 (2006).
- [13] B. Douçot, M. V. Feigel'man, L. B. Ioffe, and A. S. Ioselevich, Phys. Rev. B **71**, 024505 (2005)
- [14] S. De Liberato, C. Ciuti, I. Carusotto, Phys. Rev.

Lett. **98**, 103602 (2007); S. De Liberato *et al.*, arXiv arXiv:0906.2706.

- [15] We have found that in the case of Cooper pair quantum boxes capacitively coupled to the resonator a critical coupling exists. In that case, a much larger number of qubits is required. The corresponding Hopfield-like[9] matrix contains a finite D_k term, which can be made much smaller than Ω_k^2/ω_{12} . This point is crucial to have a quantum critical point.