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# Weak pattern matching in colored graphs: Minimizing the number of connected components 

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In the context of metabolic network analysis, Lacroix et al. ${ }^{11}$ introduced the problem of finding occurrences of motifs in vertex-colored graphs, where a motif is a multiset of colors and an occurrence of a motif is a subset of connected vertices which are colored by all colors of the motif. We consider in this paper the above-mentioned problem in one of its natural optimization forms, referred hereafter as the Min-CC problem: Find an occurrence of a motif in a vertexcolored graph, called the target graph, that induces a minimum number of connected components.

Our results can be summarized as follows. We prove the Min-CC problem to be APX-hard even in the extremal case where the motif is a set and the target graph is a path. We complement this result by giving a polynomial-time algorithm in case the motif is built upon a fixed number of colors and the target graph is a path. Also, extending recent research ${ }^{8}$, we prove the MinCC problem to be fixed-parameter tractable when parameterized by the size of the motif, and we give a faster algorithm in case the target graph is a tree. Furthermore, we prove the MiN-CC problem for trees not to be approximable within ratio $c \log n$ for some constant $c>0$, where $n$ is the order of the target graph, and to be $\mathbf{W}$ [2]-hard when parameterized by the number of connected components in the occurrence of the motif. Finally, we give an exact efficient exponential-time algorithm for the Min-CC problem in case the target graph is a tree.

## 1. Introduction

In the context of metabolic network analysis, Lacroix et al. ${ }^{11}$ introduced the following vertex colored graph problem (referred hereafter as the GraphMotif problem): Given a vertex-colored graph $G$ and a multiset of colors $\mathcal{M}$, decide whether $G$ has a connected subset of vertices which are exactly colored by $\mathcal{M}$. There, vertices correspond to chemical compounds or reactions, and each edge $\left(v_{i}, v_{j}\right)$ corresponds to an interaction between the two compounds or reactions $v_{i}$ and $v_{j}$. The vertex coloring is used to specify different chemical types or functionalities. In this scenario, connected motifs correspond to interaction-related submodules of the network which consist of a specific set of chemical compounds and reactions. A method for a rational decomposition of a metabolic network into relatively independent functional subsets is essential for a better understanding of the modularity and organization principles in the network ${ }^{5,11}$. Notice that Ideker considered a related relevant work ${ }^{10}$.

Unfortunately, it turns out that the Graph-Motif problem is $\mathbf{N P}$-complete even if the graph is a tree and the motif is actually a set ${ }^{8,11}$. Moreover, the Graph-Motif problem is fixed-parameter tractable when parameterized by the size of the motif, but $\mathbf{W}[1]$-hard when parameterized by the number of distinct colors in $\mathcal{M}^{8}$. Finally, Lacroix et al. ${ }^{11}$ gave an exact algorithm dedicated to solve small instances.

For metabolic network analysis, the Graph-Motif problem appears, however, to be too stringent. Indeed, due to measurement errors, it is often not possible to find a connected component of the graph $G$ which corresponds exactly to the motif $\mathcal{M}$. Hence one needs to relax the definition of an occurrence of a motif in a metabolic network. Therefore, aiming at dealing with inherent imprecise data, we consider in this paper the above-mentioned problem in one of its natural optimization form, referred hereafter as the Min-CC problem: Find an occurrence of a motif in a vertex-colored graph, that induces a minimum number of connected components.

The paper is organized as follows. Section 2 provides basic notations and definitions that we will use in the paper. In Section 3, we prove the Min-CC problem to be $\mathbf{A P X}$-hard even if the motif is a set and the target graph is a path. Extending recent research ${ }^{8}$, we prove in Section 4 that the Min-CC problem is fixed-parameter tractable when parameterized by the size of the motif, and we give a faster algorithm in case the target graph is a tree. In Section 5 we present a polynomial-time algorithm in case the motif is built upon a fixed number of colors and the target graph is a path. Section 6 is devoted to hardness of approximation in case the target graph is a tree
and we present in Section 7 an exact efficient exponential-time algorithm for trees. Section 8 concludes our work and suggests future directions of research.

## 2. Preliminaries

We assume readers have basic knowledge about graph theory ${ }^{6}$ and we shall only recall basic notations here. Let $G$ be a graph. We write $\mathbf{V}(G)$ for the set of vertices and $\mathbf{E}(G)$ for the set of edges. For any $V^{\prime} \subseteq \mathbf{V}(G)$, we denote by $G\left[V^{\prime}\right]$ the subgraph of $G$ induced by the vertices $V^{\prime}$, that is $G\left[V^{\prime}\right]=\left(V^{\prime}, E^{\prime}\right)$ and $(u, v) \in E^{\prime}$ iff $u, v \in V^{\prime}$ and $(u, v) \in \mathbf{E}(G)$. Let $\mathcal{M}$ be a multiset of colors, whose colors are taken from the set $\mathcal{C}=\left\{c_{1}, c_{2}, \ldots, c_{q}\right\}$. Let $G$ be a connected graph, where every vertex $u \in V(G)$ is assigned a color $\lambda(u) \in \mathcal{C}$. For any subset $V^{\prime}$ of $V$, let $C\left(V^{\prime}\right)$ be the multiset of colors assigned to the vertices in $V^{\prime}$. A subset of vertices $V^{\prime} \subseteq \mathbf{V}(G)$ is said to match a multiset of colors $\mathcal{M}$ if $C\left(V^{\prime}\right)$ is equal to $\mathcal{M}$. A color-preserving injective mapping $\theta$ of $\mathcal{M}$ to $G$ is an injective mapping $\theta: \mathcal{M} \rightarrow \mathbf{V}(G)$, such that $\lambda(\theta(c))=c$ for every $c \in \mathcal{M}$. The subgraph induced by a colorpreserving injective mapping $\theta: \mathcal{M} \rightarrow \mathbf{V}(G)$ is the subgraph of $G$ induced by the images of $\theta$ in $G$.

We are now in position to formally define the Min-CC problem we are interested in. Given a set of colors $\mathcal{C}$, a multiset (motif) $\mathcal{M}$ of size $k$ of colors from $\mathcal{C}$ and a target graph $G$ of order $n$ together with a vertexcoloring mapping $\lambda: \mathbf{V}(G) \rightarrow \mathcal{C}$, find a color preserving injective mapping $\theta: \mathcal{M} \rightarrow \mathbf{V}(G)$, i.e., $\lambda(\theta(c))=c$ for every $c \in \mathcal{M}$ that minimizes the number of connected components in the subgraph induced by $\theta$. In other words, the Min-CC problem asks to find a subset $V^{\prime} \subseteq \mathbf{V}(G)$ that matches $\mathcal{M}$, and that minimizes the number of connected components of $G\left[V^{\prime}\right]$. The Min-CC problem was proved to be NP-complete even if the target graph is a tree and the occurrence is required to be connected (the occurrence of $\mathcal{M}$ in $G$ results in one connected component) but fixed-parameter tractable in this case when parameterized by the size of the given motif ${ }^{11}$.

## 3. Hardness result for paths

In this section we show that the Min-CC problem is APX-hard (not approximable within a constant) even in the simple case where the motif $\mathcal{M}$ is a set and the target graph is a path in which each color in $\mathcal{C}$ occurs exactly twice. Our proof consists in a reduction from a restricted version of the Paintshop-For-Words problem ${ }^{2,3,15}$.

First, we need some additional definitions. Define an isogram to be a word in which no letter is used more than once. A pair isogram is a word in which each letter occurs exactly twice. A cover of size $k$ of a word $u$ is an ordered collection of words $C=\left(v_{1}, v_{2}, \ldots, v_{k}\right)$ such that $u=w_{1} v_{1} w_{2} v_{2} \ldots w_{k} v_{k} w_{k+1}$ and $v=v_{1} v_{2} \ldots v_{k}$ is an isogram The cover is called prefix (resp. suffix) if $w_{1}$ (resp. $w_{k+1}$ ) is the empty word.

A proper 2 -coloring of a pair isogram $u$ is an assignment $f$ of colors $c_{1}$ and $c_{2}$ to the letters of $u$ such that every letter of $u$ is colored with color $c_{1}$ once and colored with color $c_{2}$ once. If two adjacent letters $x$ and $y$ are colored with different colors we say that there is a color change between $x$ and $y$. For the sake of brevity, we denote a pair isogram $u$ together with a proper 2 -coloring $f$ of it as the pair $(u, f)$.

The 1-Regular-2-Colors-Paint-Shop problem is defined as follows: Given a pair isogram $u$, find a 2 -coloring $f$ of $u$ that minimizes the number of color changes in $(u, f)$. Bonsma ${ }^{2}$ proved that the 1-Regular-2-Colors-Paint-Shop problem is APX-hard. We show here how to reduce the 1-Regular-2-Colors-Paint-Shop problem to the Min-CC problem for paths. We need the following easy lemmas.

Lemma 3.1. Let $u$ be a pair isogram and $C$ be a minimum cardinality cover of $u$. Then $C$ cannot be both prefix and suffix.

Lemma 3.2. A pair isogram has a proper 2 -coloring with at most $k$ color changes iff it has a cover of size at most $\left\lceil\frac{k}{2}\right\rceil$.

Combining Lemma 3.2 with the fact that the 1-Regular-2-Colors-Paint-Shop problem is APX-hard, we state the following result.

Proposition 3.1. The following problem is APX-hard: Given a pair isogram $u$, find a minimum cardinality cover of $u$.

Corollary 3.1. The Min-CC problem is $\mathbf{A P X}$-hard even if $\mathcal{M}$ is a set and $P$ is a path in which each color appears at most twice.

## 4. Fixed-parameter algorithms

Corollary 3.1 gives us a sharp hardness result for the Min-CC problem. To complement this negative result, we first prove here that the Min-CC problem is fixed-parameter tractable ${ }^{7,9}$ when parameterized by the size of the pattern $\mathcal{M}$. The algorithm is a straightforward extension of a recent result ${ }^{8}$ and is based on the color-coding technique ${ }^{1}$. Next, we give a faster fixed-parameter algorithm in case the target graph is a tree.

### 4.1. The Min-CC problem is fixed-parameter tractable

We only sketch the fixed-parameter tractability result. Let $G$ be a graph and $k$ be a positive integer. Recall that a family $\mathcal{F}$ of functions from $\mathbf{V}(G)$ to $\{1,2, \ldots, k\}$ is perfect if for any subset $V \subseteq \mathbf{V}(G)$ of $k$ vertices there is a function $f \in \mathcal{F}$ which is injective on $V^{1}$. Let $(G, \mathcal{M})$ be an instance of the Min-CC problem, where $\mathcal{M}$ is a motif of size $k$. Then there is an occurrence of $\mathcal{M}$ in $G$, say $V \subseteq \mathbf{V}(G)$, that results in a minimum number of connected components. Furthermore, suppose we are provided with a perfect family $\mathcal{F}$ of functions from $\mathbf{V}(G)$ to $\{1,2, \ldots, k\}$. Since $\mathcal{F}$ is perfect, we are guaranteed that at least one function in $\mathcal{F}$ assigns $V$ with $k$ distinct labels. Let $f \in \mathcal{F}$ be such a function. We now turn to defining a dynamic programming table $T$ indexed by vertices of $G$ and subsets of $\{1,2, \ldots, k\}$. For any $v \in \mathbf{V}(G)$ and any $L \subseteq\{1,2, \ldots, k\}$, we define $T_{L}[v]$ to be the family of all motifs $\mathcal{M}^{\prime} \subseteq \mathcal{M},\left|\mathcal{M}^{\prime}\right|=|L|$, for which there exists an exact occurrence of $\mathcal{M}^{\prime}$ in $G$, say $V$, such that $v \in V$ and the set of (unique) labels that $f$ assigns to $V$ is exactly $L$. We need the following lemma ${ }^{8}$.

Lemma 4.1. For any labeling function $f: \mathbf{V}(G) \rightarrow\{1,2, \ldots, k\}$, there exists a dynamic programming algorithm that computes the table $T$ in $\mathcal{O}\left(2^{5 k} k n^{2}\right)$ time.

Now, denote by $\mathcal{P}$ the set of all pairs $\left(\mathcal{M}^{\prime}, L^{\prime}\right) \in \mathcal{M} \times 2^{\{1,2, \ldots, k\}}$ with $\left|\mathcal{M}^{\prime}\right|=\left|L^{\prime}\right|$ such that there exists an exact occurrence of $\mathcal{M}^{\prime}$ in $G$, say $V^{\prime}$, such that $v \in V^{\prime}$ and the set of (unique) labels that $f$ assigns to $V^{\prime}$ is exactly $L^{\prime}$. Clearly, $|\mathcal{P}| \leq 2^{2 k}$. Furthermore, by resorting to any data structure for searching and inserting that guarantees logarithmic time ${ }^{4}$ (and observing that any two pairs $\left(\mathcal{M}^{\prime}, L^{\prime}\right)$ and $\left(\mathcal{M}^{\prime \prime}, L^{\prime \prime}\right)$ can be compared in $\mathcal{O}(k)$ time $)$, one can construct the set $\mathcal{P}$ in $\mathcal{O}\left(n k^{2} 2^{2 k}\right)$ time by running through the table $T$. Our algorithm now exhaustively considers all subsets of $\mathcal{P}$ of size at most $k$ to find an occurrence of $\mathcal{M}$ in $G$ that results in a minimum number of connected components. The rationale of this approach is that two pairs $\left(\mathcal{M}^{\prime}, L^{\prime}\right)$ and $\left(\mathcal{M}^{\prime \prime}, L^{\prime \prime}\right)$ with $L^{\prime} \cap L^{\prime \prime}=\emptyset$ correspond to non-overlapping occurrences in $G$. The total time of this latter procedure is certainly upperbounded by $\sum_{i=1}^{k} k\binom{2^{2 k}}{i} \leq k^{2} 2^{2 k^{2}}$. Summing up and taking into account the time for computing the table $T$, the running time for a given $f \in \mathcal{F}$ is $\mathcal{O}\left(2^{5 k} k n^{2}+n k^{2} 2^{2 k}+k^{2} 2^{2 k^{2}}\right)$.

According to Alon et al. ${ }^{1}$, we need to use $\mathcal{O}\left(2^{\mathcal{O}(k)} \log n\right)$ functions $f: \mathbf{V}(G) \rightarrow\{1,2, \ldots, k\}$, and such a family $\mathcal{F}$ can be computed in $\mathcal{O}\left(2^{\mathcal{O}(k)} n \log n\right)$ time. For each $f \in \mathcal{F}$ we use the above procedure to determine an occurrence of $\mathcal{M}$ in $G$ that results in a minimum number of
connected components. We have thus proved the following.
Proposition 4.1. The Min-CC problem is fixed-parameter tractable when parameterized by the size of the motif.

### 4.2. A faster fixed-parameter algorithm for trees

We proved in Section 3 that the Min-CC problem is APX-hard even if the target graph is a path. To complement Proposition 4.1, we give here a dynamic programming algorithm for trees that does not rely on the colorcoding technique (approaches based on the color-coding technique usually suffer from bad running time performances).

Let $(G, \mathcal{M})$ be an instance of the Min-CC problem for trees where both $G$ and $\mathcal{M}$ are built upon a set of colors $\mathcal{C}$. Let $k=|\mathcal{M}|$ and $q=|\mathcal{C}|$. Furthermore, for ease of exposition, write $\mathbf{V}(G)=\{1,2, \ldots, n\}$ and assume $G$ is rooted at some arbitrary vertex $r(G)$.

Our dynamic programming algorithm is basically an exhaustive search procedure. The basic idea is to store - in a bottom-up fashion - for each vertex $i$ of $G$ and each submotif $\mathcal{M}^{\prime} \subseteq \mathcal{M}$ that occurs in $T(i)$, i.e., the subtree rooted at $i$, the minimum number of connected components that results in an occurrence of $\mathcal{M}^{\prime}$ in $T(i)$. More precisely, for each vertex $i$ of $G$, we compute two dynamic programming tables $X[i]$ and $Y[i]$. The dynamic programming table $X[i]$ stores all pairs $\left(\mathcal{M}^{\prime}, c\right)$, where $\mathcal{M}^{\prime} \subseteq \mathcal{M}$ is a submotif and $c$ is a positive integer, such that (1) there exists an occurrence of $\mathcal{M}^{\prime}$ in $T(i)$ that matches vertex $i$, (2) the minimum number of connected components of an occurrence of $\mathcal{M}^{\prime}$ in $T(i)$ that matches vertex $i$ is $c$. The dynamic programming table $Y[i]$ stores all pairs $\left(\mathcal{M}^{\prime}, c\right)$, where $\mathcal{M}^{\prime} \subseteq \mathcal{M}$ is a submotif and $c$ is a positive integer, such that ( $1^{\prime}$ ) there exists an occurrence of $\mathcal{M}^{\prime}$ in $T(i)$ that does not match vertex $i$, (2') the minimum number of connected components of an occurrence of $\mathcal{M}^{\prime}$ in $T(i)$ that does not match vertex $i$ is $c$.

We first claim that both $X[i]$ and $Y[i]$ contain at most $k^{q+1}$ pairs. Indeed, the number of submotifs $\mathcal{M}^{\prime} \subseteq \mathcal{M}$ is upper-bounded by $k^{q}$ and any occurrence of any submotif in any subtree of $G$ results in at most $k$ connected components. We now describe how to compute - in a bottom-up fashion - those two dynamic programming tables $X$ and $Y$.

Let $i$ be an internal vertex of $G$ and suppose that vertex $i$ has $s_{i}$ sons in the subtree $T(i)$ rooted at $i$, say $\left\{i_{1}, i_{2}, \ldots, i_{s_{i}}\right\}$. Notice that $s_{i} \geq 1$ since $i$ is an internal vertex of $G$. The entries $X[i]$ and $Y[i]$ are computed with the aid of two auxiliary tables $W_{i}$ and $V_{i}$. Table $W_{i}$ contains $s_{i}$ entries, one for
each son of vertex $i$ in the subtree rooted at $i$, that are defined as follows:

$$
\begin{aligned}
& \forall 1 \leq j \leq s_{i} \\
& W_{i}\left[i_{j}\right]=\left\{\left(\mathcal{M}^{\prime}, c, 1\right):\left(\mathcal{M}^{\prime}, c\right) \in X\left[i_{j}\right]\right\} \cup\left\{\left(\mathcal{M}^{\prime}, c, 0\right):\left(\mathcal{M}^{\prime}, c\right) \in Y\left[i_{j}\right]\right\}
\end{aligned}
$$

In other words, we merge $X\left[i_{j}\right]$ and $Y\left[i_{j}\right]$ in $W_{i}\left[i_{j}\right]$, differentiating the origin of a pair by means of a third element (an integer that is equal to 1 for $X\left[i_{j}\right]$ and 0 for $Y\left[i_{j}\right]$ ). Clearly, each entry $W_{i}\left[i_{j}\right]$ contains at most $2 k^{q+1}$ triples, and hence table $W_{i}$ on the whole contains at most $2 s_{i} k^{q+1} \leq 2 n k^{q+1}$ triples. Table $V_{i}$ also contains $s_{i}$ entries, one for each son of vertex $i$ in the subtree rooted at $i$, that are computed as follows: $V_{i}\left[i_{1}\right]=W_{i}\left[i_{1}\right]$ and

$$
\begin{aligned}
& \forall 2 \leq j \leq s_{i}, \\
& V_{i}\left[i_{j}\right]=W_{i}\left[i_{j}\right] \cup\left\{\left(\mathcal{M}^{\prime} \cup \mathcal{M}^{\prime \prime}, c^{\prime}+c^{\prime \prime}, r^{\prime}+r^{\prime \prime}\right) \subseteq \mathcal{M} \times k \times k:\right. \\
& \left.\qquad\left(\mathcal{M}^{\prime}, c^{\prime}, r^{\prime}\right) \in W_{i}\left[i_{j}\right] \text { and }\left(\mathcal{M}^{\prime \prime}, c^{\prime \prime}, r^{\prime \prime}\right) \in V_{i}\left[i_{j-1}\right]\right\}
\end{aligned}
$$

Each entry $V_{i}\left[i_{j}\right]$ contains at most $k^{q+2}$ triples, and hence table $V_{i}$ on the whole contains at most $s_{i} k^{q+2} \leq n k^{q+2}$ triples. All the needed information is stored in $V_{i}\left[i_{s_{i}}\right]$, and $X[i]$ and $Y[i]$ can be now computed as follows:

$$
\begin{aligned}
& X[i]=\left\{\left(\mathcal{M}^{\prime}, c-r+1\right):\left(\mathcal{M}^{\prime}, c, r\right) \in V_{i}\left[i_{s_{i}}\right] \text { and } r>0\right\} \\
& Y[i]=\left\{\left(\mathcal{M}^{\prime}, c\right):\left(\mathcal{M}^{\prime}, c, 0\right) \in V_{i}\left[i_{s_{i}}\right]\right\}
\end{aligned}
$$

The two entries $X[i]$ and $Y[i]$ are next filtered according to the following procedure: for each submotif $\mathcal{M}^{\prime} \subseteq \mathcal{M}$ that occurs in at least one pair of $X[i]$ (resp. $Y[i]$ ), we keep in $X[i]$ (resp. $Y[i])$ the pair $\left(\mathcal{M}^{\prime}, c\right)$ with the minimum $c$.

The base cases, i.e., vertex $i$ is a leaf, are defined as follows: $X[i]=$ $\{(\lambda(i), 1)\}$ and $Y[i]=\emptyset$. In other words, $X[i]$ contains exactly one pair $\left(\mathcal{M}^{\prime}, c\right)$, where $\mathcal{M}^{\prime}$ consists in one occurrence of the color associated to vertex $i$, and $Y[i]$ does not contain any pair. The solution for the Min-CC problem consists in finding a pair $(\mathcal{M}, c)$ in $X$ or $Y$ with minimum $c$. If such a pair cannot be found in any entry of both $X$ and $Y$, then the motif $\mathcal{M}$ does not occur in the tree $G$.

Proposition 4.2. The Min-CC problem for trees is solvable in $\mathcal{O}\left(n^{2} k^{(q+1)^{2}+1}\right)$ time, where $n$ is the order of the target graph, $k$ is the size of the motif and $q$ is the number of distinct colors.

The above result is particularly interesting in view of the fact that the Min-CC problem for trees parameterized by $q$ is $\mathbf{W}[\mathbf{1}]-$ hard $^{8}$.

## 5. A polynomial-time algorithm for paths with a bounded number of colors

We complement here the results of the two preceding sections by showing that the Min-CC problem for paths is polynomial-time solvable in case the motif is built upon a fixed number of colors. Observe, however, that each color may still have an unbounded number of occurrences in the motif.

In what follows we describe a dynamic programming algorithm for this case. The basic idea of our approach is as follows. Suppose we are left by the algorithm with the problem of finding an occurrence of a submotif $\mathcal{M}^{\prime} \subseteq \mathcal{M}$ in the subpath $G^{\prime}$ of $G$ induced by $\{i, i+1, \ldots, j\}, 1 \leq i<j \leq n$. Furthermore, suppose that any occurrence of $\mathcal{M}^{\prime}$ in $G^{\prime}$ results in at least $k^{\prime}$ connected components. This minimum number of occurrences $k^{\prime}$ can be computed as follows. Assume that we have found one leftmost connected component $C_{\text {left }}$ of the occurrence of $\mathcal{M}^{\prime}$ in $G^{\prime}$ and let $i_{2}, i \leq i_{2}<j$, be the rightmost (according to the natural order of the vertices) vertex of $C_{\text {left }}$. Let $\mathcal{M}^{\prime \prime}$ be the motif obtained from $\mathcal{M}^{\prime}$ by subtracting to each color $c_{\ell} \in \mathcal{C}$ the number of occurrences of color $c_{\ell}$ in the leftmost connected component $C_{\text {left }}$. Then the occurrence of $\mathcal{M}^{\prime}$ in $G^{\prime}$ is given by $C_{\text {left }}$ plus the occurrence of the motif $\mathcal{M}^{\prime \prime}$ in the subpath $G^{\prime \prime}$ of $G^{\prime}$ induced by $\left\{i_{2}+1, i_{2}+2, \ldots, j\right\}$, which results in $k^{\prime}-1$ connected components. From an optimization point of view, the problem thus reduces to finding a subpath $\left\{i_{1}, i_{1}+1, \ldots, i_{2}\right\}$, $i \leq i_{1} \leq i_{2}<j$, such that the occurrence of the motif $\mathcal{M}^{\prime \prime}$ modified according to the colors in $\left\{i_{1}, i_{1}+1, \ldots, i_{2}\right\}$ in the subpath induced by $\left\{i_{2}+1, i_{2}+2, \ldots, j\right\}$ results in a minimum number of connected components.

Let $(G, \mathcal{M})$ be an instance of the Min-CC problem where $G$ is a (vertexcolored) path built upon the set of colors $\mathcal{C}$. For ease of exposition, write $\mathbf{V}(G)=\{1,2, \ldots, n\}$ and $q=|\mathcal{C}|$. We denote by $m_{i}$ the number of occurrences of color $c_{i} \in \mathcal{C}$ in $\mathcal{M}$. Clearly, $\sum_{c_{i} \in \mathcal{C}} m_{i}=|\mathcal{M}|$. We now introduce our dynamic programming table $T$. Define $T\left[i, j ; p_{1}, p_{2}, \ldots, p_{q}\right], 1 \leq i \leq j \leq n$ and $0 \leq p_{\ell} \leq m_{\ell}$ for $1 \leq \ell \leq q$, to be the minimum number of connected components in the subpath of $G$ that starts at node $i$, ends at node $j$ and that covers $p_{\ell}$ occurrences of color $c_{\ell}, 1 \leq \ell \leq q$. The base conditions are as follows:

- for all $1 \leq i \leq j \leq n, T[i, j ; 0,0, \ldots, 0]=0$ and $T\left[i, i ; p_{1}, p_{2}, \ldots, p_{q}\right]=\infty$ if $\sum_{1 \leq \ell \leq q} p_{\ell}>1$,
- for all $1 \leq i \leq n, T\left[i, i ; p_{1}, p_{2}, \ldots, p_{q}\right]=\infty$ if $\sum_{1 \leq \ell \leq q} p_{\ell}=$ 1 and $\lambda(i) \neq c_{\ell}$ and $p_{\ell}=1$, and $T\left[i, i ; p_{1}, p_{2}, \ldots, p_{q}\right]=1$ if $\sum_{1 \leq \ell \leq q} p_{\ell}=1$ and $\lambda(i)=c_{\ell}$ and $p_{\ell}=1$.

The entry $T\left[i, j ; p_{1}, p_{2}, \ldots, p_{q}\right]$ of the dynamic programming table $T$ can be computed by the following recurrence

$$
\begin{equation*}
T\left[i, j ; p_{1}, p_{2}, \ldots, p_{q}\right]=\min _{i \leq i_{1} \leq i_{2}<j} T\left[i_{2}+1, j ; p_{1}^{\prime}, p_{2}^{\prime}, \ldots, p_{q}^{\prime}\right]+1 \tag{1}
\end{equation*}
$$

where each $p_{\ell}^{\prime} \geq 0$ is equal to $p_{\ell}$ minus the number of occurrences of color $c_{\ell}$ in the subpath of $G$ induced by the vertices $\left\{i_{1}, i_{1}+1, \ldots, i_{2}\right\}$. The optimal solution is clearly stored in $T\left[1, n ; p_{1}, p_{2}, \ldots, p_{q}\right]$.

We claim that our dynamic programming table $T$ contains $\mathcal{O}\left(n^{q+2}\right)$ entries. Indeed, there are $q$ colors in $\mathcal{M}$, each color $c_{i} \in \mathcal{C}$ has at most $n$ occurrences in $G$ and we have $\mathcal{O}\left(n^{2}\right)$ subpaths in $G$ to consider. We now turn to evaluating the time complexity for computing $T\left[i, j ; p_{1}, p_{2}, \ldots, p_{q}\right]$. Assuming each entry $T\left[i^{\prime}, j^{\prime} ; p_{1}^{\prime}, p_{2}^{\prime}, \ldots, p_{q}^{\prime}\right]$ with $i \leq i^{\prime} \leq j^{\prime} \leq j$ and $\mid j^{\prime}-$ $i^{\prime}\left|<|j-i|\right.$ has already been computed, $T\left[i, j ; p_{1}, p_{2}, \ldots, p_{q}\right]$ is obtained by taking a minimum number among $\mathcal{O}\left(|j-i+1|^{2}\right)=\mathcal{O}\left(n^{2}\right)$ numbers, and hence is $\mathcal{O}\left(n^{2}\right)$ time. We have thus proved the following.

Proposition 5.1. The Min-CC problem for paths is solvable in $\mathcal{O}\left(n^{q+4}\right)$ time, where $n$ is the number of vertices and $q$ is the number of colors in $\mathcal{C}$.

As an immediate consequence of the above proposition, the Min-CC problem is polynomial-time solvable in case the motif $\mathcal{M}$ is built upon a fixed number of colors and the target graph $G$ is a path.

## 6. Hardness of approximation for trees

We investigate in this section approximation issues for restricted instances of the MIN-CC problem. Unfortunately, as we shall now prove, it turns out that, even if $\mathcal{M}$ is a set and $G$ is a tree, the Min-CC problem cannot be approximated within ratio $c \log n$ for some constant $c>0$, where $n$ is the size of the target graph $G$. As a side result, we prove that the Min-CC problem is $\mathbf{W}$ [2]-hard when parameterized by the number of connected components of the occurrence of $\mathcal{M}$ in the target graph $G$.

At the core of our proof is an L-reduction ${ }^{12}$ from the Set-Cover problem. Let $I$ be an arbitrary instance of the SET-Cover problem consisting of a universe set $X(I)=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ and a collection of sets $\mathcal{S}(I)=S_{1}, S_{2}, \ldots, S_{m}$, each over $X(I)$. For each $1 \leq i \leq m$, write $t_{i}=\left|S_{i}\right|$ and denote by $e_{j}\left(S_{i}\right), 1 \leq j \leq t_{i}$, the $j$-th element of $S_{i}$. For ease of exposition, we present the corresponding instance of the Min-CC problem as a rooted tree $G$. We construct the tree $G$ as follows (see Fig. 1). Define a root $r$ and vertices $S_{1}^{\prime}, S_{2}^{\prime}, \ldots, S_{m}^{\prime}$ such that each vertex $S_{i}^{\prime}$ is connected to the root $r$. For each $S_{i}^{\prime}$ define the subtree $G\left(S_{i}^{\prime}\right)$ rooted at $S_{i}^{\prime}$


Figure 1. Construction of the corresponding instance of the Min-CC problem.
as follows: each vertex $S_{i}^{\prime}$ has a unique child $S_{i}$ and each vertex $S_{i}$ has children $e_{1}\left(S_{i}\right), e_{2}\left(S_{i}\right), \ldots, e_{t_{i}}\left(S_{i}\right)$. The set of colors $\mathcal{C}$ is defined as follows: $\mathcal{C}=\left\{c\left(S_{i}\right): 1 \leq i \leq m\right\} \cup\left\{c\left(x_{j}\right): 1 \leq j \leq n\right\} \cup\{c(r)\}$. The coloring mapping $\lambda: \mathbf{V}(G) \rightarrow \mathcal{C}$ is defined by: $\lambda\left(S_{i}\right)=\lambda\left(S_{i}^{\prime}\right)=c\left(S_{i}\right)$ for $1 \leq i \leq m$, $\lambda\left(x_{j}\right)=c\left(x_{j}\right)$ for $1 \leq j \leq n$ and $\lambda(r)=c(r)$. The motif $\mathcal{M}$ is the set defined as follows: $\mathcal{M}=\left\{c\left(S_{i}\right): 1 \leq i \leq m\right\} \cup\left\{c\left(x_{i}\right): 1 \leq i \leq n\right\} \cup\{c(r)\}$.

Proposition 6.1. For any instance $I$ of the Set-Cover problem, there exists a solution of size $h$ for $I$, i.e., a subset $\mathcal{S} \subseteq \mathcal{S}(I),|\mathcal{S}|=h$, such that $\bigcup_{S_{i} \in \mathcal{S}} S_{i}=X$, if and only if then there exists an occurrence of $\mathcal{M}$ in $G$ that results in $h+1$ connected components.

It is easily seen that the above reduction is an L-reduction ${ }^{12}$. It is known that Set-Cover cannot be approximated within ratio $c \log n$ for some constant $c>0^{14}$. Then it follows that there exists a constant $c^{\prime}>0$ such that the Min-CC for trees cannot be approximated within performance ratio $c^{\prime} \log n$, where $n$ is the number of vertices in the target graph.

As a side result, we also observe that the above reduction is a parameterized reduction. Since the Set-Cover is $\mathbf{W}$ [2]-hard when parameterized by the size of the solution ${ }^{13}$, the following result holds.

Corollary 6.1. The MIN-CC problem for trees is $\mathbf{W}[2]$-hard when parameterized by the number of connected components of the occurrence of the motif in the graph.

## 7. An exact algorithm for trees

We proved in Section 4 that the Min-CC for trees is solvable in $\mathcal{O}\left(n^{2} k^{(q+1)^{2}+1}\right)$ time, where $n$ is the order of the target tree, $k$ is the size of the motif and $q$ is the number of distinct colors. We propose here a new algorithm for this special case, which turns out not to be a fixed-parameter algorithm but has a better running time in case the motif $k$ is not that small compared to the order $n$ of the target graph. More precisely, we give an algorithm for solving the Min-CC problem for trees that runs in $\mathcal{O}\left(n^{2} 2^{\frac{2 n}{3}}\right)$, where $n$ is the order of the target tree. Due to space constraints, we skip the proof details.

Let $T$ be the target tree. For any vertex $x$ of $T$, denote by $T(x)$ the subtree of $T$ rooted at $x$. The first step of our algorithm splits the target tree in a balanced way, so that $T$ is rooted at a vertex $r$ having children, $r_{1}, r_{2}, \ldots, r_{h}$ such that none of the trees $T\left(r_{i}\right), 1 \leq i \leq h$, has order greater than $\left\lceil\frac{n}{2}\right\rceil$. Such a vertex $r$ can be found in $\mathcal{O}\left(n^{2}\right)$ time. We then construct two disjoint subsets $R_{1}$ and $R_{2}$ of $r_{1}, \ldots, r_{h}$ with the property that

$$
\frac{1}{3}|T| \leq \sum_{r_{i} \in R_{1}}\left|T\left(r_{i}\right)\right| \leq\left\lceil\frac{1}{2}|T|\right\rceil \text { and }\left\lceil\frac{1}{2}|T|\right\rceil \leq \sum_{r_{i} \in R_{2}}\left|T\left(r_{i}\right)\right|=\frac{2}{3}|T|
$$

Given $V^{\prime}$ a subset of nodes of $V$, we say that $V^{\prime}$ does not violate $\mathcal{M}$ if the multiset of colors $C\left(V^{\prime}\right)$ is a subset of $\mathcal{M}$. Given a subtree $T^{\prime}$ of $T$, we define a partial solution $F$ of Min-CC over instance $\left(T^{\prime}, \mathcal{M}\right)$ as a set of connected components of $T^{\prime}$ that does not violate the multiset $\mathcal{M}$.

The algorithm computes an optimal solution for Min-CC by first computing all the partial solutions $S_{1}$ over instance $\left(R_{1}, \mathcal{M}\right)$ and all the partial solutions $S_{2}$ over instance $\left(R_{2}, \mathcal{M}\right)$ and then merging a partial solution $F_{1}$ of $S_{1}$ and a partial solution $F_{2}$ of $S_{2}$ into a feasible solution for the Min-CC over instance $(T, \mathcal{M})$. Since there are $2^{\frac{n}{2}}$ and $2^{\frac{2 n}{3}}$ possible subsets of vertices of $R_{1}$ and $R_{2}$ respectively, it follows that the set of partial solutions over instance $\left(R_{1}, \mathcal{M}\right),\left(R_{2}, \mathcal{M}\right)$ can be computed in time $\mathcal{O}\left(2^{\frac{n}{2}}\right)$ and $\mathcal{O}\left(2^{\frac{2 n}{3}}\right)$ respectively. Then set $S_{1}$ is ordered and by binary search we can find in time $\mathcal{O}\left(n \log 2^{\frac{n}{2}}\right)=\mathcal{O}\left(n^{2}\right)$ a solution $F_{1}$ of $S_{1}$ that, merged to a solution $F_{2}$ of $S_{2}$, produces a feasible solution of Min-CC over instance ( $T$, $\mathcal{M})$. Since $\left|S_{2}\right|=\mathcal{O}\left(2^{\frac{2 n}{3}}\right)$, it follows that the overall time complexity of the algorithm is $\mathcal{O}\left(n^{2} 2^{\frac{2 n}{3}}\right)$.

## 8. Conclusion

We mention here some possible directions for future works. First, approximation issues of the Min-CC problem are widely unexplored. In particular,
is the Min-CC problem for paths approximable within a constant ? Also, most parameterized complexity issues are to be discovered. Of particular importance: is the Min-CC problem for paths $\mathbf{W}$ [1]-hard when parameterized by the number of connected components in the occurrence of the motif in the target graph?

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