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Non-parametric estimation in a semimartingale regression model. Part 1. Oracle Inequalities. *

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Abstract

This paper considers the problem of estimating a periodic function in a continuous time regression model with a general square integrable semimartingale noise. A model selection adaptive procedure is proposed. Sharp non-asymptotic oracle inequalities have been derived.

Keywords: Non-asymptotic estimation; Non-parametric regression; Model selection; Sharp oracle inequality; Semimartingale noise.

AMS 2000 Subject Classifications: Primary: 62G08; Secondary: 62G05

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1 Introduction

Consider a regression model in continuous time

$$dy_t = S(t)dt + d\xi_t, \quad 0 \leq t \leq n, \quad (1.1)$$

where S is an unknown 1-periodic $\mathbb{R} \rightarrow \mathbb{R}$ function, $S \in \mathcal{L}_2[0, n]$; $(\xi_t)_{t \geq 0}$ is a square integrable unobservable semimartingale noise such that for any function f from $\mathcal{L}_2[0, n]$ the stochastic integral

$$I_n(f) = \int_0^n f_s d\xi_s \quad (1.2)$$

is well defined with

$$\mathbf{E}I_n(f) = 0 \quad \text{and} \quad \mathbf{E}I_n^2(f) \leq \sigma^* \int_0^n f_s^2 ds \quad (1.3)$$

where σ^* is some positive constant.

An important example of the disturbance $(\xi_t)_{t \geq 0}$ is the following process

$$\xi_t = \varrho_1 w_t + \varrho_2 z_t \quad (1.4)$$

where ϱ_1 and ϱ_2 are unknown constants, $|\varrho_1| + |\varrho_2| > 0$, $(w_t)_{t \geq 0}$ is a standard Brownian motion, $(z_t)_{t \geq 0}$ is a compound Poisson process defined as

$$z_t = \sum_{j=1}^{N_t} Y_j \quad (1.5)$$

where $(N_t)_{t \geq 0}$ is a standard homogeneous Poisson process with unknown intensity $\lambda > 0$ and $(Y_j)_{j \geq 1}$ is an i.i.d. sequence of random variables with

$$\mathbf{E}Y_j = 0 \quad \text{and} \quad \mathbf{E}Y_j^2 = 1. \quad (1.6)$$

Let $(T)_{k \geq 1}$ denote the arrival times of the process $(N_t)_{t \geq 0}$, that is,

$$T_k = \inf\{t \geq 0 : N_t = k\}. \quad (1.7)$$

As is shown in Lemma A.2, the condition (1.3) holds for the noise (1.4) with $\sigma^* = \varrho_1^2 + \varrho_2^2 \lambda$.

The problem is to estimate the unknown function S in the model (1.1) on the basis of observations $(y_t)_{0 \leq t \leq n}$.

This problem enables one to solve that of functional statistics which is stated as follows. Let observations $(x^k)_{0 \leq k \leq n}$ be a segment of a sequence of independent identically distributed random processes $x^k = (x_t^k)_{0 \leq t \leq 1}$ specified on the interval $[0, 1]$, which obey the stochastic differential equations

$$dx_t^k = S(t)dt + d\xi_t^k, \quad x_0^k = x_0, \quad 0 \leq t \leq 1, \quad (1.8)$$

where $(\xi^k)_{1 \leq k \leq n}$ is an i.i.d sequence of random processes $\xi^k = (\xi_t^k)_{0 \leq t \leq 1}$ with the same distribution as the process (1.4). The problem is to estimate the unknown function $f(t) \in \mathcal{L}_2[0, 1]$ on the basis of observations $(x^k)_{1 \leq k \leq n}$. This model can be reduced to (1.1), (1.4) in the following way. Let $y = (y_t)_{0 \leq t \leq n}$ denote the process defined as :

$$y_t = \begin{cases} x_t^1, & \text{if } 0 \leq t \leq 1; \\ y_{k-1} + x_{t-k+1}^k - x_0, & \text{if } k-1 \leq t \leq k, \quad 2 \leq k \leq n. \end{cases}$$

This process satisfies the stochastic differential equation

$$dy_t = \tilde{S}(t)dt + d\tilde{\xi}_t,$$

where $\tilde{S}(t) = S(\{t\})$ and

$$\tilde{\xi}_t = \begin{cases} \xi_t^1, & \text{if } 0 \leq t \leq 1; \\ \tilde{\xi}_{k-1} + \xi_{t-k+1}^k, & \text{if } k-1 \leq t \leq k, \quad 2 \leq k \leq n; \end{cases}$$

$\{t\} = t - [t]$ is the fractional part of number t .

In this paper we will consider the estimation problem for the regression model (1.1) in $\mathcal{L}_2[0, 1]$ with the quality of an estimate \hat{S} being measured by the mean integrated squared error (MISE)

$$\mathcal{R}(\hat{S}, S) := \mathbf{E}_S \|\hat{S} - S\|^2, \quad (1.9)$$

where \mathbf{E}_S stands for the expectation with respect to the distribution \mathbf{P}_S of the process (1.1) given S ;

$$\|f\|^2 := \int_0^1 f^2(t)dt.$$

It is natural to treat this problem from the standpoint of the model selection approach. The origin of this method goes back to early seventies with the pioneering papers by Akaike [1] and Mallows [16] who proposed to introduce penalizing in a log-likelihood type criterion. The further progress has been made by Barron, Birge and Massart [2], [17] who developed a non-asymptotic model selection method which enabled one to derive non-asymptotic oracle inequalities for a gaussian non-parametric regression model with the i.i.d. disturbance. An oracle inequality yields the upper bound for the estimate risk via the minimal risk corresponding to a chosen family of estimates. Galtchouk and Pergamenshchikov [6] developed the Barron-Birge-Massart technique treating the problem of estimating a non-parametric drift function in a diffusion process from the standpoint of sequential analysis. Fourdrinier and Pergamenshchikov [5] extended the Barron-Birge-Massart method to the models with dependent observations and, in contrast to all above-mentioned papers on the model selection method, where the estimation procedures were based on the least squares estimates, they proposed to use an arbitrary family of projective estimates in an adaptive estimation procedure, and they discovered that one can employ the improved least square estimates to increase the estimation quality. Konev and Pergamenshchikov [14] applied this improved model selection method to the non-parametric estimation problem of a periodic function in a model with a coloured noise in continuous time having unknown spectral characteristics. In all cited papers the non-asymptotic oracle inequalities have been derived which enable one to establish the optimal convergence rate for the minimax risks. Moreover, in the latter paper the oracle inequalities have been found for the robust risks.

In addition to the optimal convergence rate, an important problem is that of the efficiency of adaptive estimation procedures. In order to examine the efficiency property one has to obtain the oracle inequalities in which the principal term has the factor close to unity.

The first result in this direction is most likely due to Kneip [13] who obtained, for a gaussian regression model, the oracle inequality with the factor close to unity at the principal term. The oracle inequalities of this type were obtained as well in [3] and in [4] for the inverse problems. It will be observed that the derivation of oracle inequalities in all these papers rests upon the fact that by applying the Fourier transformation one can reduce the initial model to the statistical gaussian model with independent observations. Such a transform is possible only for gaussian models with independent homogeneous observations or for the inhomogeneous ones with the known correlation

characteristics. This restriction significantly narrows the area of application of such estimation procedures and rules out a broad class of models including, in particular, widely used in econometrics heteroscedastic regression models (see, for example, [12]). For constructing adaptive procedures in the case of inhomogeneous observations one needs to amend the approach to the estimation problem. Galtchouk and Pergamenshchikov [7]-[8] have developed a new estimation method intended for the heteroscedastic regression models. The heart of this method is to combine the Barron-Birgé-Massart non-asymptotic penalization method [2] and the Pinsker weighted least square method minimizing the asymptotic risk (see, for example, [18], [19]). Combining of these approaches results in the significant improvement of the estimation quality (see numerical example in [7]). As was shown in [8] and [9], the Galtchouk-Pergamenshchikov procedure is efficient with respect to the robust minimax risk, i.e. the minimax risk with the additional supremum operation over the whole family of admissible model distributions. In the sequel [10], [11], this approach has been applied to the problem of a drift estimation in a diffusion process. In this paper we apply this procedure to the estimation of a regression function S in a semimartingale regression model (1.1). The rest of the paper is organized as follows. In Section 2 we construct the model selection procedure on the basis of weighted least squares estimates and state the main results in the form of oracle inequalities for the quadratic risks. Section 3 gives the proofs of all theorems. In Appendix some technical results are established.

2 Model selection

This Section gives the construction of a model selection procedure for estimating a function S in (1.1) on the basis of weighted least square estimates and states the main results.

For estimating the unknown function S in model (1.1), we apply its Fourier expansion in the trigonometric basis $(\phi_j)_{j \geq 1}$ in $\mathcal{L}_2[0, 1]$ defined as

$$\phi_1 = 1, \quad \phi_j(x) = \sqrt{2} Tr_j(2\pi[j/2]x), \quad j \geq 2, \quad (2.1)$$

where the function $Tr_j(x) = \cos(x)$ for even j and $Tr_j(x) = \sin(x)$ for odd j ; $[x]$ denotes the integer part of x . The corresponding Fourier coefficients

$$\theta_j = (S, \phi_j) = \int_0^1 S(t) \phi_j(t) dt \quad (2.2)$$

can be estimated as

$$\widehat{\theta}_{j,n} = \frac{1}{n} \int_0^n \phi_j(t) dy_t. \quad (2.3)$$

In view of (1.1), we obtain

$$\widehat{\theta}_{j,n} = \theta_j + \frac{1}{\sqrt{n}} \xi_{j,n}, \quad \xi_{j,n} = \frac{1}{\sqrt{n}} I_n(\phi_j) \quad (2.4)$$

where I_n is given in (1.2).

For any sequence $x = (x_j)_{j \geq 1}$, we set

$$|x|^2 = \sum_{j=1}^{\infty} x_j^2 \quad \text{and} \quad \#(x) = \sum_{j=1}^{\infty} \mathbf{1}_{\{|x_j| > 0\}}. \quad (2.5)$$

Now we impose the additional conditions on the noise $(\xi_t)_{t \geq 0}$.

C₁) *There exists some positive constant $\sigma > 0$ such that the sequence*

$$\varsigma_{j,n} = \mathbf{E} \xi_{j,n}^2 - \sigma, \quad j \geq 1,$$

for any $n \geq 1$, satisfies the following inequality

$$c_1^*(n) = \sup_{x \in \mathcal{H}, \#(x) \leq n} |B_{1,n}(x)| < \infty$$

where $\mathcal{H} = [-1, 1]^\infty$ and

$$B_{1,n}(x) = \sum_{j=1}^{\infty} x_j \varsigma_{j,n}. \quad (2.6)$$

C₂) *Assume that for all $n \geq 1$*

$$c_2^*(n) = \sup_{|x| \leq 1, \#(x) \leq n} \mathbf{E} B_{2,n}^2(x) < \infty$$

where

$$B_{2,n}(x) = \sum_{j=1}^{\infty} x_j \widetilde{\xi}_{j,n} \quad \text{with} \quad \widetilde{\xi}_{j,n} = \xi_{j,n}^2 - \mathbf{E} \xi_{j,n}^2. \quad (2.7)$$

As is stated in Theorem 2.2, Conditions \mathbf{C}_1) and \mathbf{C}_2) hold for the process (1.4). Further we introduce a class of weighted least squares estimates for $S(t)$ defined as

$$\widehat{S}_\gamma = \sum_{j=1}^{\infty} \gamma(j) \widehat{\theta}_{j,n} \phi_j, \quad (2.8)$$

where $\gamma = (\gamma(j))_{j \geq 1}$ is a sequence of weight coefficients such that

$$0 \leq \gamma(j) \leq 1 \quad \text{and} \quad 0 < \#(\gamma) \leq n. \quad (2.9)$$

Let Γ denote a finite set of weight sequences $\gamma = (\gamma(j))_{j \geq 1}$ with these properties, $\nu = \text{card}(\Gamma)$ be its cardinal number and

$$\mu = \max_{\gamma \in \Gamma} \#(\gamma). \quad (2.10)$$

The model selection procedure for the unknown function S in (1.1) will be constructed on the basis of estimates $(\widehat{S}_\gamma)_{\gamma \in \Gamma}$. The choice of a specific set of weight sequences Γ will be discussed at the end of this section. In order to find a proper weight sequence γ in the set Γ one needs to specify a cost function. When choosing an appropriate cost function one can use the following argument. The empirical squared error

$$\text{Err}_n(\gamma) = \|\widehat{S}_\gamma - S\|^2$$

can be written as

$$\text{Err}_n(\gamma) = \sum_{j=1}^{\infty} \gamma^2(j) \widehat{\theta}_{j,n}^2 - 2 \sum_{j=1}^{\infty} \gamma(j) \widehat{\theta}_{j,n} \theta_j + \sum_{j=1}^{\infty} \theta_j^2. \quad (2.11)$$

Since the Fourier coefficients $(\theta_j)_{j \geq 1}$ are unknown, the weight coefficients $(\gamma_j)_{j \geq 1}$ can not be determined by minimizing this quantity. To circumvent this difficulty one needs to replace the terms $\widehat{\theta}_{j,n} \theta_j$ by some their estimators $\widetilde{\theta}_{j,n}$. We set

$$\widetilde{\theta}_{j,n} = \widehat{\theta}_{j,n}^2 - \frac{\widehat{\sigma}_n}{n} \quad (2.12)$$

where $\widehat{\sigma}_n$ is an estimator for the quantity σ in condition \mathbf{C}_1).

For this change in the empirical squared error, one has to pay some penalty. Thus, one comes to the cost function of the form

$$J_n(\gamma) = \sum_{j=1}^{\infty} \gamma^2(j) \widehat{\theta}_{j,n}^2 - 2 \sum_{j=1}^{\infty} \gamma(j) \widetilde{\theta}_{j,n} + \rho \widehat{P}_n(\gamma) \quad (2.13)$$

where ρ is some positive constant, $\widehat{P}_n(\gamma)$ is the penalty term defined as

$$\widehat{P}_n(\gamma) = \frac{\widehat{\sigma}_n |\gamma|^2}{n}. \quad (2.14)$$

In the case when the value of σ in \mathbf{C}_1) is known, one can put $\widehat{\sigma}_n = \sigma$ and

$$P_n(\gamma) = \frac{\sigma |\gamma|^2}{n}. \quad (2.15)$$

Substituting the weight coefficients, minimizing the cost function, that is

$$\widehat{\gamma} = \operatorname{argmin}_{\gamma \in \Gamma} J_n(\gamma), \quad (2.16)$$

in (2.8) leads to the model selection procedure

$$\widehat{S}_* = \widehat{S}_{\widehat{\gamma}}. \quad (2.17)$$

It will be noted that $\widehat{\gamma}$ exists, since Γ is a finite set. If the minimizing sequence in (2.16) $\widehat{\gamma}$ is not unique, one can take any minimizer.

Theorem 2.1. *Assume that the conditions \mathbf{C}_1) and \mathbf{C}_2) hold with $\sigma > 0$. Then for any $n \geq 1$ and $0 < \rho < 1/3$, the estimator (2.17) satisfies the oracle inequality*

$$\mathcal{R}(\widehat{S}_*, S) \leq \frac{1 + 3\rho - 2\rho^2}{1 - 3\rho} \min_{\gamma \in \Gamma} \mathcal{R}(\widehat{S}_\gamma, S) + \frac{1}{n} \mathcal{B}_n^*(\rho) \quad (2.18)$$

where the risk $\mathcal{R}(\cdot, S)$ is defined in (1.9),

$$\mathcal{B}_n^*(\rho) = \Psi_n(\rho) + \frac{6\mu \mathbf{E}_S |\widehat{\sigma}_n - \sigma|}{1 - 3\rho}$$

and

$$\Psi_n(\rho) = \frac{2\sigma\sigma^*\nu + 4\sigma c_1^*(n) + 2\nu c_2^*(n)}{\sigma\rho(1 - 3\rho)}. \quad (2.19)$$

Now we check conditions \mathbf{C}_1) and \mathbf{C}_2) for the model (1.1) with the noise (1.4) to arrive at the following result.

Theorem 2.2. *Suppose that the coefficients ϱ_1 and ϱ_2 in model (1.1), (1.4), are such that $\varrho_1^2 + \varrho_2^2 > 0$ and $\mathbf{E}Y_j^4 < \infty$. Then the estimation procedure (2.17), for any $n \geq 1$ and $0 < \rho \leq 1/3$, satisfies the oracle inequality (2.18) with*

$$\sigma = \sigma^* = \varrho_1^2 + \lambda \varrho_2^2, \quad c_1^*(n) = 0,$$

and

$$\sup_{n \geq 1} c_2^*(n) \leq 4\sigma^* (\sigma^* + \varrho_2^2 \mathbf{E}Y_1^4).$$

The proofs of Theorems 2.1, 2.2 are given in Section 3.

Corollary 2.3. *Let the conditions of Theorem 2.1 hold and the quantity σ in \mathbf{C}_1) be known. Then, for any $n \geq 1$ and $0 < \rho < 1/3$, the estimator (2.17) satisfies the oracle inequality*

$$\mathcal{R}(\hat{S}_*, S) \leq \frac{1 + 3\rho - 2\rho^2}{1 - 3\rho} \min_{\gamma \in \Gamma} \mathcal{R}(\hat{S}_\gamma, S) + \frac{1}{n} \Psi_n(\rho),$$

where $\Psi_n(\rho)$ is given in (2.19).

2.1 Estimation of σ

Now we consider the case of unknown quantity σ in the condition \mathbf{C}_1). One can estimate σ as

$$\hat{\sigma}_n = \sum_{j=l}^n \hat{\theta}_{j,n}^2 \quad \text{with} \quad l = [\sqrt{n}] + 1. \quad (2.20)$$

Proposition 2.4. *Suppose that the conditions of Theorem 2.1 hold and the unknown function $S(t)$ is continuously differentiable for $0 \leq t < 1$ such that*

$$|\dot{S}|_1 = \int_0^1 |\dot{S}(t)| dt < +\infty. \quad (2.21)$$

Then, for any $n \geq 1$,

$$\mathbf{E}_S |\hat{\sigma}_n - \sigma| \leq \frac{\kappa_n(S)}{\sqrt{n}} \quad (2.22)$$

where

$$\kappa_n(S) = 4|\dot{S}|_1^2 + \sigma + \sqrt{c_2^*(n)} + \frac{4|\dot{S}|_1 \sqrt{\sigma^*}}{n^{1/4}} + \frac{c_1^*(n)}{n^{1/2}}.$$

The proof of Proposition 2.4 is given in Section 3. Theorem 2.1 and Proposition 2.4 imply the following result.

Theorem 2.5. *Suppose that the conditions of Theorem 2.1 hold and S satisfies the conditions of Proposition 2.4. Then, for any $n \geq 1$ and $0 < \rho < 1/3$, the estimate (2.17) satisfies the oracle inequality*

$$\mathcal{R}(\widehat{S}_*, S) \leq \frac{1 + 3\rho - 2\rho^2}{1 - 3\rho} \min_{\gamma \in \Gamma} \mathcal{R}(\widehat{S}_\gamma, S) + \frac{1}{n} \mathcal{D}_n(\rho), \quad (2.23)$$

where

$$\mathcal{D}_n(\rho) = 2 \Psi_n(\rho) + \frac{2\rho(1 - \rho)\mu\kappa_n(S)}{(1 - 3\rho)\sqrt{n}}.$$

2.2 Specification of weights in the selection procedure (2.17)

Now we will specify the weight coefficients $(\gamma(j))_{j \geq 1}$ in a way proposed in [7] for a heteroscedastic discrete time regression model. Consider a numerical grid of the form

$$\mathcal{A}_n = \{1, \dots, k^*\} \times \{t_1, \dots, t_m\},$$

where $t_i = i\varepsilon$ and $m = \lceil 1/\varepsilon^2 \rceil$. We assume that both parameters $k^* \geq 1$ and $0 < \varepsilon \leq 1$ are functions of n , i.e. $k^* = k^*(n)$ and $\varepsilon = \varepsilon(n)$, such that

$$\begin{cases} \lim_{n \rightarrow \infty} k^*(n) = +\infty, & \lim_{n \rightarrow \infty} \frac{k^*(n)}{\ln n} = 0, \\ \lim_{n \rightarrow \infty} \varepsilon(n) = 0 & \text{and} \quad \lim_{n \rightarrow \infty} n^\delta \varepsilon(n) = +\infty \end{cases} \quad (2.24)$$

for any $\delta > 0$. One can take, for example,

$$\varepsilon(n) = \frac{1}{\ln(n+1)} \quad \text{and} \quad k^*(n) = \sqrt{\ln(n+1)}$$

for $n \geq 1$.

For each $\alpha = (\beta, t) \in \mathcal{A}_n$, we introduce the weight sequence

$$\gamma_\alpha = (\gamma_\alpha(j))_{j \geq 1}$$

given as

$$\gamma_\alpha(j) = \mathbf{1}_{\{1 \leq j \leq j_0\}} + (1 - (j/\omega_\alpha)^\beta) \mathbf{1}_{\{j_0 < j \leq \omega_\alpha\}} \quad (2.25)$$

where $j_0 = j_0(\alpha) = \lfloor \omega_\alpha / \ln n \rfloor$,

$$\omega_\alpha = (\tau_\beta t n)^{1/(2\beta+1)} \quad \text{and} \quad \tau_\beta = \frac{(\beta+1)(2\beta+1)}{\pi^{2\beta}\beta}.$$

We set

$$\Gamma = \{\gamma_\alpha, \alpha \in \mathcal{A}_n\}. \quad (2.26)$$

It will be noted that in this case $\nu = k^*m$.

Remark 2.1. *It will be observed that the specific form of weights (2.25) was proposed by Pinsker [19] for the filtration problem with known smoothness of regression function observed with an additive gaussian white noise in the continuous time. Nussbaum [18] used these weights for the gaussian regression estimation problem in discrete time.*

The minimal mean square risk, called the Pinsker constant, is provided by the weight least squares estimate with the weights where the index α depends on the smoothness order of the function S . In this case the smoothness order is unknown and, instead of one estimate, one has to use a whole family of estimates containing in particular the optimal one.

The problem is to study the properties of the whole class of estimates. Below we derive an oracle inequality for this class which yields the best mean square risk up to a multiplicative and additive constants provided that the smoothness of the unknown function S is not available. Moreover, it will be shown that the multiplicative constant tends to unity and the additive one vanishes as $n \rightarrow \infty$ with the rate higher than any minimax rate.

In view of the assumptions (2.24), for any $\delta > 0$, one has

$$\lim_{n \rightarrow \infty} \frac{\nu}{n^\delta} = 0.$$

Moreover, by (2.25) for any $\alpha \in \mathcal{A}_n$

$$\sum_{j=1}^{\infty} \mathbf{1}_{\{\gamma_\alpha(j) > 0\}} \leq \omega_\alpha.$$

Therefore, taking into account that $A_\beta \leq A_1 < 1$ for $\beta \geq 1$, we get

$$\mu = \mu_n \leq (n/\varepsilon)^{1/3}.$$

Therefore, for any $\delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{\mu_n}{n^{1/3+\delta}} = 0.$$

Applying this limiting relation to the analysis of the asymptotic behavior of the additive term $\mathcal{D}_n(\rho)$ in (2.23) one comes to the following result.

Theorem 2.6. *Suppose that the conditions of Theorem 2.1 hold and $\dot{S} \in \mathcal{L}_1[0, 1]$. Then, for any $n \geq 1$ and $0 < \rho < 1/3$, the estimate (2.17) with the weight coefficients (2.26) satisfies the oracle inequality (2.23) with the additive term $\mathcal{D}_n(\rho)$ obeying, for any $\delta > 0$, the following limiting relation*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{D}_n(\rho)}{n^\delta} = 0.$$

3 Proofs

3.1 Proof of Theorem 2.1

Substituting (2.13) in (2.11) yields for any $\gamma \in \Gamma$

$$\text{Err}_n(\gamma) = J_n(\gamma) + 2 \sum_{j=1}^{\infty} \gamma(j) \theta'_{j,n} + \|S\|^2 - \rho \widehat{P}_n(\gamma), \quad (3.1)$$

where

$$\theta'_{j,n} = \widetilde{\theta}_{j,n} - \theta_j \widehat{\theta}_{j,n} = \frac{1}{\sqrt{n}} \theta_j \xi_{j,n} + \frac{1}{n} \widetilde{\xi}_{j,n} + \frac{1}{n} \varsigma_{j,n} + \frac{\sigma - \widehat{\sigma}_n}{n}$$

and the sequences $(\varsigma_{j,n})_{j \geq 1}$ and $(\widetilde{\xi}_{j,n})_{j \geq 1}$ are defined in conditions \mathbf{C}_1) and \mathbf{C}_2). Denoting

$$L(\gamma) = \sum_{j=1}^{\infty} \gamma(j), \quad M(\gamma) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\infty} \gamma(j) \theta_j \xi_{j,n}, \quad (3.2)$$

and taking into account the definition of the "true" penalty term in (2.15), we rewrite (3.1) as

$$\begin{aligned} \text{Err}_n(\gamma) &= J_n(\gamma) + 2 \frac{\sigma - \widehat{\sigma}_n}{n} L(\gamma) + 2M(\gamma) + \frac{2}{n} B_{1,n}(\gamma) \\ &\quad + 2\sqrt{P_n(\gamma)} \frac{B_{2,n}(e(\gamma))}{\sqrt{\sigma n}} + \|S\|^2 - \rho \widehat{P}_n(\gamma) \end{aligned} \quad (3.3)$$

where $e(\gamma) = \gamma/|\gamma|$, the functions $B_{1,n}$ and $B_{2,n}$ are defined in (2.6) and (2.7).

Let $\gamma_0 = (\gamma_0(j))_{j \geq 1}$ be a fixed sequence in Γ and $\hat{\gamma}$ be as in (2.16). Substituting γ_0 and $\hat{\gamma}$ in the equation (3.3), we consider the difference

$$\begin{aligned} \text{Err}_n(\hat{\gamma}) - \text{Err}_n(\gamma_0) &= J(\hat{\gamma}) - J(\gamma_0) + 2\frac{\sigma - \hat{\sigma}}{n} L(\hat{x}) + \frac{2}{n} B_{1,n}(\hat{x}) + 2M(\hat{x}) \\ &\quad + 2\sqrt{P_n(\hat{\gamma})} \frac{B_{2,n}(\hat{e})}{\sqrt{\sigma n}} - 2\sqrt{P_n(\gamma_0)} \frac{B_{2,n}(e_0)}{\sqrt{\sigma n}} \\ &\quad - \rho \hat{P}_n(\hat{\gamma}) + \rho \hat{P}_n(\gamma_0) \end{aligned}$$

where $\hat{x} = \hat{\gamma} - \gamma_0$, $\hat{e} = e(\hat{\gamma})$ and $e_0 = e(\gamma_0)$. Note that by (2.10)

$$|L(\hat{x})| \leq |L(\hat{\gamma})| + |L(\gamma_0)| \leq 2\mu.$$

Therefore, by making use of the condition \mathbf{C}_1) and taking into account that the cost function J attains its minimum at $\hat{\gamma}$, one comes to the inequality

$$\begin{aligned} \text{Err}_n(\hat{\gamma}) - \text{Err}_n(\gamma_0) &\leq 4\frac{|\hat{\sigma} - \sigma|}{n} \mu + \frac{2c_1^*(n)}{n} + 2M(\hat{x}) \\ &\quad + 2\sqrt{P_n(\hat{\gamma})} \frac{B_{2,n}(\hat{e})}{\sqrt{\sigma n}} - \rho \hat{P}_n(\hat{\gamma}) \\ &\quad + \rho \hat{P}_n(\gamma_0) - 2\sqrt{P_n(\gamma_0)} \frac{B_{2,n}(e_0)}{\sqrt{\sigma n}}. \end{aligned} \quad (3.4)$$

Applying the elementary inequality

$$2|ab| \leq \varepsilon a^2 + \varepsilon^{-1} b^2 \quad (3.5)$$

with $\varepsilon = \rho$ implies the estimate

$$2\sqrt{P_n(\gamma)} \frac{|B_{2,n}(e(\gamma))|}{\sqrt{\sigma n}} \leq \rho P_n(\gamma) + \frac{B_{2,n}^2(e(\gamma))}{n\sigma\rho}.$$

We recall that $0 < \rho < 1$. Therefore, from here and (3.4), it follows that

$$\begin{aligned} \text{Err}_n(\hat{\gamma}) &\leq \text{Err}_n(\gamma_0) + 2M(\hat{x}) + \frac{2B_{2,n}^*}{n\sigma\rho} + \frac{2c_1^*(n)}{n} \\ &\quad + \frac{1}{n} |\hat{\sigma} - \sigma| (|\hat{\gamma}|^2 + |\gamma_0|^2 + 4\mu) + 2\rho P_n(\gamma_0) \end{aligned}$$

where $B_{2,n}^* = \sup_{\gamma \in \Gamma} B_{2,n}^2(e(\gamma))$. In view of (2.10), one has

$$\sup_{\gamma \in \Gamma} |\gamma|^2 \leq \mu.$$

Thus, one gets

$$\begin{aligned} \text{Err}_n(\hat{\gamma}) &\leq \text{Err}_n(\gamma_0) + 2M(\hat{x}) + \frac{2B_{2,n}^*}{n\sigma\rho} + \frac{2c_1^*(n)}{n} \\ &\quad + \frac{6\mu}{n} |\hat{\sigma}_n - \sigma| + 2\rho P_n(\gamma_0). \end{aligned} \quad (3.6)$$

In view of Condition \mathbf{C}_2), one has

$$\mathbf{E}_S B_{2,n}^* \leq \sum_{\gamma \in \Gamma} \mathbf{E}_S B_{2,n}^2(e(\gamma)) \leq \nu c_2^*(n) \quad (3.7)$$

where $\nu = \text{card}(\Gamma)$.

Now we examine the first term in the right-hand side of (3.4). Substituting (2.4) in (3.2) and taking into account (1.3), one obtains that for any non-random sequence $x = (x(j))_{j \geq 1}$ with $\#(x) < \infty$

$$\mathbf{E}_S M^2(x) \leq \sigma^* \frac{1}{n} \sum_{j=1}^{\infty} x^2(j) \theta_j^2 = \sigma^* \frac{1}{n} \|S_x\|^2 \quad (3.8)$$

where $S_x = \sum_{j=1}^{\infty} x(j) \theta_j \phi_j$. Let denote

$$Z^* = \sup_{x \in \Gamma_1} \frac{nM^2(x)}{\|S_x\|^2}$$

where $\Gamma_1 = \Gamma - \gamma_0$. In view of (3.8), this quantity can be estimated as

$$\mathbf{E}_S Z^* \leq \sum_{x \in \Gamma_1} \frac{n\mathbf{E}_S M^2(x)}{\|S_x\|^2} \leq \sum_{x \in \Gamma_1} \sigma^* = \sigma^* \nu. \quad (3.9)$$

Further, by making use of the inequality (3.5) with $\varepsilon = \rho \|S_x\|$, one gets

$$2|M(x)| \leq \rho \|S_x\|^2 + \frac{Z^*}{n\rho}. \quad (3.10)$$

Note that, for any $x \in \Gamma_1$,

$$\|S_x\|^2 - \|\widehat{S}_x\|^2 = \sum_{j=1}^{\infty} x^2(j)(\theta_j^2 - \widehat{\theta}_{j,n}^2) \leq -2M_1(x) \quad (3.11)$$

where

$$M_1(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\infty} x^2(j)\theta_j\xi_{j,n}.$$

Since $|x(j)| \leq 1$ for any $x \in \Gamma_1$, one gets

$$\mathbf{E}_S M_1^2(x) \leq \sigma^* \frac{\|S_x\|^2}{n}.$$

Denoting

$$Z_1^* = \sup_{x \in \Gamma_1} \frac{nM_1^2(x)}{\|S_x\|^2},$$

one has

$$\mathbf{E}_S Z_1^* \leq \sigma^* \nu. \quad (3.12)$$

By the same argument as in (3.10), one derives

$$2|M_1(x)| \leq \rho\|S_x\|^2 + \frac{Z_1^*}{n\rho}.$$

From here and (3.11), one finds the upper bound for $\|S_x\|$, i.e.

$$\|S_x\|^2 \leq \frac{\|\widehat{S}_x\|^2}{1-\rho} + \frac{Z_1^*}{n\rho(1-\rho)}. \quad (3.13)$$

Using this bound in (3.10) gives

$$2M(x) \leq \frac{\rho\|\widehat{S}_x\|^2}{1-\rho} + \frac{Z^* + Z_1^*}{n\rho(1-\rho)}.$$

Setting $x = \widehat{x}$ in this inequality and taking into account that

$$\|\widehat{S}_{\widehat{x}}\|^2 = \|\widehat{S}_{\widehat{\gamma}} - \widehat{S}_{\gamma_0}\|^2 \leq 2(\text{Err}_n(\widehat{\gamma}) + \text{Err}_n(\gamma_0)),$$

we obtain

$$2M(\widehat{x}) \leq \frac{2\rho(\text{Err}_n(\widehat{\gamma}) + \text{Err}_n(\gamma_0))}{1-\rho} + \frac{Z^* + Z_1^*}{n\rho(1-\rho)}.$$

From here and (3.6), it follows that

$$\begin{aligned} \text{Err}_n(\widehat{\gamma}) &\leq \frac{1+\rho}{1-3\rho} \text{Err}_n(\gamma_0) + \frac{2(1-\rho)}{n(1-3\rho)} \left(\frac{B_{2,n}^*}{\sigma\rho} + c_1^*(n) + 3\mu|\widehat{\sigma} - \sigma| \right) \\ &\quad + \frac{Z^* + Z_1^*}{n\rho(1-3\rho)} + \frac{2\rho(1-\rho)}{1-3\rho} P_n(\gamma_0), \end{aligned}$$

Taking the expectation yields

$$\begin{aligned} \mathcal{R}(\widehat{S}_*, S) &\leq \frac{1+\rho}{1-3\rho} \mathcal{R}(\widehat{S}_{\gamma_0}, S) + \frac{2(1-\rho)}{n(1-3\rho)} \left(\frac{\nu c_2^*(n)}{\sigma\rho} + c_1^*(n) + 3\mu \mathbf{E}_S |\widehat{\sigma} - \sigma| \right) \\ &\quad + \frac{2\sigma^*\nu}{n\rho(1-3\rho)} + \frac{2\rho(1-\rho)}{1-3\rho} P_n(\gamma_0). \end{aligned}$$

Using the upper bound for $P_n(\gamma_0)$ in Lemma A.1, one obtains

$$\mathcal{R}(\widehat{S}_*, S) \leq \frac{1+3\rho-2\rho^2}{1-3\rho} \mathcal{R}(\widehat{S}_{\gamma_0}, S) + \frac{1}{n} \mathcal{B}_n^*(\rho),$$

where $\mathcal{B}_n^*(\rho)$ is defined in (2.18).

Since this inequality holds for each $\gamma_0 \in \Gamma$, this completes the proof of Theorem 2.1. \square

3.2 Proof of Theorem 2.2

We have to verify Conditions \mathbf{C}_1) and \mathbf{C}_2) for the process (1.4).

Condition \mathbf{C}_1) holds with $c_1^*(n) = 0$. This follows from Lemma A.2 if one puts $f = g = \phi_j$, $j \geq 1$. Now we check Condition \mathbf{C}_2). By the Ito formula and Lemma A.2, one gets

$$dI_t^2(f) = 2I_{t-}(f)dI_t(f) + \varrho_1^2 f^2(t)dt + \varrho_2^2 \sum_{0 \leq s \leq t} f^2(s)(\Delta z_s)^2$$

and

$$\mathbf{E} I_t^2(f) = \sigma^* \int_0^t f^2(t)dt.$$

Therefore, putting

$$\widetilde{I}_t(f) = I_t^2(f) - \mathbf{E} I_t^2(f),$$

we obtain

$$d\tilde{I}_t(f) = \varrho_2^2 f^2(t) dm_t + 2I_{t-}(f)f(t)d\xi_t, \quad \tilde{I}_0(f) = 0$$

and

$$m_t = \sum_{0 \leq s \leq t} (\Delta z_s)^2 - \lambda t. \quad (3.14)$$

Now we set

$$\bar{I}_t(x) = \sum_{j=1}^{\infty} x_j \tilde{I}_t(\phi_j)$$

where $x = (x_j)_{j \geq 1}$ with $\#(x) \leq n$ and $|x| \leq 1$. This process obeys the equation

$$d\bar{I}_t(x) = \varrho_2^2 \Phi_t dm_t + 2\zeta_{t-}(x)d\xi_t, \quad \bar{I}_0(x) = 0,$$

where

$$\Phi_t(x) = \sum_{j \geq 1} x_j \phi_j^2(t) \quad \text{and} \quad \zeta_t(x) = \sum_{j \geq 1} x_j I_t(\phi_j) \phi_j(t).$$

Now we show that

$$\mathbf{E} \int_0^n \bar{I}_{t-}(x) d\bar{I}_t(x) = 0. \quad (3.15)$$

Indeed, note that

$$\begin{aligned} \int_0^n \bar{I}_{t-}(x) d\bar{I}_t(x) &= \varrho_2^2 \sum_{j \geq 1} x_j \int_0^n \tilde{I}_{t-}(\phi_j) \Phi_t(x) dm_t \\ &\quad + 2 \sum_{j \geq 1} x_j \int_0^n \tilde{I}_{t-}(\phi_j) \zeta_{t-}(x) d\xi_t. \end{aligned}$$

Therefore, Lemma A.4 directly implies

$$\begin{aligned} \mathbf{E} \int_0^n \tilde{I}_{t-}(\phi_j) \Phi_t(x) dm_t &= \sum_{l \geq 1} x_l \mathbf{E} \int_0^n I_{t-}^2(\phi_j) \phi_l^2(t) dm_t \\ &\quad - \sum_{l \geq 1} x_l \mathbf{E} \int_0^n \left(\mathbf{E} I_{t-}^2(\phi_j) \right) \phi_l^2(t) dm_t = 0. \end{aligned}$$

Moreover, we note that

$$\int_0^n \tilde{I}_{t-}(\phi_j) \zeta_{t-}(x) d\xi_t = \sum_{l \geq 1} x_l \int_0^n \tilde{I}_{t-}(\phi_j) I_{t-}(\phi_l) \phi_l(t) d\xi_t$$

and

$$\begin{aligned} \int_0^n \tilde{I}_{t-}(\phi_j) I_{t-}(\phi_l) \phi_l(t) d\xi_t &= \int_0^n I_{t-}^2(\phi_j) I_{t-}(\phi_l) \phi_l(t) d\xi_t \\ &\quad - \int_0^n (\mathbf{E} I_t^2(\phi_j)) I_{t-}(\phi_l) \phi_l(t) d\xi_t. \end{aligned}$$

From Lemma A.5, it follows

$$\mathbf{E} \int_0^n \tilde{I}_{t-}(\phi_j) I_{t-}(\phi_l) \phi_l(t) d\xi_t = 0$$

and we come to (3.15). Furthermore, by the Ito formula one obtains

$$\begin{aligned} \bar{I}_n^2(x) &= 2 \int_0^n \bar{I}_{t-}(x) d\bar{I}_t(x) + 4\varrho_1^2 \int_0^n \zeta_t^2(x) dt + \varrho_2^4 \sum_{k=1}^{+\infty} \Phi_{T_k}^2(x) Y_k^4 \mathbf{1}_{\{T_k \leq n\}} \\ &\quad + 4\varrho_2^2 \sum_{k=1}^{+\infty} \zeta_{T_k-}^2(x) Y_k^2 \mathbf{1}_{\{T_k \leq n\}} + 4\varrho_2^3 \sum_{k=1}^{+\infty} \Phi_{T_k}^2(x) \zeta_{T_k-}(x) Y_k^3 \mathbf{1}_{\{T_k \leq n\}}. \end{aligned}$$

By Lemma A.3 one has $\mathbf{E}(\zeta_{T_k-} | T_k) = 0$. Therefore, taking into account (3.15), we calculate

$$\mathbf{E} \bar{I}_n^2(x) = 4\varrho_1^2 \mathbf{E} \int_0^n \zeta_t^2(x) dt + \varrho_2^4 \mathbf{E} Y_1^4 D_{1,n}(x) + 4\varrho_2^2 D_{2,n}(x), \quad (3.16)$$

where

$$D_{1,n}(x) = \sum_{k=1}^{\infty} \mathbf{E} \Phi_{T_k}^2(x) \mathbf{1}_{\{T_k \leq n\}} \quad \text{and} \quad D_{2,n}(x) = \sum_{k=1}^{\infty} \mathbf{E} \zeta_{T_k-}^2(x) \mathbf{1}_{\{T_k \leq n\}}.$$

By applying Lemma A.2, one has

$$\begin{aligned} \mathbf{E} \int_0^n \zeta_t^2(x) dt &= \sum_{i,j} x_i x_j \int_0^n \phi_i(t) \phi_j(t) \mathbf{E} I_t(\phi_i) I_t(\phi_j) dt \\ &= \sigma^* \sum_{i,j} x_i x_j \int_0^n \phi_i(t) \phi_j(t) \left(\int_0^t \phi_i(s) \phi_j(s) ds \right) dt \\ &= \frac{\sigma^*}{2} \sum_{i,j} x_i x_j \left(\int_0^n \phi_i(t) \phi_j(t) dt \right)^2 \leq n^2 \frac{\sigma^*}{2}. \end{aligned} \quad (3.17)$$

Further it is easy to check that

$$D_{1,n} = \lambda \int_0^n \Phi_t^2(x) dt = \lambda \int_0^n \left(\sum_{j \geq 1} x_j \phi_j^2(t) \right)^2 dt.$$

Therefore, taking into account that $\#(x) \leq n$ and $|x| \leq 1$, we estimate $D_{1,n}$ by applying the Cauchy-Schwartz-Bounyakovskii inequality

$$D_{1,n} \leq 4\lambda n \left(\sum_{j \geq 1} x_j \right)^2 \leq 4\lambda n \#(x) \leq 4\lambda n^2. \quad (3.18)$$

Finally, we write down the process $\zeta_t(x)$ as

$$\zeta_t(x) = \int_0^t Q_x(t, s) d\xi_s \quad \text{with} \quad Q_x(t, s) = \sum_{j \geq 1} x_j \phi_j(s) \phi_j(t).$$

By putting

$$\tilde{D}_{2,n} = \mathbf{E} \sum_{k=2}^{\infty} \mathbf{1}_{\{T_k \leq n\}} \sum_{l=1}^{k-1} Q_x^2(T_k, T_l)$$

and applying Lemma A.3 we obtain

$$\begin{aligned} D_{2,n} &= \varrho_1^2 \sum_{k=1}^{\infty} \mathbf{E} \int_0^{T_k} Q_x^2(T_k, s) ds \mathbf{1}_{\{T_k \leq n\}} + \varrho_2^2 \tilde{D}_{2,n} \\ &= \lambda \varrho_1^2 \int_0^n \int_0^t Q_x^2(t, s) ds dt + \varrho_2^2 \tilde{D}_{2,n}. \end{aligned}$$

Moreover, one can rewrite the second term in the last equality as

$$\begin{aligned} \tilde{D}_{2,n} &= \sum_{l=1}^{\infty} \mathbf{E} \mathbf{1}_{\{T_l \leq n\}} \sum_{k=l+1}^{\infty} Q_x^2(T_k, T_l) \mathbf{1}_{\{T_k \leq n\}} \\ &= \lambda^2 \int_0^n \left(\int_0^{n-s} Q_x^2(s+z, s) dz \right) ds \\ &= \lambda^2 \int_0^n \left(\int_0^t Q_x^2(t, s) ds \right) dt. \end{aligned}$$

Thus,

$$\begin{aligned} D_{2,n} &\leq (\lambda \varrho_1^2 + \lambda^2 \varrho_2^2) \int_0^n \left(\int_0^n Q_x^2(t, s) ds \right) dt \\ &= (\lambda \varrho_1^2 + \lambda^2 \varrho_2^2) n^2 = \lambda \sigma^* n^2. \end{aligned} \quad (3.19)$$

The equation (3.16) and the inequalities (3.17)–(3.18) imply the validity of condition \mathbf{C}_2) for the process (1.4). Hence Theorem 2.2. \square

3.3 Proof of Proposition 2.4

Substituting (2.4) in (2.20) yields

$$\widehat{\sigma}_n = \sum_{j=l}^n \theta_j^2 + \frac{2}{\sqrt{n}} \sum_{j=l}^n \theta_j \xi_{j,n} + \frac{1}{n} \sum_{j=l}^n \xi_{j,n}^2. \quad (3.20)$$

Further, denoting

$$x'_j = \mathbf{1}_{\{l \leq j \leq n\}} \quad \text{and} \quad x''_j = \frac{1}{\sqrt{n}} \mathbf{1}_{\{l \leq j \leq n\}},$$

we represent the last term in (3.20) as

$$\frac{1}{n} \sum_{j=l}^n \xi_{j,n}^2 = \frac{1}{n} B_{1,n}(x') + \frac{1}{\sqrt{n}} B_{2,n}(x'') + \frac{n-l+1}{n} \sigma,$$

where the functions $B_{1,n}(\cdot)$ and $B_{2,n}(\cdot)$ are defined in conditions $\mathbf{C}_1)$ and $\mathbf{C}_2)$. Combining these equations leads to the inequality

$$\begin{aligned} \mathbf{E}_S |\widehat{\sigma}_n - \sigma| &\leq \sum_{j \geq l} \theta_j^2 + \frac{2}{\sqrt{n}} \mathbf{E}_S \left| \sum_{j=l}^n \theta_j \xi_{j,n} \right| \\ &\quad + \frac{1}{n} |B_{1,n}(x')| + \frac{1}{\sqrt{n}} \mathbf{E} |B_{2,n}(x'')| + \frac{l-1}{n} \sigma. \end{aligned}$$

By Lemma A.6 and conditions $\mathbf{C}_1)$, $\mathbf{C}_2)$, one gets

$$\begin{aligned} \mathbf{E}_S |\widehat{\sigma}_n - \sigma| &\leq \sum_{j \geq l} \theta_j^2 + \frac{2}{\sqrt{n}} \mathbf{E}_S \left| \sum_{j=l}^n \theta_j \xi_{j,n} \right| \\ &\quad + \frac{c_1^*(n)}{n} + \frac{c_2^*(n)}{\sqrt{n}} + \frac{\sigma}{\sqrt{n}}. \end{aligned}$$

In view of the inequality (1.3), the last term can be estimated as

$$\mathbf{E}_S \left| \sum_{j=l}^n \theta_j \xi_{j,n} \right| \leq \sqrt{\sigma^* \sum_{j=l}^n \theta_j^2} \leq \sqrt{\sigma^*} |\dot{S}|_1 \frac{2}{\sqrt{l}}.$$

Hence Proposition 2.4. \square

4 Appendix

A.1 Property of the penalty term (2.15)

Lemma A.1. *Assume that the condition \mathbf{C}_1) holds with $\sigma > 0$. Then for any $n \geq 1$ and $\gamma \in \Gamma$,*

$$P_n(\gamma) \leq \mathbf{E}_S \text{Err}_n(\gamma) + \frac{c_1^*(n)}{n}.$$

Proof. By the definition of $\text{Err}_n(\gamma)$ one has

$$\text{Err}_n(\gamma) = \sum_{j=1}^{\infty} \left((\gamma(j) - 1)\theta_j + \gamma(j) \frac{1}{\sqrt{n}} \xi_{j,n} \right)^2.$$

In view of the condition \mathbf{C}_1) this leads to the desired result

$$\mathbf{E}_S \text{Err}_n(\gamma) \geq \frac{1}{n} \sum_{j=1}^{\infty} \gamma^2(j) \mathbf{E} \xi_{j,n}^2 = P_n(\gamma) - \frac{c_1^*(n)}{n}.$$

\square

A.2 Properties of the process (1.4)

Lemma A.2. *Let f and g be any non-random functions from $\mathcal{L}_2[0, n]$ and $(I_t(f))_{t \geq 0}$ be the process defined by (1.4). Then, for any $0 \leq t \leq n$,*

$$\mathbf{E} I_t(f) I_t(g) = \sigma^* \int_0^t f(s) g(s) ds$$

where $\sigma^* = \varrho_1^2 + \lambda \varrho_2^2$.

This Lemma is a direct consequence of Ito's formula as well as the following result.

Lemma A.3. *Let Q be a bounded $[0, \infty) \times \Omega \rightarrow \mathbb{R}$ function measurable with respect to $\mathcal{B}[0, +\infty) \otimes \mathcal{G}_k$, where*

$$\mathcal{G}_k = \sigma\{T_1, \dots, T_k\} \quad \text{with some } k \geq 2. \quad (\text{A.1})$$

Then

$$\mathbf{E} \left(I_{T_{k-}}(Q) | \mathcal{G}_k \right) = 0$$

and

$$\mathbf{E} \left(I_{T_{k-}}^2(Q) | \mathcal{G}_k \right) = \varrho_1^2 \int_0^{T_k} Q^2(s) ds + \varrho_2^2 \sum_{l=1}^{k-1} Q^2(T_l).$$

Now we will study stochastic cadlag processes $\eta = (\eta_t)_{0 \leq t \leq n}$ of the form

$$\eta_t = \sum_{l=0}^{\infty} v_l(t) \mathbf{1}_{\{T_l \leq t < T_{l+1}\}}, \quad (\text{A.2})$$

where $v_0(t)$ is a function measurable with respect to $\sigma\{w_s, s \leq t\}$ and the coefficient $v_l(t)$, $l \geq 1$, is a function measurable with respect to

$$\sigma\{w_s, s \leq t, Y_1, \dots, Y_l, T_1, \dots, T_l\}.$$

Now we show the following result.

Lemma A.4. *Let $\eta = (\eta_t)_{0 \leq t \leq n}$ be a stochastic non-negative process given by (A.2), such that*

$$\mathbf{E} \int_0^n \eta_u du < \infty.$$

Then

$$\mathbf{E} \int_0^n \eta_{u-} dm_u = 0$$

where the process $m = (m_t)$ is defined in (3.14).

Proof. Note that the stochastic integral, with respect to the martingale (3.14), can be written as

$$\begin{aligned}\int_0^n \eta_{u-} dm_u &= \sum_{0 \leq u \leq n} \eta_{u-} (\Delta z_u)^2 - \lambda \int_0^n \eta_u du \\ &= \sum_{k=1}^{+\infty} \eta_{T_k-} Y_k^2 \mathbf{1}_{\{T_k \leq n\}} - \lambda \int_0^n \eta_u du.\end{aligned}$$

Therefore, taking into account the representation (A.2), we obtain

$$\int_0^n \eta_{u-} dm_u = \Upsilon_1 - \lambda \Upsilon_2 \quad (\text{A.3})$$

where

$$\Upsilon_1 = \sum_{k=1}^{+\infty} v_{k-1}(T_k-) Y_k^2 \mathbf{1}_{\{T_k \leq n\}} \quad \text{and} \quad \Upsilon_2 = \int_0^n \eta_u du.$$

Recalling that $\mathbf{E}Y_1^2 = 1$ and $v_k \geq 0$, we calculate

$$\mathbf{E}\Upsilon_1 = \sum_{k=1}^{+\infty} \mathbf{E} v_{k-1}(T_k-) \mathbf{1}_{\{T_k \leq n\}}.$$

Moreover, the functions (v_k) are cadlag processes, therefore the Lebesgue measure of the set $\{t \in \mathbb{R}_+ : v_k(t-) \neq v_k(t)\}$ equals zero. Thus,

$$\mathbf{E} v_{k-1}(T_k-) \mathbf{1}_{\{T_k \leq n\}} = \lambda \mathbf{E} \mathbf{1}_{\{T_{k-1} \leq n\}} \int_0^{n-T_{k-1}} v_{k-1}(T_{k-1} + u) e^{-\lambda u} du.$$

This implies

$$\mathbf{E}\Upsilon_1 = \lambda \sum_{l=0}^{+\infty} \mathbf{E} \mathbf{1}_{\{T_l \leq n\}} \int_0^{n-T_l} v_l(T_l + u) e^{-\lambda u} du. \quad (\text{A.4})$$

Similarly we obtain

$$\begin{aligned}\mathbf{E}\Upsilon_2 &= \sum_{l=0}^{+\infty} \mathbf{E} \mathbf{1}_{\{T_l \leq n\}} \int_{T_l}^n v_l(t) \mathbf{1}_{\{t \leq T_{l+1}\}} dt \\ &= \sum_{l=0}^{+\infty} \mathbf{E} \mathbf{1}_{\{T_l \leq n\}} \int_0^{n-T_l} v_l(T_l + u) e^{-\lambda u} du.\end{aligned} \quad (\text{A.5})$$

Substituting (A.4) and (A.5) in (A.3) implies the assertion of Lemma A.4.
 \square

Lemma A.5. *Assume that $\mathbf{E}Y_1^4 < \infty$. Then, for any measurable bounded non-random functions f and g , one has*

$$\mathbf{E} \int_0^n I_{t-}^2(f) I_{t-}(g) g(t) d\xi_t = 0.$$

Proof. First we note that

$$\mathbf{E} \int_0^n I_{t-}^2(f) I_{t-}(g) g(t) dz_t = \mathbf{E} \sum_{j \geq 1} I_{T_j-}^2(f) I_{T_j-}(g) g(T_j) \mathbf{1}_{\{T_j \leq n\}} \mathbf{E} Y_j = 0.$$

Therefore, to prove this lemma one has to show that

$$\mathbf{E} \int_0^n I_t^2(f) I_t(g) g(t) dw_t = 0. \quad (\text{A.6})$$

To this end we represent the stochastic integral $I_t(f)$ as

$$I_t(f) = \varrho_1 I_t^w(f) + \varrho_2 I_t^z(f),$$

where

$$I_t^w(f) = \int_0^t f_s dw_s \quad \text{and} \quad I_t^z(f) = \int_0^t f_s dz_s.$$

Note that

$$\mathbf{E} |I_t^z(f)|^4 \leq M^4 \mathbf{E} Y_1^4 \mathbf{E} N_n^2 = M^4 \mathbf{E} Y_1^4 (\lambda n + \lambda^2 n^2) < \infty,$$

where

$$M = \sup_{0 \leq t \leq n} (|f(t)| + |g(t)|).$$

Therefore, taking into account that the processes (w_t) and (z_t) are independent, we get

$$\mathbf{E} \int_0^n I_t^4(f) (I_t^w(g))^2 g(t) dt < \infty,$$

i.e.

$$\mathbf{E} \int_0^n I_t^2(f) I_t^w(g) g(t) dw_t = 0.$$

Similarly, we obtain

$$\mathbf{E} \int_0^n (I_t^w(f))^2 I_t^z(g) g(t) dw_t = 0$$

and

$$\mathbf{E} \int_0^n I_t^w(f) I_t^z(f) I_t^z(g) g(t) dw_t = 0.$$

Therefore, to show (A.6) one has to check that

$$\mathbf{E} \int_0^n \eta_t dw_t = 0, \tag{A.7}$$

where

$$\eta_t = (I_t^z(f))^2 I_t^z(g) g(t).$$

Taking into account that the processes (η_t) and (w_t) are independent, we get

$$\mathbf{E} \left| \int_0^n \eta_t dw_t \right| \leq \mathbf{E} \sqrt{\int_0^n \eta_t^2 dt} \leq \sqrt{n} \mathbf{E} \sup_{0 \leq t \leq n} |\eta_t|.$$

Here, the last term can be estimated as

$$\mathbf{E} \sup_{0 \leq t \leq n} |\eta_t| \leq M^4 \left(\sum_{j=1}^{N_n} |Y_j| \right)^3 \leq M^4 \mathbf{E} |Y_1|^3 \mathbf{E} N_n^3 < \infty.$$

Hence the stochastic integral $\int_0^n \eta_t dw_t$ is an integrable random variable and

$$\mathbf{E} \int_0^n \eta_t dw_t = \mathbf{E} \mathbf{E} \left(\int_0^n \eta_t dw_t \mid \eta_t, 0 \leq t \leq n \right) = 0.$$

Thus we obtain the equality (A.7) which implies (A.6). Hence Lemma A.5.

□

A.3 Property of the Fourier coefficients

Lemma A.6. *Suppose that the function S in (1.1) is differentiable and satisfies the condition (2.21). Then the Fourier coefficients (2.2) satisfy the inequality*

$$\sup_{l \geq 2} l \sum_{j=l}^{\infty} \theta_j^2 \leq 4 |\dot{S}_1|^2.$$

Proof. In view of (2.1), one has

$$\theta_{2p} = -\frac{1}{\sqrt{2\pi p}} \int_0^1 \dot{S}(t) \sin(2\pi pt) dt .$$

and

$$\begin{aligned} \theta_{2p+1} &= \frac{1}{\sqrt{2\pi p}} \int_0^1 \dot{S}(t) (\cos(2\pi pt) - 1) dt \\ &= -\frac{\sqrt{2}}{\pi p} \int_0^1 \dot{S}(t) \sin^2(\pi pt) dt, \quad p \geq 1 . \end{aligned}$$

From here, it follows that, for any $j \geq 2$

$$\theta_j^2 \leq \frac{2}{j^2} |\dot{S}|_1^2 .$$

Taking into account that

$$\sup_{l \geq 2} l \sum_{j \geq l} \frac{1}{j^2} \leq 2 ,$$

we arrive at the desired result. \square

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