Combinatorial Hopf algebras from renormalization

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Abstract

In this paper we describe the right-sided combinatorial Hopf structure of three Hopf algebras appearing in the context of renormalization in quantum field theory: the non-commutative version of the Faà di Bruno Hopf algebra, the non-commutative version of the charge renormalization Hopf algebra on planar binary trees for quantum electrodynamics, and the non-commutative version of the Pinter renormalization Hopf algebra on any bosonic field.

We also describe two general ways to define the associative product in such Hopf algebras, the first one by recursion, and the second one by grafting and shuffling some decorated rooted trees.

1 Introduction

In the paper [13], J.-L. Loday and M. Ronco gave the definition of a right-sided combinatorial Hopf algebra (CHA) which includes the examples of Hopf algebras describing the renormalization in quantum field theory. The toy example, which inspired the work in [13], is Kreimer's Hopf algebra on rooted trees introduced in [9]. In this paper, we describe the CHA structure of three other such examples: the non-commutative version of the Faà di Bruno Hopf algebra given in [3], the non-commutative version of the charge renormalization Hopf algebra on planar binary trees given in [1] for quantum electrodynamics, and the non-commutative version of the Pinter renormalization Hopf algebra on any quantum fields, as given in [4].

Since Loday and Ronco work in the context of cofree-coassociative CHAs, we describe our examples, which are all free-associative CHAs, by considering their linear duals. A right-sided cofree-coassociative CHA is completely determined by the brace structure on the set its primitive elements. Therefore, to describe our examples, it will be sufficient to give their brace structure.

In the first section we fix the notation on right-sided cofree-coassociative CHAs and brace algebras from [13]. We also give a recursive definition of the product and show how to construct it starting from the one defined on decorated rooted trees, that is, the product defined on the right-sided CHA induced by the free brace algebra over the space of decorations. Then we fix the notation for the dual Hopf algebras, which are used in the sequel. Each example is then treated in a separate section.

2 Combinatorial Hopf algebras from dual Hopf algebras

2.1 Right-sided combinatorial Hopf algebras

In this section we recall the definition and main properties of a right-sided combinatorial Hopf algebra (r-s CHA), as given by J.-L. Loday and M. Ronco in [13], in the cofree-coassociative case.

A right-sided cofree-coassociative CHA is, up to an isomorphism, a tensor coalgebra $T(R) = \oplus_{n\geq 0} R^{\otimes n}$ with the deconcatenation coproduct, endowed with an associative product $\star$ which satisfies the right-sided condition: for any $p \geq 0$, the subspace $T^{\geq p}(R) = \oplus_{n\geq p} R^{\otimes n}$ is a right ideal of $T(R)$, that is

$$T^{\geq p}(R) \star T(R) \quad \subset \quad T^{\geq p}(R).$$

This condition is equivalent to demanding that the product $\star$ induces a right action of $T(R)$ on the quotient space $T_{<p}(R) = T(R)/T^{\geq p}(R)$, that is

$$T_{<p}(R) \star T(R) \quad \subset \quad T_{<p}(R). \tag{2.1}$$

Loday and Ronco proved that a cofree-coassociative CHA is right-sided if and only if the set of its primitive elements R is a brace algebra, that is, it is endowed with a brace product $\{\ ;\ ...,\ \}$ acting on $R \otimes T(R)$ with value in R, satisfying the brace relation

$$\big\{\{x; y_1, \dots, y_n\}; z_1, \dots, z_m\big\} = \sum \ \big\{x; z_1, \dots, \{y_1; z_{k_1}, \dots, \}, \dots, \{y_n; z_{k_n}, \dots\}, \dots, z_m\big\}, \tag{2.2}$$

for any $x, y_i, z_j \in R$. The brace product restricted to $R \otimes R^{\otimes q}$ is also denoted by M_{1q}, if one wishes to specify the number of variables on the right, or to underline that it is a special case of a multibrace product M_{pq} defined on any powers $R^{\otimes p} \otimes R^{\otimes q}$.

The brace product can be found from the associative product $\star$ by projecting the result onto the space R of co-generators. If we denote by $\pi : T(R) \longrightarrow R$ the canonical projection, this means that

$$M_{1q}(x; y_1 \cdots y_q) = \pi\big(x \star (y_1 \cdots y_q)\big),$$

for any $x \in R$ and $y_1 \cdots y_q \in R^{\otimes q}$, where we denote the tensor product in $T(R)$ by the concatenation. In particular, for any $x \in R$ we have

$$M_{10}(x; 1) \equiv M_{10}(x) = \pi(x \star 1) = x,$$

therefore $M_{10} = \mathrm{Id}$ on R.

Conversely, the product $\star$ in $T(R)$ can be reconstructed from the brace product on R, using the deconcatenation coproduct to recover $T(R)$ from R. The resulting formula of Loday and Ronco for the product is (cf. [13])

$$(x_1 \cdots x_n) \star (y_1 \cdots y_m) = \sum_{k\geq 1} \sum_{\substack{p_1+\cdots+p_k=n \\ p_1,\dots,p_k=0,1}} \sum_{\substack{q_1+\cdots+q_k=m \\ q_1,\dots,q_k\geq 0}} [M_{p_1q_1} \cdots M_{p_kq_k}](x_1 \cdots x_n; y_1 \cdots y_m), \tag{2.3}$$

where each map M_{pq} is applied to a block of p generators x and q generators y, with the following assumptions:

$$\begin{aligned} &M_{00} = 0 \qquad \text{and} \qquad M_{01} = M_{10} = \mathrm{Id}, \\ &M_{0p} = M_{p0} = 0 \qquad \text{for } p > 1, \\ &M_{pq} = 0 \qquad \text{for } p > 1 \text{ and } q \neq 0, \end{aligned} \tag{2.4}$$

and where the operators M_{1q} for $q \neq 0$ of course satisfy the brace relation (2.2).

Applying formula (2.3) requires some computations. Let us take the example of $x \star y$. For $k = 1$ we have $p_1 = q_1 = 1$ and the only term is M_{11}. For $k = 2$ we have $(p_1, p_2) = (1, 0)$ or $(p_1, p_2) = (0, 1)$ and $(q_1, q_2) = (1, 0)$ or $(q_1, q_2) = (0, 1)$. This gives us the four terms $M_{11}M_{00} + M_{10}M_{01} + M_{01}M_{10} + M_{00}M_{11}$.

Since $M_{00} = 0$, we have only two terms left $M_{10}M_{01} + M_{01}M_{10}$. For $k > 2$, all the terms are zero because each term contains a factor M_{00}. In general, the terms with $k > n + m$ are zero in the product $(x_1 \cdots x_n) \star (y_1 \cdots y_m)$. Therefore, using (2.4), we get

$$\begin{aligned} x \star y &= M_{11}(x; y) + [M_{10}M_{01}](x; y) + [M_{01}M_{10}](x; y) \\ &= M_{11}(x; y) + M_{10}(x)M_{01}(y) + M_{01}(y)M_{10}(x) = M_{11}(x; y) + xy + yx. \end{aligned}$$

Similarly, for $(xy) \star z$, we have two terms for $k = 2$: $M_{11}M_{10} + M_{10}M_{11}$ and three terms for $k = 3$: $M_{01}M_{10}M_{10} + M_{10}M_{01}M_{10} + M_{10}M_{10}M_{01}$, so that

$$(xy) \star z \quad = \quad xM_{11}(y; z) + M_{11}(x; z)y + zxy + xzy + xyz.$$

For $x \star (yz)$, we have one term for $k = 1$: M_{12}, two terms for $k = 2$: $M_{11}M_{01} + M_{01}M_{11}$ and three terms for $k = 3$: $M_{01}M_{01}M_{10} + M_{01}M_{10}M_{01} + M_{10}M_{01}M_{01}$, so that

$$x \star (yz) \quad = \quad M_{12}(x; yz) + yM_{11}(x; z) + M_{11}(x; y)z + yzx + yxz + xyz.$$

In particular then, formula (2.3) for $n = 1$ gives

$$x \star (y_1 \cdots y_m) = \sum_{i=0}^{m} \sum_{q=0}^{m-i} y_1 \cdots y_i \; M_{1q}(x; y_{i+1} \cdots y_{i+q}) \; y_{i+q+1} \cdots y_m. \tag{2.5}$$

2.2 Recursive definition of the product.

Formula (2.3) can be given in a recursive way. To do it, let us denote by $M : R \otimes T(R) \longrightarrow R$ the brace product $\{ \; ; \; , \ldots , \; \}$ induced on R by the associative product $\star$ on $T(R)$, and let us denote by $X, Y, Z...$ the generic words in $T(R)$.

Proposition 2.1 *Given the brace product M on R, the $\star$ product on $T(R)$ can be reconstructed in the following recursive way. For any $X, Y \in T(R)$, with $X, Y \notin T^0(R)$:*

$$\begin{aligned} X \star Y &= \sum_{\substack{X = X^1 X^2 \\ Y = Y^1 Y^2}} (X^1 \star Y^1) M(X^2; Y^2) \qquad (2.6) \\ &= \sum_{\substack{X = X^1 X^2 \\ Y = Y^1 Y^2}} M(X^1; Y^1)(X^2 \star Y^2) \\ &= \sum_{\substack{X = X^1 X^2 X^3 \\ Y = Y^1 Y^2 Y^3}} (X^1 \star Y^1) M(X^2, Y^2)(X^3 \star Y^3), \end{aligned}$$

where the sums run over all possible factorisations of X and Y with respect to the concatenation.

Proof. With the assumptions (2.4) on M, formula (2.3) can be expressed as

$$X \star Y = \sum_{k \geq 1} \sum_{\substack{X = X^1 \cdots X^k \\ Y = Y^1 \cdots Y^k}} M(X^1; Y^1) \cdots M(X^k; Y^k), \tag{2.7}$$

where the sum runs other all the factorisations (for the concatenation) of $X = X^1 \cdots X^k$ and $Y = Y^1 \cdots Y^k$. Note that the factors can be in $T^0(R)$ but the condition $M(1; 1) = 0$ ensures that the sum is finite. It also ensures that formula (2.6) is recursive.

If we then use this formula to express the factor $X^1 \star Y^1$ in line (2.6), we obtain the desired expression for $X \star Y$. □

In particular, if we expand the words X and Y of $T(R)$ in terms of elements x_i and y_j of R, with the notation of Section 2.1, formula (2.6) becomes

$$\begin{aligned} (x_1 \cdots x_n) \star (y_1 \cdots y_m) = \big[(x_1 \cdots x_{n-1}) \star (y_1 \cdots y_m)\big] x_n + \big[(x_1 \cdots x_n) \star (y_1 \cdots y_{m-1})\big] y_m \\ + \sum_{i=1}^{m-1} \big[(x_1 \cdots x_{n-1}) \star (y_1 \cdots y_i)\big] M_{1m-i}(x_n; y_{i+1} \cdots y_m). \end{aligned} \tag{2.8}$$

Corollary 2.2 *The brace relation (2.2) is equivalent to the identity*

$$M(M(X;Y);Z) = M(X;Y \star Z), \tag{2.9}$$

for any $X \in R$ and $Y, Z \in T(R)$.

Proof. If we call $X = x$, $Y = y_1 \cdots y_n$ and $Z = z_1 \cdots z_m$, the left hand-side of the brace relation (2.2) is simply $M(M(X;Y);Z)$. The right-hand side, instead, becomes

$$\sum_{k \geq 1} \sum_{\substack{Y = Y^1 \cdots Y^k \\ Z = Z^1 \cdots Z^k}} M(X; M(Y^1; Z^1) \cdots M(Y^k; Z^k)),$$

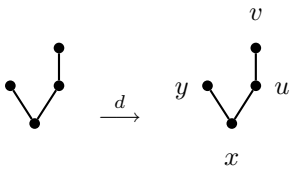
where we use the assumptions (2.4) on M. From (2.7) it then follows that the right-hand side is equal to $M(X; Y \star Z)$. □

Remark 2.3 The right-sided combinatorial Hopf algebras, as well as the shuffle and the quasishuffle Hopf algebras, are examples of more general cofree-coassociative combinatorial Hopf algebras $T(R)$ introduced by Loday and Ronco, where R is a so-called multibrace algebra. See [13] and the references therein for details. In all these examples, as well as in the general case, the product $\star$ on $T(R)$ can be defined recursively from the multibrace product on R. The expressions (2.7), (2.6) and (2.9), with $X \in T(R)$, are valid in the most general case as well as in all the examples.

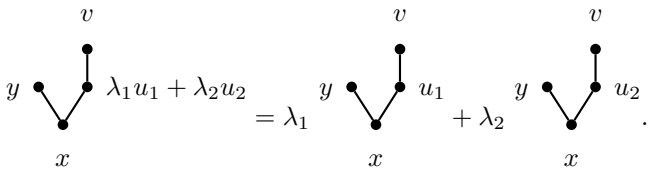
2.3 Right-sided Hopf algebras and trees

Let $T(R)$ be a right-sided CHA such that the brace algebra R is a finite dimensional or a graded vector space. In the first part of this section we forget the brace structure on R and construct the free brace algebra on R using planar rooted trees. In the second part we relate the free brace structure on R with the original one, and deduce a way to describe the associative product $\star$ on $T(R)$ starting from the one defined on trees.

Let $\mathcal{T}$ denote the set of plane rooted trees and for any non-empty tree t denote by $V(t)$ the set of its vertices and by $|t|$ the cardinality of $V(t)$. If d is a map from $V(t)$ to $R^{\otimes |t|}$, then the pair (t, d) is a plane tree with decorations in R. For example, the following tree with 4 vertices is decorated by the given map d:



For any $t \in \mathcal{T}$, we denote by R^t the linear span of all the decorated trees of shape t, modulo the obvious linear relations coming from R, such as:



Then the vector space

$$R^{\mathcal{T}} = \bigoplus_{t \in \mathcal{T}} R^t$$

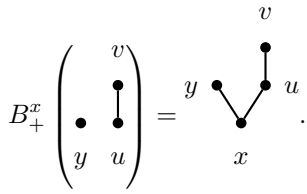
contains the decorated trees of any shape, and the elements of the tensor space $T(R^{\mathcal{T}})$ are called decorated forests. From now on, we denote by the concatenation the tensor product between trees, and we simply denote by t a decorated tree, omitting the decoration.

We now define the structure of a brace algebra on $R^{\mathcal{T}}$, and consequently that of a right-sided CHA on $T(R^{\mathcal{T}})$, using the results of section 2.2. Therefore, we endow $T(R^{\mathcal{T}})$ with the free (deconcatenation) coproduct and we suppose that the product $\star$ is defined through the brace product M on $R^{\mathcal{T}}$.

To define the brace product, consider, for any $x \in R$, the linear map

$$B_+^x : T(R^{\mathcal{T}}) \longrightarrow R^{\mathcal{T}}$$

which maps a forest $f = t_1 \cdots t_n$ into the tree $t = B_+^x(f)$ obtained by joining the ordered trees $t_1, ..., t_n$ to a new root decorated by x. For instance

B_+^x 

Any tree t can be written as $B_+^x(f)$. If t is made only of the root, decorated by x, the forest f is the formal unit of $T(R^{\mathcal{T}})$, that is $1 \in (R^{\mathcal{T}})^0$.

Theorem 2.4 *Let $M : R^{\mathcal{T}} \otimes T(R^{\mathcal{T}}) \longrightarrow R^{\mathcal{T}}$ be the linear map defined on any decorated tree $t = B_+^x(f_1)$ and any decorated forest f_2 by*

$$M(B_+^x(f_1); f_2) = B_+^x(f_1 \star f_2), \tag{2.10}$$

where $\star$ is the product induced by M on $T(R^{\mathcal{T}})$ according to Eq. (2.7) and with the assumptions (2.4).

Then M is a brace product on $R^{\mathcal{T}}$, and consequently $\star$ is an associative product on $T(R^{\mathcal{T}})$ which makes it into a right-sided CHA.

Proof. Assuming conditions (2.4) on the maps M_{pq}, it suffices to show that M satisfies the brace identity (2.9), that is

$$M(M(t_1; f_2); f_3) = M(t_1; f_2 \star f_3)$$

for any tree t_1 and any forests f_2, f_3. If t_1, f_2, f_3 have respectively n_1, n_2 and n_3 vertices, we prove the brace identity by induction on the total number of vertices $n_1 + n_2 + n_3$.

Since t_1 is a tree, it has at least one vertex (the root) and therefore $n_1+n_2+n_3 \geq 1$. If $n_1+n_2+n_3 = 1$, only t_1 is a tree, namely the single root, and f_2, f_3 are scalars. Because $1 \star 1 = 1$ and $M_{10} = M_{01} = \mathrm{Id}$, both terms in the equality produce t_1 and therefore coincide.

Now suppose that the brace identity holds for $n_1 + n_2 + n_3 \leq n$, for a given $n \geq 1$. According to the previous sections, this implies that the product $\star$ is associative up to the same number n of vertices. (In fact, the associativity of $\star$ does not depend on the definition (2.10) of M and was proved in general by Loday and Ronco.) Then consider a tree and two forests with a total number of vertices $n_1 + n_2 + n_3 = n + 1$. Using the definition of M and the associativity of $\star$ up to n, for $t_1 = B_+^x(f_1)$ we obtain

$$\begin{aligned} M(M(B_+^x(f_1); f_2); f_3) &= M(B_+^x(f_1 \star f_2); f_3) \\ &= B_+^x((f_1 \star f_2) \star f_3) \qquad \text{(Recursion)} \\ &= B_+^x(f_1 \star (f_2 \star f_3)) \\ &= M(B_+^x(f_1); f_2 \star f_3). \end{aligned}$$

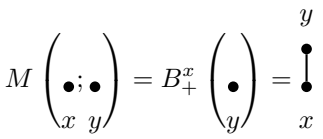
□

Lemma 2.5 *The brace product M on $R^{\mathcal{T}}$ can be expressed in terms of graftings of trees: if $t \in R^{\mathcal{T}}$ and $s_1 \cdots s_n \in T(R^{\mathcal{T}})$, then $M(t; s_1 \cdots s_n)$ is the sum of the terms obtained by grafting the trees $s_1, ...s_n$ onto the vertices of t, in all the possible ways which preserve the order of the s_i's from left to right.*

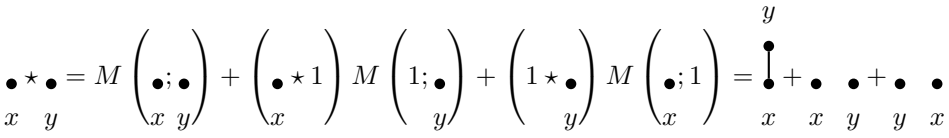
Similarly, the product $\star$ on $T(R^{\mathcal{T}})$ is the sum of terms mixing all possible shuffles and all possible graftings of the trees on the right-hand side onto the vertices of the trees on the left hand-side.

Proof. We prove it by induction on the size of the forests, given by the total number of vertices. For this, we need to treat at the same time M and $\star$, which is given by (2.6).

The induction starts with the first non-trivial operations:



is the grafting of the second $\bullet$ on the first one. The claim is obtained because this is the only possible grafting. Similarly, according to (2.6), the product



is exactly made of the only possible grafting and the two allowed shuffles.

Now suppose that $M(t_1; f_2)$ is made of all possible ordered graftings of the trees composing the forest f_2 on the vertices of the tree t_1, for t_1 and f_2 having a total number of vertices $n_1 + n_2 \leq n$, for some $n \geq 1$. Similarly, suppose that $f_1 \star f_2$ is made of the terms mixing all possible shuffles of the trees of f_1 with those of f_2, and of graftings of the trees of f_2 over the vertices of those contained in f_1, for f_1 and f_2 having a total number of vertices $n_1 + n_2 \leq n$.

Then, consider t_1 and f_2 with $n_1 + n_2 = n + 1$. Applying the definition (2.10) for $t_1 = B_+^x(f_1)$, we see that

$$M(B_+^x(f_1); f_2) = B_+^x(f_1 \star f_2)$$

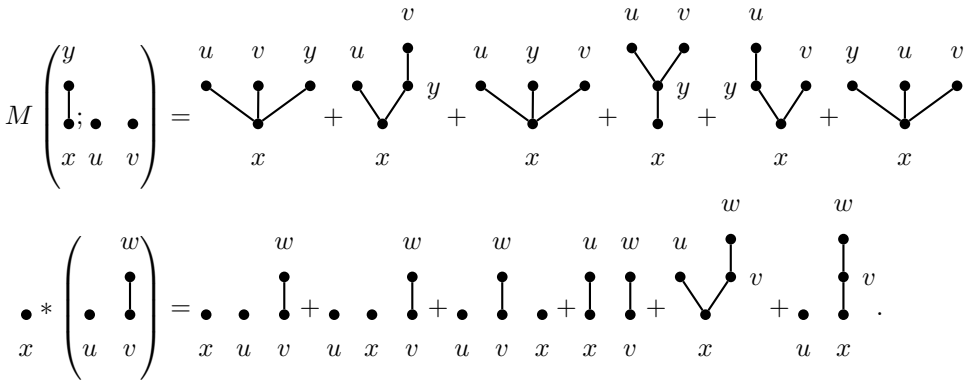
is the grafting of $f_1 \star f_2$ onto the original root. The claim is then guaranteed by the inductive hypothesis on $f_1 \star f_2$.

Similarly, consider t_1 and f_2 with $n_1 + n_2 = n + 1$. Applying formula (2.6) we see that

$$f_1 \star f_2 = \sum_{\substack{f_1 = f_1' f_2'' \\ f_2 = f_1' f_2''}} (f_1' \star f_2') M(f_1''; f_2'')$$

contains exactly all possible shuffles mixed with all possible graftings. This is guaranteed by the inductive hypothesis on the factors $f_1' \star f_2'$ and $M(f_1''; f_2'')$, and the fact that the sum runs over all decompositions of f_1 and f_2 into two factors, including the trivial ones. □

Here are some examples of graftings and shuffles:



According to the results of Loday and Ronco, in particular paragraph 3.14 in [13], we can conclude that

Corollary 2.6 *The brace algebra $R^{\mathcal{T}}$, endowed with M, is the free brace algebra over R.*

Remark 2.7 The algebras $R^{\mathcal{T}}$ and $T(R^{\mathcal{T}})$, or their duals, or their non-planar and commutative versions, appeared in really many recent works. We quote, but the list is surely not exhaustive, the works by R. Grossman and R. G. Larson [7], F. Chapoton [5], L. Foissy [6], D. Guin et J.-M. Oudom [8], M. Ronco [15] and J.-L. Loday and M. Ronco [13].

Now suppose that R itself is a brace algebra, and therefore $T(R)$ is a right-sided CHA. We are going to show that the $\star$ product defined in $T(R)$ can be found by computing its lifting to $T(R^{\mathcal{T}})$.

Let ι be the linear morphism from $T(R)$ to $T(R^{\mathcal{T}})$ defined by $\iota(1) = 1$ and

$$\iota(x_1 \cdots x_n) = B_+^{x_1}(1) \cdots B_+^{x_n}(1) = \underset{x_1}{\bullet} \cdots \underset{x_n}{\bullet}.$$

It is clear that ι preserves the concatenation in $T(R)$ and $T(R^{\mathcal{T}})$, and that it identifies R with $R^{\bullet} \subset R^{\mathcal{T}}$.

Note that the map ι does not preserve the brace product, in fact

$$M\left(\underset{x}{\bullet}; \underset{y_1}{\bullet} \cdots \underset{y_q}{\bullet}\right) = \underset{x}{\overset{y_1}{\bullet}} \cdots \underset{x}{\overset{y_q}{\bullet}} \neq \iota(M(x; y_1 \cdots y_q)) = \underset{M(x; y_1 \cdots y_q)}{\bullet},$$

and therefore it does not preserve the product $\star$ in $T(R)$ and $T(R^{\mathcal{T}})$. We show that ι has an inverse map which allows to compute the product $\star$ in $T(R)$ starting from that in $T(R^{\mathcal{T}})$.

Theorem 2.8 *Let μ the linear morphism from $T(R^{\mathcal{T}})$ to $T(R)$ defined recursively by*

1. $\mu(1) = 1$,
2. $\mu(t_1 \dots t_n) = \mu(t_1) \dots \mu(t_n)$,
3. $\mu\big(B_+^x(s_1 \dots s_q)\big) = M_{1q}\big(x; \mu(s_1 \dots s_q)\big)$.

Then, for any $X, Y \in T(R)$, we have

$$X \star Y = \mu\big(\iota(X) \star \iota(Y)\big).$$

Proof. The proof is straightforward and relies on the recursive formulas in brace algebras and on the fact that $\mu \circ \iota = Id_{T(R)}$. If $X = x_1 \cdots x_p$ and $Y = y_1 \cdots y_q$ then

$$X \star Y = \sum_{\substack{X = X^1 X^2 \\ Y = Y^1 Y^2}} (X^1 \star Y^1) M(X^2; Y^2)$$

Since R is a brace algebra, either $X^2 = 1$ or $X^2 = x_p$.

- If $X^2 = 1$, then
$$M(1, Y^2) = Y^2 = \mu(\iota(Y^2)) = \mu\big(M(1; \iota(Y^2)\big).$$
- If $X^2 = x_p$, then
$$M(x_p, Y^2) = \mu(B_+^{x_p}(\iota(Y^2))) = \mu\big(M(\iota(x_p); \iota(Y^2)))\big).$$

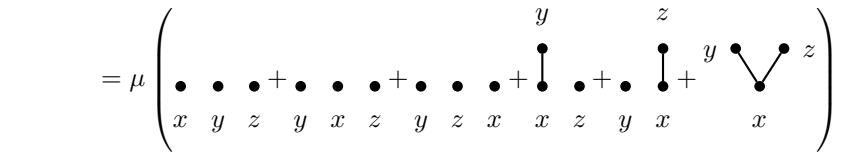
Recursively, we get

$$\begin{aligned} X \star Y &= \sum_{\substack{X = X^1 X^2 \\ Y = Y^1 Y^2}} (X^1 \star Y^1) M(X^2; Y^2) \\ &= \sum_{\substack{X = X^1 X^2 \\ Y = Y^1 Y^2}} \mu\big(\iota(X^1) \star \iota(Y^1)\big) \mu\big(M(\iota(X^2); \iota(Y^2)))\big) \\ &= \mu\left(\sum_{\substack{\iota(X) = \iota(X^1)\iota(X^2) \\ \iota(Y) = \iota(Y^1)\iota(Y^2)}} \big(\iota(X^1) \star \iota(Y^1)\big) M\big(\iota(X^2); \iota(Y^2)\big) \right) \\ &= \mu\big(\iota(X) \star \iota(Y)\big). \end{aligned}$$

□

This result means that one can compute the $\star$ product in $T(R)$ with the help of the $\star$ product in $T(R^{\mathcal{T}})$. This last product, given by shuffle and grafting of trees, is quite easy to handle. Let us illustrate the application with an example:

$$x \star (yz) = \mu\left(\begin{array}{c} \bullet \\ x \end{array} \star \left(\begin{array}{cc} \bullet & \bullet \\ y & z \end{array}\right)\right)$$



$$= xyz + yxz + yzx + M_{11}(x;y)z + yM_{11}(x;z) + M_{12}(x;yz).$$

This computation can be compared with that made in section 2.1.

The algebra $T(R^{\mathcal{T}})$ is the dual of Foissy's Hopf algebra of decorated plane trees, cf. [6]. Foissy's Hopf algebra on rooted trees has a "universal property", in the sense that it is endowed with the Hochschild cocycle operator B_+. Note that the property that we show for $T(R^{\mathcal{T}})$ is not the dual property of that one.

2.4 Right-sided CHAs from dual Hopf algebras

In the next sections we determine some right-sided CHAs $T(R)$ from the dualization of given Hopf algebras $T(V)$ where the vector space V is itself graded. In this case, the vector space R coincides with the dual space V^*, and we give a general tool to compute the brace structure on R starting from the coproduct on $T(V)$. This tool is then used in the examples presented in the rest of the paper.

Let $\mathcal{H}$ be a graded Hopf algebra that is free as an algebra, finite dimensional in all degrees, and with generators which are themselves graded. If we denote by V be the vector space spanned by all the generators of $\mathcal{H}$ (infinite but countable many), the Hopf algebra can be given as $\mathcal{H} = T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$.

The dual Hopf algebra $\mathcal{H}^*$ is defined as follows. As a vector space, it is the graded linear dual vector space of $\mathcal{H}$, that is $\mathcal{H}^* = T(V^*)$, where V^* is the graded linear dual vector space of V. If we denote by $\{v_n, n \geq 1\}$ a generic basis of V, that is a set of generators of $\mathcal{H}$, the element v_n^* dual to each v_n is defined by the pairing $\langle v_n^* | v_m \rangle = \delta_{n,m}$. Therefore, the dual of a generic basis element $v_{n_1} \cdots v_{n_i}$ in $\mathcal{H}$ is the element $v_{n_1}^* \cdots v_{n_i}^*$ in $\mathcal{H}^*$ defined by the pairing

$$\langle v_{n_1}^* \cdots v_{n_i}^* | v_{m_1} \cdots v_{m_j} \rangle \quad = \quad \delta_{i,j} \langle v_{n_1}^* | v_{m_1} \rangle \ldots \langle v_{n_i}^* | v_{m_i} \rangle.$$

As a coalgebra, $\mathcal{H}^*$ inherits the coproduct Δ^* dual to the product of $\mathcal{H}$, that is, such that

$$\left\langle \Delta^*(v_{n_1}^* \cdots v_{n_i}^*) | (v_{m_1} \cdots v_{m_j}) \otimes (v_{p_1} \cdots v_{p_k}) \right\rangle = \left\langle v_{n_1}^* \cdots v_{n_i}^* | v_{m_1} \cdots v_{m_j} v_{p_1} \cdots v_{p_k} \right\rangle.$$

Since $\mathcal{H}$ is free and unital as an algebra, $\mathcal{H}^*$ is cofree and counital as a coalgebra, that is, Δ^* is the deconcatenation coproduct

$$\Delta^*(v_n^*) = v_n^* \otimes 1 + 1 \otimes v_n^*,$$

$$\Delta^*(v_{n_1}^* \cdots v_{n_k}^*) = \sum_{i=0}^{k} v_{n_1}^* \cdots v_{n_i}^* \otimes v_{n_{i+1}}^* \cdots v_{n_k}^*,$$

and with the counit $\varepsilon = 1^*$. In this expression, 1 is not (yet) a unit, it denotes the generator of $(V^*)^{\otimes 0}$. The set of primitive elements of $\mathcal{H}^*$ is then obviously the dual V^* of the set of generators of $\mathcal{H}$.

Finally, as an algebra, $\mathcal{H}^*$ inherits the product $\star$ dual to the coproduct Δ of $\mathcal{H}$, that is, the product such that

$$\left\langle (v_{n_1}^* \cdots v_{n_i}^*) \star (v_{m_1}^* \cdots v_{m_j}^*), v_{p_1} \cdots v_{p_k} \right\rangle = \left\langle v_{n_1}^* \cdots v_{n_i}^* \otimes \ v_{m_1}^* \cdots v_{m_j}^*, \Delta(v_{p_1} \cdots v_{p_k}) \right\rangle.$$

The right-sided condition (2.1) for the Hopf algebra $T(V^*)$ is equivalent to require that the coproduct Δ on $T(V)$ induces a right coaction of the Hopf algebra $T(V)$ on the subspace $T^{\leq p}(V) = \oplus_{n\leq p} V^{\otimes n}$, that is

$$\Delta\big(T^{\leq p}(V)\big) \quad \subset \quad T^{\leq p}(V) \otimes T(V). \tag{2.11}$$

Explicitly, this means that for any element $a = v_{n_1} \cdots v_{n_p} \in V^{\otimes p}$, the coproduct $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ (in Sweedler's notation) produces terms with $a_{(1)} \in T^{\leq p}(V)$. Since the coproduct is an algebra morphism, and the product is free, it is sufficient that this property holds for $p = 1$: for any generator $v \in V$, the coproduct $\Delta(v) = \sum v_{(1)} \otimes v_{(2)}$ produces terms with $v_{(1)} \in T^{\leq 1}(V) = K \oplus V$ (cf. [13], Section 3.6).

Finally, the brace product on V^* is the projection of $\star$ onto V^*. The projection $\pi : T(V^*) \longrightarrow V^*$, in this case, is the map

$$\pi(a^*) = \sum_m \langle a^* | v_m \rangle \ v_m^*,$$

for any $a^* \in T(V^*)$. Therefore, the maps $M_{1q} : V^* \otimes (V^*)^{\otimes q} \longrightarrow V^*$ are simply given by

$$M_{1q}(v_n^*; a^*) = \sum_m \langle v_n^* \star a^* | v_m \rangle \ v_m^* = \sum_m \langle v_n^* \otimes a^* | \Delta v_m \rangle \ v_m^*. \tag{2.12}$$

This sum if finite. In fact, let us denote by $|a|$ [resp. $|a^*|$] the degree of the elements in the graded Hopf algebra $T(V)$ [resp. $T(V^*)$]. We recall that if $|v_n|$ denotes the degree of a generic basis element of V, then the degree of an element $v_{n_1} \cdots v_{n_p} \in T(V)$ is given by $|v_{n_1} \cdots v_{n_p}| = |v_{n_1}| + \cdots + |v_{n_p}|$. Then the sum over m in Formula (2.12) is limited by the fact that $|v_m^*| = |v_m| = |v_n^*| + |a^*|$.

3 The right-sided combinatorial structure of the Hopf algebra $(\mathcal{H}^{\mathrm{dif,nc}})^*$

Let $\mathcal{H}^{\mathrm{dif,nc}}$ be the non-commutative Hopf algebra of formal diffeomorphisms, as defined in [16] or [3] (where it was denoted by $\mathcal{H}^{\mathrm{dif}}$). It is the graded and connected Hopf algebra $\mathbb{Q}\langle v_1, v_2, ...\rangle = T(V)$ on the generators $V = \mathrm{Span}\{v_1, v_2, ...\}$ graded by $|v_n| = n$, considered with the free (tensor) product, and endowed with the coproduct

$$\Delta^{\mathrm{dif}}(v_n) = \sum_{m=0}^{n} v_m \otimes \sum_{\substack{k_0,k_1,...,k_m \geq 0 \\ k_0+k_1+\cdots+k_m = n-m}} v_{k_0} v_{k_1} \dots v_{k_m}, \tag{3.1}$$

where we set $v_0 = 1$, and with the counit $\varepsilon(v_n) = \delta_{n,0}$. This Hopf algebra is a free-associative CHA, and the formula (3.1) clearly says that the coproduct Δ^{dif} satisfies the right-sided condition (2.11).

Let $(\mathcal{H}^{\mathrm{dif,nc}})^* = T(V^*)$ be its dual Hopf algebra, as described in section 2, with primitive elements in $V^* = \mathrm{Span}\{v_1^*, v_2^*, ...\}$. Then $(\mathcal{H}^{\mathrm{dif,nc}})^* = T(V^*)$ is a right-sided cofree-coassociative CHA, and moreover a dendriform algebra, and V^* is a brace algebra.

Proposition 3.1 *The brace product on V^* which induces the CHA structure on $(\mathcal{H}^{\mathrm{dif,nc}})^*$ is given by*

$$\{v_n^*; v_{m_1}^* \cdots v_{m_q}^*\} = M_{1q}(v_n^*; v_{m_1}^* \cdots v_{m_q}^*) = \binom{n+1}{q} \ v_{n+m_1+\cdots+m_q}^*.$$

Proof. According to eq. (2.12), for any $v_n \in V$ and any $a \in \mathcal{H}^{\mathrm{dif,nc}}$, we have

$$M_{1q}(v_n^*; a^*) = \sum_{m=1}^{\infty} \langle v_n^* \otimes a^* \mid \Delta^{\mathrm{dif}} v_m \rangle \ v_m^*.$$

If we put $v_0 = 1$, the coproduct of v_m becomes [3]

$$\Delta^{\mathrm{dif}} v_m = \sum_{k=0}^{m} v_k \otimes Q^{(k)}_{m-k}(v),$$

with

$$Q^{(k)}_{m-k}(v) = \sum_{l=1}^{k+1} \binom{k+1}{l} \sum_{\substack{j_1,\ldots,j_l>0 \\ j_1+\cdots+j_l=m-k}} v_{j_1} \ldots v_{j_l}.$$

Therefore,

$$M_{1q}(v_n^*; a^*) = \sum_{m=1}^{\infty} \langle a^* \mid Q^{(n)}_{m-n}(v) \rangle \; v_m^*.$$

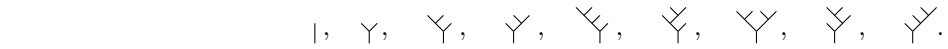
We take $a = v_{m_1} \ldots v_{m_q}$ and we evaluate $\langle v^*_{m_1} \ldots v^*_{m_q} \mid Q^{(n)}_{m-n}(v) \rangle$. From the definition of $Q^{(n)}_{m-n}(v)$, we must have $l = q$, $j_1 = m_1, \ldots, j_q = m_q$. Therefore, $m - n = j_1 + \cdots + j_q = m_1 + \cdots + m_q$ and

$$M_{1q}(v_n^*; v^*_{m_1} \ldots v^*_{m_q}) = \binom{n+1}{q} \; v^*_{n+m_1+\cdots+m_q}.$$

□

4 The right-sided combinatorial structure of the Hopf algebra $(\mathcal{H}^{\alpha,nc})^*$

Let Y be the set of planar binary rooted trees, that is, planar graphs without loops and a preferred external edge called the root. For instance,



For a tree t, we denote by $|t|$ the number of its internal vertices. We denote by Y_n the set of planar binary trees with n internal vertices, so that $Y = \bigcup_{n=0}^{\infty} Y_n$.

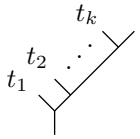
Let $\mathcal{H}^{\alpha,nc}$ be the non-commutative lift of the Hopf algebra $\mathcal{H}^{\alpha}$ introduced in [1, 2] to describe the renormalization of the electric charge in the perturbative expansion of quantum electrodynamics based on planar binary trees. It is the graded and connected Hopf algebra $\mathbb{Q}Y$ spanned by all planar binary trees, endowed with the product *over*, cf. [11]: given two planar binary trees $s, t \neq |$, the tree s over t, denoted by s/t, is the tree obtained by grafting the root of s over the left-most leaf of t, that is,

This product is not commutative, and the root tree $|$ is the unit. Moreover, any tree can be decomposed as the over product of subtrees which have nothing branched on their left-most leaf. If we set $v(t) =$  to denote these trees, the algebra $\mathcal{H}^{\alpha,nc}$ is in fact isomorphic to the free algebra $T(V)$, where $V = \mathrm{Span}\{v(t), t \in Y\}$.

The coproduct $\Delta^{\alpha} : \mathcal{H}^{\alpha,nc} \longrightarrow \mathcal{H}^{\alpha,nc} \otimes \mathcal{H}^{\alpha,nc}$ can be described in an elegant way in the form proposed by P. Palacios in her Master Thesis [14]. For this, we need some notation. Let us denote by $\vee : Y_n \times Y_m \longrightarrow Y_{n+m+1}$ the operation which grafts two trees on a new root, that is

and call dressed comb the tree $\gamma(t_1,\dots,t_k)$ recursively given by

$$\gamma(t) = t \vee \, | \, = \; t \;\curlyvee\;, \qquad \gamma(t_1,t_2,\dots,t_k) = t_1 \vee \gamma(t_2,\dots,t_k) =$$



.

Then, any tree $t \neq |$ can be written as $t = \gamma(t_1,t_2,\dots,t_k)$ for some suitable trees t_i.

The coproduct Δ^α is the unital algebra homomorphism defined on the generators as

$$\begin{aligned}
\Delta^\alpha \curlyvee &= | \otimes \curlyvee + \curlyvee \otimes | \,, \\
\Delta^\alpha v(t) &= | \otimes v(t) + \sum v(\gamma(t_{1\,(1)}, t_{2\,(1)}, \dots, t_{k\,(1)})) \otimes t_{1\,(2)}/t_{2\,(2)}/\cdots/t_{k\,(2)}, \qquad (4.1)
\end{aligned}$$

where $t = \gamma(t_1,t_2,\dots,t_k)$ and $\sum t_{i\,(1)} \otimes t_{i\,(2)} = \Delta^\alpha(t_i)$ is the Sweedler notation for the coproduct applied to each subtree t_i.

Finally, the counit $\varepsilon : \mathcal{H}^{\alpha,nc} \longrightarrow \mathbb{Q}$ is the unital algebra morphism with value $\varepsilon(v(t)) = 0$ on the generators.

Let $(\mathcal{H}^{\alpha,nc})^* = T(V^*)$ be its dual Hopf algebra, with primitive elements given by $V^* = \mathrm{Span}\{v(t^*), t \in Y\}$, where t^* is the dual form of the tree t. Then $(\mathcal{H}^{\alpha,nc})^*$ is a right-sided cofree-coassociatibe CHA, and V^* is a brace algebra. We describe here its brace structure.

Theorem 4.1 *The brace product M_{1q} on V^* which induces the CHA structure on $(\mathcal{H}^{\mathrm{dif,nc}})^*$ is given for any $q > 0$ by*

$$M_{1q}(\curlyvee^*; v(s_1)^* \cdots v(s_q)^*) = 0$$

and for any $t \neq |$ by

$$M_{1q}(v(t)^*; v(s_1)^* \cdots v(s_q)^*) = \sum v\big(\gamma({p^*}_{(1)}, \dots, {p^*}_{(k)})\big)^*,$$

where for $t = \gamma(t_1,\dots,t_k)$ we set $p^* = (t_1^* \cdots t_l^*) \star (v(s_1)^* \cdots v(s_q)^*)$, and where we use the Sweedler notation

$$\sum {p^*}_{(1)} \otimes {p^*}_{(2)} \otimes \cdots \otimes {p^*}_{(k)} = (\Delta^*)^{k-1}\, p^*$$

for the deconcatenation coproduct Δ^* applied $k-1$ times to p^*.

Proof. According to Eq. (2.12), if we denote $a^* = v(s_1)^* \cdots v(s_q)^*$, the brace product on V^* can be computed as

$$\begin{aligned}
M_{1q}\big(v(t)^*; a^*\big) &= \sum_{u \in Y} \big\langle v(t)^* \star a^* | v(u) \big\rangle \; v(u)^* = \sum_{u \in Y} \big\langle v(t)^* \otimes a^* | \Delta^\alpha v(u) \big\rangle \; v(u)^* \\
&= \big\langle v(t)^* \otimes a^* | \Delta^\alpha v(\,|\,) \big\rangle \; v(\,|\,)^* + \sum_{u \neq |} \big\langle v(t)^* \otimes a^* | \Delta^\alpha v(u) \big\rangle \; v(u)^*. \qquad (4.2)
\end{aligned}$$

The first term gives

$$\big\langle v(t)^* \otimes a^* | \curlyvee \otimes | + | \otimes \curlyvee \big\rangle \; \curlyvee^* = 0,$$

because $\langle a^* \mid \,|\,\rangle = 0$ for $q > 0$, and $\langle v(t)^* \mid \,|\,\rangle = 0$ for any t.

Then we suppose that $u \neq |$ and we evaluate the second term of (4.2) using the expression (4.1) for $\Delta^\alpha v(u)$. The first part gives

$$\big\langle v(t)^* \otimes a^* \mid \, | \otimes v(u) \big\rangle = 0$$

for any choice of t.

The second part gives

$$\begin{aligned}\big\langle v(t)^* \otimes a^* | v(\gamma(u_{1(1)}, u_{2(1)}, \dots, u_{l(1)})) \otimes u_{1(2)}/u_{2(2)}/\cdots/u_{l(2)} \big\rangle \\ = \big\langle v(t)^* | v(\gamma(u_{1(1)}, u_{2(1)}, \dots, u_{l(1)})) \big\rangle \big\langle a^* | u_{1(2)}/u_{2(2)}/\cdots/u_{l(2)} \big\rangle, \end{aligned} \tag{4.3}$$

where we write u as $\gamma(u_1, ..., u_l)$. The only primitive element $v(t)$ which can not be of the form $v\big(\gamma(u_{1(1)}, u_{2(1)}, \dots, u_{k(1)})\big)$ is $v(\,|\,) = \curlyvee$. In this case, the brace product is obviously

$$M_{1q}\big(\curlyvee^*; a^*\big) = 0$$

for any $q > 0$. Finally let us consider the case $t \neq |$. Then $t = \gamma(t_1, \dots, t_k)$, and the first pairing of (4.3) is non-zero only if

$$\gamma(u_{1(1)}, u_{2(1)}, \dots, u_{l(1)}) = t = \gamma(t_1, \dots, t_k),$$

that is, if and only if $l = k$ and

$$u_{1(1)} = t_1 \ , \quad u_{2(1)} = t_2 \ , \quad \dots \ , \quad u_{k(1)} = t_k \ .$$

Therefore we have

$$\begin{aligned} M_{1q}(v(t)^*; a^*) &= \sum_{u_1,\dots,u_k} \big\langle v(t)^* \mid v\big(\gamma(u_{1(1)}, \dots, u_{k(1)})\big)\big\rangle \big\langle a^* \mid u_{1(2)}/\cdots/u_{l(2)}\big\rangle v\big(\gamma(u_1, \dots, u_k)\big)^* \\ &= \sum_{u_1,\dots,u_k} \big\langle t_1^* \cdots t_k^* \mid u_{1(1)} \cdots u_{k(1)}\big\rangle \big\langle a^* \mid u_{1(2)}/\cdots/u_{l(2)}\big\rangle v\big(\gamma(u_1, \dots, u_k)\big)^* \\ &= \sum_{u_1,\dots,u_k} \big\langle t_1^* \cdots t_k^* \otimes a^* \mid \Delta^\alpha(u_1/\cdots/u_k)\big\rangle v\big(\gamma(u_1, \dots, u_k)\big)^* \\ &= \sum_{u_1,\dots,u_k} \big\langle (t_1^* \cdots t_k^*) \star a^* \mid u_1/\cdots/u_k\big\rangle v\big(\gamma(u_1, \dots, u_k)\big)^*. \end{aligned}$$

Now, note that the deconcatenation coproduct Δ^* in $(\mathcal{H}^{\alpha,nc})^*$ is dual to the product / in $\mathcal{H}^{\alpha,nc}$, that is

$$\big\langle a^* \ | b/c \big\rangle = \big\langle \Delta^*(a^*) \ | b \otimes c \big\rangle.$$

Let us call $p^* = (t_1^* \cdots t_k^*) \star a^*$. If we apply $k-1$ times the deconcatenation coproduct Δ^* to p^*, and we use the Sweedler notation $\sum {p^*}_{(1)} \otimes {p^*}_{(2)} \otimes \cdots \otimes {p^*}_{(l)} = (\Delta^*)^{k-1}\ p^*$, we obtain the final result

$$\begin{aligned} M_{1q}(v(t)^*; a^*) &= \sum_{u_1,\dots,u_k} \sum \big\langle {p^*}_{(1)} \otimes {p^*}_{(2)} \otimes \cdots \otimes {p^*}_{(k)} \mid u_1 \otimes \cdots \otimes u_k\big\rangle v\big(\gamma(u_1, \dots, u_k)\big)^* \\ &= \sum v\big(\gamma({p^*}_{(1)}, \dots, {p^*}_{(k)})\big)^*. \end{aligned}$$

□

Examples 4.2 The simplest example is given by $M_{11}(\curlyvee^*; \curlyvee^*)$. We have $\curlyvee = v(\curlyvee) = v(\gamma(\,|\,))$, so that $k = 1$ and $t_1 = |$. Therefore, $p^* = |^* \star \curlyvee^* = \curlyvee^*$ and

$$M_{11}(\curlyvee^*; \curlyvee^*) = v(\gamma(\curlyvee^*)) = v(\curlyvee^*) = \begin{array}{c} \curlyvee \\ \curlyvee \end{array}^* .$$

Similarly

$$M_{11}(\curlyvee^*; \curlyvee^*) = v(\gamma(\curlyvee^*)) = v(\begin{array}{c} \curlyvee \\ \curlyvee \end{array}^*) = \begin{array}{c} \curlyvee \\ \curlyvee \\ \curlyvee \end{array}^* .$$

Another simple example is given by $M_{11}(\begin{array}{c} \curlyvee \\ \curlyvee \end{array}^*; \curlyvee^*)$. We have $\begin{array}{c} \curlyvee \\ \curlyvee \end{array} = v(\gamma(\curlyvee))$, so that $k = 1$ and $t_1 = \curlyvee$. Since for any t formula (2.12) gives

$$\curlyvee^* \star v(t)^* = v(t)^* \curlyvee^* + \curlyvee^* v(t)^* + M_{1q}(\curlyvee^*; v(t)^*) = v(t)^* \curlyvee^* + \curlyvee^* v(t)^*,$$

because $M_{1q}(\curlyvee^*; v(t)^*) = 0$, we get $p^* = \curlyvee^* \star \curlyvee^* = 2\ \curlyvee^{*}$, and finally

$$M_{11}(\ \curlyvee^{*}\ ; \curlyvee^*) = 2v(\gamma(\ \curlyvee\ ))^* = 2v(\ \curlyvee\ )^* = 2\ \curlyvee^{*}.$$

Now let us compute $M_{11}(\ \curlyvee^{*}\ ; \curlyvee^*)$. We have $\curlyvee^{\curlyvee} = v(\gamma(\ |\ ,\ |\ ))$ so that $k = 2$ and $t_1 = t_2 = |$. Since $|^*|^* = |^*$, we have $p^* = |^* \star \curlyvee^* = \curlyvee^*$ and therefore

$$M_{11}(\ \curlyvee^{\curlyvee^{*}}; \curlyvee^*) = v(\gamma(\curlyvee,\ |\ ))^* + v(\gamma(\ |\ , \curlyvee))^* = v(\ \curlyvee^{\curlyvee}\ )^* + v(\ \curlyvee^{\curlyvee}\ )^* =\ \curlyvee^{\curlyvee^{*}} + \curlyvee^{\curlyvee^{*}}.$$

Finally, let us compute an example of M_{1q} with $q > 1$. We consider $M_{12}(\ \curlyvee^{*}; \curlyvee^*\curlyvee^*)$. We have $\curlyvee^{\curlyvee} = v(\gamma(\ |\ ))$, so that $k = 1$ and $t_1 = |$. Therefore, $p^* = |^* \star (\curlyvee^*\curlyvee^*) =\ \curlyvee^{*}$ and

$$M_{12}(\ \curlyvee^{*}; \curlyvee^*\curlyvee^*) = v(\gamma(\ \curlyvee\ ))^* = v(\ \curlyvee^{\curlyvee}\ )^* =\ \curlyvee^{\curlyvee^{*}}.$$

Similarly

$$\begin{aligned} M_{12}(\ \curlyvee^{\curlyvee^{*}}; \curlyvee^*\curlyvee^*) &= v(\gamma(\ \curlyvee\ ,\ |\ ))^* + v(\gamma(\curlyvee, \curlyvee))^* + v(\gamma(\ |\ ,\ \curlyvee\ ))^* \\ &= v(\ \curlyvee^{\curlyvee}\ )^* + v(\ \curlyvee^{\curlyvee}\ )^* + v(\ \curlyvee^{\curlyvee}\ )^* =\ \curlyvee^{\curlyvee^{*}} + \curlyvee^{\curlyvee^{*}} + \curlyvee^{\curlyvee^{*}}. \end{aligned}$$

Remark 4.3 The brace structure can be described as follows:

$$M_{1q}(v(t)^*; v(s_1)^* \cdots v(s_q)^*) = \sum_u v(u)^*,$$

where the sum runs over any tree u obtained by branching the trees $v(s_i)$ or any block $v(s_i)/\cdots/v(s_j)$ on the \-leaves of t or inside the \-branches of t, by preserving the order from left to right of the trees $v(s_i)$ and of the \-branches of t.

5 The right-sided combinatorial structure of the Hopf algebra $T(\overline{T}(B))^*$

In the paper [4], C. Brouder and W. Schmitt introduced a non-commutative version $\mathcal{H}$ of a renormalization Hopf algebra inspired by Pinter's work on the Epstein-Glaser renormalization of quantum fields on the configuration space. In this section we describe the brace structure on the set of primitive elements of its dual Hopf algebra $\mathcal{H}^*$.

As an algebra, $\mathcal{H}$ is the tensor algebra $T(V)$ on a set of generators $V = \overline{T}(B) = \oplus_{n\geq 1} B^{\otimes n}$ which is the augmentation ideal of the tensor algebra over a given bialgebra B.

In order to apply the dualization procedure described in Section 2, we have to assume that the bialgebra B admits at most a countable basis, and that this has been fixed. Let us denote by $\{x_n\}$ the chosen basis of B. We also denote the product (in B) by a dot, namely $x \cdot y$ for any $x, y \in B$, and the coproduct (in B) through the Sweedler notation $\delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

In $V = \overline{T}(B)$, we denote the tensor products by tuples. Therefore the elements $v_{n_1,\ldots,n_k} = (x_{n_1}, \ldots, x_{n_k})$ form a basis of V. On V we consider two coproducts. The first one is the extention of the coproduct on B as an algebra morphism, namely

$$\delta(x, y, \ldots, z) = \sum (x_{(1)}, y_{(1)}, \ldots, z_{(1)}) \otimes (x_{(2)}, y_{(2)}, \ldots, z_{(2)}).$$

For a generic element v of V, we use the same Sweedler notation $\delta(v) = \sum v_{(1)} \otimes v_{(2)}$ as on B. The second coproduct is the (reduced) deconcatenation coproduct

$$\Delta(x, y, \ldots, z) = (x) \otimes (y, \ldots, z) + (x, y) \otimes (\ldots, z) + \cdots + (x, y, \ldots) \otimes (z).$$

For this coproduct we use a modified Sweedler notation $\Delta(v) = \sum\ v_{[1]} \otimes v_{[2]}$.

Finally, in $\mathcal{H} = T(V)$, we denote the tensor product by the concatenation. A generic element in $\mathcal{H}$ is then of the form

$a = uv \cdots w$, where $u, v, ..., w \in V$, and for any elements a, b we denote their product (in $T(V)$) by ab. Both coproducts δ and Δ can be extended to $\mathcal{H}$. The renormalization coproduct is then the algebra homomorphism defined on a generator $v \in V$ as

$$\Delta^{\mathcal{H}}(v) = \sum_{n=1}^{\infty} \sum\ \big(\mu(v_{[1](1)}), \ldots, \mu(v_{[n](1)})\big) \otimes v_{[1](2)} \cdots v_{[n](2)}, \tag{5.1}$$

where $\sum\ v_{[1]} \otimes \cdots \otimes v_{[n]} = \Delta^{n-1}(v)$ is the result of the deconcatenation applied $n-1$ times to v, and $\mu : \overline{T}(B) \longrightarrow B$, $\mu(x, y, ..., z) = x \cdot y \cdot \ldots \cdot z$ is the product of B extended to many factors.

Note that for any $v \in V$, the left tensor factor $\big(\mu(v_{[1](2)}), \ldots, \mu(v_{[n](2)})\big)$ of (5.1) belongs to V (in fact it belongs to $B^{\otimes n}$), while the right factor $v_{[1](1)} \cdots v_{[n](1)}$ belongs to $V^{\otimes n}$, therefore $\Delta^{\mathcal{H}}$ can not be defined in V.

Note also that any monomial of elements of B in $v \in V$ has finite length. Since the deconcatenation of v vanishes if repeated more then the length of v, the sum over n in formula (5.1) is indeed finite. For example, let us compute the simplest coproducts, for x, y and z in B:

$$\begin{aligned}
\Delta^{\mathcal{H}}(x) &= \sum (x_{(1)}) \otimes (x_{(2)}), \\
\Delta^{\mathcal{H}}(x, y) &= \sum (x_{(1)}, y_{(1)}) \otimes (x_{(2)})(y_{(2)}) + \sum (x_{(1)} \cdot y_{(1)}) \otimes (x_{(2)}, y_{(2)}), \\
\Delta^{\mathcal{H}}(x, y, z) &= \sum (x_{(1)}, y_{(1)}, z_{(1)}) \otimes (x_{(2)})(y_{(2)})(z_{(2)}) + \sum (x_{(1)}, y_{(1)} \cdot z_{(1)}) \otimes (x_{(2)})(y_{(2)}, z_{(2)}) \\
&\quad + \sum (x_{(1)} \cdot y_{(1)}, z_{(1)}) \otimes (x_{(2)}, y_{(2)})(z_{(2)}) + \sum (x_{(1)} \cdot y_{(1)} \cdot z_{(1)}) \otimes (x_{(2)}, y_{(2)}, z_{(2)}).
\end{aligned}$$

It is important to remark that, if $v = (x_{n_1}, \ldots, x_{n_l})$, then $\Delta^{\mathcal{H}} v$ is a sum of elements of $B^{\otimes k}$ tensorized by elements of $\overline{T}(B)^{\otimes k}$, for $k = 1$ to l.

We now consider the dual Hopf algebra $\mathcal{H}^* = T(V^*)$, where V^* is the graded dual space $\overline{T}(B^*)$. Here B^* is the dual bialgebra of B, with basis elements x_m^* such that $\langle x_m^* \mid x_n \rangle = \delta_{m,n}$. The product and coproduct in B^* are defined in terms of those of B by the standard duality

$$\begin{aligned}
\langle x^* \cdot y^*, z \rangle &= \langle x^* \otimes y^*, \delta z \rangle, \\
\langle \delta x^*, y \otimes z \rangle &= \langle x^*, y \cdot z \rangle.
\end{aligned}$$

In $V^* = \overline{T}(B^*)$ we have $(x_1, \ldots, x_n)^* = (x_1^*, \ldots, x_n^*)$ and therefore

$$\big\langle (x_1^*, \ldots, x_n^*) \mid (y_1, \ldots, y_m) \big\rangle = \delta_{n,m} \langle x_1^* \mid y_1 \rangle \cdots \langle x_n^* \mid y_n \rangle.$$

As we said, $\mathcal{H}^*$ is a right-sided cofree-coassociative CHA, with deconcatenation coproduct Δ^* and product $\star$. In this section we describe the brace product induced by $\star$ on the set V^* of primitive elements of $\mathcal{H}^*$.

Theorem 5.1 *The brace product M_{1q} on $V^* = \overline{T}(B^*)$ which induces the CHA structure on $\mathcal{H}^*$ is given on the elements $v^* = (x^1, ..., x^n)^*$ and $u_1^*, ..., u_q^*$, for any $q > 0$, by*

$$M_{1q}(v^*; u_1^* \cdots u_q^*) = 0, \quad \text{if } n \neq q,$$

and, for $n = q$, by

$$M_{1q}(v^*; u_1^* \cdots u_q^*) = \big(x^1{}_{(1)} \cdot y_1^1\ , ...,\ x^1{}_{(k_1)} \cdot y_1^{k_1}\ , ...,\ x^q{}_{(1)} \cdot y_q^1\ , ...,\ x^q{}_{(k_q)} \cdot y_q^{k_q}\big)^*,$$

where $u_i^* = (y_i^1, ..., y_i^{k_i})^*$, for $i = 1, ..., q$.

Proof. If $n \neq q$, we already remarked in the expression (5.1) of the coproduct that no term can occur.

If $n = q$, consider $v = (x^1, ..., x^q) \in B^{\otimes q}$ and $u_1, \ldots, u_q$ in $B^{\otimes k_1}, \ldots, B^{\otimes k_q}$, respectively. Set $a = u_1 \ldots u_q \in V^{\otimes q}$. Then

$$M_{1q}(v^*; a^*) \quad = \quad \sum_k \sum_{w \in B^{\otimes k}} \langle v^* \otimes a^* \mid \Delta^{\mathcal{H}} w \rangle w^* = \sum_k \sum_{w \in B^{\otimes k}} \sum \langle v^* \otimes a^* \mid w_{(1)} \otimes w_{(2)} \rangle w^*,$$

where the sums run over the generators w of $B^{\otimes k}$, that is, elements of the form $w = (z^1, \ldots, z^k)$ where z^i are generators of B, and where $\sum w_{(1)} \otimes w_{(2)} = \delta(w)$ denotes the coproduct of B.

As a consequence, the term corresponding to w appears in $M_{1q}(v^*; a^*)$ if $w_{(1)} = v$ and $w_{(2)} = a$. The equality $w_{(1)} = v = (x^1, \ldots, x^q)$ can be rewritten

$$\begin{aligned} z^1_{(1)} \cdot \ldots \cdot z^{k_1}_{(1)} &= x^1, \\ z^{k_1+1}_{(1)} \cdot \ldots \cdot z^{k_1+k_2}_{(1)} &= x^2, \\ &\ldots \\ z^{k_1+\cdots+k_{q-1}+1}_{(1)} \cdot \ldots \cdot z^{k}_{(1)} &= x^q. \end{aligned}$$

The equality $w_{(2)} = a = u_1 \ldots u_q$ can be rewritten

$$\begin{aligned} (z^1_{(2)}, \ldots, z^{k_1}_{(2)}) &= u_1, \\ (z^{k_1+1}_{(2)}, \ldots, z^{k_1+k_2}_{(2)}) &= u_2, \\ &\ldots \\ (z^{k_1+\cdots+k_{q-1}+1}_{(2)}, \ldots, z^{k}_{(2)}) &= u_q. \end{aligned}$$

The second equality means that $k = k_1 + \cdots + k_q$. Because of this special form, each block can be treated independently.

We consider the first block and we write $k = k_1$ and $u_1 = (y_1, \ldots, y_k)$. We have now, for the first block and after putting $x = x^1$,

$$\begin{aligned} X &= \sum_{z^1, \ldots, z^k} \langle (x^*) \otimes (y_1, \ldots, y_k)^*, (z^1_{(1)} \cdot \ldots \cdot z^k_{(1)}) \otimes (z^1_{(2)}, \ldots, z^k_{(2)}) \rangle (z^1, \ldots, z^k)^* \\ &= \sum_{z^1, \ldots, z^k} \langle (x^*), (z^1_{(1)} \cdot \ldots \cdot z^k_{(1)}) \rangle \langle (y_1, \ldots, y_k)^*, (z^1_{(2)}, \ldots, z^k_{(2)}) \rangle (z^1, \ldots, z^k)^* \\ &= \sum_{z^1, \ldots, z^k} \langle x^*_{(1)}, z^1_{(1)} \rangle \ldots \langle x^*_{(k)}, z^k_{(1)} \rangle \langle y^*_1, z^1_{(2)} \rangle \ldots \langle y^*_k, z^k_{(2)} \rangle (z^1, \ldots, z^k)^*, \end{aligned}$$

where we used the fact that the coproduct in B^* is the dual of the product in B. Now we use the fact that the product in B^* is the dual of the coproduct in B, and obtain

$$\begin{aligned} X &= \sum_{z^1, \ldots, z^k} \langle x^*_{(1)} \cdot y^*_1, z^1 \rangle \ldots \langle x^*_{(k)} \cdot y^*_k, z^k \rangle (z^1, \ldots, z^k)^* \\ &= \sum (x_{(1)} \cdot y_1, \ldots, x_{(k)} \cdot y_k)^*. \end{aligned}$$

The same is done for each block and we finally find

$$M_{1q}(v^*; a^*) \quad = \quad (x^1_{(1)} \cdot y^1_1, \ldots, x^1_{(k_1)} \cdot y^1_{k_1}, \ldots, x^q_{(1)} \cdot y^q_1, \ldots, x^q_{(k)} \cdot y^q_{k_q})^*.$$

□

Examples 5.2 Examples of M_{11} are:

$$\begin{aligned} M_{11}\big((x^*); (y^*)\big) &= (x^* \cdot y^*), \\ M_{11}\big((x^*); (y^*, z^*)\big) &= \sum (x^*{}_{(1)} \cdot y^*, x^*{}_{(2)} \cdot z^*), \\ M_{11}\big((x^*); (y^*_1, \ldots, y^*_n)\big) &= (x^*{}_{(1)} \cdot y^*_1, \ldots, x^*{}_{(n)} \cdot y^*_n). \end{aligned}$$

Examples of M_{12} are:

$$\begin{aligned}
M_{12}\big((x,y)^*;(s)^*(t)^*\big) &= (x\cdot s, y\cdot t)^*,\\
M_{12}\big((x,y)^*;(s)^*(t_1,t_2)^*\big) &= \sum (x\cdot s, y_{(1)}\cdot t_1, y_{(2)}\cdot t_2)^*,\\
M_{12}\big((x,y)^*;(s_1,s_2)^*(t)^*\big) &= \sum (x_{(1)}\cdot s_1, x_{(2)}\cdot s_2, y\cdot t)^*,\\
M_{12}\big((x^*,y^*);(s_1^*,\ldots,s_m^*)(t_1^*,\ldots,t_n^*)\big) &= \sum ({x^*}_{(1)}\cdot s_1^*,\ldots,{x^*}_{(m)}\cdot s_m^*, {y^*}_{(1)}\cdot t_1^*,\ldots,{y^*}_{(n)}\cdot t_n^*).
\end{aligned}$$

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