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► **To cite this version:**

Susumu Ariki, Nicolas Jacon, Cédric Lecouvey. Factorization of the canonical bases for higher level Fock spaces. 2010. hal-00417495v2

HAL Id: hal-00417495

<https://hal.science/hal-00417495v2>

Preprint submitted on 4 Feb 2010 (v2), last revised 15 Oct 2010 (v4)

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FACTORIZATION OF THE CANONICAL BASES FOR HIGHER LEVEL FOCK SPACES

SUSUMU ARIKI, NICOLAS JACON AND CÉDRIC LECOUCVEY

ABSTRACT. The level l Fock space admits canonical bases \mathcal{G}_e and \mathcal{G}_∞ . They correspond to $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -module structures. We establish that the transition matrices relating these two bases are unitriangular with coefficients in $\mathbb{N}[v]$. Restriction to the highest weight modules generated by the empty l -partition then gives a natural quantization of a theorem by Geck and Rouquier on the factorization of decomposition matrices which are associated to Ariki-Koike algebras.

1. INTRODUCTION

In the classification of finite complex reflection groups by Shephard and Todd [28], there is a single infinite family of groups $G(lp, p, n)$ parametrized by the triplets $(l, p, n) \in \mathbb{N}^3$ and 34 other “exceptional” groups. If $p = 1$, the group $G(l, 1, n)$ is the wreath product of the cyclic group of order l with the symmetric group S_n . It generalizes both the Weyl group of type A_{n-1} (corresponding to the case $l = 1$) and the Weyl group of type B_n ($l = 2$). To $G(l, 1, n)$ we may associate its Hecke algebra over the ring $A := \mathbb{C}[q^{\pm 1}, Q_1^{\pm 1}, \dots, Q_l^{\pm 1}]$, where (q, Q_1, \dots, Q_l) is an $l + 1$ -tuple of indeterminates. This algebra can be seen as a deformation of the group algebra of $G(l, 1, n)$ and has applications to the modular representation theory of finite reductive groups (see for example the survey [26]). As an A -algebra, it has the set of generators $\{T_0, \dots, T_{n-1}\}$ such that the defining relations are

$$\prod_{i=1}^l (T_0 - Q_i) = 0, \quad (T_i - q)(T_i + 1) = 0, \quad i = 1, \dots, n-1$$

and the braid relations of type B_n . We denote this algebra by \mathcal{H}_A . If we extend the scalars of \mathcal{H}_A to $K = \mathbb{C}(q, Q_1, \dots, Q_l)$, the field of fractions of A , we obtain the algebra $\mathcal{H}_K := K \otimes_A \mathcal{H}_A$ whose representation theory is well understood. For example, we know how to classify the irreducible representations, what are their dimensions etc (see [2], [15]). The theory is far more difficult in the modular case. Let $\theta : A \rightarrow \mathbb{C}$ be a ring homomorphism and let $\mathcal{H}_\mathbb{C} := \mathbb{C} \otimes_A \mathcal{H}_A$ be the associated Hecke algebra. Due to results of Dipper and Mathas [7], one can reduce various important problems to the case when $\theta(q) = \eta_e := \exp(\frac{2i\pi}{e})$ is a e -th root of unity, for $e \in \mathbb{Z}_{\geq 2}$, and $\theta(Q_j) = \eta_e^{s_j}$, for $j = 1, \dots, l$, where $(s_1, \dots, s_l) \in \mathbb{Z}^l$. An important object of study in the modular case is the decomposition map. As \mathcal{H}_A is a cellular algebra [14], the decomposition map may be defined as follows. Let $V_K \in \text{Irr}(\mathcal{H}_K)$. Then there exists a specific \mathcal{H}_A -module V_A , which is called a *cell module*, such that $V_K = K \otimes_A V_A$. We can then associate to V_K the $\mathcal{H}_\mathbb{C}$ -module $V_\mathbb{C} = \mathbb{C} \otimes_A V_A$.¹

¹In fact, it is shown in [11] that we may choose V_A more general than the cell module and the decomposition map is still well defined.

This gives a well defined map at the level of Grothendieck groups $R_0(\mathcal{H}_K)$ (resp. $R_0(\mathcal{H}_{\mathbb{C}})$) of finitely generated \mathcal{H}_K -modules (resp. $\mathcal{H}_{\mathbb{C}}$ -modules):

$$\begin{aligned} d_{\theta} : R_0(\mathcal{H}_K) &\rightarrow R_0(\mathcal{H}_{\mathbb{C}}) \\ [V_K] &\mapsto [V_{\mathbb{C}}] \end{aligned}$$

and the associated decomposition matrix, which we denote by D_e , encodes this map. There exist algorithms to compute the map d_{θ} , but it remains difficult to describe it in general. One useful tool here is a result by Geck and Rouquier [12], which gives information on the matrix D_e by factorizing the decomposition map. Let $\theta^q : A \rightarrow \mathbb{C}(q)$ be the specialization map defined by $\theta^q(Q_i) = q^{s_i}$, for $i = 1, \dots, l$. Denote by $\mathcal{H}_{\mathbb{C}(q)} := \mathbb{C}(q) \otimes_A \mathcal{H}_A$ the associated Hecke algebra. As above, we have the decomposition map

$$d_{\theta^q} : R_0(\mathcal{H}_K) \rightarrow R_0(\mathcal{H}_{\mathbb{C}(q)})$$

and the associated decomposition matrix D_{∞} . Then [12, Prop. 2.12] implies the following.

Theorem 1.1 (Geck-Rouquier). *There exists a unique \mathbb{Z} -linear map*

$$d_{\theta^q}^{\theta} : R_0(\mathcal{H}_{\mathbb{C}(q)}) \rightarrow R_0(\mathcal{H}_{\mathbb{C}})$$

such that the following diagram commutes:

$$\begin{array}{ccc} R_0(\mathcal{H}_K) & \xrightarrow{d_{\theta}} & R_0(\mathcal{H}_{\mathbb{C}}) \\ & \searrow d_{\theta^q} & \nearrow d_{\theta^q}^{\theta} \\ & R_0(\mathcal{H}_{\mathbb{C}(q)}) & \end{array}$$

Thus, we have the factorization $D_e = D_{\infty}.D_{\infty}^e$ of the decomposition matrices, where D_{∞}^e is the decomposition matrix for $d_{\theta^q}^{\theta}$. We shall call D_{∞}^e the *relative decomposition matrix*. This result shows that a part of the representation theory of $\mathcal{H}_{\mathbb{C}}$ does not depend on e but only on the representation theory of $\mathcal{H}_{\mathbb{C}(q)}$, which is “easier” to understand (for example, there are closed formulae for the entries of D_{∞} when $l = 2$ [25]). An example of its application is that one may give explicit relationship among various classifications of simple modules arising from the theory of canonical basic sets in type B_n [18].

In view of the Fock space theory, which is now standard in the study of Hecke algebras, Theorem 1.1 naturally leads to several questions. As noted above, there is an algorithm for computing the decomposition matrices of $\mathcal{H}_{\mathbb{C}}$ and $\mathcal{H}_{\mathbb{C}(q)}$. This algorithm relies on the first author’s proof of the Lascoux-Leclerc-Thibon conjecture [24].² His theorem asserts that D_e (resp. D_{∞}) is equal to the evaluation at $v = 1$ of the matrix $D_e(v)$ (resp. $D_{\infty}(v)$) which is obtained by expanding the canonical basis in a highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -module (resp. $\mathcal{U}_v(\widehat{\mathfrak{sl}}_{\infty})$ -module) into linear combination of the standard basis of a Fock space. Thus, Theorem 1.1 implies the existence of a matrix D_{∞}^e such that $D_e(1) = D_{\infty}(1).D_{\infty}^e$. The entries of $D_e(v)$ and $D_{\infty}(v)$ are known to be in $\mathbb{N}[v]$, i.e. polynomials with nonnegative integer coefficients. Hence it is natural to ask:

²See [1].

(Q1) Does the matrix D_∞^e have a natural quantization? Namely, is there a matrix $D_\infty^e(v)$ with entries in $\mathbb{N}[v]$ such that

$$D_e(v) = D_\infty(v) \cdot D_\infty^e(v) ?$$

(Q2) If $D_\infty^e(v)$ is known to exist, find a practical algorithm to compute $D_\infty^e(v)$.

In other words, we ask if the matrix of the canonical basis for $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -modules factorizes through the matrix of the canonical basis for $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules.

Highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -modules and $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules are realized as irreducible components in Fock spaces of higher level. By Uglov's results [29], these Fock spaces also admit canonical bases. So the above questions also make sense for the matrices $\Delta_e(v)$ and $\Delta_\infty(v)$ which are associated to the canonical bases of the whole Fock space. Thus, instead of (Q1), we ask whether there exists a matrix $\Delta_\infty^e(v)$ with entries in $\mathbb{N}[v]$ such that

$$\Delta_e(v) = \Delta_\infty(v) \cdot \Delta_\infty^e(v).$$

The matrix $\Delta_\infty^e(v)$ is expected to have several interpretations. Observe that recent conjectures and results [3], [4], [5] show that $D_e(v)$ and $D_\infty(v)$ are graded decomposition matrices of Hecke algebras. Thus, $D_\infty^e(v)$ might be interpreted as a graded analogue of D_∞^e in this setting. According to conjectures of Yvonne [30] and Rouquier [27, §6.4], $\Delta_e(1)$ and $\Delta_\infty(1)$ are expected to be decomposition matrices of a generalized η_e and q -Schur algebras, respectively. Thus, $\Delta_\infty^e(v)$ might have the similar interpretation as $D_\infty^e(v)$ as well.

In another direction, we interpret the factorization $D_e = D_\infty \cdot D_\infty^e$ in the context of parabolic BGG categories in the last section. This second interpretation should also have graded version, which is independent of the first.³

In this paper, we answer positively to the questions (Q1) and (Q2). We first show the existence of the matrices $D_\infty^e(v)$ and $\Delta_\infty^e(v)$ with entries in $\mathbb{Z}[v]$. In fact $D_\infty^e(v)$ is a submatrix of $\Delta_\infty^e(v)$ and we provide an efficient algorithm for computing it (and thus an algorithm for computing D_∞^e). Then, we prove that the entries of $\Delta_\infty^e(v)$ are in $\mathbb{N}[v]$. More precisely, we show that they can be expressed as sum of products of structure constants in the affine Hecke algebras of type A with respect to the Kazhdan-Lusztig basis and its generalization by Grojnowski-Haiman [13].

Let us briefly summarize the main ingredients of our proofs. The Fock space theory developed in [20] and the notion of canonical bases for these Fock spaces introduced in [29] make apparent strong connections between the representation theories of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$. They permit us to prove the existence of a matrix $\Delta_\infty^e(v)$ with entries in $\mathbb{Z}[v]$ such that $\Delta_e(v) = \Delta_\infty(v) \cdot \Delta_\infty^e(v)$. This factorization can be regarded as an analogue, at the level of canonical bases, of the compatibility of the crystal graphs structures established in [19]. It is achieved by introducing a new partial order on the set of l -partitions, which does not depend on e ⁴, and has the property that $\Delta_e(v)$ and $\Delta_\infty(v)$ are simultaneously unitriangular. The compatibility between the $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -module structures on the Fock space then implies the factorization $\Delta_e(v) = \Delta_\infty(v) \cdot \Delta_\infty^e(v)$. To show the positivity, recall that the coefficients of the matrices $\Delta_\infty(v)$ and $\Delta_e(v)$ are expressed by parabolic Kazhdan-Lusztig polynomials of the affine Hecke algebras of type A [29]. We see in a simpler manner than [29] how the parabolic Kazhdan-Lusztig polynomials are

³Note that Hecke algebras are not positively graded.

⁴Thus, it differs from that used in [29].

related to the entries of $\Delta_\infty(v)$ and $\Delta_e(v)$, for a fixed pair of l -partitions. The positivity result then follows from this and the positivity of the structure constants of the affine Hecke algebra.

2. BACKGROUND ON FOCK SPACES AND CANONICAL BASES

We refer to [21] and to [1] for a detailed review on the canonical and crystal basis theory. [10, §7] also gives a nice survey on modular representation theory of Hecke algebras. Let v be an indeterminate, $e > 1$ an integer, and $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ the quantum group of type $A_{e-1}^{(1)}$. It is an associative $\mathbb{Q}(v)$ -algebra with Chevalley generators e_i, f_i, t_i, t_i^{-1} , for $i \in \mathbb{Z}/e\mathbb{Z}$, and ∂ . We refer to [29, §2.1] for the precise definition. The bar involution $\overline{}$ is the ring automorphism of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ such that $\overline{v} = v^{-1}$, $\overline{\partial} = \partial$ and,

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i \text{ and } \overline{t_i} = t_i, \quad \text{for } i \in \mathbb{Z}/e\mathbb{Z}.$$

We denote by $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ the subalgebra generated by $\{e_i, f_i, t_i, t_i^{-1} \mid i \in \mathbb{Z}/e\mathbb{Z}\}$. By slightly abuse of notation, we identify the elements of $\mathbb{Z}/e\mathbb{Z}$ with their corresponding labels in $\{0, \dots, e-1\}$ when there is no risk of confusion. Write $\{\Lambda_0, \dots, \Lambda_{e-1}\}$ for the set of fundamental weights of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$, and δ for the null root. Let $l \in \mathbb{Z}_{\geq 1}$ and consider $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$, which we call a *multicharge*. We set

$$\mathbf{s} = (s_1(\bmod e), \dots, s_l(\bmod e)) \in (\mathbb{Z}/e\mathbb{Z})^l$$

and $\Lambda_{\mathbf{s}} := \Lambda_{s_1(\bmod e)} + \dots + \Lambda_{s_l(\bmod e)}$. We denote by $V_e(\mathbf{s})$ the integrable irreducible $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ -module with highest weight $\Lambda_{\mathbf{s}}$.

Similarly, let $\mathcal{U}_v(\mathfrak{sl}_\infty)$ be the quantum group of type A_∞ . It is an associative $\mathbb{Q}(v)$ -algebra with Chevalley generators E_j, F_j, T_j, T_j^{-1} , for $j \in \mathbb{Z}$. We denote its bar involution by the same $\overline{}$, which is the ring automorphism of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ such that $\overline{v} = v^{-1}$ and,

$$\overline{E_j} = E_j, \quad \overline{F_j} = T_j \text{ and } \overline{T_j} = T_j, \quad \text{for } j \in \mathbb{Z}.$$

Write $\{\omega_j, j \in \mathbb{Z}\}$ for its set of fundamental weights. To $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$, we associate the dominant weight $\Lambda_{\mathbf{s}} := \omega_{s_1} + \dots + \omega_{s_l}$. We denote by $V_\infty(\mathbf{s})$ the integrable irreducible $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -module with highest weight $\Lambda_{\mathbf{s}}$.

2.1. Fock spaces. Let $\Pi_{l,n}$ be the set of l -partitions with rank n , that is, the set of sequences $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ of partitions such that $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + \dots + |\lambda^{(l)}| = n$. Set $\Pi_l = \cup_{n \geq 0} \Pi_{l,n}$. The *Fock space* \mathcal{F} of level l is a $\mathbb{Q}(v)$ -vector space which has the set of all l -partitions as the given basis, so that we write

$$\mathcal{F} = \bigoplus_{\boldsymbol{\lambda} \in \Pi_l} \mathbb{Q}(v)\boldsymbol{\lambda}.$$

The Fock space \mathcal{F} may be endowed with a structure of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -modules. Let $\boldsymbol{\lambda}$ be an l -partition (identified with its Young diagram). Then, the *nodes* of $\boldsymbol{\lambda}$ are the triplets $\gamma = (a, b, c)$ where $c \in \{1, \dots, l\}$ and a, b are the row and column indices of the node γ in $\lambda^{(c)}$, respectively. The *content* of γ is the integer $c(\gamma) = b - a + s_c$ and the *residue* $\text{res}(\gamma)$ of γ is the element of $\mathbb{Z}/e\mathbb{Z}$ such that

$$(1) \quad \text{res}(\gamma) \equiv c(\gamma) \pmod{e}.$$

For $i \in \mathbb{Z}/e\mathbb{Z}$, we say that γ is an i -node of λ when $\text{res}(\gamma) \equiv i \pmod{e}$. Similarly for $j \in \mathbb{Z}$, we say that γ is a j -node of λ when $c(\gamma) = j$. We say that a node γ is *removable* when $\gamma = (a, b, c) \in \lambda$ and $\lambda \setminus \{\gamma\}$ is an l -partition, and *addable* when $\gamma = (a, b, c) \notin \lambda$ and $\lambda \cup \{\gamma\}$ is an l -partition.

Let $i \in \mathbb{Z}/e\mathbb{Z}$. In the sequel, we follow the convention of [29]. We define a total order on the set of i -nodes of λ . Consider two nodes $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ in λ . We define the order by

$$\gamma_1 \prec_s \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2) \text{ or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 < c_2. \end{cases}$$

Let λ and μ be two l -partitions of rank n and $n+1$ such that $[\mu] = [\lambda] \cup \{\gamma\}$ where γ is an i -node. Define

$$\begin{aligned} (2) \quad N_i^{\succ}(\lambda, \mu) &= \#\{\text{addable } i\text{-nodes } \gamma \text{ of } \lambda \text{ such that } \gamma' \succ_s \gamma\} \\ &\quad - \#\{\text{removable } i\text{-nodes } \gamma' \text{ of } \mu \text{ such that } \gamma' \succ_s \gamma\}, \\ N_i^{\prec}(\lambda, \mu) &= \#\{\text{addable } i\text{-nodes } \gamma' \text{ of } \lambda \text{ such that } \gamma' \prec_s \gamma\} \\ &\quad - \#\{\text{removable } i\text{-nodes } \gamma' \text{ of } \mu \text{ such that } \gamma' \prec_s \gamma\}, \\ (3) \quad N_i(\lambda) &= \#\{\text{addable } i\text{-nodes of } \lambda\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \lambda\} \\ \text{and } M_0(\lambda) &= \#\{0\text{-nodes of } \lambda\}. \end{aligned}$$

Theorem 2.1. [20] *Let $\mathbf{s} \in \mathbb{Z}^l$. The Fock space \mathcal{F} has a structure of an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -module $\mathcal{F}_e^{\mathbf{s}}$ defined by*

$$\begin{aligned} e_i \lambda &= \sum_{\text{res}([\lambda]/[\mu])=i} v^{-N_i^{\prec}(\mu, \lambda)} \mu, & f_i \lambda &= \sum_{\text{res}([\mu]/[\lambda])=i} v^{N_i^{\succ}(\lambda, \mu)} \mu, \\ t_i \lambda &= v^{N_i(\lambda)} \lambda & \text{and } \partial \lambda &= -(\Delta + M_0(\lambda)) \lambda, \end{aligned}$$

for $i \in \mathbb{Z}/e\mathbb{Z}$, where Δ is the rational number defined in [20, Thm 2.1]. The module structure on $\mathcal{F}_e^{\mathbf{s}}$ depends on \mathbf{s} and e .

We may consider \mathcal{F} as a $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ -module by restriction. We denote it by the same $\mathcal{F}_e^{\mathbf{s}}$ by abuse of notation.

Let $j \in \mathbb{Z}$. For l -partitions λ and μ of rank n and $n+1$ such that $[\mu] = [\lambda] \cup \{\gamma\}$ where γ is a j -node, we define $N_j^{\succ}(\lambda, \mu)$, $N_j^{\prec}(\lambda, \mu)$ and $N_j(\lambda)$ as in (2) except that we consider j -nodes, for $e = \infty$, instead of i -nodes, for e finite.

Theorem 2.2. [20] *Let $\mathbf{s} \in \mathbb{Z}^l$. The Fock space \mathcal{F} has a structure of an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_\infty)$ -module $\mathcal{F}_\infty^{\mathbf{s}}$ defined by*

$$\begin{aligned} E_j \lambda &= \sum_{c([\lambda]/[\mu])=j} v^{-N_j^{\prec}(\mu, \lambda)} \mu, & F_j \lambda &= \sum_{c([\mu]/[\lambda])=j} v^{N_j^{\succ}(\lambda, \mu)} \mu, \\ T_j \lambda &= v^{N_j(\lambda)} \lambda, \end{aligned}$$

for $j \in \mathbb{Z}$. The module structure on $\mathcal{F}_e^{\mathbf{s}}$ depends on \mathbf{s} .

The following result is implicit in [20, Prop 3.5].

Proposition 2.3. *The $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}'_v(\mathfrak{sl}_\infty)$ -module structures $\mathcal{F}_e^{\mathfrak{s}}$ and $\mathcal{F}_\infty^{\mathfrak{s}}$ are compatible in the sense that we may write the action of e_i, f_i and t_i , for $i \in \mathbb{Z}/e\mathbb{Z}$, as follows:*

$$\begin{aligned} e_i &= \sum_{j \in \mathbb{Z}, j \equiv i \pmod{e}} \left(\prod_{r \geq 1} T_{j-re}^{-1} \right) E_j, \\ f_i &= \sum_{j \in \mathbb{Z}, j \equiv i \pmod{e}} \left(\prod_{r \geq 1} T_{j+re} \right) F_j, \\ t_i &= \prod_{j \in \mathbb{Z}, j \equiv i \pmod{e}} T_j. \end{aligned}$$

Remark 2.4. The infinite sums and products in the proposition reduce in fact to finite ones since the number of nodes in λ is finite.

The empty multipartition \emptyset is a highest weight vector in $\mathcal{F}_e^{\mathfrak{s}}$ and $\mathcal{F}_\infty^{\mathfrak{s}}$ of weight $\Lambda_{\mathfrak{s}}$ and $\Lambda_{\mathfrak{s}}$, respectively. We will identify $V_e(\mathfrak{s})$ and $V_\infty(\mathfrak{s})$ with $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e) \cdot \emptyset$ and $\mathcal{U}'_v(\mathfrak{sl}_\infty) \cdot \emptyset$, respectively. Thanks to this identification and the previous proposition, it follows that $V_\infty(\mathfrak{s})$ is endowed with the structure of a $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ -module and $V_e(\mathfrak{s})$ coincides with the $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ -irreducible component of $V_\infty(\mathfrak{s})$ with highest weight vector \emptyset .

2.2. Uglov's canonical bases. We now briefly recall Uglov's plus canonical basis of the Fock spaces. Let $\mathbb{A}(v)$ be the ring of rational functions which has no pole at $v = 0$. Set

$$\begin{aligned} \mathcal{L} &:= \bigoplus_{n \geq 0} \bigoplus_{\lambda \in \Pi_{l,n}} \mathbb{A}(v)\lambda \quad \text{and} \\ \mathcal{B} &:= \{\lambda \pmod{v\mathcal{L}} \mid \lambda \in \Pi_l\}. \end{aligned}$$

Theorem 2.5. [9] *The pair $(\mathcal{L}, \mathcal{B})$ is a crystal basis for $\mathcal{F}_e^{\mathfrak{s}}$ and $\mathcal{F}_\infty^{\mathfrak{s}}$.*

Note that although the crystal lattice \mathcal{L} and the basis \mathcal{B} of $\mathcal{L}/v\mathcal{L}$ are the same for $\mathcal{F}_e^{\mathfrak{s}}$ and $\mathcal{F}_\infty^{\mathfrak{s}}$, the induced crystal structures \mathcal{B}_e and \mathcal{B}_∞ on \mathcal{B} do not coincide. The crystal structure \mathcal{B}_e is obtained as follows. Let λ be an l -partition, and $i \in \mathbb{Z}/e\mathbb{Z}$. We consider the set of addable and removable i -nodes of λ . We read the nodes in the increasing order with respect to $\prec_{\mathfrak{s}}$, and let w_i be the resulting word of the nodes. If a removable i -node appears just before an addable i -node, we delete both and continue the same procedure as many times as possible. In the end, we reach a word \tilde{w}_i of nodes such that the first p nodes are addable and the last q nodes are removable, for some $p, q \in \mathbb{N}$. If $p > 0$, let γ be the rightmost addable i -node in \tilde{w}_i . The node γ is called the *good i -node* of λ . Then, the crystal \mathcal{B}_e may be read off from its crystal graph:

- vertices: l -partitions whose nodes are colored with residues.
- edges: $\lambda \xrightarrow{i} \mu$ if and only if μ is obtained by adding a good i -node to λ .

We denote by $B_e(\mathfrak{s})$ the connected component of \mathcal{B}_e which contains the highest weight vertex \emptyset . We may identify $B_e(\mathfrak{s})$ with the crystal graph of $V_e(\mathfrak{s})$. The crystal graph of \mathcal{B}_∞ is obtained in the similar manner: we use j -nodes ($j \in \mathbb{Z}$), for $e = \infty$, instead of i -nodes, for e finite. We may also identify the crystal graph of $V_\infty(\mathfrak{s})$ with $B_\infty(\mathfrak{s})$, the connected component of \mathcal{B}_∞ which contains the highest weight vertex \emptyset .

Let $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. We denote

$$\mathcal{U}_v(\mathfrak{g}) = \begin{cases} \mathcal{U}'_v(\widehat{\mathfrak{sl}}_e) & (\text{if } e < \infty) \\ \mathcal{U}'_v(\mathfrak{sl}_\infty) & (\text{if } e = \infty) \end{cases}$$

for short. We define a $\mathbb{Z}[v]$ -lattice $\mathcal{L}_{\mathbb{Z}}$ of \mathcal{L} by

$$\mathcal{L}_{\mathbb{Z}} := \bigoplus_{n \geq 0} \bigoplus_{\lambda \in \Pi_{l,n}} \mathbb{Z}[v]\lambda.$$

In [29], Uglov introduced a bar involution $\overline{}$ on $\mathcal{F}_e^{\mathfrak{s}}$, which is defined by

$$\overline{u \cdot f} = \overline{u} \cdot \overline{f}, \text{ for } u \in \mathcal{U}_v(\mathfrak{g}) \text{ and } f \in \mathcal{F}_e^{\mathfrak{s}}, \text{ and } \overline{\emptyset} = \emptyset.$$

Theorem 2.6. [29] *Let $\mathfrak{s} \in \mathbb{Z}^l$ and $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. There exists a unique basis $\mathcal{G}_e(\mathfrak{s}) = \{G_e(\lambda, \mathfrak{s}) \mid \lambda \in \Pi_l\}$ of $\mathcal{F}_e^{\mathfrak{s}}$ such that the basis elements are characterized by the following two conditions.*

- (1) $\overline{G_e(\lambda, \mathfrak{s})} = G_e(\lambda, \mathfrak{s})$,
- (2) $G_e(\lambda, \mathfrak{s}) = \lambda \pmod{v\mathcal{L}_{\mathbb{Z}}}$.

The basis $\mathcal{G}_e(\mathfrak{s})$ is called the *plus canonical basis* of $\mathcal{F}_e^{\mathfrak{s}}$. It strongly depends on $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. The next theorem is to identify the Kashiwara-Lusztig canonical basis of $V_e(\mathfrak{s})$ with a subset of $\mathcal{G}_e(\mathfrak{s})$.

Theorem 2.7. [29] *Let $\mathfrak{s} \in \mathbb{Z}^l$ and $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Define*

$$\mathcal{G}_e^{\circ}(\mathfrak{s}) = \{G_e(\lambda, \mathfrak{s}) \in \mathcal{G}_e(\mathfrak{s}) \mid G_e(\lambda, \mathfrak{s}) \in V_e(\mathfrak{s})\}.$$

Then $\mathcal{G}_e^{\circ}(\mathfrak{s})$ coincides with the canonical basis of the irreducible highest weight $\mathcal{U}_v(\mathfrak{g})$ -module $V_e(\mathfrak{s})$. Moreover, $G_e(\lambda, \mathfrak{s}) \in \mathcal{G}_e^{\circ}(\mathfrak{s})$ if and only if $\lambda \in B_e(\mathfrak{s})$.

3. COMPATIBILITY OF CANONICAL BASES

In this section, we prove that each $G_e(\lambda, \mathfrak{s})$ may be expanded into $\mathbb{Z}[v]$ -linear combination of the canonical basis $\mathcal{G}_{\infty}(\mathfrak{s})$. A crucial observation for the proof is that we may define a partial order on multipartitions which is independent of e . Then, the transition matrix becomes unitriangular with respect to the partial order.

3.1. Ordering multipartitions. First of all, we introduce the dominance order in general setting. Let $k \in \mathbb{N}$ and $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{Q}^k$, $\mathbf{v} = (v_1, \dots, v_k) \in \mathbb{Q}^k$. Then, we write $\mathbf{u} \triangleright \mathbf{v}$ if $\mathbf{u} \neq \mathbf{v}$ and

$$\sum_{s=1}^a u_s \geq \sum_{s=1}^a v_s, \text{ for } a = 1, \dots, k.$$

We fix a decreasing sequence $1 > \alpha_0 > \alpha_1 > \dots > \alpha_{l-1} > 0$ of rational numbers. Then, for each $\lambda \in \Pi_{l,n}$, we read the rational numbers

$$\lambda_j^{(i)} - j + s_i - \alpha_i, \text{ for } j = 1, \dots, n + s_i \text{ and } i = 1, \dots, l,$$

in the decreasing order and denote the resulting sequence by $\gamma(\lambda) \in \mathbb{Q}^k$ where $k = \sum_{i=1}^l s_i + nl$.

Definition 3.1. Let $\lambda, \mu \in \Pi_{l,n}$. Then we write $\lambda \succ \mu$ if $\gamma(\lambda) \triangleright \gamma(\mu)$.

One can check that this defines a partial order which does not depend on the choice of the sequence $1 > \alpha_1 > \dots > \alpha_l > 0$, nor e .

Theorem 3.2.

- (1) For each $\lambda \in \Pi_{l,n}$, there exist polynomials $\Delta_{\lambda,\mu}^e(v) \in \mathbb{Z}[v]$, for $\mu \in \Pi_{l,n}$, such that we have the unitriangular expansion

$$G_e(\lambda, \mathbf{s}) = \lambda + \sum_{\lambda > \mu} \Delta_{\lambda,\mu}^e(v) \mu.$$

- (2) For each $\lambda \in \Pi_{l,n}$, there exist polynomials $\Delta_{\lambda,\mu}^\infty(v) \in \mathbb{Z}[v]$, for $\mu \in \Pi_{l,n}$, such that we have the unitriangular expansion

$$G_\infty(\lambda, \mathbf{s}) = \lambda + \sum_{\lambda > \mu} \Delta_{\lambda,\mu}^\infty(v) \mu.$$

- (3) For each pair $(\lambda, \mu) \in \Pi_{l,n} \times \Pi_{l,n}$, $\Delta_{\lambda,\mu}^e(v)$ and $\Delta_{\lambda,\mu}^\infty(v)$ are expressed by certain parabolic Kazhdan-Lusztig polynomials. In particular, they are polynomials with nonnegative integer coefficients.

We do not prove the unitriangularity in (1) and (2) here, since the proof is exactly the same as in [17] except that one has to modify the order slightly. The main ingredient is the unitriangularity of the matrix of the bar involution on $\mathcal{F}_e^{\mathbf{s}}$ and $\mathcal{F}_\infty^{\mathbf{s}}$ with respect to our partial order $>$. This follows from the explicit description of the bar involution obtained in [29, Prop 3.16] and after an easy but quite long combinatorial analysis. The third part is a result of Uglov [29]. Uglov proved that the coefficients $\Delta_{\lambda,\mu}^e(v)$ are expressed by parabolic Kazhdan-Lusztig polynomials as we will see in Section 5. By results of Kashiwara and Tanisaki [22], this notably implies that they have nonnegative integer coefficients.

Remark 3.3. The order $>$ does not coincide with the partial order used by Uglov in [29]. His partial order depends on e , so that he could not use a common partial order in the statements (1) and (2) of Theorem 3.2. On the other hand, we have used the common partial order $>$ there.

As a direct consequence, we have the following theorem :

Theorem 3.4. For each $\lambda \in \Pi_l$, we may expand $G_e(\lambda, \mathbf{s})$ as follows.

$$(4) \quad G_e(\lambda, \mathbf{s}) = \sum_{\mu \in \Pi_l} d_{\lambda,\mu}(v) G_\infty(\mu, \mathbf{s})$$

where :

- $d_{\lambda,\lambda}(v) = 1$,
- $d_{\lambda,\mu}(v) \in v\mathbb{Z}[v]$ if $\lambda \neq \mu$
- $d_{\lambda,\mu}(v) \neq 0$ only if $\lambda \succeq \mu$.

Proof. It follows from Theorem 3.2(1) and (2). □

Corollary 3.5. For $\lambda \in B_e(\mathbf{s})$, the formula (4) has the form

$$(5) \quad G_e(\lambda, \mathbf{s}) = \sum_{\mu \in B_\infty(\mathbf{s})} d_{\lambda,\mu}(v) G_\infty(\mu, \mathbf{s}).$$

Proof. We have already observed that $V_e(\mathbf{s})$ may be regarded as a $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ -submodule of $V_\infty(\mathbf{s})$ which shares the common highest weight vector \emptyset . Thus, we may expand $G_e(\lambda, \mathbf{s}) \in \mathcal{G}_e^\circ(\mathbf{s})$ on the basis $\mathcal{G}_\infty^\circ(\mathbf{s}) \subset \mathcal{G}_\infty(\mathbf{s})$, and Theorem 3.4 implies (5). □

Definition 3.6. We define

$$\begin{aligned}\Delta_e(v) &= (\Delta_{\lambda, \mu}^e(v))_{\lambda \in \Pi_l, \mu \in \Pi_l}, \\ \Delta_\infty(v) &= (\Delta_{\lambda, \mu}^\infty(v))_{\lambda \in \Pi_l, \mu \in \Pi_l}, \\ \Delta_\infty^e(v) &= (d_{\lambda, \mu}(v))_{\lambda \in \Pi_l, \mu \in \Pi_l}.\end{aligned}$$

They depend on \mathbf{s} . Then, we have

$$\Delta_e(v) = \Delta_\infty(v)\Delta_\infty^e(v).$$

We also define the following submatrices

$$\begin{aligned}D_e(v) &= (\Delta_{\lambda, \mu}^e(v))_{\lambda \in B_e(\mathbf{s}), \mu \in B_\infty(\mathbf{s})}, \\ D_\infty(v) &= (\Delta_{\lambda, \mu}^\infty(v))_{\lambda \in B_e(\mathbf{s}), \mu \in B_\infty(\mathbf{s})}, \\ D_\infty^e(v) &= (d_{\lambda, \mu}(v))_{\lambda \in B_e(\mathbf{s}), \mu \in B_\infty(\mathbf{s})}.\end{aligned}$$

Then we have $D_e(v) = D_\infty(v)D_\infty^e(v)$.

Remark 3.7. If $l = 1$, then the matrix $D_\infty(v)$ is the identity and $D_\infty^e(v) = D_e(v)$.

4. COMPUTATION OF $\Delta_\infty^e(v)$ AND $D_\infty^e(v)$

Before proceeding further, we explain algorithmic aspects for computing $\Delta_\infty^e(v)$ and $D_\infty^e(v)$. As $\Delta_\infty^e(v) = \Delta_\infty^{-1}(v)\Delta_e(v)$, we start with computing $\Delta_\infty(v)$ and $\Delta_e(v)$. Two algorithms are already proposed: one by Uglov and the other by Yvonne. Both use a natural embedding of the Fock spaces $\mathcal{F}_e^{\mathbf{s}}$ into the space of semi-infinite wedge products and compute the canonical bases $\mathcal{G}_e(\mathbf{s})$ and $\mathcal{G}_\infty(\mathbf{s})$.

The algorithm described by Uglov [29] needs steps to compute straightening laws of the wedge products, which soon starts to require enormous resources for the computation. It occurs especially in the case when the differences between two consecutive entries of \mathbf{s} are large.

Yvonne's algorithm is much more efficient but it requires subtle computation related to the commutation relations of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e) \otimes \mathcal{H} \otimes \mathcal{U}_{-v-1}(\widehat{\mathfrak{sl}}_l)$ on the space of semi-infinite wedge products, where \mathcal{H} is the Heisenberg algebra. We do not pursue this direction and refer to [31] for complete description of this algorithm.

Once $\mathcal{G}_e(\mathbf{s})$ and $\mathcal{G}_\infty(\mathbf{s})$ are computed, we can efficiently compute $\Delta_\infty^e(v)$ from them: see §4.1 below.

The computation of $D_\infty^e(v)$ is easier. One may compute it directly from the canonical bases $\mathcal{G}_e^o(\mathbf{s})$ and $\mathcal{G}_\infty^o(\mathbf{s})$ and we may compute the canonical bases by the algorithms proposed in [24] or [16]. The algorithm given in [16] was originally suited for multicharges \mathbf{s} such that $0 \leq s_0 \leq s_1 \leq \dots \leq s_{l-1} < e$. However, we will see in §4.2 that it also computes the canonical bases $\mathcal{G}_e^o(\mathbf{s})$ and $\mathcal{G}_\infty^o(\mathbf{s})$ (and thus the matrix $D_\infty^e(v)$) for arbitrary multicharge \mathbf{s} . Observe that this only uses $\mathcal{U}_v(\mathfrak{g})$ -module structure of the Fock space.

4.1. A general procedure. Assume that we have computed the canonical bases $\mathcal{G}_e(\mathbf{s})$ and $\mathcal{G}_\infty(\mathbf{s})$. Using the unitriangularity of the decomposition matrices, one can obtain $\Delta_\infty^e(v)$ directly from the relation $\Delta_\infty^e(v) = \Delta_\infty^{-1}(v)\Delta_e(v)$. This can be done efficiently by applying the procedure below.

- (1) Let $\lambda \in \Pi_{l,n}$. We know by Theorem 3.4 that $G_e(\lambda, \mathbf{s})$ may be expanded on $\mathcal{G}_\infty(\mathbf{s})$. We denote

$$\Lambda(\lambda) := \{\mu \in \Pi_{l,n} \mid d_{\lambda, \mu}(v) \neq 0\}.$$

Our aim is to find the members of $\Lambda(\lambda)$, and determine $d_{\lambda, \mu}(v)$ when μ is a member. Set $\lambda^0 := \lambda$. Then λ^0 is a member and $d_{\lambda, \lambda^0}(v) = 1$.

- (2) Let $k \in \mathbb{N}$. Suppose that we already know k members $\lambda^0, \dots, \lambda^{k-1}$ of $\Lambda(\lambda)$ and the polynomials $d_{\lambda, \lambda^i}(v)$, for $i = 0, \dots, k-1$. Then, we expand

$$G_e(\lambda, \mathbf{s}) - \sum_{i=0}^{k-1} d_{\lambda, \lambda^i}(v) G_\infty(\lambda^i, \mathbf{s})$$

into linear combination of the standard basis of l -partitions and write

$$\sum_{\mu \in \Lambda(\lambda) \setminus \{\lambda^0, \dots, \lambda^{k-1}\}} d_{\lambda, \mu}(v) G_\infty(\mu, \mathbf{s}) = \sum_{\nu \in \Pi_{l, n}} b_\nu(v) \nu.$$

We have $b_\nu(v) \in \mathbb{Z}[v]$ by Theorem 3.2. If the right hand side is zero, we are done. Otherwise, let λ^k be a maximal l -partition in $\{\nu \in \Pi_{l, n} \mid b_\nu(v) \neq 0\}$, with respect to the partial order \succ .

- (3) Consider $\mu \in \Lambda(\lambda) \setminus \{\lambda^0, \dots, \lambda^{k-1}\}$ which satisfies $\mu \succ \lambda^k$. If such μ does not exist, then we have

$$\lambda^k \in \Lambda(\lambda) \setminus \{\lambda^0, \dots, \lambda^{k-1}\}.$$

Otherwise let μ^k be maximal among them. If μ^k appears in $G_\infty(\mu, \mathbf{s})$, for $\mu \in \Lambda(\lambda) \setminus \{\lambda^0, \dots, \lambda^{k-1}\}$, then $\mu \succeq \mu^k \succ \lambda^k$, so that the maximality implies $\mu = \mu^k$. Since μ^k appears in $G_\infty(\mu^k, \mathbf{s})$, it follows that $b_{\mu^k}(v) \neq 0$, which is impossible by the maximality of λ^k and $\mu^k \succ \lambda^k$. Hence, λ^k is a maximal element of $\Lambda(\lambda) \setminus \{\lambda^0, \dots, \lambda^{k-1}\}$. Therefore, λ^k does not appear in $G_\infty(\mu, \mathbf{s})$, for $\mu \in \Lambda(\lambda) \setminus \{\lambda^0, \dots, \lambda^k\}$, and it follows that $d_{\lambda, \lambda^k}(v) = b_{\lambda^k}(v)$.

- (4) We increment k and go to (2).

4.2. The computation of $\mathcal{G}_e^\circ(\mathbf{s})$ and $\mathcal{G}_\infty^\circ(\mathbf{s})$. Let $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Assume first that $0 \leq s_1 \leq s_2 \leq \dots \leq s_l < e$. It is proved in [24] and [16] that one may construct a sequence of elements in $\mathbb{Z}/e\mathbb{Z}$

$$(6) \quad \underbrace{k_1, \dots, k_1}_{u_1}, \underbrace{k_2, \dots, k_2}_{u_2}, \dots, \underbrace{k_s, \dots, k_s}_{u_s},$$

for each $\lambda \in B_e(\mathbf{s})$, such that if we define

$$A_e(\lambda, \mathbf{s}) := f_{k_1}^{(u_1)} \dots f_{k_s}^{(u_s)}. \emptyset \in V_e(\mathbf{s})$$

then

$$\mathcal{A}_e(\mathbf{s}) = \{A_e(\lambda, \mathbf{s}) \mid \lambda \in B_e(\mathbf{s})\}$$

is a basis of $V_e(\mathbf{s})$. It is easy to obtain the coefficients $\gamma_{\lambda, \mu}(v) \in \mathbb{Z}[v, v^{-1}]$ in the expansion

$$(7) \quad G_e(\lambda, \mathbf{s}) = \sum_{\mu \in B_e(\mathbf{s})} \gamma_{\lambda, \mu}(v) A_e(\mu, \mathbf{s}).$$

When $e \in \mathbb{Z}_{\geq 2}$, there is a natural action of the (extended) affine symmetric group W on \mathbb{Z}^l such that

$$\mathcal{B}^l := \{(s_1, \dots, s_l) \in \mathbb{Z}^l \mid 0 \leq s_1 \leq \dots \leq s_l < e\}$$

is a fundamental domain for this action. Hence, for any $\mathbf{v} := (v_1, \dots, v_l) \in \mathbb{Z}^l$, there exist $\mathbf{s} := (s_1, \dots, s_l) \in \mathcal{B}^l$ and $w \in W$ such that $\mathbf{v} = w.\mathbf{s}$. Since \mathbf{v} and \mathbf{s} yield

the same dominant weight, we have an isomorphism $\phi_{\mathbf{s}, \mathbf{v}}$ from $V_e(\mathbf{s})$ to $V_e(\mathbf{v})$. We can assume that $\phi_{\mathbf{s}, \mathbf{v}}(\emptyset) = \emptyset$. For each $\boldsymbol{\lambda} \in B_e(\mathbf{s})$, we set

$$A_e(\boldsymbol{\lambda}, \mathbf{v}) = f_{k_1}^{(r_1)} \cdots f_{k_s}^{(r_s)} \cdot \emptyset \in V_e(\mathbf{v}),$$

where the pairs (k_a, r_a) are defined by (6). Then we have $\phi_{\mathbf{s}, \mathbf{v}}(A_e(\boldsymbol{\lambda}, \mathbf{s})) = A_e(\boldsymbol{\lambda}, \mathbf{v})$. By the uniqueness of the crystal basis on $V_e(\mathbf{v})$ proved by Kashiwara, we also have $\phi_{\mathbf{s}, \mathbf{v}}(G_e(\boldsymbol{\lambda}, \mathbf{s})) = G_e(\varphi_{\mathbf{s}, \mathbf{v}}(\boldsymbol{\lambda}), \mathbf{v})$, where $\varphi_{\mathbf{s}, \mathbf{v}}$ is the crystal isomorphism from $B_e(\mathbf{s})$ to $B_e(\mathbf{v})$.⁵ By applying $\phi_{\mathbf{s}, \mathbf{v}}$ to (7), we obtain

$$G_e(\boldsymbol{\nu}, \mathbf{v}) = \sum_{\boldsymbol{\mu} \in B_e(\mathbf{s})} \gamma_{\varphi_{\mathbf{s}, \mathbf{v}}^{-1}(\boldsymbol{\nu}), \boldsymbol{\mu}}(v) A_e(\boldsymbol{\mu}, \mathbf{v}),$$

for $\boldsymbol{\nu} \in B_e(\mathbf{v})$, and it follows that

$$\mathcal{G}_e(\mathbf{v}) = \left\{ \sum_{\boldsymbol{\mu} \in B_e(\mathbf{s})} \gamma_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(v) A_e(\boldsymbol{\mu}, \mathbf{v}) \mid \boldsymbol{\lambda} \in B_e(\mathbf{s}) \right\}.$$

Hence, the algorithms in [24] and [17] compute the canonical basis $\mathcal{G}_e(\mathbf{v})$ for any multicharge $\mathbf{v} = (v_1, \dots, v_l) \in \mathbb{Z}^l$. Applying the general procedure in §4.2 restricted to the canonical bases $\mathcal{G}_e^\circ(\mathbf{v})$ and $\mathcal{G}_\infty^\circ(\mathbf{v})$, we may compute $D_\infty^e(v)$.

Remark 4.1. Another algorithm is recently proposed by Fayers [8] for computing the canonical basis of the highest weight $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ -modules which is realized in the tensor product of level one Fock spaces.

4.3. Example. We set $e = 2$. Then the matrix $D_e(v)$ of the canonical basis of the $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -module $V_e(0, 0)$ is:

$$\begin{pmatrix} 1 & \cdot & \cdot \\ v & \cdot & \cdot \\ v & 1 & \cdot \\ v^2 & v & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & v \\ v & v^2 & \cdot \\ v^2 & v^3 & \cdot \\ v^2 & \cdot & \cdot \\ v^3 & \cdot & \cdot \end{pmatrix}$$

where dots mean 0. The matrix $D_\infty(v)$ of the canonical basis of the $\mathcal{U}_v(\mathfrak{sl}_\infty)$ -module $V_\infty(0, 0)$ is:

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ v & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & v & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & v & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & v & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & v \end{pmatrix}$$

⁵See [19] for a combinatorial description of $\varphi_{\mathbf{s}, \mathbf{v}}$.

The matrix $D_\infty^e(v)$ obtained from our algorithm is:

$$\begin{pmatrix} 1 & \cdot & \cdot \\ v & 1 & \cdot \\ \cdot & \cdot & 1 \\ v & v^2 & \cdot \\ v^2 & \cdot & \cdot \end{pmatrix}$$

and one can check that we have

$$D_e(v) = D_\infty(v) \cdot D_\infty^e(v).$$

5. POSITIVITY OF THE COEFFICIENTS IN $d_{\lambda,\mu}(v)$

The aim of this section is to study the entries of the matrix $D_\infty(v)$. The main result asserts that they are polynomials with nonnegative integer coefficients.

5.1. Some combinatorial preliminaries. A *1-runner abacus* is a subset A of \mathbb{Z} such that $-k \in A$ and $k \notin A$ for all large enough $k \in \mathbb{N}$. To visualize a 1-runner abacus, we view \mathbb{Z} as a horizontal runner and place a bead on the k -th position, for each $k \in A$. Thus, the runner is full of beads on the far left and no beads on the far right. For $l \geq 1$, an *l -runner abacus* is a l -tuple of 1-runner abaci. Let \mathcal{A}^l be the set of l -runners abaci. For each pair of an l -partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ and a multicharge $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$, we associate the l -runner abacus

$$a(\lambda, \mathbf{s}) := \{(\lambda_i^{(d)} + s_d + 1 - i, d) \mid i \geq 1, 1 \leq d \leq l\},$$

which is a subset of $\mathbb{Z} \times [1, l]$. One checks easily that the map

$$(\lambda, \mathbf{s}) \in \Pi_l \times \mathbb{Z}^l \mapsto a(\lambda, \mathbf{s}) \in \mathcal{A}^l$$

is bijective. To describe the embedding of Fock spaces into the space of semi-infinite wedge products and then cut semi-infinite wedge products to finite wedge products, we need to introduce a bijective map $\tau_l : \Pi \times \mathbb{Z} \cong \mathcal{A} \rightarrow \mathcal{A}^l \cong \Pi_l \times \mathbb{Z}^l$.

Definition 5.1. Let $\tau_l : \mathbb{Z} \rightarrow \mathbb{Z} \times [1, l]$ be the bijective map defined by

$$k \mapsto (\phi(k), d(k)),$$

where $k = c(k) + e(d(k) - 1) + elm(k)$ such that

$$c(k) \in [1, e], \quad d(k) \in [1, l] \quad \text{and} \quad m(k) \in \mathbb{Z},$$

and $\phi(k) = c(k) + em(k)$. Then, it defines $\tau_l : \Pi \times \mathbb{Z} \cong \mathcal{A} \rightarrow \mathcal{A}^l \cong \Pi_l \times \mathbb{Z}^l$ by

$$A \mapsto \tau_l(A) = \{(\phi(k), d(k)) \mid k \in A\} \in \mathcal{A}^l,$$

for $A \in \mathcal{A}$.

Remark 5.2.

- (1) If $(\lambda, \mathbf{s}) = \tau_l(\lambda, s)$ then $s = s_1 + \dots + s_l$.
- (2) To read off the multicharge $\mathbf{s} = (s_1, \dots, s_l)$ from the l -runner abacus, we proceed as follows: if the left adjacent position of a bead on a runner is vacant, we move the bead to the left to occupy the vacant position, and we repeat this procedure as many times as possible. Then, s_d is the column number of the rightmost bead of the d -th runner.

Example 5.3. Let $e = 2$ and $l = 3$. Then the preimage of

$$(\boldsymbol{\lambda}, \mathbf{s}) = (((1.1), (1.1), (1)), (0, 0, -1))$$

is $(\boldsymbol{\lambda}, \mathbf{s}) = ((4.3.3.2.1), -1)$.

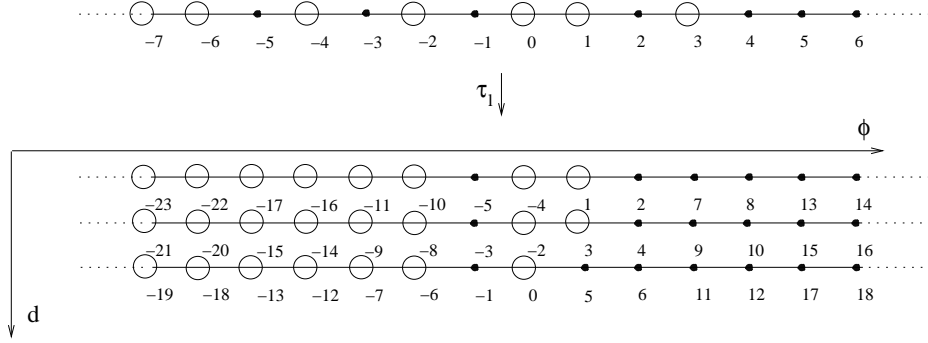


FIGURE 1. Computation of the bijection τ_l using abaci.

Now, $(\boldsymbol{\lambda}, \mathbf{s}) = \tau_l^{-1}(\boldsymbol{\lambda}, \mathbf{s})$ has the 1-runner abacus

$$a(\boldsymbol{\lambda}, \mathbf{s}) = \{(k_i := \lambda_i + s + 1 - i) \mid i \geq 1\},$$

and the semi-infinite sequence (k_1, k_2, \dots) defines a semi-infinite wedge product.

We fix a sufficiently large r such that $\lambda_i = 0$, for $i \geq r$. Then $(\boldsymbol{\lambda}, \mathbf{s})$ is determined by the finite sequence $\mathbf{k} := (k_1, \dots, k_r)$. For example, $((4.3.3.2.1), -1)$ is determined by $\mathbf{k} = (3, 1, 0, -2, -4, -6, -7)$. We write $\mathbf{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \mathbf{s})$ by abuse of notation. Then they give the wedge basis in the space of finite wedge products Λ^r , which will be introduced in a different guise in §5.2.

We read the beads $\tau_l(k_1), \dots, \tau_l(k_r)$ on the l -runner abacus $a(\boldsymbol{\lambda}, \mathbf{s})$ from right to left, starting with the l -th runner, and obtain a permutation $w(\mathbf{k}) = (w_1, \dots, w_r)$ of \mathbf{k} . In our example, we have $w(\mathbf{k}) = (0, -6, -7, 3, -2, 1, -4)$.

Definition 5.4. Let $\tau_l(w_i) = (\zeta_i, b_i)$, for $1 \leq i \leq r$, that is, ζ_i and b_i are the column number and the row number of the bead $\tau_l(w_i)$ on the l -runner abacus $a(\boldsymbol{\lambda}, \mathbf{s})$, respectively. Then, we define

$$\zeta(\boldsymbol{\lambda}) = (\zeta_1, \dots, \zeta_r) \quad \text{and} \quad b(\boldsymbol{\lambda}) = (b_1, \dots, b_r).$$

Example 5.5. In our example, we have

$$\zeta(\boldsymbol{\lambda}) = (0, -2, -3, 1, 0, 1, 0) \quad \text{and} \quad b(\boldsymbol{\lambda}) = (3, 3, 3, 2, 2, 1, 1).$$

We will need the $\zeta(\boldsymbol{\lambda})$ and $b(\boldsymbol{\lambda})$ when we express $\Delta_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^e(v)$ in Kazhdan-Lusztig polynomials. In this respect, the following remark is also important.

Remark 5.6. Suppose that we have fixed $\boldsymbol{\lambda}$ and \mathbf{s} . Then, $\zeta(\boldsymbol{\lambda})$ and $b(\boldsymbol{\lambda})$ do not depend on e . Indeed, they are determined by the location of the beads in $a(\boldsymbol{\lambda}, \mathbf{s})$, which are computed from $\boldsymbol{\lambda}$ and \mathbf{s} . On the other hand, $\mathbf{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \mathbf{s})$ does depend on e .

Let $P = \mathbb{Z}^r$ and W the affine symmetric group which is the semidirect product of the symmetric group S_r and the normal subgroup P . W acts $\beta = (\beta_1, \dots, \beta_r) \in P$ on the right by

$$\begin{aligned}\beta \cdot s_i &= (\beta_1 \cdots, \beta_{i+1}, \beta_i, \dots, \beta_r), \text{ for } 1 \leq i \leq r-1, \text{ and} \\ \beta \cdot \mu &= \beta + e\mu, \text{ for } \mu \in P.\end{aligned}$$

Then

$$A^r = \{a = (a_1, \dots, a_r) \in P \mid 1 \leq a_1 \leq \cdots \leq \cdots a_r \leq e\}$$

is a fundamental domain for the action. We denote the stabilizer of $a \in A^r$ by W_a . It is clear that W_a is a subgroup of S_r . Let w_a be the maximal element of W_a . We denote by aW and aS_r the set of minimal length coset representatives in $W_a \backslash W$ and $W_a \backslash S_r$, respectively.

In the similar manner, W acts $\beta = (\beta_1, \dots, \beta_r) \in P$ on the left by

$$\begin{aligned}s_i \cdot \beta &= (\beta_1 \cdots, \beta_{i+1}, \beta_i, \dots, \beta_r), \text{ for } 1 \leq i \leq r-1, \text{ and} \\ \mu \cdot \beta &= \beta + l\mu, \text{ for } \mu \in P.\end{aligned}$$

Then

$$B^r = \{b = (b_1, \dots, b_r) \in P \mid l \geq b_1 \geq \cdots \geq \cdots b_r \geq 1\}$$

is a fundamental domain for the action. We denote the stabilizer of $b \in B^r$ by W_b , its maximal element by w_b , and the set of minimal length coset representatives in W/W_b and S_r/W_b by W^b and S_r^b , respectively.

Write $k = c(k) + e(d(k) - 1) + elm(k)$ and $\phi(k) = c(k) + em(k)$, for $k \in \mathbb{Z}$, as before, and define

$$\begin{aligned}c(\mathbf{k}) &= (c(k_1), \dots, c(k_r)), \\ d(\mathbf{k}) &= (d(k_1), \dots, d(k_r)), \\ m(\mathbf{k}) &= (m(k_1), \dots, m(k_r)), \\ \phi(\mathbf{k}) &= (\phi(k_1), \dots, \phi(k_r)),\end{aligned}$$

for $\mathbf{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \mathbf{s}) \in \mathbb{Z}^r$. Then,

- there exist $a(\mathbf{k}) \in A^r$ and $u(\mathbf{k}) \in {}^a(\mathbf{k})S_r$ such that $c(\mathbf{k}) = a(\mathbf{k}) \cdot u(\mathbf{k})$.
- there exist $b(\mathbf{k}) \in B^r$ and $v(\mathbf{k}) \in S_r^{b(\mathbf{k})}$ such that $d(\mathbf{k}) = v(\mathbf{k}) \cdot b(\mathbf{k})$.

It is clear that $b(\mathbf{k}) = b(\boldsymbol{\lambda})$. We define $\zeta(\mathbf{k}) := \phi(\mathbf{k}) \cdot v(\mathbf{k})$. Then, comparing it with

$$b(\mathbf{k}) = v(\mathbf{k})^{-1} \cdot d(\mathbf{k}) = d(\mathbf{k}) \cdot v(\mathbf{k}),$$

we have $\zeta(\mathbf{k}) = \zeta(\boldsymbol{\lambda})$. In the sequel, we will use the notation $b(\boldsymbol{\lambda}), \zeta(\boldsymbol{\lambda})$ to make apparent that they do not depend on e . From the definitions, we have

$$\zeta(\boldsymbol{\lambda}) = a(\mathbf{k}) \cdot u(\mathbf{k})v(\mathbf{k}) + e(m(\mathbf{k}) \cdot v(\mathbf{k})),$$

which shows that $\zeta(\boldsymbol{\lambda})$ belongs to $a(\mathbf{k})W$.

Example 5.7. With $\mathbf{k} = (3, 1, 0, -2, -4, -6, -7)$, $e = 2$ and $l = 3$, we obtain

$$\begin{aligned} c(\mathbf{k}) &= (1, 1, 2, 2, 2, 2, 1), \\ d(\mathbf{k}) &= (2, 1, 3, 2, 1, 3, 3), \\ m(\mathbf{k}) &= (0, 0, -1, -1, -1, -2, -2), \\ \phi(\mathbf{k}) &= (1, 1, 0, 0, 0, -2, -3), \\ a(\mathbf{k}) &= (1, 1, 1, 2, 2, 2, 2), \\ b(\mathbf{k}) &= b(\boldsymbol{\lambda}) = (3, 3, 3, 2, 2, 1, 1), \\ \zeta(\mathbf{k}) &= \zeta(\boldsymbol{\lambda}) = (0, -2, -3, 1, 0, 1, 0). \end{aligned}$$

5.2. Expression of the coefficients $\Delta_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^e(v)$ in terms of KL-polynomials.

In [29], Uglov proved that $\Delta_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^e(v)$ are expressed by parabolic Kazhdan-Lusztig polynomials. Let H be the extended affine Hecke algebra of the symmetric group S_r . Namely, it is generated by T_1, \dots, T_{r-1} and X^λ , for $\lambda \in \bigoplus_{i=1}^r \mathbb{Z}\epsilon_i$, such that the defining relations are

$$(T_i - v^{-1})(T_i + v) = 0, \quad X^\lambda T_i = T_i X^{s_i \lambda} + (v - v^{-1}) \frac{X^{s_i \lambda} - X^\lambda}{1 - X^{\alpha_i}}$$

and the Artin braid relations for T_1, \dots, T_{r-1} . We want to introduce the space of finite wedge products. Consider $a \in A^r$ and $b \in B^r$. We define W_a, W_b, w_a, w_b as in §5.1. The subgroups W_a and W_b define parabolic subalgebras H_a and H_b of the affine Hecke algebra H . If we denote

$$J = \{i \mid 1 \leq i \leq r-1, b_i = b_{i+1}\},$$

then the parabolic subgroup W_J is nothing but the subgroup W_b . Let 1_a^+ and 1_b^- be such that

$$(8) \quad \begin{aligned} 1_a^+ T_i &= v^{-1} 1_a^+ & (s_i \in W_a) \\ T_i 1_b^- &= -v 1_b^- & (s_i \in W_b). \end{aligned}$$

Then, the space of finite wedges Λ^r is the direct sum of $\Lambda^r(a, b) = 1_a^+ \otimes_{H_a} H \otimes_{H_b} 1_b^-$, for $a \in A^r$ and $b \in B^r$.

Definition 5.8. Let $\xi \in P$. Then, there are unique $a \in A^r$ and $x \in {}^a W$ such that $\xi = ax$. We denote this x by $x(\xi)$.

We say that ξ is J -dominant and write $\xi \in P_b^{++}$, if $\xi_i > \xi_{i+1}$ whenever $b_i = b_{i+1}$. Note that $\zeta(\boldsymbol{\lambda}) \in P_b^{++}$, for $\boldsymbol{\lambda} \in \Pi_l$. If $\xi \in P_b^{++}$ then [29, Prop. 3.8] implies that $x(\xi)s < x(\xi)$ in the Bruhat order, for $s \in W_b$, so that $x(\xi)w_b$ is the minimal length coset representative of $W_a x(\xi)W_b$.

By [29, Lem. 3.19, Prop. 3.20], the wedge basis of $\Lambda^r(a, b)$ is given by

$$\{|\boldsymbol{\lambda}\rangle = 1_a^+ \otimes T_{x(\zeta(\boldsymbol{\lambda}))w_b} \otimes 1_b^- = (-v)^{-\ell(w_b)} 1_a^+ \otimes T_{x(\zeta(\boldsymbol{\lambda}))} \otimes 1_b^- \mid \zeta(\boldsymbol{\lambda}) \in aW\}.$$

Let $x = x(\xi)w_b$, for $\xi \in aW \cap P_b^{++}$. Then, by the ordinary Kazhdan-Lusztig theory,

$$C'_{w_a x} = v^{\ell(w_a x)} \sum_{y \in W} P_{y, w_a x}(v^{-2}) v^{-\ell(y)} T_y$$

is bar-invariant. As $W \simeq W_a \times \{x(\eta) \mid \eta \in aW\}$, we have

$$\begin{aligned} C'_{w_a x} &= v^{\ell(w_a x)} \sum_{\eta \in aW} \sum_{z \in W_a} P_{zx(\eta), w_a x}(v^{-2}) v^{-\ell(z) - \ell(x(\eta))} T_z T_{x(\eta)} \\ &= v^{\ell(w_a x)} \sum_{\eta \in aW} \sum_{z \in W_a} P_{w_a x(\eta), w_a x}(v^{-2}) v^{-\ell(z) - \ell(x(\eta))} T_z T_{x(\eta)}. \end{aligned}$$

Denote

$$C_e^+(\boldsymbol{\lambda}) = \frac{v^{-\ell(w_a)}}{\sum_{z \in W_a} v^{-2\ell(z)}} 1_a^+ \otimes C'_{w_a x} \otimes 1_b^-$$

where $x = x(\xi)w_b$ and $\xi = \zeta(\boldsymbol{\lambda})$. Then, using (8), we have that

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\eta \in aW} v^{\ell(x) - \ell(x(\eta))} P_{w_a x(\eta), w_a x}(v^{-2}) 1_a^+ \otimes T_{x(\eta)} \otimes 1_b^-$$

is bar-invariant, where the bar-involution on $\Lambda^r(a, b)$ is defined by

$$\overline{1_a^+ \otimes h \otimes 1_b^-} = 1_a^+ \otimes \bar{h} \otimes 1_b^-.$$

Now, we rewrite $C_e^+(\boldsymbol{\lambda})$ into the expression ⁶

$$\begin{aligned} C_e^+(\boldsymbol{\lambda}) &= \sum_{\eta \in aW \cap P_b^{++}} \sum_{u \in W_b} v^{\ell(x) - \ell(x(\eta)w_b u)} P_{w_a x(\eta)w_b u, w_a x}(v^{-2}) (-v)^{\ell(u)} 1_a^+ \otimes T_{x(\eta)w_b} \otimes 1_b^-. \end{aligned}$$

Recall that

$$P_{w_a x(\eta)w_b, w_a x(\xi)w_b}^{J, -1}(v^{-2}) = \sum_{u \in W_b} (-1)^{\ell(u)} P_{w_a x(\eta)w_b u, w_a x(\xi)w_b}(v^{-2})$$

is a parabolic Kazhdan-Lusztig polynomial. These polynomials were introduced by Deodhar [6]. As $x = x(\xi)w_b$ and $v^{\ell(x) - \ell(x(\eta)w_b u)} = v^{\ell(x(\xi)) - \ell(x(\eta)) - \ell(u)}$, we have

$$(9) \quad C_e^+(\boldsymbol{\lambda}) = \sum_{\eta \in aW \cap P_b^{++}} v^{\ell(x(\xi)) - \ell(x(\eta))} P_{w_a x(\eta)w_b, w_a x(\xi)w_b}^{J, -1}(v^{-2}) 1_a^+ \otimes T_{x(\eta)w_b} \otimes 1_b^-$$

and it satisfies the defining properties of the plus canonical basis. Hence, we recover Uglov's result. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_{l, n}$. Choose $r \in \mathbb{N}$ as before, and define $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^r$ by

$$\mathbf{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \mathbf{s}) \quad \text{and} \quad \mathbf{l} = \tau_l^{-1}(\boldsymbol{\mu}, \mathbf{s}).$$

Define $a(\mathbf{k}), a(\mathbf{l})$ and $b(\boldsymbol{\lambda}), b(\boldsymbol{\mu})$ as before, and set $\xi = \zeta(\boldsymbol{\lambda})$ and $\eta = \zeta(\boldsymbol{\mu})$.

Theorem 5.9. *With the above notation, we have*

- (1) *If $a(\mathbf{k}) \neq a(\mathbf{l})$, or $b(\boldsymbol{\lambda}) \neq b(\boldsymbol{\mu})$ then $\Delta_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^\xi(v) = 0$.*
- (2) *If $a(\mathbf{k}) = a(\mathbf{l}) = a \in A^r$ and $b(\boldsymbol{\lambda}) = b(\boldsymbol{\mu}) = b \in B^r$, then*

$$(10) \quad \Delta_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^\xi(v) = v^{\ell(x(\xi)) - \ell(x(\eta))} P_{w_a x(\eta)w_b, w_a x(\xi)w_b}^{J, -1}(v^{-2}).$$

By Remark 5.6 we derive the corollary below.

Corollary 5.10. *In the formula (10), b, η and ξ do not depend on e .*

⁶If $\eta \in P_b^+ \setminus P_b^{++}$ the corresponding term is zero. See Equation (13) in [29].

Now assuming that $\mathbf{s} \in \mathbb{Z}^l$ and $\boldsymbol{\lambda} \in \Pi_l$ are fixed, we increase e . Then $\xi = \zeta(\boldsymbol{\lambda})$ is independent of e , and, if e is sufficiently large then there exist $\tilde{x}(\xi) \in S_r$ and $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_r)$ which are also independent of e such that

$$\tilde{a}_1 \leq \dots \leq \tilde{a}_r, \quad \tilde{x}(\xi) \in \tilde{a}S_r \quad \text{and} \quad \xi = \tilde{a}\tilde{x}(\xi).$$

Then, under this stabilization assumption, we have

$$\Delta_{\boldsymbol{\lambda}, \boldsymbol{\mu}}^\infty(v) = v^{\ell(\tilde{x}(\zeta(\boldsymbol{\lambda}))) - \ell(\tilde{x}(\zeta(\boldsymbol{\mu})))} P_{w_{\tilde{a}\tilde{x}(\zeta(\boldsymbol{\mu}))} w_b, w_{\tilde{a}\tilde{x}(\zeta(\boldsymbol{\lambda}))} w_b}^{J, -1}(v^{-2}),$$

for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_l$. We have $\zeta(\boldsymbol{\lambda}) \in \tilde{a}S_r$, for $\boldsymbol{\lambda} \in \Pi_l$, as before.

Let $\xi, \eta \in P_b^{++}$. Then Theorem 3.4 implies that there exist polynomials

$$d_{\gamma\xi}(v) \in \mathbb{Z}[v], \quad \text{for } \gamma \in P_b^{++},$$

such that, if we define $\tilde{a} \in aW \cap P^-$ by $\eta S_r = \tilde{a}S_r$, then we have the equality

$$\begin{aligned} v^{\ell(x(\xi)) - \ell(x(\eta))} P_{w_a x(\eta) w_b, w_a x(\xi) w_b}^{J, -1}(v^{-2}) = \\ \sum_{\gamma \in \tilde{a}S_r \cap P_b^{++}} v^{\ell(\tilde{x}(\gamma)) - \ell(\tilde{x}(\eta))} P_{w_{\tilde{a}\tilde{x}(\eta)} w_b, w_{\tilde{a}\tilde{x}(\gamma)} w_b}^{J, -1}(v^{-2}) d_{\gamma\xi}(v). \end{aligned}$$

Define a linear map $\psi : \Lambda(\tilde{a}, b) \hookrightarrow \Lambda(a, b)$ by

$$1_{\tilde{a}}^+ \otimes T_{\tilde{x}(\xi) w_b} \otimes 1_b^- \mapsto 1_a^+ \otimes T_{x(\xi) w_b} \otimes 1_b^- = 1_a^+ \otimes T_{x(\tilde{a})} T_{\tilde{x}(\xi) w_b} \otimes 1_b^-.$$

Then, in view of (9), the above equality is equivalent to

$$(11) \quad C_e^+(\boldsymbol{\lambda}) = \sum_{\tilde{a} \in aW \cap P^-} \sum_{\gamma := \zeta(\boldsymbol{\mu}) \in \tilde{a}S_r \cap P_b^{++}} d_{\gamma\xi}(v) \psi(C_\infty^+(\boldsymbol{\mu})),$$

where

$$(12) \quad C_\infty^+(\boldsymbol{\mu}) = \frac{v^{-\ell(w_{\tilde{a}})}}{\sum_{u \in W_{\tilde{a}}} v^{-2\ell(u)}} 1_{\tilde{a}}^+ \otimes C'_{w_{\tilde{a}\tilde{x}(\zeta(\boldsymbol{\mu}))}} \otimes 1_b^-.$$

5.3. Proof of the positivity. The idea of the proof is to expand $C_e^+(\boldsymbol{\lambda})$ into a linear combination of $\psi(C_\infty^+(\boldsymbol{\mu}))$ and compare it with (11). The famous positivity result of the multiplicative structure constants with respect to the Kazhdan-Lusztig basis and its generalization in [13] then yields the desired positivity. Recall the basis

$$C'_w = v^{\ell(w)} \sum_{y \in W} P_{y, w}(v^{-2}) v^{-\ell(y)} T_y.$$

For $y \in W$, we write $y = y'y''$, where $y'' \in S_r$ and y' is the minimal length coset representative of yS_r . Then we define

$$(13) \quad U_y = T_{y'} C'_{y''}.$$

It is clear that we may write

$$C'_w = \sum_{y \in W} A_{y, w}(v) U_y,$$

where $A_{y, w}(v) \in \mathbb{Z}[v, v^{-1}]$. By [13, Cor. 3.9], we have $A_{y, w}(v) \in \mathbb{N}[v, v^{-1}]$.

Let $w = w_a x(\xi) w_b$. Then, by the bijection

$$W \simeq W_a \times W_a \backslash W / S_r \times S_r,$$

$U_{ux(\gamma)} = T_u U_{x(\gamma)}$, for $u \in W_a$. As $s_i w < w$, for $s_i \in W_a$,

$$\begin{aligned} v^{-1}C'_w &= T_i C'_w = \sum_{s_i y > y} A_{y,w}(v) U_{s_i y} + \sum_{s_i y < y} A_{y,w}(v) ((v^{-1} - v)U_y + U_{s_i y}) \\ &= \sum_{s_i y < y} (A_{s_i y, w}(v) + (v^{-1} - v)A_{y,w}(v)) U_y + \sum_{s_i y > y} A_{s_i y, w}(v) U_y. \end{aligned}$$

Thus, if $s_i y > y$ then $A_{s_i y, w}(v) = v^{-1}A_{y,w}(v)$. It implies that if $x \in W_a$ and y is the minimal length coset representative of $W_a y$ then $A_{xy, w}(v) = v^{-\ell(x)}A_{y,w}(v)$. In particular, we have

$$(14) \quad \left(\sum_{x \in W_a} v^{-\ell(x)} T_x \right) \left(\sum_{\eta \in aW} A_{x(\eta), w}(v) U_{x(\eta)} \right) = C'_w.$$

Hence, for any $\lambda \in \Pi_l$, the plus canonical basis is given by

$$\begin{aligned} (15) \quad C_e^+(\lambda) &= \frac{v^{-\ell(w_a)}}{\sum_{x \in W_a} v^{-2\ell(x)}} 1_a^+ \otimes C'_w \otimes 1_b^- \\ &= \sum_{\eta \in aW} v^{-\ell(w_a)} A_{x(\eta), w_a x(\xi)w_b}(v) 1_a^+ \otimes U_{x(\eta)} \otimes 1_b^- \\ &= \sum_{\tilde{a} \in aW \cap P^-} \sum_{z \in S_r} v^{-\ell(w_a)} A_{x(\tilde{a})z, w_a x(\xi)w_b}(v) 1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- \end{aligned}$$

where the second equality follows from (14) and the third from (13). Note that $W_{\tilde{a}} = S_r \cap x(\tilde{a})^{-1}W_a x(\tilde{a})$ by $\tilde{a} = ax(\tilde{a})$. This allows us to write

$$1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- = \frac{1}{\sum_{u \in W_{\tilde{a}}} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} \left(\sum_{u \in W_{\tilde{a}}} v^{-\ell(u)} T_u \right) C'_z \otimes 1_b^-.$$

As the left multiplication by $\sum_{u \in W_{\tilde{a}}} v^{-\ell(u)} T_u$ gives the subspace of dimension $|S_r|/|W_{\tilde{a}}|$ in the Hecke algebra $H(S_r)$, it has the basis $\{C'_{w_{\tilde{a}}y} \mid y \in W_{\tilde{a}} \setminus S_r\}$. By the positivity of the structure constants, we may write

$$\left(\sum_{u \in W_{\tilde{a}}} v^{-\ell(u)} T_u \right) C'_z = \sum_{y \in W_{\tilde{a}} \setminus S_r} B_{y,z}(v) C'_{w_{\tilde{a}}y}$$

such that $B_{y,z}(v) \in \mathbb{N}[v, v^{-1}]$. Thus,

$$1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- = \sum_{\gamma \in \tilde{a}S_r} B_{\tilde{x}(\gamma), z}(v) \frac{1}{\sum_{u \in W_{\tilde{a}}} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} C'_{w_{\tilde{a}}\tilde{x}(\gamma)} \otimes 1_b^-.$$

For each $\gamma \in \tilde{a}S_r$, define

$$(16) \quad d'_{\gamma, \xi}(v) = v^{-\ell(w_a)} \sum_{z \in S_r} v^{\ell(w_{\tilde{a}})} A_{x(\tilde{a})z, w_a x(\xi)w_b}(v) B_{\tilde{x}(\gamma), z}(v).$$

Then, $d'_{\gamma,\xi}(v) \in \mathbb{N}[v, v^{-1}]$ and

$$\begin{aligned} & \sum_{z \in S_r} v^{-\ell(w_a)} A_{x(\bar{a})z, w_a x(\xi)w_b}(v) 1_a^+ \otimes T_{x(\bar{a})} C'_z \otimes 1_b^- \\ &= \sum_{\gamma \in \bar{a}S_r} d'_{\gamma,\xi}(v) \frac{v^{-\ell(w_a)}}{\sum_{u \in W_{\bar{a}}} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\bar{a})} C'_{w_{\bar{a}}\tilde{x}(\gamma)} \otimes 1_b^- \\ &= \sum_{\gamma \in \bar{a}S_r \cap P_b^+} \sum_{t \in W_b} d'_{\gamma w_b t, \xi}(v) \frac{v^{-\ell(w_a)}}{\sum_{u \in W_{\bar{a}}} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\bar{a})} C'_{w_{\bar{a}}\tilde{x}(\gamma)w_b t} \otimes 1_b^-. \end{aligned}$$

If $x s_i < x$, for some $s_i \in W_b$, then

$$v^{-1} C'_x \otimes 1_b^- = C'_x T_i \otimes 1_b^- = -v C'_x \otimes 1_b^-$$

and $C'_x \otimes 1_b^- = 0$. Thus, by using the last expression in (15), we derive

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\bar{a} \in aW \cap P^-} \sum_{\gamma \in \bar{a}S_r \cap P_b^{++}} d'_{\gamma w_b, \xi}(v) \frac{v^{-\ell(w_a)}}{\sum_{u \in W_{\bar{a}}} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\bar{a})} C'_{w_{\bar{a}}\tilde{x}(\gamma)w_b} \otimes 1_b^-.$$

By using (12), this can also be rewritten

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\bar{a} \in aW \cap P^-} \sum_{\gamma \in \bar{a}S_r \cap P_b^{++}} d'_{\gamma w_b, \xi}(v) \psi(C_\infty^+(\boldsymbol{\mu}))$$

with $\tilde{x}(\gamma)w_b = \zeta(\boldsymbol{\mu})$. Hence, comparing it with (11), we derive $d_{\gamma\xi}(v) = d'_{\gamma w_b, \xi}(v)$. Since $d'_{\gamma w_b, \xi}(v) \in \mathbb{N}[v, v^{-1}]$, we have obtained the desired positivity result:

Theorem 5.11. *The polynomials $d_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(v)$ which appear in (4) have nonnegative integer coefficients.*

5.4. The case $v = 1$. The proof of the positivity we have obtained, does not properly yield a geometric interpretation of the coefficients $d_{\gamma\xi}(v)$. The purpose of this section is to show that their specializations $d_{\gamma\xi}(1)$ may be interpreted as branching coefficients. Let us rewrite the right action in more coordinate free manner. For this, we consider

$$\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_r(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ is the Kac-Moody Lie algebra of type $A_{r-1}^{(1)}$. Then the fundamental weights $\Lambda_0, \dots, \Lambda_{r-1}$ remain linearly independent on

$$\mathfrak{h}' = \bigoplus_{i=0}^{r-1} \mathbb{C}\alpha_i^\vee$$

and we may write its dual space as follows.

$$\mathfrak{h}'^* = \mathfrak{h}^* / \mathbb{C}\delta = \bigoplus_{i=0}^{r-1} \mathbb{C}\Lambda_i.$$

We identify the weight lattice P of $\mathfrak{sl}_r(\mathbb{C})$ with the set of level zero integral weights in \mathfrak{h}'^* by

$$P = \frac{\bigoplus_{i=1}^r \mathbb{Z}\epsilon_i}{\mathbb{Z}(\epsilon_1 + \dots + \epsilon_r)} = \bigoplus_{i=1}^{r-1} \mathbb{Z}(\Lambda_i - \Lambda_0) \subseteq \mathfrak{h}'^*$$

where $\xi = \sum_{i=1}^r \xi_i \epsilon_i \mapsto \sum_{i=1}^{r-1} (\xi_i - \xi_{i+1})(\Lambda_i - \Lambda_0)$.⁷ For $\xi \in P$, we define

$$\hat{\xi} = -\xi + e\Lambda_0 \in \mathfrak{h}^*.$$

The Weyl group action on \mathfrak{h}^* preserves $P + e\Lambda_0$ and if we write $w\hat{\xi} = -w\xi + e\Lambda_0$, for $w \in W$, then

$$s_i \xi = \xi_{i+1} \epsilon_i + \xi_i \epsilon_{i+1} + \sum_{j \neq i, i+1} \xi_j \epsilon_j,$$

for $1 \leq i \leq r-1$, and

$$s_0 \xi = (\xi_r - e)\epsilon_1 + (\xi_1 + e)\epsilon_r + \sum_{j \neq 1, r} \xi_j \epsilon_j.$$

Thus, $\xi \cdot w := w^{-1}\xi$, for $\xi \in P$ and $w \in W$, is the right action of W .

Let $J \subset \{1, \dots, r-1\}$ and μ the composition of r defined by J . Write $\mathfrak{p}_\mu(\mathbb{C})$ for the parabolic subalgebra of \mathfrak{g} defined by μ and $\mathfrak{l}_\mu(\mathbb{C})$ for the standard Levi subalgebra of $\mathfrak{p}_\mu(\mathbb{C})$. For $\zeta \in P_b^{++}$, we denote by $V(w_b \hat{\zeta})$ the finite dimensional irreducible $\mathfrak{l}_\mu(\mathbb{C}) \oplus \mathbb{C}c$ -module with highest weight $w_b \hat{\zeta} - \rho$, where ρ is such that $\langle \rho, \alpha_i^\vee \rangle = 1$, for $0 \leq i \leq r-1$. Thus, the canonical central element c acts as the scalar $e - r$. We view it as a $\mathfrak{p}_\mu(\mathbb{C}) \oplus \mathbb{C}c$ -module. Then, through the evaluation homomorphism

$$\mathfrak{p}_\mu = \{X \in \mathfrak{sl}_r(\mathbb{C}[t]) \mid X|_{t=0} \in \mathfrak{p}_\mu(\mathbb{C})\} \oplus \mathbb{C}c \rightarrow \mathfrak{p}_\mu(\mathbb{C}) \oplus \mathbb{C}c$$

we may view it as a \mathfrak{p}_μ -module as well. We define the following \mathfrak{g}' -module.

$$M_\mu(w_b \hat{\zeta}) = U(\mathfrak{g}') \otimes_{U(\mathfrak{p}_\mu)} V(w_b \hat{\zeta}).$$

If $X \in \mathfrak{p}_\mu$, then

$$Xu \otimes v = [X, u] \otimes v + u \otimes Xv \quad (u \in U(\mathfrak{g}'), v \in V(w_b \hat{\zeta})).$$

Hence $M_\mu(w_b \hat{\zeta})$ is isomorphic to the tensor product representation of the adjoint representation on $U(\mathfrak{g}')$ and $V(w_b \hat{\zeta})$ as a \mathfrak{p}_μ -module. Thus, $M_\mu(w_b \hat{\zeta})$ is an integrable \mathfrak{p}_μ -module.

We denote by $M(\eta)$ the Verma \mathfrak{g}' -module with highest weight $\eta - \rho$. Then, by the Weyl character formula, we have

$$M_\mu(w_b \hat{\zeta}) = \sum_{u \in W_b} (-1)^{\ell(u)} M(uw_b \hat{\zeta}).$$

We consider the following maximal parabolic subalgebra of \mathfrak{g}' .

$$\mathfrak{g}'_0 = \mathfrak{sl}_r(\mathbb{C}[t]) \oplus \mathbb{C}c \subseteq \mathfrak{g}'.$$

We define

$$M_0(w_b \hat{\zeta}) = U(\mathfrak{g}') \otimes_{U(\mathfrak{g}'_0)} L(w_b \hat{\zeta})$$

where $L(w_b \hat{\zeta})$ is the irreducible highest weight \mathfrak{g}'_0 -module whose highest weight is $w_b \hat{\zeta} - \rho$. Now, observe that $\langle \tilde{a}, \alpha_i^\vee \rangle \leq 0$, for $1 \leq i \leq r-1$, and we have

$$-uw_b \zeta = -uw_b \tilde{x}(\zeta)^{-1} \tilde{a}$$

⁷We drop “modulo $\mathbb{Z}(\epsilon_1 + \dots + \epsilon_r)$ ” by abuse of notation.

such that $w_{\tilde{a}}\tilde{x}(\zeta)w_bu^{-1}$ is the maximal length coset representative of $W_{\tilde{a}}\tilde{x}(\zeta)w_bu^{-1}$. Thus, by the Kazhdan-Lusztig conjecture for $\mathfrak{sl}_r(\mathbb{C})$,⁸

$$M(uw_b\hat{\zeta}) = \sum_{\gamma \in \tilde{a}S_r} P_{w_{\tilde{a}}\tilde{x}(\zeta)w_bu^{-1}, w_{\tilde{a}}\tilde{x}(\gamma)w_b}(1)M_0(w_b\hat{\gamma}),$$

for $u \in W_b$, where $\zeta \in \tilde{a}S_r$. This implies that

$$\begin{aligned} M_\mu(w_b\hat{\zeta}) &= \sum_{u \in W_b} (-1)^{\ell(u)} M(uw_b\hat{\zeta}) \\ &= \sum_{\gamma \in \tilde{a}S_r} P_{w_{\tilde{a}}\tilde{x}(\zeta)w_b, w_{\tilde{a}}\tilde{x}(\gamma)w_b}^{J,-1}(1)M_0(w_b\hat{\gamma}). \end{aligned}$$

By the integrality as a \mathfrak{p}_μ -module, we have

$$M_\mu(w_b\hat{\zeta}) = \sum_{\gamma \in \tilde{a}S_r \cap P_b^{++}} P_{w_{\tilde{a}}\tilde{x}(\zeta)w_b, w_{\tilde{a}}\tilde{x}(\gamma)w_b}^{J,-1}(1)M_0(w_b\hat{\gamma}).$$

Note also that $\hat{a} = -\sum_{i=1}^{r-1} (a_i - a_{i+1})(\Lambda_i - \Lambda_0) + e\Lambda_0$ satisfies

$$\langle \hat{a}, \alpha_i^\vee \rangle = \begin{cases} a_{i+1} - a_i \geq 0 & (1 \leq i \leq r-1) \\ e + a_1 - a_r \geq 1 > 0 & (i = 0) \end{cases}$$

and we have

$$uw_b\hat{\zeta} = uw_bx(\zeta)^{-1}\hat{a}$$

such that $w_ax(\zeta)w_bu^{-1}$ is the maximal length coset representative of $W_ax(\zeta)w_bu^{-1}$, for $u \in W_b$. Thus, by the Kazhdan-Lusztig conjecture again, this time for \mathfrak{g} ,

$$M(uw_b\hat{\zeta}) = \sum_{\xi \in aW} P_{w_ax(\zeta)w_bu^{-1}, w_ax(\xi)w_b}(1)L(w_b\hat{\xi}),$$

for $u \in W_b$. This implies that

$$\begin{aligned} M_\mu(w_b\hat{\zeta}) &= \sum_{u \in W_b} (-1)^{\ell(u)} M(uw_b\hat{\zeta}) \\ &= \sum_{\xi \in aW} P_{w_ax(\zeta)w_b, w_ax(\xi)w_b}^{J,-1}(1)L(w_b\hat{\xi}). \end{aligned}$$

By the integrality as a \mathfrak{p}_μ -module again, we obtain

$$M_\mu(w_b\hat{\zeta}) = \sum_{\xi \in aW \cap P_b^{++}} P_{w_ax(\zeta)w_b, w_ax(\xi)w_b}^{J,-1}(1)L(w_b\hat{\xi}).$$

Therefore, if we write

$$M_0(w_b\hat{\gamma}) = \sum_{\xi \in aW \cap P_b^{++}} d_{\gamma\xi} L(w_b\hat{\xi}),$$

for $d_{\gamma\xi} \in \mathbb{N}$, in other words $[M_0(w_b\hat{\gamma}) : L(w_b\hat{\xi})] = d_{\gamma\xi}$, we have

$$P_{w_ax(\zeta)w_b, w_ax(\xi)w_b}^{J,-1}(1) = \sum_{\gamma \in \tilde{a}S_r \cap P_b^{++}} P_{w_{\tilde{a}}\tilde{x}(\zeta)w_b, w_{\tilde{a}}\tilde{x}(\gamma)w_b}^{J,-1}(1)d_{\gamma\xi}.$$

Hence, $d_{\gamma\xi}(1)$ is understood as the multiplicity $[M_0(w_b\hat{\gamma}) : L(w_b\hat{\xi})]$. It would be desirable to understand $d_{\gamma\xi}(v)$ in terms of Jantzen filtration. In the case when W_b

⁸See [23, Thm. 1.1(i)]. We derive the formula first in the trivial central character case, and then apply the translation functor and use [23, Prop. 3.6, Prop. 3.8].

is trivial, we expect that the Verma module is rigid and Jantzen conjecture holds.

Acknowledgements. The first author is partly supported by the Grant-in-Aid for Scientific Research (B) (No. 20340004), Japan Society for the Promotion of Science. The second author is supported by "Agence nationale de la recherche" ANR JC-07-1923-39. The third author is supported by "Agence Nationale de la Recherche" ANR-09-JCJC-0102-01.

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