Some stochastic process without birth, linked to the mean curvature flow.

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Abstract

Using Huisken results about the mean curvature flow on a strictly convex hypersurface, and Kendall-Cranston coupling, we will build a stochastic process without birth, and show that there exists a unique law of such process. This process has many similarities with the circular Brownian motions studied by Émery, Schachermayer, and Arnaudon. In general, this process is not a stationary process, it is linked with some differential equation without initial condition. We will show that this differential equation has a unique solution up to a multiplicative constant.

1 Tools and first properties

Let M be a Riemannian compact n-manifold without boundary, which is smoothly embedded in $\mathbb{R}^{n+1}$, and $n \geq 2$. Denote by F_0 the embedding function:

$$F_0 : M \hookrightarrow \mathbb{R}^{n+1}.$$

Consider the flow defined by:

$$\begin{cases} \partial_t F(t,x) = -H_\nu(t,x)\vec{\nu}(t,x) \\ F(0,x) = F_0(x). \end{cases} \tag{1.1}$$

Let $M_t = F(t,M)$, we identify M with M_0 and F_0 with Id. In the previous equation (1.1), $\nu(t,x)$ is the outer unit normal at $F(t,x)$ on M_t, and $H_\nu(t,x)$ is the mean curvature at $F(t,x)$ on M_t in the direction $\nu(t,x)$, (i.e. $H_\nu(x) =$ trace $(S_\nu(x))$ where S_ν is the second fundamental form, for definition see [20]).

Remark : In this paper we take this point of view of the mean curvature flow (see [14] for existence, and related result). Many other authors give a different point of view for this equation. The viscosity solution (see [11],[9],[10],[12],[8]) generalizes the solution after the explosion time and gives a uniqueness solution which is also contained in Brakke family of solutions and passes the singularity.

We will just look at the smooth solution until the explosion time.

As usual we call M_t the motion by mean curvature. For self-completeness, we include a proof of the next lemma, although it is well-known.

Lemma 1.1 *Let (M, g) be a Riemannian manifold isometrically embedded in $\mathbb{R}^{n+1}$. Denote ι the isometry:*

$$(M, g) \overset{\iota}{\hookrightarrow} \mathbb{R}^{n+1}.$$

then:

$$\forall x \in M, \quad \Delta\iota(x) = -H_\nu(x)\vec{\nu}(x). \tag{1.2}$$

Where Δ is the Laplace-Beltrami operator associated to the metric g.

proof : By the flatness of target manifold, we have

$$\Delta\iota(x) = \begin{pmatrix} \Delta\iota^1(x) \\ \vdots \\ \Delta\iota^{n+1}(x) \end{pmatrix}$$

and $\Delta\iota^j(x) = \sum_{i=1}^n \frac{d}{dt^2}\big|_{t=0}\iota^j(\gamma_i(t))$, where $\gamma_i(t)$ is a geodesic in M such that $\gamma_i(0) = x$ and $\dot\gamma_i(0) = A_i$ and A_i is a orthogonal basis of T_xM. By definition of a geodesic we obtain:

$$\Delta\iota(x) \perp T_{\iota(x)}(\iota(M)),$$

so there exists a function β such that $\Delta\iota(x) = \beta(x)\vec{\nu}(x)$. We compute β as follows:

$$\begin{aligned}
\beta(x) &= \langle \Delta\iota(x), \vec{\nu}(x)\rangle \\
&= \textstyle\sum_{i=1}^n \langle \frac{d}{dt^2}\big|_{t=0}\iota(\gamma_i(t)), \vec{\nu}(x)\rangle \\
&= \textstyle\sum_{i=1}^n \langle \nabla^{\mathbb{R}^n}_{\dot\iota(\gamma_i(t))}\iota(\dot\gamma_i(t))\big|_{t=0}, \vec{\nu}(x)\rangle \\
&= \textstyle\sum_{i=1}^n -\langle \iota(\gamma_i(t)), \nabla^{\mathbb{R}^n}_{\dot\iota(\gamma_i(t))}\vec{\nu}\rangle\big|_{t=0}, \text{ metric connection} \\
&= \textstyle\sum_{i=1}^n -\langle \iota(\dot\gamma_i(t)), (\nabla^{\mathbb{R}^n}_{\dot\iota(\gamma_i(t))}\vec{\nu})^\top\rangle\big|_{t=0} \\
&= -\text{ trace } (S_\nu(x)).
\end{aligned}$$

□

To give a parabolic interpretation of this equation (1.1), let us define a family of metrics $g(t)$ on M which is the pull-back by $F(t, .)$ of the induced metric on M_t. Using the previous lemma we rewrite the equation as in ([14]):

$$\begin{cases} \partial_t F(t, x) = \Delta_t F(t, x) \\ F(0, x) = F_0(x) \end{cases} \tag{1.3}$$

where Δ_t is the Laplace-Beltrami operator associated to the metric $g(t)$.

Remark : Sometimes we will use a probabilistic convention, consisting in putting $\frac{1}{2}$ before the Laplacian (which just changes the time and makes the calculus more synthetic), sometimes we will use geometric convention.

We call T_c the explosion time of the mean curvature flow, let $T < T_c$, and $g(t)$ be the family of metrics defined above. Let $(W^i)_{1\leq i\leq n}$ be a $\mathbb{R}^n$-valued Brownian motion. We recall from [4] the definition of the $g(t)$-Brownian motion in M started at x, denoted by $g(t)$-BM(x):

Definition 1.2 *Let us take a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ and a $C^{1,2}$-family $g(t)_{t\in[0,T[}$ of metrics over M. A M-valued process $X(x)$ defined on $\Omega\times[0,T[$ is called a $g(t)$ Brownian motion in M started at $x \in M$ if $X(x)$ is continuous, adapted and for every smooth function f,*

$$f(X_s(x)) - f(x) - \frac{1}{2}\int_0^s \Delta_t f(X_t(x))dt$$

is a local martingale vanishing at 0.

We give a proposition which yields a characterization of mean curvature flow by the $g(t)$ Brownian motion.

Proposition 1.3 *Let M be an n-dimensional manifold isometrically embedded in $\mathbb{R}^{n+1}$. Consider the application:*

$$F : [0,T[\times M \to \mathbb{R}^{n+1}$$

such that $F(t,.)$ are diffeomorphisms, and the family of metrics $g(t)$ over M, which is the pull-back by $F(t,.)$ of the induced metric on $M_t = F(t,M)$. Then the following items are equivalent:

i) $F(t,.)$ is a solution of mean curvature flow

ii) $\forall x_0 \in M$, $\forall T \in [0,T_c[$, let $\tilde{g}_t^T = \frac{1}{2}g_{T-t}$ and $X^T(x_0)$ be a $(\tilde{g}_t^T)_{t\in[0,T]}$-BM($x_0$), then:

$$Y_t^T = F(T-t, X_t^T(x_0))$$

is a local martingale in $\mathbb{R}^{n+1}$.

proof : By definition we have a sequence of isometries:

$$F(t,.) : (M, g_t) \tilde{\to} M_t \hookrightarrow \mathbb{R}^{n+1}$$

Let $x_0 \in M$ and $T \in [0,T_c[$ and $X^T(x_0)$ a $(\tilde{g}_t^T)_{t\in[0,T]}$-BM($x_0$). We just compute the Itô differential of:

$$Y_t^{T,i} = F^i(T-t, X_t^T(x_0)),$$

that is to say:

$$\begin{array}{rcl} d(Y_t^{T,i}) & = & -\frac{\partial}{\partial t}F^i(T-t, X_t^T(x_0))dt + d(F^i_{T-t}(X_t^T(x_0)) \\ & \underset{d\mathcal{M}}{\equiv} & -\frac{\partial}{\partial t}F^i(T-t, X_t^T(x_0))dt + \frac{1}{2}\Delta_{\tilde{g}_t}F^i_{T-t}(X_t^T(x_0))dt \\ & \underset{d\mathcal{M}}{\equiv} & -\frac{\partial}{\partial t}F^i(T-t, X_t^T(x_0))dt + \Delta_{g_{T-t}}F^i_{T-t}(X_t^T(x_0))dt \\ & \underset{d\mathcal{M}}{\equiv} & 0. \end{array}$$

Therefore Y_t^T is a local martingale.

Let us show the converse. Let $x_0 \in M$ and $T \in [0, T_c[$ and $X^T(x_0)$ a $(\tilde{g}_t^T)_{t\in[0,T]}$-BM$(x_0)$, $Y_t^{T,i}$ is a local martingale so almost surely, for all $t \in [0,T]$:

$$-\frac{\partial}{\partial t}F^i(T-t, X_t^T(x_0))dt + \Delta_{g_{T-t}}F^i_{T-t}(X_t^T(x_0))dt = 0$$

so that for all $s \in [0,T]$, by integrating we get

$$\int_0^s -\frac{\partial}{\partial t}F^i(T-t, X_t^T(x_0))dt + \Delta_{g_{T-t}}F^i_{T-t}(X_t^T(x_0))dt = 0$$

the continuity of every $g(t)$-Brownian motion yields,

$$-\frac{\partial}{\partial t}F^i(T, x_0) + \Delta_{g_T}F^i_T(x_0) = 0.$$

□

In order to apply this proposition, we give an estimation of the explosion time. It is also a consequence of a maximum principle, which is explicitly contained in the $g(t)$-Brownian Motion.

The quadratic covariation of Y_t^T is given by:

Proposition 1.4 *Let Y_t^T be defined as before, then the quadratic covariation of Y_t^T for the usual scalar product in $\mathbb{R}^{n+1}$ is:*

$$\langle dY_t^T, dY_t^T\rangle = 2n\mathbb{1}_{[0,T]}(t)dt$$

proof : Let $/\!\!/_{0,t}^T$ be the parallel transport above X_t^T, it is shown in [4] that it is an isometry :

$$/\!\!/_{0,t}^T : (T_{X_0}M, \tilde{g}(0)) \longmapsto (T_{X_t}M, \tilde{g}(t)).$$

Let $(e_i)_{1\le i\le n}$ be a orthonormal basis of $(T_{X_0}M, \tilde{g}(0))$, and $(W^i)_{1\le i\le n}$ be the $\mathbb{R}^n$-valued Brownian motion such that (e.g. [4], [2]):

$$*dW_t = /\!\!/_{0,t}^{T,-1} * dX_t^T,$$

and in the Itô's sense:

$$dX_t^T = /\!\!/_{0,t}^T e_i dW_t^i.$$

Hence

$$\begin{aligned}
\langle dY_t^T, dY_t^T \rangle &= \langle d(F_{T-t}(X_t^T(x_0))), d(F_{T-t}(X_t^T(x_0))) \rangle \\
&= \langle d(X_t^T(x_0)), d(X_t^T(x_0)) \rangle_{g_{T-t}} \\
&= \langle d(X_t^T(x_0)), d(X_t^T(x_0)) \rangle_{2\tilde{g}_t} \\
&= \langle \sum_{i=1}^n /\!\!/_{0,t}^T e_i dW^i, \sum_{j=1}^n /\!\!/_{0,t}^T e_j dW^j \rangle_{2\tilde{g}_t} \\
&= \sum_{i=1}^n \langle /\!\!/_{0,t}^T e_i, /\!\!/_{0,t}^T e_i \rangle_{2\tilde{g}_t} dt \\
&= \sum_{i=1}^n 2dt \\
&= 2ndt.
\end{aligned}$$

To go from the first to the second line, we have used the fact that F_{T-t} is a isometry, for the last step we used the isometry of the parallel transport. □

Remark : Up to convention we recover the same martingale as in [21].
An immediate corollary of Proposition 1.4 is the following result, which appears in [14] and [11].

Corollary 1.5 *Let M be a compact Riemannian n-manifold and T_c the explosion time of the mean curvature flow, then: $T_c \leq \frac{diam(M_0)^2}{2n}$*

proof : Recall that the mean curvature flow stays in a compact region, like the smallest ball which contain M_0, this result is clear in the strictly convex starting manifold and can be found in a general setting using P.L Lions viscosity solution (e.g. theorem 7.1 in [11]).
For all $T \in [0, T_c[$ take the previous notation. So by the above recall that:

$$\| Y_t^T \| \leq diam(M_0),$$

then Y_t^T is a true martingale. And

$$\| Y_t^T \|^2 - \langle Y^T, Y^T \rangle_t$$

is also a true martingale. Hence:

$$\mathbb{E}[\| Y_0^T \|^2] + 2nT \leq diam(M_0)^2,$$

we obtain

$$T \leq \frac{diam(M_0)^2}{2n}.$$

□

2 Tightness, and first example on the sphere

We now define $(\tilde{g}^{T_c})_{t\in]0,T_c]}$-BM in a general setting. When the initial manifold M_0 is a sphere we use the conformality of the metric, to show that after a deterministic change of time such process is a $]-\infty, T_c]$ Brownian motion on the sphere (for existence and definition see [6] and [1]). In the next section, we will give a general result of uniqueness when the initial manifold M_0 is strictly convex.

Definition 2.1 *Let M be an n-dimensional strictly convex manifold (i.e. with a strictly positive definite second fundamental form), $F(t,.)$ the smooth solution of the mean curvature flow, $(M, g(t))$ the family of metrics constructed by pull-back (as in 1.3) and T_c the explosion time. We define a family of processes as follow:* $\forall \epsilon \in]0, T_c]$

$$X_t^\epsilon(x_0) = \begin{cases} x_0 \text{ if } 0 < t \leq \epsilon \\ \\ BM(\epsilon, x_0)_t \text{ if } \epsilon \leq t \leq T_c \end{cases}$$

where $BM(\epsilon, x_0)_t$ denotes a $\frac{1}{2}g(T_c - t)$ Brownian motion that starts at x_0 at time ϵ, and

$$Y_t^\epsilon(x_0) = \begin{cases} F(T_c - \epsilon, x_0) \text{ if } 0 \leq t \leq \epsilon \\ \\ F(T_c - t, X_t^\epsilon(x_0)) \text{ if } \epsilon \leq t \leq T_c. \end{cases}$$

Remark : We proceed as before because, at the time T_c, there is no more metric. Huisken shows in [14] that in this case:

$$\exists \mathcal{D} \in \mathbb{R}^{n+1}, \quad s.t. \quad \forall x_0 \in M, \qquad \lim_{s \to T_c} F(s, x_0) = \mathcal{D}$$

Proposition 2.2 *With the same notation as the above definition, there exists at least one martingale Y^1 in the adherence (for the weak convergence) of $(Y_\cdot^\epsilon(x_0))_{\epsilon>0}$ when ϵ goes to 0. Also, every adherence value is a martingale.*

proof : We have:

$$\begin{cases} dY_t^\epsilon(x_0) = 0 \text{ if t } \leq \epsilon \\ \\ dY_t^\epsilon(x_0) = d\mathcal{M} \text{ if t } \geq \epsilon. \end{cases}$$

Where $d\mathcal{M}$ is an Itô differential of some martingale. This defines a family of martingales. With the same computation as in proposition 1.4, we get:

$$\langle dY_t^\epsilon, dY_t^\epsilon \rangle_{\mathbb{R}^{n+1}} = 2n 1_{]\epsilon, T_c]}(t) dt \leq 2n dt.$$

Also by the above remark Y_0^ϵ is tight, hence $(Y_\cdot^\epsilon(x_0))_{\epsilon>0}$ is tight. As usual, Prokhorov's theorem implies that one adherence value exists. We also use Huisken [14] (for the strictly convex manifold) to yield:

$$\| Y^\epsilon \| \leq diam(M_0). \tag{2.1}$$

By proposition 1-1 in [16] page 481, and the fact that (Y^ϵ) are martingales we conclude that all adherence values of (Y^ϵ) are martingales with respect to the filtration that they generate. □

Remark : The above proposition is also valid for arbitrary M that are isometrically embedded in $\mathbb{R}^{n+1}$. Just because the bound 2.1 is also a consequence of theorem 7.1 in [11].
We will now derive the tightness of X_t^ϵ from those of (Y^ϵ). This purpose will be completed by the next lemma 2.4.
Recall some results of [14], if M_0 is a strictly convex manifold then M_t is also strictly convex, and $\forall 0 \leq t_1 < t_2 < T_c$, $M_{t_2} \subset int(M_{t_1})$, where int is the interior of the bounded connected component. Hence there is a foliation on $\overline{int(M_0)}$:

$$\bigsqcup_{t\in[0,T_c[} M_t,$$

where $\bigsqcup$ stand for the disjoint union.

Definition 2.3 *We note:*

$$\mathcal{C}^f(]0,T_c],\mathbb{R}^{n+1}) = \{\gamma \in \mathcal{C}(]0,T_c],\mathbb{R}^{n+1}), s.t. \quad \gamma(t) \in M_{T_c-t}\}.$$

Noted that $\mathcal{C}^f(]0,T_c],\mathbb{R}^n)$ is a closed set of $\mathcal{C}(]0,T_c],\mathbb{R}^n)$ for the Skorokhod topology.

Lemma 2.4 *Let M an n-dimensional strictly convex manifold, $F(t,.)$ the smooth solution of the mean curvature flow and T_c the explosion time. Then*

$$F : [0,T_c[\times M \quad \longrightarrow \quad \bigsqcup_{t\in[0,T_c[} M_t \ ,$$

is a diffeomorphism in the sense of manifold with boundary. And,

$$\begin{array}{rcl} \Psi : \mathcal{C}^f(]0,T_c],\mathbb{R}^n) & \longrightarrow & \mathcal{C}(]0,T_c],M) \\ \gamma & \longmapsto & t \mapsto F^{-1}(T_c - t, \gamma(t)) \end{array}$$

is continuous for the different Skorokhod topologies. To define the Skorokhod topology in $\mathcal{C}(]0,T_c],M)$ we could use the initial metric $g(0)$ on M.

proof : It is clear that F is smooth as a solution of a parabolic equation [14], and this result has been used above. Its differential is given at each point by:

$$\forall (t,x) \in [0,T_c[\times M, \quad \forall v \in T_xM$$

$$DF(t,x)(\frac{\partial}{\partial_t}, v) = \frac{\partial}{\partial_t}F(t,x) \oplus DF_t(x)(v)$$

where $\frac{\partial}{\partial_t}F(t,x) = -H(t,x)\overrightarrow{\nu}(t,x)$, here $\oplus$ stands for $+$ and means that we cannot cancel the sum without cancelling each term. Since there is no ambiguity we write $H(t,x)$ for $H_\nu(t,x)$. Recall that $H(t,x) > 0$.

For the second part of this lemma, we remark that for $0 \leq \delta < T_c$

$$F^{-1} : \bigsqcup_{t\in[0,\delta]} M_t \longrightarrow [0,\delta] \times M$$

is Lipschitz (use the bound of the differential on a compact).

Recall also that a sequence converges to a continuous function for Skorokhod topology if and only if it converges to this function locally uniformly. We will now show the continuity of Ψ. Take a sequence α_m in $\mathcal{C}^f(]0,T_c], \mathbb{R}^{n+1})$ and $\alpha \in \mathcal{C}^f(]0,T], \mathbb{R}^{n+1})$ such that $\alpha_m \longrightarrow \alpha$ for the Skorokhod topology.
Then for all A compact set in $]0,T_c]$, $\parallel \alpha_m - \alpha \parallel_A \longrightarrow 0$, where $\parallel f \parallel_A = \sup_{t\in A} \parallel f(t) \parallel$.

Let A be a compact set in $]0,T_c]$, then there exists a Lipschitz constant C_A of F^{-1} in $\bigsqcup_{t\in A} M_t$, such that for all t in A,

$$d_{g(o)}(F^{-1}(\alpha_m(t)), F^{-1}(\alpha(t))) \leq C_A \parallel \alpha_m(t) - \alpha(t) \parallel,$$

where $d_{g(o)}(x,y)$ is the distance in M beetwen x and y for the metric $g(0)$. We also define $d_{g(o),A}(f,g) = \sup_{t\in A} d_{g(o)}(f(t),g(t))$, where f, g are M-valued function. We get:

$$d_{g(o),A}(\Psi(\alpha_m), \Psi(\alpha) \leq C_A \parallel \alpha_m - \alpha \parallel_A .$$

So $\Psi(\alpha_m) \longrightarrow \Psi(\alpha)$ uniformly in all compact, so for the Skorokhod topology in $\mathcal{C}(]0,T_c], M)$. □

Let:

$$\tilde{Y}^\epsilon_t = (Y^\epsilon_t - Y^\epsilon_0) + (Y^\epsilon_0 \mathbb{1}_{[\epsilon,T_c]}(t) + \mathbb{1}_{[0,\epsilon]}(t)F(T_c - t, x_o)).$$

Proposition 2.2 gives the tightness of $Y^\epsilon_t - Y^\epsilon_0$, and $Y^\epsilon_0 \mathbb{1}_{[\epsilon,T_c]}(t) + \mathbb{1}_{[0,\epsilon]}(t)F(T_c - t, x_o)$ is a non-random sequence of functions that converges uniformly, hence $\tilde{Y}^\epsilon$ is tight. For strictly positive time t,

$$X^\epsilon_t = F^{-1}(T_c - t, \tilde{Y}^\epsilon_t).$$

The previous lemma 2.4 yields the tightness of X^ϵ. Hence we have shown that:

$$\forall \varphi = (\epsilon_k)_k \to 0, \quad \exists X^\varphi_{]0,T_c]}, \qquad X^{\epsilon_k}_{]0,T_c]} \xrightarrow{\mathcal{L}} X^\varphi_{]0,T_c]} \text{ for an extracted sequence.}$$

Proposition 2.5 *Let $\varphi = (\epsilon_k)_k \to 0$, and $X^{\varphi}_{]0,T_c]}$, s.t. $X^{\epsilon_k}_{]0,T_c]} \xrightarrow{\mathcal{L}} X^{\varphi}_{]0,T_c]}$. Then $X^{\varphi}_{]0,T_c]}$ is a $\frac{1}{2}g(T_c - t)$-BM in the following sense:*

$$\forall \epsilon > 0 \quad X^{\varphi}_{[\epsilon,T_c]} \stackrel{\mathcal{L}}{=} BM(\epsilon, X^{\varphi}_{\epsilon})$$

.

proof : Let $\epsilon > 0$ then for large k:

$$\begin{cases} X^{\epsilon_k} \text{ is a } BM(\epsilon, X^{\epsilon_k}_{\epsilon}) \text{ after time } \epsilon \text{ , by Markov property} \\ \\ \text{and let } X \text{ be a } BM(\epsilon, X^{\varphi}_{\epsilon}) \text{ after time } \epsilon \end{cases}$$

We want to show that $X = X^{\varphi}$ after ϵ . So for sketch of the proof:

$$X^{\epsilon_k} \xrightarrow[k\to\infty]{\mathcal{L}} X^{\varphi}$$

$$\text{so } X^{\epsilon_k}_{\epsilon} \xrightarrow[k\to\infty]{\mathcal{L}} X^{\varphi}_{\epsilon},$$

we use the Skorokhod theorem, to have a L_2-convergence in a larger probability space:

$$X^{'\epsilon_k}_{\epsilon} \xrightarrow[k\to\infty]{L_2,a.s.} X^{'\varphi}_{\epsilon},$$

with $X^{'\epsilon_k}_{\epsilon} \stackrel{\mathcal{L}}{=} X^{\epsilon_k}_{\epsilon}$ and $X^{'\varphi}_{\epsilon} \stackrel{\mathcal{L}}{=} X^{\varphi}_{\epsilon}$. We use convergence of solution of S.D.E with initial conditions converging in L_2 (e.g. in Stroock and Varadhan [22]), to get:

$$BM(\epsilon, X^{'\epsilon_k}_{\epsilon}) \xrightarrow[k\to\infty]{\mathcal{L}} BM(\epsilon, X^{'\varphi}_{\epsilon}),$$

$$BM(\epsilon, X^{'\epsilon_k}_{\epsilon}) \stackrel{\mathcal{L}}{=} X^{\epsilon_k}_{[\epsilon,T_c]},$$

$$BM(X^{'\varphi}_{\epsilon}) \stackrel{\mathcal{L}}{=} BM(\epsilon, X^{\varphi}_{\epsilon}).$$

We use that

$$X^{\epsilon_k} \xrightarrow[k\to\infty]{\mathcal{L}} X^{\varphi}$$

to conclude, after identification of the limit:

$$X = BM(\epsilon, X^{\varphi}_{\epsilon}) \stackrel{\mathcal{L}}{=} X^{\varphi}_{[\epsilon,T_c]}.$$

Hence the process X^{φ} is a $\frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM in the above sense, we call it "without birth" .

□

We now show that, in the sphere case, the $\frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM is, after a change of time, nothing else than a $BM(g(0))_{]-\infty,0]}$, this will give uniqueness in law of such process.

Proposition 2.6 *Let $g(t)$ be a family of metrics which comes from a mean curvature flow on the sphere. Then the $\tilde{g}(u) = \frac{1}{2}g(T_c - u)_{u\in]0,T_c]}$-BM is unique in law.*

proof : Let R_0 be the radius of the first sphere. Then $T_c = \frac{R_0^2}{2n}$, and by direct computation we obtain:

$$F(t,x) = \frac{\sqrt{R_0^2 - 2nt}}{R_0}x,$$

$$g(t) = \frac{R_0^2 - 2nt}{R_0^2}g(0).$$

So for all $f \in \mathcal{C}^\infty(S)$ we have:

$$\Delta_{g(t)}f = \frac{R_0^2}{R_0^2 - 2nt}\Delta_{g(0)}f$$

and

$$\begin{aligned}\nabla^{g(s)}df(X_i,X_j) &= f_{ij} - \Gamma_{ij}^k(s,.)f_k \\ &= f_{ij} - \Gamma_{ij}^k(0,.)f_k \text{ because the metrics are homothetic} \\ &= \nabla^{g(0)}df(X_i,X_j).\end{aligned}$$

Let X be a $\frac{1}{2}g(T_c-u)_{u\in]0,T_c]}$-BM. For all $f \in \mathcal{C}^\infty(S)$, $u \in]0,T_c]$ and for all $T_c > t \geq u$ we have:

$$\begin{aligned}f(X_t) - f(X_u) &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_u^t \Delta_{\tilde{g}(s)}f(X_s)ds \\ &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_u^t \nabla^{\tilde{g}(s)}df(*dX,*dX) \\ &\overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_u^t \nabla^{g(0)}df(*dX,*dX)\end{aligned}$$

$$df(X)_{]0,T_c]} \overset{d\mathcal{M}}{\equiv} \frac{1}{2}\nabla^{g(0)}df(*dX,*dX),$$

hence $X_{]0,T_c]}$ is a $g(0)$- martingale. From [4]:

$$df(X_t(x)) = \langle \nabla^{\tilde{g}(t)}f, /\!\!/_{0,t}v_i\rangle_{\tilde{g}(t)}dW^i + \frac{1}{2}\Delta_{\tilde{g}(t)}(f)(X_t(x))dt, \tag{2.2}$$

with abusive notation (because we have no starting point, to get sense we have to take the conditional expectation at a time before t).
It follows from (2.2):

$$df(X_t(x)) = \| \nabla^{\tilde{g}(t)}f(X_t(x)) \|_{\tilde{g}(t)} dB_t + \frac{1}{2}(\frac{R_0^2}{R_0^2 - 2n(T_c-t)})\Delta_0 f(X_t(x))dt,$$

where B_t is some real-valued Brownian motion. With help of the first computation,

$$df(X_t) = \sqrt{\frac{R_0^2}{nt}} \| \nabla^{g(0)}f(X_t) \|_{g(0)} dB_t + \frac{1}{2}(\frac{R_0^2}{nt})\Delta_0 f(X_t)dt.$$

Now consider the solution of:

$$\varphi'(t)R_0^2 = n(\varphi(t)) \text{ such that } \varphi(0) = T_c.$$

i.e. the function

$$\varphi(t) = T_c e^{\frac{t}{2T_c}}.$$

We get that $X_{\varphi(t)} = (BM_{g(0)})_t$. According to the usual characterization of a Brownian motion [7].

So by this deterministic change of time, and by the uniqueness in law of a $(BM_{g(0)})_{]-\infty,0]}$ on the sphere, we get the uniqueness in law of a $\frac{1}{2}g(T_c-u)_{u\in]0,T_c]}$-BM on a sphere. □

We have essentially used the conformality of this family of metric, that does not change the martingale family. Even if the beginning manifold is strictly convex, this is not the case in general. But we will see, in the next section, that the result is also true.

3 Kendall-Cranston Coupling

In this section the manifold M is compact and strictly convex. The goal in this section is to prove the uniqueness in law of the $g(T_c - t)$-BM. This section will be cut in two parts, the first will be a geometric result inspired by the work of Huisken, the second will be an adaptation of the Kendall-Cranston coupling. We will, by a deterministic change of time, transform a $g(T_c - t)$-BM (the existence of which comes from proposition 2.5) into a $\tilde{g}(t)_{]-\infty,0]}$-BM which has good geometric properties.

Remark : In the two last sections in [14], Huisken considers, like Hamilton for the Ricci flow, the normalized mean curvature flow. That consists in dilating the manifolds M_t by a coefficient to obtain constant volume manifolds. He obtains a positive coefficient of dilation $\psi(t)$ that satisfies the following property.

Theorem 3.1 *[14]*

For all $t \in [0, T_c[$, define $\tilde{F}(.,t) = \psi(t)F(.,t)$ such that $\int_{\tilde{M}_t} d\tilde{\mu}_t = |M_0|$, and $\tilde{t}(t) = \int_0^t \psi^2(\tau)d\tau$, then there exist several positive constants δ, C such that:

i) $\tilde{T}_c = \infty$

ii) $\tilde{H}_{max}(\tilde{t}) - \tilde{H}_{min}(\tilde{t}) \leq Ce^{-\delta\tilde{t}}$

iii) $|\frac{\partial}{\partial\tilde{t}}\tilde{g}_{ij}(\tilde{t})| \leq Ce^{-\delta\tilde{t}}$

iv) $\tilde{g}_{ij}(\tilde{t}) \to \tilde{g}_{ij}(\infty)$ *when* $\tilde{t} \to \infty$ *uniformly, for the* $C^\infty -$ *topology, and the convergence is exponentially fast.*

v) $\tilde{g}(\infty)$ is a metric such that $(M, \tilde{g}(\infty))$ is a sphere.

We will now give the change of time propositions.

Proposition 3.2 *Let $\psi : [0, T_c[\longrightarrow]0, \infty[$ as above, $\tilde{t}$ defined by:*

$$\begin{array}{rccc} \tilde{t} : [0, T_c[& \longrightarrow & [0, \infty[\\ t & \longmapsto & \int_0^t \psi^2(\tau) d\tau, \end{array}$$

for all $t \in [0, \infty[$, define

$$\tilde{g}(t) = \psi^2(\tilde{t}^{-1}(t)) g(\tilde{t}^{-1}(t)),$$

where $g(t)$ is the family of metrics coming from a mean curvature flow, and X_t is a $g(t)$-BM . Then:

$$t \longmapsto X_{\tilde{t}^{-1}(t)} \text{ is a } \tilde{g}(t)\text{-BM defined on } [0, \infty[.$$

proof :
Let $f \in \mathcal{C}^\infty(M)$:

$$\begin{array}{rcl} f(X_{\tilde{t}^{-1}(t)}) & \overset{\mathcal{M}}{\equiv} & \frac{1}{2} \int_0^{\tilde{t}^{-1}(t)} \Delta_{g(s)} f(X_s) ds \\ & \overset{\mathcal{M}}{\equiv} & \frac{1}{2} \int_0^t \Delta_{g(\tilde{t}^{-1}(s))} f(X_{\tilde{t}^{-1}(s)}) (\tilde{t}^{-1})'(s) ds \\ & \overset{\mathcal{M}}{\equiv} & \frac{1}{2} \int_0^t \Delta_{\frac{1}{(\tilde{t}^{-1})'(s)} g(\tilde{t}^{-1}(s))} f(X_{\tilde{t}^{-1}(s)}) ds. \end{array}$$

Using

$$\psi^2(\tilde{t}^{-1}(s))(\tilde{t}^{-1})'(s) = 1,$$

we obtain:

$$\frac{1}{(\tilde{t}^{-1})'(s)} g(\tilde{t}^{-1}(s)) = \tilde{g}(s).$$

□

Proposition 3.3 *Let $X_t^{T_c}$, with $t \in]0, T_c]$, be a $g(T_c - t)$-BM. Let τ be defined by:*

$$\begin{array}{rccc} \tau :]0, T_c] & \longrightarrow &]-\infty, 0] \\ t & \longmapsto & -\tilde{t}(T - t). \end{array}$$

Let $\tilde{g}(t)$ be defined by:

$$\tilde{g}(t) = \psi^2(T_c - \tau^{-1}(t)) g(T_c - \tau^{-1}(t)) \ \forall t \in]-\infty, 0].$$

Then:

$$t \mapsto X^{T_c}_{\tau^{-1}(t)} \text{ is a } \tilde{g}(t)\text{-BM.}$$

proof :
Let $f \in \mathcal{C}^\infty(M)$ and $s < t$,

$$\begin{aligned} f(X^{T_c}_{\tau^{-1}(t)}) - f(X^{T_c}_{\tau^{-1}(s)}) &\overset{\mathcal{M}}{\equiv} \tfrac{1}{2}\int_{\tau^{-1}(s)}^{\tau^{-1}(t)} \Delta_{g(T_c-u)} f(X^{T_c}_u)du \\ &\overset{\mathcal{M}}{\equiv} \tfrac{1}{2}\int_s^t \Delta_{g(T_c-\tau^{-1}(u))} f(X^{T_c}_{\tau^{-1}(u)})(\tau^{-1}(u))'(s)du \\ &\overset{\mathcal{M}}{\equiv} \tfrac{1}{2}\int_s^t \Delta_{\frac{1}{(\tau^{-1})'(u)}g(T_c-\tau^{-1}(u))} f(X^{T_c}_{\tau^{-1}(u)})du. \end{aligned}$$

We have $-\tilde{t}(T_c - \tau^{-1}(u)) = u$, and

$$(\tau^{-1})'(u)\psi^2(T_c - \tau^{-1}(u)) = 1.$$

We obtain

$$f(X^{T_c}_{\tau^{-1}(t)}) - f(X^{T_c}_{\tau^{-1}(s)}) \overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_s^t \Delta_{\psi^2(T_c-\tau^{-1}(u))g(T_c-\tau^{-1}(u))} f(X^{T_c}_{\tau^{-1}(u)})du$$

i.e.

$$f(X^{T_c}_{\tau^{-1}(t)}) - f(X^{T_c}_{\tau^{-1}(s)}) \overset{\mathcal{M}}{\equiv} \frac{1}{2}\int_s^t \Delta_{\tilde{g}(u)} f(X^{T_c}_{\tau^{-1}(u)})du.$$

□

Remark : By the above theorem 3.1, we know that $\tilde{g}(t)$ tends to a sphere metric as t goes to $-\infty$. The above proposition transforms "two" $g(T_c - t)$-BM into "two" $\tilde{g}$-BM so we will use the standardization of the metric into sphere metric and also the large time interval to perform the coupling.

Let τ_x be a plane in T_xM and $g(t)$ be a metric over M, we denote by $K(t, \tau_x)$ the sectional curvature of the plane τ_x according to the metric $g(t)$. We will now give a few geometric lemmas that will be used later, for simplicity we will take positive times.

Lemma 3.4 *Let $g(t)$ be a family of metrics on a manifold M, and $g(\infty)$ a metric that makes M into a sphere, suppose that:*

i) $g(t) \longrightarrow g(\infty)$ *uniformly, when* $t \longrightarrow \infty$ *for the* $C^\infty -$*topology exponentially fast, i.e.:* $\forall n \in \mathbb{N}, \forall$ *multi-indices* $(i_1, ..., i_k)$ *such that* $\sum i_k = n, \exists C_n, \delta_n > 0$, *such that:*

$$|\frac{\partial^n}{\partial X_{i_1}..X_{i_k}} g_{ij}(t) - \frac{\partial^n}{\partial X_{i_1}..X_{i_k}} g_{ij}(\infty)| \leq C_n e^{-\delta_n t}$$

ii) $\exists \delta, C^1 > 0$ *such that* $|\frac{\partial}{\partial t} g_{ij}(t)| \leq C^1 e^{-\delta t}$

iii) $vol_{g(t)}(M) = vol_{g(0)}(M)$

Then:

for all $\epsilon > 0$ *, there exists* $T \in [0, \infty[$, $\exists C, cst, cst_1 \in \mathbb{R}^+$ *and* $c_n(cst, V) > 0$ *such that,* $\forall t \in [T, \infty[$ *the following conditions are satisfied:*

i) for all x *in* M *and for all plane* $\tau_x \subset T_xM$, $\quad \mid K(t, \tau_x) - cst \mid \leq \epsilon$.

ii) $|\rho_t - \rho_\infty|_{M\times M} \leq cst_1 e^{-\delta t}$.

iii) $\rho'_t(x, y) := \frac{d}{dt}\rho_t(x, y) \leq C$ *in a compact CC of* $M \times M$,

where the constant cst, comes from the radius of M *with respect to* $g(\infty)$, $\rho_t(x, y)$ *is the distance between* x *and* y *for the metric* $g(t)$, *and*

$$CC = \{(x, y) \in M \times M, s.t. \quad \rho_t(x, y) \leq min(\frac{\pi}{2\sqrt{(cst + \epsilon)}}, c_n(cst, V)), \forall t > T\}.$$

proof :

Let us prove i).

Curvatures are functions of second order derivatives of the metric tensor. We give the definitions of curvatures tensors, to make this point clear. Conventions are as in [20],[18],[17], in particular, we use Einstein's summation convention.. For a metric connection without torsion (Levi-Civita connection), we recall standard definitions:

-the Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij})$$

-the (3,1) Riemann tensor:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

-the (4,0) curvature tensor:

$$R_m(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$$

-the sectional curvature:

$$K(X, Y) = \frac{R_m(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y \rangle^2}$$

We see that the sectional curvature depends on the metric and its derivatives up to order two, so $\forall x \in M$, for all plane $\tau_x \subset T_xM$,

$$\lim_{t\to\infty} K(t, \tau_x) = cst.$$

Also, for all $\epsilon > 0$, there exists T such that $\forall t > T$, for all x in M and for all plane $\tau_x \subset T_xM$,

$$\mid K(t, \tau_x) - cst \mid \leq \epsilon.$$

For the third point iii):
for $(x, y) \in CC$, where CC is defined above, we will show that we have the uniqueness of minimal $g(t)$-geodesic from x to y, for all time $t > T$, because we have the well-known Klingenberg's result (e.g. [13] page 158) about injectivity radius of compact manifold whose sectional curvature is bounded above. To use Klingenberg's lemma, we have to bound the shortest length of a closed geodesic. We will use Cheeger's theorem page 96 [3]. Since by the convergence of the metric, we have the convergence of the Ricci curvature, we obtain that they are bounded by the same constant. We obtain, using Myers' theorem that all diameters are then bounded above. The volumes are constant so bounded below, all sectional curvatures of M are bounded in absolute value from above. So by Cheeger's theorem there exists a constant $c_n(K, d, V) > 0$ that bounds the length of smooth closed geodesics. Hence, for large time, using Klingenberg's lemma, we get a uniform bound , in time, of the injectivity radius (i.e $min(\frac{\pi}{2\sqrt{(cst+\epsilon)}}, c_n(cst, V))$).

So for all $t > T$, there exist only one $g(t)$-geodesic between x and y, we denote it γ^t. Let $E(\gamma^t) = \int_0^1 \langle \dot{\gamma}^t(s), \dot{\gamma}^t(s) \rangle_{g(t)} ds$ be the energy of the geodesic where $\dot{\gamma}^t(s) = \frac{\partial}{\partial s}\gamma^t(s)$, $\rho_t^2(x, y) = E(\gamma^t)$. We compute:

$$\begin{aligned}
2(\tfrac{\partial}{\partial t}|_{t=t_0}\rho_t(x,y))(\rho_t(x,y)) &= \tfrac{\partial}{\partial t}|_{t=t_0} E(\gamma^t) \\
&= \int_0^1 \langle \dot{\gamma}^{t_0}(s), \dot{\gamma}^{t_0}(s) \rangle_{\frac{\partial}{\partial t}|_{t=t_0} g(t)} ds \\
&+ 2\int_0^1 \langle D_t|_{t=t_0} \tfrac{\partial}{\partial s}\gamma^t(s), \tfrac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)} ds \\
&= \int_0^1 \langle \dot{\gamma}^{t_0}(s), \dot{\gamma}^{t_0}(s) \rangle_{\frac{\partial}{\partial t}|_{t=t_0} g(t)} ds \\
&+ 2\int_0^1 \langle D_s \tfrac{\partial}{\partial t}|_{t=t_0}\gamma^t(s), \tfrac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)} ds
\end{aligned}$$

Let $X = \frac{\partial}{\partial t}|_{t=t_0}\gamma^t(s)$ be a vector field such that $X(x) = 0_{T_xM}, X(y) = 0_{T_yM}$, because we do not change the beginning and terminal point. The covariant derivative is computed with the Levi-Civita connection associated to $g(t_0)$. Hence we obtain:

$$\int_0^1 \langle D_s \frac{\partial}{\partial t}|_{t=t_0}\gamma^t(s), \frac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)} ds = \int_0^1 \langle \nabla_{\dot{\gamma}^{t_0}(s)} X, \frac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)} ds,$$

also:

$$\langle \nabla_{\dot{\gamma}^{t_0}(s)} X, \frac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)} = \frac{\partial}{\partial s}\langle X, \frac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)},$$

because the connection is metric and γ^{t_0} is a $g(t_0)$-geodesic. Hence

$$\int_0^1 \frac{\partial}{\partial s}\langle X, \frac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)} ds = [\langle X, \frac{\partial}{\partial s}\gamma^{t_0}(s) \rangle_{g(t_0)}]_0^1 = 0.$$

Finally, we obtain:

$$\frac{\partial}{\partial t}|_{t=t_0}\rho_t(x,y) = \frac{1}{2\rho_{t_0}(x,y)}\int_0^1 \langle \dot\gamma^{t_0}(s), \dot\gamma^{t_0}(s)\rangle_{\frac{\partial}{\partial t}|_{t=t_0}g(t)} ds. \tag{3.1}$$

We will now control the second term in the previous equation. By the exponential convergence of the metric, we could assume that the time is in the compact interval $[0,1]$. The manifold is compact, so we have a finite family of charts (indeed, we may assume that we have two charts, because the manifold has a metric which turns it into a sphere). The support of this chart could be taken to be relatively compact, and in this chart we can take the Euclidien metric i.e $\langle \partial_i, \partial_j\rangle_E = \delta_i^j$. This is not in general a metric on M. For the simplicity of expression, after taking the minimum over all charts we may assume that we just have one chart. Let S_1 be a sphere in $\mathbb{R}^n$ with the Euclidean metric. The functional:

$$\begin{array}{rcl} [0,1]\times S_1 \times M & \longrightarrow & \mathbb{R} \\ (t,v,x) & \longmapsto & g_{ij}(t,x)v_iv_j \end{array}$$

reaches its minimum $C>0$, so:

$$\|T\|_E \leq C^{-1}\|T\|_{g(t)}, \forall t\in[0,1], \forall T\in TM.$$

Hence, for the equation (3.1) we get the estimate:

$$\begin{array}{rcl} \left|\frac{\partial}{\partial t}|_{t=t_0}\rho_t(x,y)\right| & \leq & \frac{1}{2\rho_{t_0}(x,y)}C^1e^{-\delta t_0}\int_0^1 \left|\langle \dot\gamma^{t_0}(s),\dot\gamma^{t_0}(s)\rangle_E\right| ds \\ & \leq & \frac{1}{2\rho_{t_0}(x,y)}C^1(C)^{-1}e^{-\delta t_0}\int_0^1 \left|\langle \dot\gamma^{t_0}(s),\dot\gamma^{t_0}(s)\rangle_{g(t_0)}\right| ds \\ & \leq & \frac{1}{2}C^1(C)^{-1}e^{-\delta t_0}. \end{array}$$

This expression is clearly bounded.

For the second point ii),
let $x,y\in M$ take γ_∞ be a $g(\infty)$-geodesic that joins x to y. Then we have, on the one hand,

$$\begin{array}{rcl} \rho_t^2(x,y)-\rho_\infty^2(x,y) & \leq & \int_0^1\langle\dot\gamma_\infty(s),\dot\gamma_\infty(s)\rangle_{g(t)-g(\infty)}ds \\ & \leq & Cste^{-\delta t}\int_0^1\|\dot\gamma_\infty(s)\|^2_{g(\infty)}ds \\ & \leq & Cste^{-\delta t}diam^2_{g(\infty)}(M); \end{array}$$

where the constant changes and depends on the previous constant. On the other hand, we have:

$$\begin{array}{rcl} \rho_\infty^2(x,y)-\rho_t^2(x,y) & \leq & \int_0^1\langle\dot\gamma^t(s),\dot\gamma^t(s)\rangle_{g(\infty)-g(t)}ds \\ & \leq & Cste^{-\delta t}\int_0^1\|\dot\gamma^t(s)\|^2_{g(t)}ds \\ & \leq & Cste^{-\delta t}diam^2_{g(t)}(M) \\ & \leq & cst_1e^{-\delta t}, \end{array}$$

for some constant cst_1 , and we use Myers theorem for the last inequality to get a uniform upper bound of the diameter (because all Ricci curvature are uniformly bounded).
We get exponential convergence of the length.
□

We will now show uniqueness in law of a $g(T_c - t)$-BM. By proposition 3.3, this uniqueness is equivalent to uniqueness in law of a $\tilde{g}(t)_{]-\infty,0]}$-BM. This family of metrics, $\tilde{g}(t)$, satisfies:

$$\tilde{g}(t) \longrightarrow \tilde{g}(-\infty) \text{ for the } C^\infty\text{-topology.}$$

Let Z^1, Z^2 be two $\tilde{g}$-$\text{BM}_{]-\infty,0]}$ and $N << T$ where T is the time of the lemma 3.4, i.e the time up to which all bounds of the lemma are under control. Geometry before this time is similar to the geometry of the sphere. So the result of uniqueness in law for Brownian motion defined in a product probability space, indexed by $\mathbb{R}$ in a compact manifold (e.g. [6],[1]) could give the heuristics to our results. As we can see in [4] the $g(t)$-stochastic development and the $g(t)$-horizontal lift of a $g(t)$-BM is well defined.
We will consider a new process $Z^3_{N,t}$ equal in law to Z^2 after N and equal to Z^2 before. In the sequel we will note Z^3_t for $Z^3_{N,t}$. The construction, after time N, will be given by localization in a stochastic interval.

Let $T_0^N = N$, and for all $t \leq N$, $Z^3_{N,t} = Z^2_t$.
1) we will let Z^3_t evolve independently of Z^1_t i.e. Z^3_t is a $g(T_0^N + .)$-BM which starts at $Z^3_{T_0^N}$ and the $\mathbb{R}^n$-valued Brownian motion that drives Z^3_t will be independent with the one that drives Z^1_t.

Let $T_1^N = (N + \frac{1}{2}) \wedge inf\{t > T_0^N,\ \rho_t(Z^1_t, Z^3_t) \leq \frac{\frac{\pi}{\sqrt{cst+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4}\} \wedge T$. The constant ϵ is just taken to be small enough.

Let $C_N = inf\{t > N,\ Z^1_t = Z^3_t\}$.
2) At time T_1^N:

- if $\rho_{T_1^N}(Z^1_{T_1^N}, Z^3_{T_1^N}) \leq \frac{\frac{\pi}{\sqrt{cst+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4}$, these two points ($Z^3_{T_1^N}$ and $Z^1_{T_1^N}$) are close enough to make mirror coupling. The distance between these two points is strictly less than the injectivity radius $i_{g(t)}(M)$, hence we have uniqueness of the geodesic that joins these two points. After T_1^N and before C_N, we build Z^3_t as the $g(T_1^N + .)$-BM that starts at $Z^3_{T_1^N}$, and solves:

$$*dZ^3_t = U^3_t * d((U^3_t)^{-1} m^t_{Z^1_t, Z^3_t} U^1_t e_i dW^i_t)$$

and after C_N,

$$Z^3_t = Z^1_t, \quad C_N \leq t,$$

where U_t^3 is the horizontal lift of Z_t^3, to be correct we have to express a system of stochastic differential equations as in Kendall [19], U_t^1 is the horizontal lift of Z_t^1, and dW_t^i are Brownian motion that drives Z_t^1, the mirror map $m_{x,y}^t$ consists in transporting a vector along the unique minimal $g(t)$-geodesic that joins x to y and then reflecting it in the hyperplane of $(T_yM, g(t))$ which is perpendicular to the incoming geodesic.

By isometry property of the horizontal lift of the $g(t)$-BM (see [4]),

$$(U_t^3)^{-1} m_{Z_t^1, Z_t^3}^t U_t^1 dW_t^i,$$

is an $\mathbb{R}^n$-valued Brownian motion.

Let $T_2^N = (T_1^N + \frac{1}{2}) \wedge inf\{t > T_1^N,\ \rho_t(Z_t^1, Z_t^3) > \frac{\frac{\pi}{\sqrt{cst+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{2}\} \wedge T \wedge C_N$.

- if $\rho_{T_1^N}(Z^1_{T_1^N}, Z^3_{T_1^N}) > \frac{\frac{\pi}{\sqrt{cst+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4}$ then $T_2^N = T_1^N$.

Iterate step 1 and 2 successively (changing T_0^N by T_2^N and T_1^N by T_3^N in step 1, changing T_1^N by T_3^N and T_2^N by T_4^N in step 2 ..., after time T if we have no coupling, we let Z^3 evolve independently of Z_t^1 until the end), we build by induction the process Z_t^3 and a sequence of stopping times. We sketch it as:

- if $C_N < T$

$$T_0^N \overset{\text{independent}}{\longrightarrow} T_1^N \overset{coupling}{\longrightarrow} T_2^N \overset{independent}{\longrightarrow} T_3^N \overset{coupling}{\longrightarrow} T_4^N \ ...\ C_N \overset{Z_t^3 = Z_t^1}{\longrightarrow} 0$$

- if $C_N > T$

$$T_0^N \overset{\text{independent}}{\longrightarrow} T_1^N \overset{coupling}{\longrightarrow} T_2^N \overset{independent}{\longrightarrow} T_3^N \overset{coupling}{\longrightarrow} T_4^N \ ...\ T \overset{independent}{\longrightarrow} 0$$

Proposition 3.5 *The two processes Z^3 and Z^2 are equal in law.*

proof : It is clear that before N the two processes are equal so equal in law. After:
$Z_N^3 = Z_N^2$.

$$\begin{cases} *dZ_t^3 = \sum_i U_t^3 e_i * dB^i, \text{ when } t \in [T_{2k}^N, T_{2k+1}^N] \text{ and } T_{2k+1}^N \leq C_N \\ *dZ_t^3 = \sum_i U_t^3 * d((U_t^3)^{-1} m_{Z_t^1, Z_t^3}^t U_t^1) e_i dW_t^i, \text{ when } t \in [T_{2k+1}^N, T_{2k+2}^N], \text{ and } T_{2k+2}^N \leq C_N \\ Z_t^3 = Z_t^1, C_N \leq t \end{cases}$$

We write:

$$\begin{aligned} Z_t^3 &= \sum_{k=0}^{\infty} \mathbb{1}_{[T_k^N, T_{k+1}^N]}(t) * dZ_t^3 \\ &= \sum_{k:even} .... + \sum_{k:odd} \end{aligned}$$

Let $f \in C^\infty(M)$ then we have:
for even k:

$$df(\mathbb{1}_{[T_k^N, T_{k+1}^N]}(t) * dZ_t^3) \overset{d\mathcal{M}}{=} \frac{1}{2}\mathbb{1}_{[T_k^N, T_{k+1}^N]}(t)\Delta_{\tilde{g}(t)} f(Z_t^3)dt$$

for odd k:

$$df(\mathbb{1}_{[T_k^N, T_{k+1}^N]}(t) * dZ_t^3) \quad = \quad \frac{1}{2}\mathbb{1}_{[T_k^N, T_{k+1}^N]}\Delta_{\tilde{g}(t)} f(Z_t^3)dt$$

So Z^3 and Z^2 are two diffusions with the same starting distribution and the same generator, hence they are equal in law. For the gluing with Z^1 after C_N this is just the strong Markov property for (t, Z). □

Proposition 3.6 *There exists $\alpha > 0$ such that:*

$$\mathbb{P}(T_1^N - N < \frac{1}{2}) > \alpha$$

proof : By the C^∞-convergence of the metric we get:

$$\forall t < T, |\Delta_{\tilde{g}(t)} f - \Delta_{\tilde{g}(-\infty)} f| \leq \tilde{C} e^{\delta t}$$

where the constant comes from Theorem 3.1, and the derivative of f up to order two. We also obtain, by lemma 3.4, for a constant ϵ_2 that will be fixed below:

$$|\rho_t - \rho_{-\infty}| \leq \epsilon_2.$$

Over the sphere $(M, \tilde{g}(-\infty))$, we have by ordinary comparison theorem:

$$\Delta_{\tilde{g}(-\infty)} \rho_{-\infty}(x) \leq (n) cot(\rho_{-\infty}(x)).$$

We can suppose after normalization that the radius of the sphere $(M, \tilde{g}(-\infty))$ is one, $Radius_{-\infty}(M) = 1$ (i.e. $cst = 1$) in 3.4. We deduce from above that:

$$\Delta_{\tilde{g}(t)} \rho_{-\infty}(x) \leq (n) cot(\rho_{-\infty}(x)) + \tilde{C} e^{\delta t}.$$

In $[N, T_1^N[$, we have $\rho_t(Z_t^1, Z_t^3) > \frac{\frac{\pi}{\sqrt{1+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4}$ so:

$$\frac{\frac{\pi}{\sqrt{1+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4} - \epsilon_2 \leq \rho_t(Z_t^1, Z_t^3) - \epsilon_2 \leq \rho_{-\infty}(Z_t^1, Z_t^3) \leq \pi$$

We can choose ϵ, ϵ_2 such that, $\frac{\frac{\pi}{\sqrt{1+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4} - \epsilon_2 \geq \beta > 0$. We obtain:

$$cot(\rho_{-\infty}(Z_t^1, Z_t^3)) \leq cot(\beta),$$

and

$$\Delta_{\tilde{g}(t)} \rho_{-\infty}(Z_t^1, .)(Z_t^3) \leq (n) cot(\beta) + \tilde{C} e^{\delta T},$$

(recall $T << 0$) The progression of Z^3 and Z^1 are independent between $[N, T_1^N]$ hence:

$$(Z_t^1, Z_t^3) \text{ is a diffusion with generator } \frac{1}{2}(\Delta_{\tilde{g}(t),1} + \Delta_{\tilde{g}(t),2})$$

i.e.

$$d\rho_{-\infty}(Z_t^1, Z_t^3) = dM_t + \frac{1}{2}(\Delta_{\tilde{g}(t)} \rho_{-\infty}(Z_t^1, .)(Z_t^3) + \Delta_{\tilde{g}(t)} \rho_{-\infty}(., Z_t^3)(Z_t^1))dt$$

where M_t is a local martingale, so

$$d\rho_{-\infty}(Z_t^1, Z_t^3) \leq dM_t + (cot(\frac{\pi}{8}) + \tilde{C} e^{\delta T})dt.$$

.

Let us compute the quadratic variation of this local martingale, i.e:

$$d\langle M, M \rangle_t = d\rho_{-\infty}(Z_t^1, Z_t^3) d\rho_{-\infty}(Z_t^1, Z_t^3),$$

with:

$$d\rho_{-\infty}(Z_t^1, Z_t^3) = d\rho_{-\infty}(Z_t^1, .) * dZ_t^3 + d\rho_{-\infty}(., Z_t^3) * dZ_t^1. \tag{3.2}$$

Let $\gamma_{-\infty}(Z_t^3, Z_t^1)(s)$ be the minimal $\tilde{g}(-\infty)$-geodesic beetwen Z_t^3 and Z_t^1 that exists and is unique almost everywhere because $Cut_{-\infty}(M)$ is a null measure subspace. We denote:

$$v_t^1 = \frac{\dot{\gamma}_{-\infty}(Z_t^3, Z_t^1)(0)}{\|\dot{\gamma}_{-\infty}(Z_t^3, Z_t^1)(0)\|_{\tilde{g}(-\infty)}}.$$

We complete v_t^1 with v_t^j to get a $\tilde{g}(-\infty)$-orthonormal basis. We rewrite $*dZ_t^3$ as:

$$\begin{aligned} *dZ_t^3 &= \sum U_t^3 e_i * dB^i \\ &= \sum_{i,j} \langle U_t^3 e_i, v_t^j \rangle_{\tilde{g}(-\infty)} v_t^j * dB^i \end{aligned}$$

Hence by Gauss lemma, we obtain:

$$\begin{aligned} d\rho_{-\infty}(Z_t^1, .) * dZ_t^3 &= \sum d\rho_{-\infty}(Z_t^1, .) U_t^3 e_i * dB^i \\ &= \sum_{i,j} d\rho_{-\infty}(Z_t^1, .) \langle U_t^3 e_i, v_t^j \rangle_{\tilde{g}(-\infty)} v_t^j * dB^i \\ &= \sum_{i} d\rho_{-\infty}(Z_t^1, .) \langle U_t^3 e_i, v_t^1 \rangle_{\tilde{g}(-\infty)} v_t^1 * dB^i \\ &= \sum_{i} \langle U_t^3 e_i, v_t^1 \rangle_{\tilde{g}(-\infty)} * dB^i. \end{aligned}$$

It follows that:

$$(d\rho_{-\infty}(Z_t^1,.)*dZ_t^3)(d\rho_{-\infty}(Z_t^1,.)*dZ_t^3)=\sum_i\langle U_t^3e_i,v_t^1\rangle^2_{\tilde{g}(-\infty)}dt.$$

By the exponential convergence of the metric,

$$\langle U_t^3e_i,v_t^1\rangle_{\tilde{g}(-\infty)}\geq\langle U_t^3e_i,v_t^1\rangle_{\tilde{g}(t)}-\tilde{C}e^{\delta T},$$

hence:

$$\begin{aligned}
\sum_i\langle U_te_i,v_t^1\rangle^2_{\tilde{g}(-\infty)} &\geq \sum_i\langle U_te_i,v_t^1\rangle^2_{\tilde{g}(t)}-2\tilde{C}e^{\delta T}\sum_i\langle U_te_i,v_t^1\rangle_{\tilde{g}(t)}+n(\tilde{C}e^{\delta T})^2\\
&= \|v_t^1\|^2_{\tilde{g}(t)}-2\tilde{C}e^{\delta T}\sum_i\langle U_te_i,v_t^1\rangle_{\tilde{g}(t)}+n(\tilde{C}e^{\delta T})^2\\
&\geq \|v_t^1\|^2_{\tilde{g}(t)}-2\tilde{C}e^{\delta T}n\|v_t^1\|_{\tilde{g}(t)}+n(\tilde{C}e^{\delta T})^2 \text{ Schwartz}\\
&\geq (\|v_t^1\|_{\tilde{g}(-\infty)}-\tilde{C}e^{\delta T})^2-2\tilde{C}e^{\delta T}n(\|v_t^1\|_{\tilde{g}(-\infty)}+\tilde{C}e^{\delta T})\\
&+ n(\tilde{C}e^{\delta T})^2\\
&\geq 1-\tilde{C}e^{\delta T}(2-\tilde{C}e^{\delta T}+2(n+n\tilde{C}e^{\delta T})-n\tilde{C}e^{\delta T})\\
&\geq \frac{1}{2}\text{ for a small enough T.}
\end{aligned}$$

The independence of Z_t^1 et Z_t^3 gives,

$$\begin{aligned}
d\langle M_t,M_t\rangle &= (d\rho_{-\infty}(Z_t^1,.)*dZ_t^3)(d\rho_{-\infty}(Z_t^1,.)*dZ_t^3)\\
&\quad +(d\rho_{-\infty}(.,Z_t^3))*dZ_t^1)(d\rho_{-\infty}(.,Z_t^3))*dZ_t^1)
\end{aligned}$$

hence

$$d\langle M_t,M_t\rangle\geq 1dt.$$

For simplicity write $\theta=\frac{\frac{\pi}{\sqrt{1+\epsilon}}\wedge\frac{C_n(d,K,cst-\epsilon)}{2}}{4}$, it follows from (3.2) that:

$$\begin{aligned}
&\mathbb{P}(T_1^N-N<\frac{1}{2})\\
&=\mathbb{P}(\exists t\in[N,N+1/2]\text{ s.t. }\rho_t(Z_t^1,Z_t^3)\leq\theta\\
&\geq\mathbb{P}(\exists t\in[N,N+1/2]\text{ s.t. }\rho_{-\infty}(Z_t^1,Z_t^3)\leq\theta-\epsilon_2)\\
&\geq\mathbb{P}(\exists t\in[N,N+1/2]\text{ s.t. }\pi+M_t+(cot(\beta)+\tilde{C}e^{\delta T})(t-N)\leq\theta-\epsilon_2)\\
&\geq\alpha>0.
\end{aligned}$$

For the last step, we use the usual comparison theorem for stochastic processes (e.g. Ikeda and Watanabe [15]). □

We will now show that the coupling can occur between $[T_1^N, T_2^N]$ in a time smaller than $\frac{1}{2}$.

Proposition 3.7 *There exists $\tilde{\alpha} > 0$ such that:*

$$\mathbb{P}(C_N < (T_1^N + \frac{1}{2}) \wedge T_2^N) > \tilde{\alpha}.$$

proof : Between the two times T_1^N and T_2^N, we have mirror coupling between Z_t^1 and Z_t^3. As in [19, 5] we have:

$$d\rho_t(Z_t^1, Z_t^3) = \rho_t'(Z_t^1, Z_t^3)dt + 2d\beta_t + \frac{1}{2}\sum_{i=2}^{n} I^t(J_i^t, J_i^t)dt$$

$$*dZ_t^3 = U_t^3 * d((U_t^3)^{-1} m_{Z_t^1, Z_t^3}^t U_t^1 e_i dW_t^i),$$

Where:
-β_t is a standard real Brownian motion.
-$\gamma_t(Z_t^1, Z_t^3)(s)$ the minimal $\tilde{g}(t)$ geodesic between Z_t^1 and Z_t^3.
-$(\dot{\gamma}(Z_t^1, Z_t^3)(0), e_i(t))$ a $\tilde{g}(t)$-orthonormal basis of $T_{Z_t^1}M$.
- $J_i^t(s)$ the Jacobi field along γ_t for the metric $\tilde{g}(t)$, with initial condition $J_i^t(0) = e_i(t)$ and $J_i^t(\rho_t(Z_t^1, Z_t^3)) = /\!\!/_{\rho_t(Z_t^1, Z_t^3)}^{t, \gamma_t} e_i(t)$ i.e. the parallel transport for the metric $\tilde{g}(t)$ along γ_t, that is an orthogonal Jacobi field .
-I^t is the index bilinear form for the metric $\tilde{g}(t)$.
Between the times T_1^N and T_2^N, we have:

$$\rho_t(Z_t^1, Z_t^3) \leq \frac{\frac{\pi}{\sqrt{cst+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{2}$$

So by 3.4, there exists a constant C such that:

$$\rho_t'(x, y) \leq C.$$

We have to show that between the times T_1^N and T_2^N,

$$\sum_{i=2}^{n} I^t(J_i^t, J_i^t)$$

is bounded above. We note $r = \rho_t(Z_t^1, Z_t^3)$, and γ for γ^t. Let $G(s)$ be a real-valued function and K_i^t be the orthogonal vector field over γ defined by:

$$K_i^t(s) = G(s)(/\!\!/_t^{\gamma_t} e_i(t))(s)$$

where $G(0) = G(r) = 1$. We have:

$$\|\nabla^t_{\frac{\partial}{\partial s}} K^t_i(s)\|^2_{\tilde{g}(t)} = (\dot{G})^2.$$

By the index lemma (e.g. [20]), we deduce:

$$I^t(J^t_i, J^t_i) \leq I^t(K^t_i, K^t_i),$$

and

$$I^t(K^t_i, K^t_i) = \int_0^r \langle D_s K^t_i, D_s K^t_i \rangle_{\tilde{g}(t)} - R_{m,\tilde{g}(t)}(K^t_i, \dot{\gamma}, \dot{\gamma}, K^t_i) dt,$$

where $R_{m,\tilde{g}(t)}$ denote the $(4,0)$ curvature tensor associated to the metric $\tilde{g}(t)$. Hence:

$$\begin{aligned}
\sum_{i=2}^n I^t(K^t_i, K^t_i) &= \sum_{i=2}^n \int_0^r \langle D_s K^t_i, D_s K^t_i \rangle_{\tilde{g}(t)} - R_{m,\tilde{g}(t)}(K^t_i, \dot{\gamma}, \dot{\gamma}, K^t_i) ds \\
&= \sum_{i=2}^n \int_0^r \|\nabla^t_{\frac{\partial}{\partial s}} K_i(s)\|^2_{\tilde{g}(t)} - R_{m,\tilde{g}(t)}(K^t_i, \dot{\gamma}, \dot{\gamma}, K^t_i) ds \\
&= \int_0^r (n-1)(\dot{G})^2 - (G)^2 Ric_{\tilde{g}(t)}(\dot{\gamma}, \dot{\gamma}) ds \\
&\leq (n-1) \int_0^r ((\dot{G})^2 - (G)^2 (\frac{1-\epsilon}{n-1})) ds.
\end{aligned}$$

For performing the computation, we impose to G to satisfy the O.D.E:

$$\left\{ \begin{array}{l} G(0) = G(r) = 1 \\ \ddot{G} + (\frac{1-\epsilon}{n-1}) G = 0 \end{array} \right.$$

We notice that:

$$(\dot{G})^2 - (G)^2 (\frac{1-\epsilon}{n-1})) = (G\dot{G})',$$

and the solution of this O.D.E is given by the function:

$$G(s) = cos(\sqrt{\frac{1-\epsilon}{n-1}} s) + \frac{1 - cos(\sqrt{\frac{1-\epsilon}{n-1}} r)}{sin(\sqrt{\frac{1-\epsilon}{n-1}} r)} sin(\sqrt{\frac{1-\epsilon}{n-1}} s).$$

This function does not explode for r in $[0, \frac{\pi}{2\sqrt{\frac{1-\epsilon}{n-1}}}]$, and,

$$(\dot{G})(r) - (\dot{G})(0) = -2\sqrt{\frac{1-\epsilon}{n-1}}\, tan(\frac{\sqrt{\frac{1-\epsilon}{n-1}} r}{2}).$$

Hence

$$\sum_{i=2}^{n} I^t(J_i^t, J_i^t) \leq -2(n-1)\sqrt{\frac{1-\epsilon}{n-1}}\, tan(\frac{\sqrt{\frac{1-\epsilon}{n-1}}r}{2}) \leq 0.$$

We get:

$$d\rho_t(Z_t^1, Z_t^3) \leq Cdt + 2d\beta_t.$$

After conditioning by $\mathcal{F}_{T_1^N}$ we get the following computation:

$$\begin{aligned}
&\mathbb{P}(C_N < (T_1^N + \tfrac{1}{2}) \wedge T_2^N) \\
&= \mathbb{P}(\exists t \in [(T_1^N, (T_1^N + \tfrac{1}{2}) \wedge T_2^N] \text{ s.t. } \rho_t(Z_t^1, Z_t^3) = 0) \\
&\geq \mathbb{P}\Bigg(\exists t \in [0, \tfrac{1}{2}] \text{ s.t. } Ct + 2\beta_t + \frac{\frac{\pi}{\sqrt{1+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4} = 0 \\
&\text{ and } \sup_{0\leq s\leq t}(Cs + 2\beta_s + \frac{\frac{\pi}{\sqrt{1+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{4}) < \frac{\frac{\pi}{\sqrt{1+\epsilon}} \wedge \frac{C_n(d,K,cst-\epsilon)}{2}}{2} \Bigg) \\
&\geq \tilde{\alpha} > 0.
\end{aligned}$$

□

Remark : A better $\tilde{\alpha}$ could be found with a martingale of the type $e^{a\beta_t - \frac{a^2}{2}t}$.

Theorem 3.8 *Let (M, g) be a compact, strictly convex hypersurface isometrically embedded in $\mathbb{R}^{n+1}$, $n \geq 2$, and $(M, g(t))$ the family of metrics constructed by the mean curvature flow (as in 1.3). There exists a unique $g(T_c - t)$-BM in law.*

proof : Let X_t^1 and X_t^2 two $g(T_c - t)$-BM , by a deterministic change of time we get two $\tilde{g}(t)$-BM that we note Z_t^1 and Z_t^2. Let $N \leq T << 0$, as above we build $Z_{N,t}^3$, we obtain $Z_{N,t}^3 = Z_t^2$ in law. Let $\tilde{k} = E(T - N)$, where $E(t)$ is the integer part of t. We have by construction:

$$\mathbb{P}(\exists t \in [N, T], \text{ s.t. } Z_{N,t}^3 = Z_t^1) \geq \mathbb{P}(\exists t \in [T_0^N, T_{2\tilde{k}}^N], \text{ s.t. } Z_{N,t}^3 = Z_t^1).$$

Let $\mathcal{F}$ be the natural filtration generated by the two processes, by propositions 3.6, 3.7 and strong Markov property we obtain:

$$\begin{aligned}
&\mathbb{P}(\exists t \in [N, T_2^N] \text{ s.t. } Z_{N,t}^3 = Z_t^1) \\
&\geq \mathbb{P}(T_1^N < \tfrac{1}{2} + N; C_N < (T_1^N + \tfrac{1}{2}) \wedge T_2^N) \\
&= \mathbb{E}[\mathbb{P}(C_N \leq (T_1^N + \tfrac{1}{2}) \wedge T_2^N | \mathcal{F}_{T_1^N}) \mathbb{1}_{T_1^N \leq \frac{1}{2}+N}] \\
&\geq \tilde{\alpha}\mathbb{E}[\mathbb{1}_{T_1^N \leq \frac{1}{2}+N}] \\
&\geq \alpha\tilde{\alpha} > 0.
\end{aligned}$$

By successive conditioning (by $\mathcal{F}_{T_{2\tilde{k}-2}}$, ...) we get:

$$\mathbb{P}(\nexists t \in [T_0^N, T_{2\tilde{k}}^N] \text{ s.t. } Z_{N,t}^3 = Z_t^1) \leq (1 - \alpha\tilde{\alpha})^{\tilde{k}}.$$

Let $f_1...f_m \in \mathcal{B}_b(M)$ (bounded Borel functions) and $t < t_1 < ... < t_m \leq 0$,

$$\begin{aligned}
&|\mathbb{E}[f_1(Z^1_{t_1})...f_m(Z^1_{t_m}) - f_1(Z^2_{t_1})...f_m(Z^2_{t_m})]| \\
&\quad = |\mathbb{E}[f_1(Z^1_{t_1})...f_m(Z^1_{t_m}) - f_1(Z^3_{N,t_1})...f_m(Z^3_{N,t_m})]| \\
&\quad \leq \mathbb{E}[|f_1(Z^1_{t_1})...f_m(Z^1_{t_m}) - f_1(Z^3_{N,t_1})...f_m(Z^3_{N,t_m})|\mathbb{1}_{Z^1_t \neq Z^3_{N,t}}] \\
&\quad \leq 2\|f_1\|_\infty ... \|f_m\|_\infty \mathbb{P}(Z^1_t \neq Z^3_{N,t}) \\
&\quad = 2\|f_1\|_\infty ... \|f_m\|_\infty \mathbb{P}(\nexists u \in [N,t], \text{ s.t. } Z^1_u = Z^3_{N,u}) \\
&\quad \leq 2\|f\|_\infty ... \|f_m\|_\infty (1 - \alpha\tilde{\alpha})^{E(t-N)}
\end{aligned}$$

We get the result by sending N to $-\infty$. □

As application, we give uniqueness of a solution of a differential equation without initial condition.

Corollary 3.9 *Let (M, g) be a compact, strictly convex hypersurface isometrically embedded in $\mathbb{R}^{n+1}$, $n \geq 2$, and $(M, g(t))$ the family of metrics constructed by the mean curvature flow (as in 1.3). Then the following equation has a unique solution in $]0, T_c]$, where T_c is the explosion time of the mean curvature flow.*

$$\begin{cases} \frac{\partial}{\partial t} h(t,y) + H^2(T_c - t, y)h(t,y) = \frac{1}{2}\Delta_{g(T_c-t)} h(t,y) \\ \int_M h(T_c, y) d\mu_0 = 1 \end{cases} \tag{3.3}$$

proof : Existence: let $X^{T_c}_{]0,T_c]}$ be a $g(T_c - t)$-BM with law at time t, $h(t,y)d\mu_{T_c-t}$. Then the law satisfies the equation (3.3), it is a consequence of a Green formula (compare with the similar computation for the Ricci flow in [4] section 2).

Uniqueness: let $\tilde{h}$ be a solution of (3.3), and ν_k be a non-increasing sequence in $]0..T_c]$ such that $\lim_{k\to\infty} \nu_k = 0$. Take a M-valued random variable $\tilde{X}^{\nu_k} \sim \tilde{h}_{\nu_k} d\mu_{T_c-\nu_k}$, define the process:

$$\overline{X}^{\nu_k}_t = \begin{cases} \tilde{X}^{\nu_k} & \text{for } t \in]0..\nu_k] \\ g(T_c - t)\text{-BM}(\tilde{X}^{\nu_k}) & \text{for } t \in [\nu_k..T_c] \end{cases}$$

By the similar argument as in section 2, we deduce the tightness of the sequence $\overline{X}^{\nu_k}$, let $\overline{X}$ be a limit of a extracted sequence (also noted by ν_k). It is easy to see (by uniqueness of a solution of S.D.E, and of P.D.E with starting function) that $\overline{X}^{\nu_{k'}}_{(.)} \overset{\mathcal{L}}{=} \overline{X}^{\nu_k}_{(.)}$ for times greater than ν_k and $k' \geq k$. Sending k' to infinity, we obtain $\overline{X}_{(.)} \overset{\mathcal{L}}{=} \overline{X}^{\nu_k}_{(.)}$ for times greater than ν_k. Note also that for $t \geq \nu_k$

$$\overline{X}^{\nu_k}_{(.)} \overset{\mathcal{L}}{=} g(T_c - .)\text{-BM}(\overline{X}^{\nu_k}_t) \overset{\mathcal{L}}{=} g(T_c - .)\text{-BM}(\overline{X}_t).$$

Hence $\overline{X}$ is a $g(T_c - t)_{]0,T_c]}$ Brownian motion. For $t \geq \nu_k$ we have

$$\overline{X}_t \overset{\mathcal{L}}{=} \overline{X}^{\nu_k}_t \sim \tilde{h}_t d\mu_{T_c-t}.$$

By uniqueness in law of such process, we get the uniqueness of the solution, hence $h = \tilde{h}$. □

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